数值差商公式研究*

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摘　要　分别针对低度光滑函数和充分光滑函数, 给出其数值差商公式的余项估计, 然后推导出若干超收敛的数值差商公式并给出其余项的Lagrange表示.

关键词　数值差商公式　余项估计　超收敛

当节点$x_0, \cdots, x_k$的间距很小时, 一个函数f在其上的差商$f[x_0, \cdots, x_k]$是很难直接利用f的函数值进行计算的. 这时以f的Hermite插值多项式的差商来近似代替f的差商便是一个有效的途径. 插值多项式中虽然也有在另一组节点$a_0, \cdots, a_n$上的差商计算问题，但我们可以把这组节点的间距取得足够大, 以便使计算能顺利进行. 这是数值微分概念和技术的自然拓广, 下列公式不妨称之为数值差商公式:

$$f[x_0, \cdots, x_k] = \sum_{n=k}^{n} f[a_0, \cdots, a_\nu] \cdot \omega_\nu[x_0, \cdots, x_k] + R[x_0, \cdots, x_k], \tag{0.1}$$

其中

$$\omega_\nu(x) := \prod_{i=0}^{\nu-1}(x - a_i), \quad \nu = 0, 1, \cdots, n+1, \tag{0.2}$$

而

$$R(x) := f(x) - \sum_{\nu=0}^{n} f[a_0, \cdots, a_\nu]\omega_\nu(x) \tag{0.3}$$

是插值余项.

关于插值余项的差商, 即数值差商公式(0.1)的余项, 我们[1]有下列命题.

命题1　设$x_0, \cdots, x_k$和$a_0, \cdots, a_n$同是$\mathbb{R}$或$\mathbb{C}$中的数列, $k \le n$. 则对满足$0 \le m \le k$的任

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意整数m有

$$\begin{aligned} R[x_0,\cdots,x_k] &= \sum_{\nu=0}^{m-1} f[x_0,\cdots,x_\nu,a_0,\cdots,a_n]\cdot\omega_{n+1}[x_\nu,\cdots,x_k] \\ &+ \sum_{\nu=m}^{k} f[x_0,\cdots,x_\nu,a_0,\cdots,a_{n+m-\nu}](x_\nu-a_{n+m-\nu})\omega_{n+m-\nu}[x_\nu,\cdots,x_k]; \end{aligned} \tag{0.4}$$

又对满足$k\le m\le n$的任意整数m有

$$\begin{aligned} R[x_0,\cdots,x_k] &= \sum_{\nu=0}^{k} f[x_0,\cdots,x_\nu,a_0,\cdots,a_{m-\nu}](x_\nu-a_{m-\nu})\omega_{m-\nu}[x_\nu,\cdots,x_k] \\ &- \sum_{\nu=m+1}^{n} f[a_0,\cdots,a_\nu]\cdot\omega_\nu[x_0,\cdots,x_k]. \end{aligned} \tag{0.5}$$

对于(0.4)置$m=0$, 则给出de Boor[2]的公式. 此公式中, 若$a_0=\cdots=a_k=x$, 则又给出Floater[3]的公式. 公式(0.4)和(0.5)中若$x_0=\cdots=x_k=x$, 则分别给出王兴华[4]和王兴华和杨义群[5]的公式, 其中当$m=0$的(0.4)也是Dokken和Lyche[6]的公式.

本文将根据数值计算的实际需要，利用这个命题分别针对低度光滑函数和充分光滑函数, 给出数值差商公式余项的估计, 并且推导出若干超收敛的数值差商公式及其余项的Lagrange表示.

下文中我们将用到关于Newton插值基函数的差商的下列等式（参见文献[1, 2]）:

$$\omega_{n+1}[x_0,\cdots,x_k]=(x_0-a_n)\omega_n[x_0,\cdots,x_k]+\omega_n[x_1,\cdots,x_k], \tag{0.6}$$

$$\sum_{\nu=m}^{k}(x_\nu-a_{n+m-\nu})\omega_{n+m-\nu}[x_\nu,\cdots,x_k]=\omega_{n+1}[x_m,\cdots,x_k] \quad (0\le m\le k). \tag{0.7}$$

1　低度光滑情形的余项估计

余项的阶随函数光滑程度变化的情形通常称为低度光滑的情形. 关于低度光滑情形的余项估计, 我们有下列定理.

