Bivariate Splines Applied to Variational Model for Image Processing

by

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(Under the direction of Professor Ming-jun Lai)

Abstract

In this work, we use bivariate splines to find the approximation of the solutions to three variational models, the ROF model, TV-L^p model and the Chan-Vese Active Contour model. We start by showing first variational models have solutions in the spline space, and the solutions are unique and stable. And then we go on to prove that the solutions in the spline space approximate the solution in the Sobolev space or the BV space, according to the shape of the domain. Finally, we study the discretization of the Chan-Vese Active Contour Model in the level set setting. Numerical examples in image processing based on finite difference are included.

Index words: Bivariate Splines, Variational Model, ROF model, TV-L^p model, Level Set, Active Contour, Image Processing.

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Chapter 1

Introduction

Many problems in computer vision, e.g., image processing can be formulated as minimization problems. Among them an influential one is the ROF model proposed by Rudin, Osher and Fatemi in [44], which solves the following minimization problem

$$\min_{u\in BV(\Omega)} \int_\Omega |\nabla u| dx + \frac{1}{\lambda}\int_\Omega |u-f|^2 dx.$$

The ROF model has been studied extensively. See [44], [1], [12], [5], [10], [23], and many references in [47]. Many numerical methods have be proposed to to find the numerical approximation. Wang and Lucier have shown in [53] that if $\Omega = [0,1]^2$ the unit square, and f is a two dimensional matrix of size $k \times k$, the solution u, a $k \times k$ matrix, to the following minimization problem.

$$E_k(u) = \sum_{i,j=0}^{k-1} h^2 |(\nabla u)_{ij}| + \frac{1}{2\lambda}\sum_{i,j=0}^{k-1} h^2 (u_{ij} - g_{ij})^2, \quad h = \frac{1}{k},$$

converges to the solution of the solution of ROF model in BV space as $h \to 0$. However, their proof can not be easily extended to the case when Ω is a non rectangular polygon. To solve this problem, we propose to use spline function to solve the ROF model. A bivariate spline function is a piece-wise polynomial function defined on a triangulation of a polygonal domain with enough smoothness. The main reason we use splines is due to their capability to approximate functions on complicated regions and their accuracy of evaluation, which is critical in image resizing and inpainting. Our main contribution in this dissertation is we show that the minimizer in a a finite dimensional space, such as bivariate spline space, converges to the minimizer in (1)the space of bounded variation $BV(\Omega)$, when the domain Ω is a rectangle, (2)the Sobolev space $W^{1,1}(\Omega)$, when Ω is a polygon. Moreover, we give an iterative algorithm to compute the bivariate spline approximation and prove the convergence.

For the image denoising problem, we assume the original image $u_0 : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is corrupted by a white noise $\epsilon \sim N(0, \sigma^2)$, such that the corrupted image $f = u_0 + \epsilon$. In statistic settings, by the maximum likelyhood method, the best estimation of σ^2 is $\int_\Omega |u - f|^2$. This gives one reason we use the L^2 term as the fidelity term in the ROF model. However, if we generalize the distribution of the error term ϵ from normal distribution to the whole Laplacian distribution family, then by the maximum likelyhood method, the fidelity term also needs to be changed adapted. That is the reason why we introduce a more general model–the TV-L^p model, in which the fidelity term is a L^p term. Similar to the method we use to study the ROF model, we also use the minimizer in a finite dimensional space, e.x, the spline space, to approximate the minimizer in the BV space. A similar approximation property is deduced. Finally, we give an iterative algorithm to compute the bivariate spline approximation and extablish the convergence.

It is known when the size of the triangulation is small enough, the spline function can approximate a discontinuous function well, the computation time increases simultaneously. Therefore, in some applications, e.g., image denoising, we might want to decompose the image into several regions in each of which the image are sufficiently smooth, instead of finding the spline approximation on the whole image domain. The active contour method proposed in [15] by Tony F. Chan and Luminita A. Vese is another energy-based method which has its application in image segmentation. In this dissertation, we deduce the Euler-Lagrange equation of the Active Contour Model and give a finite difference scheme to find the numerical solution. Several numerical examples of image segmentation and the consequent triangulations based on these segmentations are given at the end of the chapter.

This dissertation is divided into five chapters. In Chapter 2, we review some preliminary knowledge. First we review the properties of the bivariate splines we use in our numerical analysis. Next we give an analysis of the underling reason to use the ROF model for image processing, e.g., image denoising. Then we review some properties of the BV space used in the ROF model, especially some important results concerning the well-posedness of the minimization problem. Finally in this chapter, we review some basic mathematical tools used in this dissertation, such as mollification, subdifferential convex function, and some basic inequalities. In Chapter 3 and Chapter 4, we show our main results of this dissertation: the analysis of the ROF and TV-L^p model in the finite dimensional space. Finally, in Chapter 5 we explain about the Chan-Vese Active Contour method we mentioned above for image segmentation.