定理1　设$a_0,\cdots,a_n,x_0,\cdots,x_k\in[a,b], 0\le k\le n$. 对任意适合$k\le m\le n$的整数$m$, 如果$f\in C^m[a,b]$, 则

$$\big|R[x_0,\cdots,x_k]\big|\le\frac{2^{n-m+1}n^k}{k!m!}\cdot\frac{h^{n-k}}{\prod_{j=m+1}^{n}h_j}\omega(f^{(m)},h_{m+1}), \tag{1.1}$$

式中$h:=b-a$, $h_j:=\min\{\bar a_{i+j}-\bar a_i: 0\le i,i+j\le n\}$, 而$\bar a_0\le\cdots\le\bar a_n$是$a_0\cdots,a_n$的重排, $\omega(f^{(m)},\cdot)$是$f^{(m)}$的连续模.

证 于(0.5)置$m=k-1$, 该式是

$$\begin{aligned} R[x_0,\cdots,x_k] &= f[x_0,\cdots,x_k](x_k-a_{-1})\omega_{-1}[x_k] \\ &- \sum_{\nu=k}^{n} f[a_0,\cdots,a_\nu]\cdot\omega_\nu[x_0,\cdots,x_k]. \end{aligned} \tag{1.2}$$

若命

$$(x_k-a_{-1})\omega_{-1}[x_k]=\omega_0(x_k)=1, \tag{1.3}$$

则(1.2)不仅有意义，而且显然是正确的. 因此我们可以在命题1的式(0.5)中用$m-1$代替m, 得到

$$\begin{aligned} &R[x_0,\cdots,x_k] \\ &=\sum_{\nu=0}^{k}\big\{f[x_0,\cdots,x_\nu,a_0,\cdots,a_{m-\nu-1}]-f[a_0,\cdots,a_m]\big\}(x_\nu-a_{m-\nu-1})\cdot\omega_{m-\nu-1}[x_\nu,\cdots,x_k] \\ &-\sum_{\nu=m+1}^{n} f[a_0,\cdots,a_\nu]\cdot\omega_\nu[x_0,\cdots,x_k]. \end{aligned} \tag{1.4}$$

此式对适合$k\le m\le n$的m是正确的. 如同文献[5]所指出的, 对$0\le\nu\le k$有

$$\begin{aligned} &\big|f[x_0,\cdots,x_\nu,a_0,\cdots,a_{m-\nu-1}]-f[a_0,\cdots,a_m]\big| \\ &\le\frac{1}{m!}\omega(f^{(m)},h)\le\frac{2}{m!}\cdot\frac{h}{h_{m+1}}\omega(f^{(m)},h_{m+1}), \end{aligned}$$

而对$m+1\le\nu\le n$有

$$\big|f[a_0,\cdots,a_\nu]\big|\le\frac{2^{\nu-m}}{m!}\cdot\frac{\omega(f^{(m)},h_{m+1})}{\prod\limits_{j=m+1}^{\nu}h_j}.$$

此外, 显然存在$\eta\in[a,b]$使

$$\begin{aligned} |\omega_\nu[x_0,\cdots,x_k]| &=\left|\frac{1}{k!}\omega_\nu^{(k)}(\eta)\right|=\left|\sum_{0\le i_1<\cdots<i_k\le\nu-1}\ \prod_{i\ne i_1,\cdots,i_k}(\eta-a_i)\right| \\ &\le\sum_{0\le i_1<\cdots<i_k\le\nu-1}\ \prod_{i\ne i_1,\cdots,i_k}h=\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}h^k}h^\nu=\binom{\nu}{k}h^{\nu-k}, \end{aligned}$$

同样地

$$\begin{aligned} &\left|\sum_{\nu=0}^{k}(x_\nu-a_{m-\nu-1})\omega_{m-\nu-1}[x_\nu,\cdots,x_k]\right|\le\sum_{\nu=0}^{k}h\left|\frac{1}{(k-\nu)!}\omega_{m-\nu-1}^{(k-\nu)}(\eta_\nu)\right| \\ &\le\sum_{\nu=0}^{k}h\binom{m-\nu-1}{k-\nu}h^{m-k-1}=\binom{m}{k}h^{m-k}. \end{aligned}$$

所以由(1.4)得到

$$\begin{aligned}|R[x_0,\cdots,x_k]| &\le \frac{2}{m!}\frac{\omega(f^{(m)},h_{m+1})h}{h_{m+1}}\cdot\binom{m}{k}h^{m-k}\\ &+\sum_{\nu=m+1}^{n}\frac{2^{\nu-m}}{m!}\frac{\omega(f^{(m)},h_{m+1})}{\prod\limits_{j=m+1}^{\nu}h_j}\cdot\binom{\nu}{k}h^{\nu-k}.\end{aligned}$$

对$m+1\le\nu\le n$,

$$\left.\begin{array}{r}\begin{pmatrix} m\\ k\end{pmatrix}\dfrac{h}{h_{m+1}}\\ \\ \begin{pmatrix}\nu\\ k\end{pmatrix}\dfrac{h^{\nu-m}}{\prod\limits_{j=m+1}^{\nu}h_j}\end{array}\right\}\le\frac{n^k}{k!}\frac{h^{n-m}}{\prod\limits_{j=m+1}^{n}h_j}.$$

所以

$$\begin{aligned}|R[x_0,\cdots,x_k]| &\le \left(2+\sum_{\nu=m+1}^{n}2^{\nu-m}\right)\frac{n^k}{k!m!}\cdot\frac{h^{n-k}}{\prod\limits_{j=m+1}^{n}h_j}\omega(f^{(m)},h_{m+1})\\ &= \frac{2^{n-m+1}n^k}{k!m!}\frac{h^{n-k}}{\prod\limits_{j=m+1}^{n}h_j}\omega(f^{(m)},h_{m+1}).\end{aligned}$$

定理证毕.

于定理1中置$x_0=\cdots=x_k=x$, 我们得到低度光滑情形数值微商公式的余项估计的结果（见文献[5]）, 这个结果回答了胡祖炽教授在中国数学会计算数学学会第一届年会（1979, 广州）的大会报告中提出的问题.

2 充分光滑情形的余项估计

根据定理1, 如果$f\in C^n[a,b]$且$f^{(n)}$满足Lipschitz条件, 则对于比较均匀的节点$a_0,\cdots,a_n$有

$$R[x_0,\cdots,x_k]=O(h^{n+1-k}). \tag{2.1}$$

一般来说, 这已是余项的最高阶, 即饱和阶. 因为我们有下列定理.

定理2 设$a_0,\cdots,a_n,x_0,\cdots,x_k\in[a,b]$, $0\le k\le n$. 对任意适合$0\le m\le k$的整数m, 如果$f\in C^{n+m+1}[a,b]$, 则有

$$R[x_0,\cdots,x_k]=\sum_{\nu=0}^{m-1}f[x_0,\cdots,x_\nu,a_0,\cdots,a_n]\cdot\omega_{n+1}[x_\nu,\cdots,x_k]+O(h^{n+m+1-k}), \tag{2.2}$$

这里$h=b-a$. 详言之

$$|O(h^{n+m+1-k})| \leq \binom{n+1}{k-m}\frac{\|f^{(n+m+1)}\|_\infty}{(n+m+1)!}h^{n+m+1-k}. \tag{2.3}$$

证 用命题1的式(0.4)来证. 我们有

$$f[x_0,\cdots,x_\nu,a_0,\cdots,a_{n+m-\nu}] = \frac{f^{(n+m+1)}(\xi_\nu)}{(n+m+1)!} \quad (\xi_\nu \in [a,b]),$$

$$|\omega_{n+m-\nu}[x_\nu,\cdots,x_k]| \leq \binom{n+m-\nu}{n+m-k}h^{n+m-k}.$$

所以

$$\begin{aligned}&\left|\sum_{\nu=m}^{k} f[x_0,\cdots,x_\nu,a_0,\cdots,a_{n+m-\nu}](x_\nu-a_{n+m-\nu})\omega_{n+m-\nu}[x_\nu,\cdots,x_k]\right| \\ &\leq \frac{\|f^{(n+m+1)}\|_\infty}{(n+m+1)!}h^{n+m-k+1}\sum_{\nu=m}^{k}\binom{n+m-\nu}{n+m-k}.\end{aligned}$$

但

$$\sum_{\nu=m}^{k}\binom{n+m-\nu}{n+m-k} = \binom{n+1}{n+m-k+1} = \binom{n+1}{k-m}.$$

所以定理的结论成立.