Chapter 2

Preliminary

2.1 Bivariate Splines

In this section we outline some basic properties and theories of bivariate splines which will be used in our application to the ROF model and TV-L^p model. We refer to [36] for most spline results presented in this section. Let Ω be a polynomial domain in $\mathbb{R}^2$. Let $\triangle := \{t_1, ..., t_N\}$ be a collection of triangles such that $\Omega = \bigcup_{i=1}^N t_i$ and if a pair of triangles in $\triangle$ intersect, then their intersection is either a common vertex or a common edge. For each t, we write $|t|$ for the length of its longest edge, and ρ_t for the radius of the largest disk that can be inscribed in t. We call the ratio $\kappa_t := \dfrac{|t|}{\rho_t}$ the shape parameter of t, $|\triangle| := \max\{|t|, t \in \triangle\}$ the size of the triangle. Finally, denote $\rho_\triangle := \max\{\rho_t, t \in \triangle\}$.

Definition 2.1.1 *(β-Quasi-Uniform Triangulation). Let $0 < \beta < infty$. A triangulation $\triangle$ is a β-quasi-uniform triangulation provided that*

$$\frac{|\triangle|}{\rho_\triangle} \leq \beta.$$

Definition 2.1.2 *(Spline Space). Fix $r \geq 0$ and $d > r$. Let $C^r(\Omega)$ be the class of all rth* *continuously differentiable functions over* Ω. *We call*

$$S^r_d(\triangle) = \{s \in C^r(\Omega), \quad s|_t \in \mathcal{P}_d, \forall t \in \triangle\}$$

the spline space of degree d *and smoothness* r *over triangulation* $\triangle$, *where* $\mathcal{P}_d$ *is the space of all polynomials of degree* $\leq d$ *and* t *is a triangle in* $\triangle$.

When working with polynomials on triangulations, the barycentric coordinates are more handy than the Cartesian coordinates. Let $t = \langle (x_1, y_1), (x_2, y_2), (x_3, y_3) \rangle = \langle v_1, v_2, v_3 \rangle$ be a non-degenerate triangle. Then any point $v := (x, y)$ has a unique representation of the form $v = b_1 v_1 + b_2 v_2 + b_3 v_3$ with $1 = b_1 + b_2 + b_3$. The number b_1, b_2, b_3 are called barycentric coordinates of the point v with respect to the triangle t. We use the Bernstein-Bézier polynomials to form a basis for polynomials over a given triangle. Therefore we can write any polynomial of degree d over a single triangle uniquely in terms of Bernstein-Bézier polynomials. We call this the B-form of a polynomial.

Definition 2.1.3 *(Bernstein-Bézier Polynomials). A Bernstein-Bézier polynomial of degree* d *is defined by*

$$B^d_{ijk}(x, y) = \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k,$$

where i, j, k *are non-negative integers with* $i + j + k = d$.

Theorem 2.1.1 *The set*

$$\mathcal{B}^d := \{B^d_{ijk}\}_{i+j+k=d}$$

of Bernstein-Bézier polynomials is a basis for the space of polynomials $\mathcal{P}_d$.

Definition 2.1.4 *(B-Form). Let* $s \in \mathcal{P}_d$ *satisfy*

$$s|_t = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(x, y).$$

Next we explain about the conditions on which polynomials over each single triangle can be connected smoothly to form our splines on Ω.

Theorem 2.1.2 *(Smoothness). Let $t = \langle v_1, v_2, v_3\rangle$ and $\tilde{t} = \langle v_4, v_3, v_2\rangle$ be triangles sharing the edge $e := \langle v_2, v_3\rangle$. Let*

$$p(v) = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(v)$$

and

$$\tilde{p}(v) = \sum_{i+j+k=d} \tilde{c}_{ijk} \tilde{B}^d_{ijk}(v)$$

where $\{B^d_{ijk}\}$ and $\{\tilde{B}^d_{ijk}\}$ are Bernstein-Bézier polynomials associated to t and $\tilde{t}$ respectively. Suppose u is any direction not parallel to e. Denote D^n_u the nth directional derivative in u. Then

$$D^n_u p(v) = D^n_u \tilde{p}(v),$$

$v \in e$, $n = 0, ..., r$ if and only if

$$\tilde{c}_{ijk} = \sum_{\nu+\mu+\kappa=n} c_{\nu,k+\mu,j+\kappa} B^n_{\nu\mu\kappa}(v_4)$$

for $j + k = d - n$ and $n = 0, ..., r$.

Let c be the coefficient vector of the B-form of our spline. In practice we can write the smoothness conditions in a linear system $Hc = 0$, where H is a rectangular matrix determined by the conditions imposed in Theorem 2.1.2.

Next we review some calculus facts of spline functions.

Theorem 2.1.3 *(Integration). Let p be a polynomial written in B-form over a triangle t*

with coefficients $c_{ijk}, i+j+k=d$. *Then*

$$\int_t p(x,y)dxdy = \frac{A_t}{\binom{d+2}{2}} \sum_{i+j+k=d} c_{ijk},$$

where A_t *is the area of* t.