所证定理表明, 在

$$f[x_0,a_0,\cdots,a_n]\neq 0, \quad \omega_{n+1}[x_0,\cdots,x_k]\neq 0 \tag{2.4}$$

的一般情形下, 存在常数$C>0$使

$$R[x_0,\cdots,x_k] \sim Ch^{n+1-k}. \tag{2.5}$$

所以如果对某个$m>0$有

$$\sum_{\nu=0}^{m-1} f[x_0,\cdots,x_\nu,a_0,\cdots,a_n]\cdot\omega_{n+1}[x_\nu,\cdots,x_k]=0, \tag{2.6}$$

则对充分光滑的f, 有

$$R[x_0,\cdots,x_k] = O(h^{n+m+1-k}). \tag{2.7}$$

这种情形谓之“超收敛”.

3 超收敛的公式及其余项的Lagrange表示

下面的定理给出很广泛的一类超收敛的数值差商公式.

定理3 设$x_0,\cdots,x_k$和$a_0,\cdots,a_n$是关于一个共同的对称中心c对称分布的两组节点, 并且$n-k$是一个非负偶数, 则

$$f[x_0,\cdots,x_k]=H_n[x_0,\cdots,x_k]+R[x_0,\cdots,x_k] \tag{3.1}$$

是一个超收敛的数值差商公式:

$$R[x_0,\cdots,x_k]=O(h^{n+2-k}), \tag{3.2}$$

这里H_n是f以$a_0,\cdots,a_n$为节点的n次Hermite插值多项式. 又设

$$\min\{|a_i-c|:a_i\neq c,0\leq i\leq n\}\geq\max\{|x_i-c|:x_i\neq c,0\leq i\leq k\}, \tag{3.3}$$

且$f\in C^{n+2}[a,b]$, 这里$a_0,\cdots,a_n\in[a,b]$且$k=1,2,3,4$, 则存在$\xi\in[a,b]$使

$$R[x_0,\cdots,x_k]=\frac{f^{(n+2)}(\xi)}{(n+2)!}\omega_{n+1}[\tilde{x}_0,\cdots,\tilde{x}_{k-1}], \tag{3.4}$$

这里$\tilde{x}_0,\cdots,\tilde{x}_{k-1}$是$x_0,\cdots,x_k$中的任意$k$个点.

证 不妨设两组节点均无重节点. 根据定理条件易证

$$\omega_{n+1}[x_0,\cdots,x_k]=\sum_{i=0}^{k}\frac{\omega_{n+1}(x_i)}{\prod\limits_{j\neq i}(x_i-x_j)}=0. \tag{3.5}$$

所以

$$\begin{aligned}R[x_0,\cdots,x_k]&=\sum_{\nu=1}^{k}f[x_0,\cdots,x_\nu,a_0,\cdots,a_{n+1-\nu}]\cdot(x_\nu-a_{n+1-\nu})\omega_{n+1-\nu}[x_\nu,\cdots,x_k]\\&=O(h^{n+2-k}).\end{aligned}$$

再证(3.4). 首先由(3.5)可知, $\omega_{n+1}[\tilde{x}_0,\cdots,\tilde{x}_{k-1}]$与$\tilde{x}_0,\cdots,\tilde{x}_{k-1}$从$\{x_0,\cdots,x_k\}$选取的取法无关. 我们现以$k=3,4$为例来证(3.4). 对$k=3$, 这时$n$为奇数. 设

$$a_1<a_3<\cdots<a_n<x_0<x_2<c<x_3<x_1<a_{n-1}<\cdots<a_2<a_0.$$

这样, 我们有

$$(x_3-a_{n-2})\omega_{n-2}=\omega_{n-1}(x_3),$$

$$(x_2-a_{n-1})\omega_{n-1}[x_2,x_3]=0.$$

又由式(0.6),

$$\begin{aligned}(x_1-a_n)\omega_n[x_1,x_2,x_3] &= (x_1-a_n)\{(x_1-a_{n-1})\omega_{n-1}[x_1,x_2,x_3]+\omega_{n-1}[x_2,x_3]\}\\ &= (x_1-a_n)(x_1-a_{n-1})\frac{\omega_{n-1}[x_1,x_3]-\omega_{n-1}[x_2,x_3]}{x_1-x_2}\\ &= \frac{(x_1-a_n)(a_{n-1}-x_1)}{(x_1-x_2)(x_1-x_3)}\left\{\omega_{n-1}(x_3)-\omega_{n-1}(x_1)\right\}.\end{aligned}$$