Theorem 2.1.4 *(Inner Product).*

$$\int_t B_{ijk}(x,y)B_{\nu\mu\kappa}(x,y)dxdy = A_t \frac{\binom{i+\nu}{i}\binom{j+\mu}{j}\binom{k+\kappa}{k}}{\binom{2d}{d}\binom{2d+2}{2}}.$$

Scattered data fitting using splines by discrete least square and Lagrange multiplier method are a typical application of splines. Given scattered data $\{(x_i, y_i, f(x_i, y_i)), i = 1, ..., N\}$ where N is a relatively large number. Let Ω be the convex hull of the given data location and $\triangle$ a triangulation of Ω. We look for $l_f \in S_d^r(\triangle)$ such that

$$\sum_{i=1}^{N} |l_f(x_i, y_i) - f(x_i, y_i)|^2 = \min_{s \in S_d^r(\triangle)} |l(x_i, y_i) - f(x_i, y_i)|^2 \qquad (2.1)$$

subject to $Hc = 0$.

Here $\mathbf{c}$ is the coefficient vector of the B-form of spline function, and $H\mathbf{c} = 0$ is the smoothness conditions imposed in the Theorem 2.1.2.

If the data locations are evenly distributed over $\triangle$ with respect to d, then for each triangle t, the matrix $B_t := [B_{ijk}^d(v_l)|_t, i+j+k=d, v_l \in t]$ is of full rank. Since there are $\binom{d+2}{2}$ such combinations of triples (i,j,k) subject to $0 \le i,j,k < \infty$ and $i+j+k=d$, the size of B_t is M_t-by-$\binom{d+2}{2}$, where M_t is the number of points v_l's in t. Let $B := diag(B_t, t \in \triangle)$, then any spline function $s \in S_d^r(\Omega)$ can be written as $s = B\mathbf{c}$ for some coefficient vector $\mathbf{c}$. Suppose there are R triangles in $\triangle$, then $B^\top B$ is a full rank square matrix of size $R\binom{d+2}{2}$-by-$R\binom{d+2}{2}$. To solve the constrained discrete least square problem we use the Lagrange multiplier method. Let

$$L(\mathbf{c}) := \sum_{i=1}^{N} |l(x_i - y_i) - f(x_i, y_i)|^2 = \|B\mathbf{c} - \mathbf{f}\|^2$$

where $\mathbf{f}$ is the vector of $f(x_i, y_i)$'s, and

$$G(\mathbf{c}, \alpha) = L(\mathbf{c}, \alpha) = L(\mathbf{c}, \alpha) + \alpha^\top H\mathbf{c}.$$

By the Lagrange multiplier method, we solve

$$\frac{\partial}{\partial \mathbf{c}} G(\mathbf{c}, \alpha) = 2B^\top B\mathbf{c} - 2B'\mathbf{f} + H^\top \alpha = 0$$

$$\frac{\partial}{\partial \alpha} G(\mathbf{c}, \alpha) = H\mathbf{c} = 0.$$

This is equivalent to solve the linear system

$$\begin{pmatrix} H^\top & 2B^\top B \\ 0 & H \end{pmatrix} \begin{pmatrix} \alpha \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} 2B^\top \mathbf{f} \\ 0 \end{pmatrix}.$$

One of the main reasons we want to use splines is because they have a nice property: the optimal approximation order. We denote by $W^{k,p}(\Omega)$ the Sobolev space of locally summable functions $u : \Omega \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \leq k$, $D^\alpha u$ exists in weak sense and belongs to $L^p(\Omega)$. Define $|f|_{\Omega,m,p}$ the L^p norm of the m^{th} derivatives of f over Ω that is,

$$|u|_{\Omega,k,p} := \left(\sum_{\nu+\mu=k} \|D_1^\nu D_2^\mu u\|_{\Omega,p}^p \right)^{1/p}, \quad for\ 1 \leq p < \infty, \tag{2.2}$$

and define $\|f\|_{\Omega,p} = \left(\frac{1}{A_\Omega} \int_\Omega |f(x)|^q dx \right)^{1/p}$.

We first use the so-called Markov inequality to compare the size of the derivative of a polynomial with the size of the polynomial itself on a given triangle t. (See [36] for a proof.)

Theorem 2.1.5 *Let $t := \langle v_1, v_2, v_3 \rangle$ be a triangle, and fix $1 \le q \le \infty$. Then there exists a constant K depending only on d such that for every polynomial $p \in \mathcal{P}_d$, and any nonnegative integers α and β with $0 \le \alpha + \beta \le d$,*

$$\|D_1^\alpha D_2^\beta p\|_{t,q} \le \frac{K}{\rho_t^{\alpha+\beta}} \|p\|_{t,q}, \quad 0 \le \alpha + \beta \le d, \tag{2.3}$$

where ρ_t denotes the radius of the largest circle inscribed in t. Here $\|\cdot\|_{t,p}$ is the L_p norm over t.

Next we have the following approximation property (cf. [35] and [36]):

Theorem 2.1.6 *Assume $d \ge 3r + 2$ and let $\triangle$ be a triangulation of Ω. Then there exists a quasi-interpolatory operator $Qf \in S_d^r(\triangle)$ mapping $f \in L_1(\Omega)$ into $S_d^r(\triangle)$ such that Qf achieves the optimal approximation order: if $f \in W^{m+1,p}(\Omega)$,*

$$\|D_1^\alpha D_2^\beta (Qf - f)\|_{\Omega,p} \le C|\triangle|^{m+1-\alpha-\beta} |f|_{\Omega,m+1,p} \tag{2.4}$$

for all $\alpha + \beta \le m + 1$ with $0 \le m \le d$. Here the constant C depends only on the degree d and the smallest angle $\theta_\triangle$ and may be dependent on the Lipschitz condition on the boundary of Ω.