显然

$$|\omega_{n-1}(x_3)|>|\omega_{n-1}(x_1)|,$$

所以$(x_1-a_n)\omega_n[x_1,x_2,x_3]$与$\omega_{n-1}(x_3)$同号. 因此由(0.7),

$$\begin{aligned}R[x_0,x_1,x_2,x_3] &= \sum_{\nu=1}^{3} f[x_0,\cdots,x_\nu,a_0,\cdots,a_{n+1-\nu}]\cdot(x_\nu-a_{n+1-\nu})\omega_{n+1-\nu}[x_\nu,\cdots,x_3]\\ &= \frac{f^{(n+2)}(\xi)}{(n+2)!}\sum_{\nu=1}^{3}(x_\nu-a_{n+1-\nu})\omega_{n+1-\nu}[x_\nu,\cdots,x_3]\\ &= \frac{f^{(n+2)}(\xi)}{(n+2)!}\omega_{n+1}[x_1,x_2,x_3].\end{aligned}$$

对$k=4$, n为偶数, 设

$$a_1<a_3<\cdots<a_{n-1}<x_1<x_3<c=x_0=a_n<x_4<x_2<a_{n-2}<\cdots<a_2<a_0.$$

这时

$$\begin{aligned}(x_\nu-a_{n+1-\nu})\omega_{n+1-\nu}[x_\nu,\cdots,x_4] &= 0\quad(\nu=1,3).\\ (x_4-a_{n-3})\omega_{n-3}(x_4) &= \omega_{n-2}(x_4),\\ (x_2-a_{n-1})\omega_{n-1}[x_2,x_3,x_4] &= (x_2-a_{n-1})\{(x_2-a_{n-2})\omega_{n-2}[x_2,x_3,x_4]+\omega_{n-2}[x_3,x_4]\}\\ &= (x_2-a_{n-1})(x_2-a_{n-2})\frac{\omega_{n-2}[x_2,x_4]-\omega_{n-2}[x_3,x_4]}{x_2-x_3}\\ &= \frac{(x_2-a_{n-1})(a_{n-2}-x_2)}{(x_2-x_3)(x_2-x_4)}\{\omega_{n-2}(x_4)-\omega_{n-2}(x_2)\}.\end{aligned}$$

因为$|\omega_{n-2}(x_4)|>|\omega_{n-2}(x_2)|$, 所以与$\omega_{n-2}(x_4)$同号. 同样得到(3.4). 定理证毕.

猜测 式(3.4)对所有正整数k成立.

支持这个猜测的事实, 除了定理3所指出的$k=1,2,3,4$之外, Brodskii[7]亦曾证明当$x_0=\cdots=x_k=x$时对所有正整数k成立.

引入中心差商的记号

$$\begin{aligned}f^n[x;d] &:= f[x-\tfrac{n}{2}d,x-\tfrac{n-2}{2}d,\cdots,x+\tfrac{n-2}{2}d,x+\tfrac{n}{2}d]\\ &= \frac{1}{n!d^n}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f(x+\tfrac{2i-n}{2}d),\end{aligned}\tag{3.6}$$

我们由定理3得到

推论1 设$f \in C^{n+2}[x-\frac{n}{2}d, x+\frac{n}{2}d]$, $d \geq \varepsilon > 0$. 则对奇数n有

$$\begin{aligned} f^1[x;\varepsilon] &= f^1[x;d] - \sum_{i=2}^{\frac{n+1}{2}} (-1)^i f^{2i-1}[x;d] \cdot 2^{-2i+2} \prod_{j=1}^{i-1} ((2j-1)^2 d^2 - \varepsilon^2) \\ &+ (-1)^{\frac{n+1}{2}} \frac{f^{(n+2)}(\xi)}{(n+2)!} 2^{-n-1} \prod_{j=1}^{\frac{n+1}{2}} ((2j-1)^2 d^2 - \varepsilon^2), \end{aligned} \tag{3.7}$$

对偶数n有

$$\begin{aligned} f^2[x;\varepsilon] &= f^2[x;d] - \sum_{i=2}^{\frac{n}{2}} (-1)^i f^{2i}[x;d] \prod_{j=1}^{i-1} (j^2 d^2 - \varepsilon^2) \\ &+ (-1)^{\frac{n}{2}} \frac{f^{(n+2)}(\xi)}{(n+2)!} \prod_{j=1}^{\frac{n}{2}} (j^2 d^2 - \varepsilon^2). \end{aligned} \tag{3.8}$$

证 先设n为奇数. 把H_n写成在点$x+td$以

$$x+\tfrac{d}{2}, x-\tfrac{d}{2}, x+\tfrac{3}{2}d, x-\tfrac{3}{2}d, \cdots, x+\tfrac{n}{2}d, x-\tfrac{n}{2}d$$

为节点的Gauss向前插值公式的形式:

$$\begin{aligned} &G_n(x+td) = f_{\frac{1}{2}} + f_0^1 \cdot (t-\tfrac{1}{2}) + \frac{f_{\frac{1}{2}}^2}{2!}(t^2-\tfrac{1}{4}) + \frac{f_0^3}{3!}(t^2-\tfrac{1}{4})(t-\tfrac{3}{2}) + \cdots \\ &+\frac{f_{\frac{1}{2}}^{n-1}}{(n-1)!}(t^2-\tfrac{1}{4})(t^2-\tfrac{9}{4})\cdots(t^2-(\tfrac{n-2}{2})^2) + \frac{f_0^n}{n!}(t^2-\tfrac{1}{4})(t^2-\tfrac{9}{4})\cdots(t^2-(\tfrac{n-2}{2})^2)(t-\tfrac{n}{2}), \end{aligned}$$

式中

$$f_{i+\frac{1}{2}}^0 := f(x+(i+\tfrac{1}{2})d), \quad f_i^m := f_{i+\frac{1}{2}}^{m-1} - f_{i-\frac{1}{2}}^{m-1}.$$

此时

$$\omega_{n+1}(x+td) = (t^2-\tfrac{1}{4})(t^2-\tfrac{9}{4})\cdots(t^2-\tfrac{n^2}{4})d^{n+1}.$$

由是我们得到

$$\begin{aligned} G_n^1[x;\varepsilon] &= \frac{f_0^1}{d} + \frac{f_0^3}{3!d^3}\tfrac{\varepsilon^2-d^2}{4} + \cdots + \frac{f_0^n}{n!d^n}\tfrac{(\varepsilon^2-d^2)(\varepsilon^2-9d^2)\cdots(\varepsilon^2-(n-2)^2d^2)}{2^{n-1}} \\ &= f^1[x;d] - f^3[x;d]\tfrac{d^2-\varepsilon^2}{4} + \cdots + (-1)^{\frac{n-1}{2}} f^n[x;d]\tfrac{(d^2-\varepsilon^2)(9d^2-\varepsilon^2)\cdots((n-2)^2d^2-\varepsilon^2)}{2^{n-1}}, \\ \omega_{n+1}(x+\tfrac{\varepsilon}{2}) &= (-1)^{\frac{n+1}{2}} \tfrac{(d^2-\varepsilon^2)(9d^2-\varepsilon^2)\cdots(n^2d^2-\varepsilon^2)}{2^{n+1}}. \end{aligned}$$

这就得到(3.7).

又设n为偶数. 把H_n写成在点$x+td$以

$$x, x+d, x-d, \cdots, x+\tfrac{n}{2}d, x-\tfrac{n}{2}d$$

的节点的Gauss向前插值公式的形式:

$$
\begin{aligned}
G_n(x+td) &= f_0 + f_{\frac{1}{2}}^1 t + \frac{f_0^2}{2!}t(t-1) + \frac{f_{\frac{1}{2}}^3}{3!}t(t^2-1) + \frac{f_0^4}{4!}t(t^2-1)(t-2) + \cdots \\
&+ \frac{f_0^n}{n!}t(t^2-1)(t^2-4)\cdots(t^2-(\tfrac{n}{2}-1)^2)(t-\tfrac{n}{2}),
\end{aligned}
$$