We suppose that the data locations $\mathcal{Q} = \{x_i, i = 1, \cdots, n\}$ satisfy the conditions(cf. [31]), that for every $s \in S_d^r(\Omega)$ and every triangle $t \in \triangle$, there exist a positive constant F_1, independent of s and t, such that

$$F_1 \|s\|_{t,\infty} \le (\sum_{v \in \mathcal{V} \cap t} s(v)^2)^{1/2}. \tag{2.5}$$

And let F_2 be the largest number of data sites in a triangle $t \in \triangle$. That is, we have

$$\left\{ \sum_{v \in \mathcal{V} \cap t} s(v)^2 \right\}^{1/2} \leq F_2 \|s\|_{t,\infty}. \tag{2.6}$$

Then we have the following approximation property of the least square minimizer l_f of (2.1).

Theorem 2.1.7 *Suppose that $d \geq 3r+2$ and $\triangle$ is a β quasi-uniform triangulation, and there exist two positive constants F_1 and F_2 such that (2.1) and (2.6) are satisfied. Then there exists a constant C depending on d and β such that for every function f in Sobolev space $W^{m+1,\infty}(\Omega)$ with $0 \leq m \leq d$ such that*

$$\|f - l_f\|_{\Omega,\infty} \leq C \frac{F_2}{F_1} |\triangle|^{m+1} |f|_{\Omega,m+1,\infty}.$$

2.2 Energy Models for Image Restoration

Variational and PDE-based approaches have been well studied in image restoration problem. One early model is introduced in 1977 by Tikhonov and Arsenin to find the minimize of the following functional.

$$F(u) = \int_\Omega |u - u_0|^2 dx + \lambda \int_\Omega |\nabla u|^2 dx, \tag{2.7}$$

where ∇ is the standard gradient. The first term in $F(u)$ measures the fidelity to the data. The second is a regularization term. We search for a u that best fits the data so that its gradient is small. The parameter λ is a positive weighting constant. The minimization problem admits a unique solution in the functional space

$$W^{1,2}(\Omega) = \{u \in L^2(\Omega); \nabla u \in L^2(\Omega)\},$$

characterized by the Euler-Lagrange equation

$$u - u_0 - \lambda \triangle u = 0 \tag{2.8}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \ (n \text{ is the outward normal to } \partial\Omega)$$

However this is not a good solution to image restoration problem, because the Laplacian operator has very strong isotropic smoothing properties which annihilate the noise as well as edges as it evolves. In (2.7), the regularization term penalizes too much the gradients corresponding at edges. One may then decrease p in order to preserve the edges as much as possible. That gives one reason why we should discuss the following model, with $1 \leq p < 2$.

$$E(u) = \int_\Omega |u - u_0|^2 dx + \lambda \int_\Omega |\nabla u|^p dx. \tag{2.9}$$

Furthermore, Rudin, Osher, and Fatemi [43, 44] proposed to use the L^1 norm to measure the magnitude of u, that is to find the minimizer in the $W^{1,1}(\Omega)$ space. Let us study the following energy (cf. [6, 7]) for the influence of the smoothing term.

$$E(u) = \frac{1}{2} \int_\Omega |u_0 - u|^2 dx + \lambda \int_\Omega \phi(|\nabla u|) dx, \tag{2.10}$$

which is characterized by the Euler-Lagrange equation:

$$u - u_0 - \lambda \mathrm{div} \left(\frac{\phi'(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0. \tag{2.11}$$

It can be shown that its diffusion term $\mathrm{div}\left(\frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u\right)$ term can be decomposed into a weighted sum of normal and tangent directional second derivative to the contour lines(lines along which the intensity is constant). More precisely, for each point x where $|\nabla u(x)| \neq 0$, its normal direction is characterized by the vector $N(x) = \frac{\nabla u(x)}{|\nabla u(x)|} = \frac{1}{|\nabla u(x)|}(u_x, u_y)$ and tangent direction by $T(x) = \frac{1}{|\nabla u(x)|}(-u_y, u_x)$, $|T(x)| = 1$, $T(x)$ orthogonal to $N(x)$. We denote by u_{TT} and u_{NN} the second derivatives of u in T-direction and N-direction respectively:

$$u_{TT} = T^\top \nabla^2 u T, \quad u_{NN} = N^\top \nabla^2 u N.$$

Here

$$\nabla^2 u = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix}.$$

The projection operator $P_N = NN^\top$, $P_T = TT^\top$. And since $N \cdot T = 0$, we have $P_N + P_T = I$, where I is the identity matrix. It follows that

$$\begin{aligned} u_{NN} + u_{TT} &= T^\top \nabla^2 u T + N^\top \nabla^2 u N = \mathrm{tr}(T^\top \nabla^2 u_T + N^\top \nabla^2 u_N) \\ &= \mathrm{tr}(\nabla^2 u (TT^\top + NN^\top)) = \mathrm{tr}(\nabla^2 u) = \Delta u, \end{aligned}$$

or in another words,

$$u_{NN} + u_{TT} = \mathrm{div}(\nabla u).$$

We can go on to show that the curvature of u

$$\begin{aligned}
\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) &= \nabla\left(\frac{1}{|\nabla u|}\right)\cdot\nabla u + \frac{1}{|\nabla u|}\operatorname{div}(\nabla u) \\
&= -\frac{1}{|\nabla u|^2}\nabla|\nabla u|\cdot\nabla u + \frac{1}{|\nabla u|}(u_{TT}+u_{NN}) \\
&= \frac{1}{|\nabla u|}\left(-\nabla^2 u\frac{\nabla u}{|\nabla u|}\cdot\frac{\nabla u}{|\nabla u|} + u_{TT}+u_{NN}\right) \\
&= \frac{1}{|\nabla u|}(-u_{NN}+u_{TT}+u_{NN}) = \frac{u_{TT}}{|\nabla u|}.
\end{aligned}$$

Now we can rewrite the diffusion term of (2.11) as a weighted sum of u_{NN} and u_{TT}:

$$\begin{aligned}
&\operatorname{div}\left(\frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u\right) \\
=\ &\nabla(\phi'(|\nabla u|))\cdot\frac{\nabla u}{|\nabla u|} + \phi'(|\nabla u|)\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \\
=\ &\phi''(|\nabla u|)\left(\nabla^2 u\frac{\nabla u}{|\nabla u|}\cdot\frac{\nabla u}{|\nabla u|}\right) + \phi'(|\nabla u|)\frac{u_{TT}}{|\nabla u|} \\
=\ &\phi''(|\nabla u|)u_{NN} + \frac{\phi'(|\nabla u|)}{|\nabla u|}u_{TT}.
\end{aligned}$$

In a neighborhood of an edge C, the image presents a strong gradient. If we want to preserve the edge, it is preferable to diffuse along C(in the T-direction) and not across it. In another word, we would like to annihilate the diffusion in the N-direction. That is:

$$\lim_{s\to+\infty}\frac{\phi'(s)}{s} = \beta > 0,$$

and

$$\lim_{s\to+\infty}\phi''(s) = 0.$$

However, it is possible that both $\frac{\phi'(s)}{s}$ and $\phi''(s)$ will approaches to 0 as $s\to\infty$ so we can assume $\frac{\phi'(s)}{s}$ and $\phi''(s)$ both converge to zero but at different rates:

$$\lim_{s\to+\infty}\frac{\phi''(s)}{\phi'(s)/s}=0.$$

Suppose $\phi(s)=s^z$ then we have

$$\lim_{s\to+\infty}\frac{z(z-1)s^{z-2}s}{zs^{z-1}}=z-1=0$$

$$\phi(s)\simeq s$$

To summarize, the assumption imposed on $\phi(s)$ are

$$\begin{cases}\phi:[0,\infty)\mapsto[0,\infty);\\ \phi''(0)>0;\\ \phi(s)\simeq s, \text{when } s\to+\infty\end{cases}$$

For example, $\phi(s)=|s|$, and $\phi(s)=\sqrt{1+s^2}$ are such two suitable candidates. Especially, when $\phi(s)=\sqrt{1+s^2}$, the smoothing term in the energy model is the surface area of s, and the corresponding diffusion term $\operatorname{div}\left(\frac{\nabla s}{\sqrt{1+|\nabla s|^2}}\right)$ is the curvature of surface function s.

2.3 Bounded Variational Penalty Methods

In previous section, we solve the minimization problem in $W^{1,1}(\Omega)$ space. However, in a lot of applications the quantity we study can be discontinuous across hypersurfaces, or edges in image processing. Therefore the classical Sobolev space is sometimes too smooth for these applications. One approach is to relax the problem to the BV space, i.e. to solve the following unconstrained minimization problem in BV space

$$\min_{u\in BV} J(u) + \frac{1}{\lambda}\int_\Omega \|u-f\|^2 dx. \tag{2.12}$$

where $J(u) = \int_\Omega |\nabla u| dx$ is the total variation of u. In [46], a slightly more general penalty functional than the BV seminorm is considered, denoted J_ϵ. For sufficiently smooth u, J_ϵ is defined by

$$J_\epsilon(u) = \int_\Omega \sqrt{\epsilon + |\nabla u|^2} dx, \tag{2.13}$$

where $\epsilon \geq 0$. A variation definition of J_ϵ that extends (2.13) to nonsmooth function u is given in [1]. Also in [1], the existence and uniqueness of (2.12) are discussed as well as the effect of taking small $\epsilon > 0$ rather than $\epsilon = 0$.

Let us review some properties of BV functions. Let Ω be a convex region in $\mathbb{R}^d$, $d = 1, 2, 3$, whose boundary $\partial\Omega$ is Lipschitz continuous.

Definition 2.3.1 *(BV Seminorm). Let*

$$J_0(u) = \sup_{\mathbf{v}\in\mathcal{V}} \int_\Omega (-u div \mathbf{v}) dx,$$

where the set of test functions

$$\mathcal{V} := \{\mathbf{v} \in C_0^1(\Omega; \mathbb{R}^d) \, : \, |\mathbf{v}| \leq 1 \, for \; all \; x \in \Omega)\}.$$

We call $J_0(u)$ the BV seminorm, or total variation of u.