式中

$$
f_i^0 := f(x+id), \quad f_{i+\frac{1}{2}}^m := f_{i+1}^{m-1} - f_i^{m-1}.
$$

此时

$$
\omega_{n+1}(x+td) = t(t^2-1)(t^2-4)\cdots(t^2-\tfrac{n^2}{4})d^{n+1}.
$$

由是我们得到

$$
\begin{aligned}
G^2[x;\varepsilon] &= \frac{f_0^2}{2!d^2} + \frac{f_0^4}{4!d^4}(\varepsilon^2-d^2) + \cdots + \frac{f_0^n}{n!d^n}(\varepsilon^2-d^2)(\varepsilon^2-4d^2)\cdots(\varepsilon^2-(\tfrac{n}{2}-1)^2d^2) \\
&= f^2[x;d] + \cdots + (-1)^{\frac{n}{2}-1}f^n[x;d](d^2-\varepsilon^2)(4d^2-\varepsilon^2)\cdots((\tfrac{n}{2}-1)^2d^2-\varepsilon^2), \\
\omega_{n+1}[x,x+\varepsilon] &= (-1)^{\frac{n}{2}}(d^2-\varepsilon^2)(4d^2-\varepsilon^2)\cdots(\tfrac{n^2}{4}d^2-\varepsilon^2).
\end{aligned}
$$

这就得到(3.8). 证毕.

推论2 设$f \in C^{n+2}[x-\frac{n}{2}, x+\frac{n}{2}d]$, $d \geq 3\varepsilon > 0$. 则对奇数$n \geq 3$有

$$
\begin{aligned}
f^3[x;\varepsilon] &= f^3[x;d] \\
&+ \sum_{i=3}^{\frac{n+1}{2}}(-1)^i f^{2i-1}[x;d]\cdot 2^{-2i+4}\sum_{l=1}^{i-1}\prod_{j=1}^{l-1}((2j-1)^2d^2-9\varepsilon^2)\prod_{j=l+1}^{i-1}((2j-1)^2d^2-\varepsilon^2) \\
&+ (-1)^{\frac{n+3}{2}}\frac{f^{(n+2)}(\xi)}{(n+2)!}2^{-n+1}\sum_{l=1}^{\frac{n+1}{2}}\prod_{j=1}^{l-1}((2j-1)^2d^2-9\varepsilon^2)\prod_{j=l+1}^{\frac{n+1}{2}}((2j-1)^2d^2-\varepsilon^2),
\end{aligned}
\tag{3.9}
$$

对偶数$n \geq 4$有

$$
\begin{aligned}
f^4[x;\varepsilon] &= f^4[x;d] \\
&+ \sum_{i=3}^{\frac{n}{2}}(-1)^i f^{2i}[x;d]\cdot\sum_{l=1}^{i-1}\prod_{j=1}^{l-1}(j^2d^2-4\varepsilon^2)\prod_{j=l+1}^{i-1}(j^2d^2-\varepsilon^2) \\
&+ (-1)^{\frac{n+2}{2}}\frac{f^{(n+2)}(\xi)}{(n+2)!}\sum_{l=1}^{\frac{n}{2}}\prod_{j=1}^{l-1}(j^2d^2-4\varepsilon^2)\prod_{j=l+1}^{\frac{n}{2}}(j^2d^2-\varepsilon^2).
\end{aligned}
\tag{3.10}
$$

证 设n为奇数, 我们有

$$\begin{aligned}G_n^3[x;\varepsilon] &= \frac{1}{2\varepsilon^2}\{G_n^1[x;3\varepsilon]-G_n^1[x;\varepsilon]\}\\ &= f^3[x;d]-f^5[x;d]\tfrac{5(d^2-\varepsilon^2)}{2}+f^7[x;d]\tfrac{7(d^2-\varepsilon^2)(37d^2-13\varepsilon^2)}{16}\\ &- f^9[x;d]\tfrac{(d^2-\varepsilon^2)(3229d^4-1706d^2\varepsilon^2+205\varepsilon^4)}{16}\\ &+ f^{11}[x;d]\tfrac{11(d^2-\varepsilon^2)(9d^2-\varepsilon^2)(181d^2-61\varepsilon^2)(59d^2-11\varepsilon^2)}{256}+\cdots\\ &+ (-1)^{\frac{n+1}{2}}f^n[x;d]\frac{1}{2^{n-3}}\sum_{l=1}^{\frac{n-1}{2}}\prod_{j=1}^{l-1}((2j-1)^2d^2-9\varepsilon^2)\prod_{j=l+1}^{\frac{n-1}{2}}((2j-1)^2d^2-\varepsilon^2),\end{aligned}$$