If $u \in C^1(\Omega)$, one can show using integration by parts that

$$J_0(u) = \int_\Omega |\nabla u| dx,$$

because

$$-\int_{\Omega} u\text{div}(\mathbf{v})dx + \int_{\partial\Omega} u \underbrace{(\mathbf{v}\cdot\mathbf{n})}_{=0} dx = \int_{\Omega} \nabla u \cdot \mathbf{v} dx. \tag{2.14}$$

Then take the supremum over $\mathcal{V}$, we have

$$\sup_{\mathbf{v}\in\mathcal{V}} \int_{\Omega} \nabla u \cdot \mathbf{v} dx = \int_{\Omega} |\nabla u| dx.$$

Definition 2.3.2 *(BV Space). The space of functions of bounded variation on Ω is defined by*

$$BV(\Omega) = \{u \in L^1(\Omega) : J_0(u) \leq \infty\}.$$

The BV norm is given by

$$\|u\|_{BV} := \|u\|_1 + J_0(u).$$

We should notice that the $BV(\Omega)$ is complete, therefore a Banach space. And it is easy to see $W^{1,1}(\Omega) \subset BV(\Omega) \subset L^1(\Omega)$. Since $W^{1,1}$ is dense in L^1, for any $u \in BV(\Omega)$, there exists a sequence in $W^{1,1}(\Omega)$ converging to u in L^1. The following theorem shows that every function u in $BV(\Omega)$ can be approximated, in certain sense, by C^∞ functions, and consequently by $W^{1,1}$ functions.

Theorem 2.3.1 *(cf. [28]) Let $u \in BV(\Omega)$. Then there exists a sequence $\{u_j\}$ in $C^\infty(\Omega)$ such that*

$$\lim_{j\to\infty} \int_{\Omega} |u_j - u| dx = 0,$$

and

$$\lim_{j\to\infty} \int_{\Omega} |\nabla u_j| dx = \int_{\Omega} |\nabla u| dx.$$

Since $C^\infty(\Omega)$ is dense in $W^{1,1}(\Omega)$ with respect to L^1 topology, the above theorem holds for $\{u_j\}$ in $W^{1,1}(\Omega)$ also.

Remark. The above theorem also shows that, if using $W^{1,1}$ seminorm as the regulization term, the minimization problem (2.12) might not admit a solution, because the convergent sequence might converge to a minimizer in the BV space, not in $W^{1,1}$. Because $W^{1,1}$ is a proper space of BV.

Now, we give a definition of J_ϵ for nonsmooth functions according to [1]. Let us identify the convex functional $f(\mathbf{x}) = \sqrt{|\mathbf{x}|^2 + \epsilon}$ with its second conjugate, or Fenchel transform

$$\sqrt{\epsilon + |\mathbf{x}|^2} = \sup\{\mathbf{x} \cdot \mathbf{y} + \sqrt{\epsilon(1 - |\mathbf{y}|^2)} \ : \ |\mathbf{y}| \leq 1\}, \tag{2.15}$$

the supremum being attained for $y = \dfrac{\mathbf{x}}{\sqrt{\epsilon + |\mathbf{x}|^2}}$. Define

$$J_\epsilon(u) := \sup_{\mathbf{v} \in \mathcal{V}} \int_\Omega \left(-u\text{div}(\mathbf{v}) + \sqrt{\epsilon(1 - |\mathbf{v}|^2)} \right) dx. \tag{2.16}$$

Theorem 2.3.2 *(cf.[1]) If $u \in W^{1,1}(\Omega)$, then*

$$J_\epsilon(u) = \int_\Omega \sqrt{\epsilon + |\nabla u|^2} dx.$$

Proof. Integration by parts gives

$$\int_\Omega \left(-u\text{div}(\mathbf{v}) + \sqrt{\epsilon(1 - |\mathbf{v}|^2)} \right) dx = \int_\Omega \left(\nabla u \cdot \mathbf{v} + \sqrt{\epsilon(1 - |\mathbf{v}|^2)} \right) dx$$

It follows from (2.15) that

$$\int_\Omega \left(-u \ \text{div}(\mathbf{v}) + \sqrt{\epsilon(1 - |\mathbf{v}|^2)} \right) dx \leq \int_\Omega \sqrt{\epsilon + |\nabla u|^2} dx.$$

Consequently, $J_\epsilon(u) \leq \int_\Omega \sqrt{\epsilon + |\nabla u|^2} dx$. On the other hand, take $\bar{\mathbf{v}} = \dfrac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}}$, and observe that

$$\int_\Omega \left(-u\text{div}(\bar{\mathbf{v}}) + \sqrt{\epsilon(1-|\bar{\mathbf{v}}|^2)}\right) dx$$

and $\bar{\mathbf{v}} \in C(\Omega; \mathbb{R}^d$ with $|\bar{\mathbf{v}}(x)| \leq 1$ for all $x \in \Omega$. By multiplying $\bar{\mathbf{v}}$ a suitable characteristic function compactly supported in Ω and then mollifying, one can obtain $v \in \mathcal{V} \cap C_0^\infty(\Omega)$ for which $\int_\Omega \left(-u\text{div}(\mathbf{v}) + \sqrt{\epsilon(1-|\mathbf{v}|^2)}\right) dx$ is arbitrarily close to $\int_\Omega \sqrt{\epsilon + |\nabla u|^2}$. ■

Theorem 2.3.3 *(Convexity). For $\epsilon \geq 0$, J_ϵ is convex.*

Proof. Let $0 \leq \gamma \leq 1$ and $u_1, u_2 \in L^p(\omega)$. For any $\mathbf{v} \in \mathcal{V}$,

$$\begin{aligned}
& \int_\Omega \left(\gamma u_1 + (1-\gamma)u_2)\text{div}(v) + \sqrt{\epsilon(1-|\mathbf{v}|^2)}\right) dx \\
= \;& \gamma \int_\Omega \left(u_1\text{div}(v) + \sqrt{\epsilon(1-|\mathbf{v}|^2)}\right) dx + (1-\gamma)\int_\Omega \left(u_2\text{div}(v) + \sqrt{\epsilon(1-|\mathbf{v}|^2)}\right) dx \\
\leq \;& \gamma J_\epsilon(u_1) + (1-\gamma)J_\epsilon(u_2).
\end{aligned}$$

Taking the supremum in the top line over $\mathbf{v} \in \mathcal{V}$ gives the convexity of J_ϵ. ■

The next theorem shows that both J_0 and J_ϵ are effective in $BV(\Omega)$, and J_0 is the pointwise limit of J_0.

Theorem 2.3.4 *(c.f. [1]) (i) For any $\epsilon > 0$ and $u \in L^1(\Omega)$, $J_0(u) \leq \infty$ if and only if $J_\epsilon \leq \infty$; (ii) For any $u \in BV(\Omega)$,*

$$\lim_{\epsilon \to 0} J_\epsilon(u) = J_0(u).$$

Define the BV-bounded set

$$\mathcal{D} := \{u \in BV(\Omega),\ \|u\|_{BV} \leq B\},$$

for some constant $B > 0$. Let d be the dimension of the Euclidean space. Now, we discuss the relatively compactness of the BV-bounded set $\mathcal{D}$ in $L^p(\Omega)$, the lower semicontinuity and coerciveness of J_ϵ, which are critical for the discussion of the existence and uniqueness of minimizer for (2.12).

Theorem 2.3.5 *(c.f. [1]) If $\mathcal{D}$ is a BV-bounded set, then $\mathcal{D}$ is relatively compact in $L^p(\Omega)$ for $1 \leq p \leq \dfrac{d}{d-1}$. $\mathcal{S}$ is bounded, and hence weakly compact for dimensions $d \geq 2$, in $L^p(\Omega)$ for $p = \dfrac{d}{d-1}$.*

Theorem 2.3.6 *(c.f. [1]) For any $\epsilon \geq 0$, J_ϵ is weakly lower semicontinuous with respect to L^p topology for $1 \leq p < \infty$.*

Proof. Let u_n weakly converges to $\bar{u}$ in $L^p(\Omega)$. For any $\mathbf{v} \in \mathcal{V}$, $\text{div}\mathbf{v} \in C(\Omega)$ and hence,

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega \left(-\bar{u}\text{div}(\mathbf{v}) + \sqrt{\epsilon(1-|\mathbf{v}|^2)}\right) dx &= \lim_{n\to\infty} \int_\Omega \left(-u_n\text{div}(\mathbf{v}) + \sqrt{\epsilon(1-|\mathbf{v}|^2)}\right) dx \\
&= \liminf_{n\to\infty} \int_\Omega \left(-u_n\text{div}(\mathbf{v}) + \sqrt{\epsilon(1-|\mathbf{v}|^2)}\right) dx \\
&\leq J_\epsilon(u_n).
\end{aligned}$$

Take the supremum over $\mathbf{v} \in \mathcal{V}$ gives $J_\epsilon(\bar{u}) \leq \liminf_{n\to\infty} J_\epsilon(u_n)$ ■

We call a functional F BV-coercive if

$$F(u) \to \infty \text{ whenever } \|u\|_{BV} \to \infty.$$

Next we explain about the other two important properties of the energy functional in (2.12) for the existence and uniqueness of the problem.

Lemma 2.3.1 *(c.f. [1]) Let*

$$F(u) = J_\epsilon(u) + \frac{1}{\lambda}\|u - f\|_2^2.$$

with $\epsilon \geq 0$, then $F(u)$ is weakly lower semicontinuous and BV-coercive.

Finally, we have the existence and uniqueness of solution to (2.12) in the following theorem.

Theorem 2.3.7 *(c.f. [1]) Suppose that F is BV-coercive. If $1 \leq p < \dfrac{d}{d-1}$ and F is semicontinuous, then problem*

$$\min_{u \in L^p(\Omega)} F(u)$$

has a solution. If in addition $p = \dfrac{d}{d-1}$, dimension $d \geq 2$, and F is weakly lower semicontinuous, then a solution also exists. In either case, the solution is unique if F is strictly convex.