$$\begin{aligned}&\omega_{n+1}[x+\tfrac{\varepsilon}{2},x-\tfrac{\varepsilon}{2},x+\tfrac{3\varepsilon}{2}]=\tfrac{1}{2\varepsilon}\omega_{n+1}[x+\tfrac{\varepsilon}{2},x+\tfrac{3\varepsilon}{2}]\\ &=(-1)^{\frac{n+1}{2}}\cdot\frac{1}{2^{n+2}\varepsilon^2}\left\{(d^2-9\varepsilon^2)(9d^2-9\epsilon^2)\cdots(n^2d^2-9\epsilon^2)-(d^2-\varepsilon^2)(9d^2-\varepsilon^2)\cdots(n^2d^2-\varepsilon^2)\right\}\\ &=(-1)^{\frac{n+3}{2}}\frac{1}{2^{n-1}}\sum_{l=1}^{\frac{n+1}{2}}\prod_{j=1}^{l-1}((2j-1)^2d^2-9\varepsilon^2)\prod_{j=l+1}^{\frac{n+1}{2}}((2j-1)^2d^2-\varepsilon^2).\end{aligned}$$

这样就得到(3.9). 又设n为偶数, 我们有

$$\begin{aligned}G_n^4[x;\varepsilon] &= \frac{1}{3\varepsilon^2}\{G_n^2[x;2\varepsilon]-G_n^2[x;\varepsilon]\}\\ &= f^4[x;d]-f^6[x;d]\cdot5(d^2-\varepsilon^2)+f^8[x;d]\cdot7(d^2-\varepsilon^2)(7d^2-3\varepsilon^2)\\ &- f^{10}[x;d]\cdot5(d^2-\varepsilon^2)(4d^2-\varepsilon^2)(41d^2-17\varepsilon^2)\\ &+ f^{12}[x;d]\cdot11(d^2-\varepsilon^2)(4d^2-\varepsilon^2)(479d^4-270d^2\varepsilon^2+31\varepsilon^4)+\cdots\\ &+ (-1)^{\frac{n}{2}}f^n[x;d]\cdot\sum_{l=1}^{\frac{n}{2}-1}\prod_{j=1}^{l-1}(j^2d^2-4\varepsilon^2)\prod_{j=l+1}^{\frac{n}{2}-1}(j^2d^2-\varepsilon^2),\end{aligned}$$

$$\begin{aligned}&\omega_{n+1}[x,x+\varepsilon,x-\varepsilon,x+2\varepsilon]=\frac{1}{3\varepsilon}\omega_{n+1}[x,x+\varepsilon,x+2\varepsilon]\\ &=\frac{1}{3\varepsilon^2}\{\omega_{n+1}[x,x+2\varepsilon]-\omega_{n+1}[x,x+\varepsilon]\}\\ &=(-1)^{\frac{n}{2}}\frac{1}{3\varepsilon^2}\{(d^2-4\varepsilon^2)(4d^2-4\varepsilon^2)\cdots(\tfrac{n^2}{4}d^2-4\varepsilon^2)-(d^2-\varepsilon^2)(4d^2-\varepsilon^2)\cdots(\tfrac{n^2}{4}d^2-\varepsilon^2)\}\\ &=(-1)^{\frac{n+2}{2}}\sum_{l=1}^{\frac{n}{2}}\prod_{j=1}^{l-1}(j^2d^2-4\varepsilon^2)\prod_{j=l+1}^{\frac{n}{2}}(j^2d^2-\varepsilon^2).\end{aligned}$$

这样就得到(3.10). 证毕.

于推论1和推论2中置$\varepsilon=0$, 我们即得超收敛的数值微分公式. 在由推论2得到的数值微分公式中置$n=3,4$, 所得结果与Floater[3]得到的公式是一致的.

数值微分公式在迭代法[8]和样条分析[9]中是很有用的. 文献[4]和[5]的结果也曾被推广至多元情形[10,11]. 数值差商公式也有类似的推广[1], 其应用则有待进一步研究.

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