2.4 Mollifier and Mollification

Definition 2.4.1 *Define $\eta \in C^\infty(\Omega)$, by*

$$\eta(x) := \begin{cases} C\,exp\left(\frac{1}{|x|^2-1}\right), & if\ |x| < 1 \\ 0, \quad if\ |x| \geq 1. \end{cases} \tag{2.17}$$

the constant $C > 0$ selected so that

$$\int_\Omega \eta dx = 1.$$

For each $\epsilon > 0$, set

$$\eta_\epsilon(x) := \frac{C}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right)$$

we call η the standard mollifier. The function η_ϵ are C^∞ and satisfy

$$\int_\Omega \eta_\epsilon = 1, \quad spt(\eta_\epsilon) \subset B(0, \epsilon).$$

Definition 2.4.2 *If $f : \Omega_\epsilon \to \mathbb{R}$ is locally integrable define its mollification*

$$f^\epsilon := \eta_\epsilon * f \quad in\ \Omega.$$

That is

$$f^\epsilon(x) = \int_{\Omega_\epsilon} \eta_\epsilon(x-y)f(y)dy = \int_{B(0,\epsilon)} \eta_\epsilon(y)f(x-y)dy$$

for $x \in \Omega$.

Theorem 2.4.1 *(Properties of mollifiers) (i) $f^\epsilon \in C^\infty(\Omega_\epsilon)$.*

(ii) $f^\epsilon \to f$ a.e. as $\epsilon \to 0$.

(iii) If $f \in C(\Omega)$, then $f^\epsilon \to f$ uniformly on compact subsets of Ω.

(iv) If $1 \le p \le \infty$ and $f \in L^P_{loc}(\Omega)$, then $f^\epsilon \to f$ in $L^P_{loc}(\Omega)$.

Lemma 2.4.1 *(cf. [52]) If $u \in BV(\Omega)$, then*

$$\int_\Omega |\nabla u^\epsilon| \le \int_\Omega |\nabla u|.$$

2.5 Convex function and Subdifferential

Since the functionals we study are convex, we shall discuss some basic properties of convex function here. First we give the definition of subdifferential functions.

Definition 2.5.1 *If $F : U \to R$ is a convex function defined on a convex open set in the Euclidean space $\mathbb{R}^n$, a vector v in that space is called a subgradient at a point u_0 in U if for any u in U one has*

$$F(u) - F(u_0) \ge \langle v, u - u_0 \rangle.$$

The set of subgradients at u_0 is called the subdifferential at u_0 and is denoted $\partial F(u)$.

Then we have the following characterization:

$$u^* \in \partial F(u) \quad \text{if and only if} \quad F(u) \text{ is finite and} \tag{2.18}$$
$$\langle v-u, u^*\rangle + F(u) \le F(v), \quad \forall v \in V.$$

Now we have the well-known non expensive property for the minimizer of a convex functional.

Theorem 2.5.1 *.Given*

$$I(u,f) := G(u) + \frac{1}{\lambda}\int_\Omega |u-f|^2 dx, \tag{2.19}$$

where G is a convex functional. If u_f and u_g are the minimizers of $I(u,f)$ and $I(u,g)$, for some given functions f and g in $L_2(\Omega)$, then

$$\|u_f - u_g\|_2 \le \|f-g\|_2.$$

Proof. Since G is convex, by (2.18) we have

$$\langle \partial G(u_f), u_f - u_g\rangle \ge G(u_f) - G(u_g),$$

and

$$\langle \partial G(u_g), u_f - u_g\rangle \le G(u_f) - G(u_g).$$

where ∂G is the subderivative of G. It follows that

$$\langle \partial G(u_f) - \partial G(u_g), u_f - u_g\rangle \ge 0. \tag{2.20}$$

The minimizers of $I(u, f)$ and $I(u, g)$ are characterized by the Euler-Lagrangue equations

$$\partial G(u_f) = \frac{f - u_f}{\lambda},$$

and

$$\partial G(u_g) = \frac{g - u_g}{\lambda}$$

respectively. It follows that

$$\partial G(u_f) - \partial G(u_g) + \frac{u_f - u_g}{\lambda} = \frac{f - g}{\lambda}.$$

Then take the inner product with $u_f - u_g$ on both sides, we have

$$\langle \partial G(u_f) - \partial G(u_g), u_f - u_g \rangle + \frac{1}{\lambda}\|u_f - u_g\|_2^2 = \frac{1}{\lambda}\langle f - g, u_f - u_g \rangle.$$

Recall that the first term on the left hand-side above is non-negative, therefore we have

$$\frac{1}{\lambda}\|u_f - u_g\|_2^2 \leq \frac{1}{\lambda}\langle f - g, u_f - u_g \rangle \leq \frac{1}{\lambda}\|f - g\|_2\|u_f - u_g\|_2.$$

That is

$$\|u_f - u_g\|_2 \leq \|f - g\|.$$

■ This shows that if g is very close to f, the minimizer u_g is very close to the minimizer u_f.

2.6 Inequalities

In our work, we will use the following inequalities.

Cauchy-Schwarz ineqaulity: $\|xy\| \leq \|x\|\|y\|$.

Höder's inequality: Assume $1 \le p, q \le \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then if $u \in L^p\Omega$, $v \in L^q(\Omega)$, we have

$$\int_\Omega |uv| dx \le \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Jensen's inequality: Assume that $f : \mathbb{R} \to \mathbb{R}$ is convex, and $\Omega \subset \mathbb{R}^N$ is open and bounded. Let $u : \Omega \to \mathbb{R}$ be integrable. Then

$$f\left(⨍_\Omega u dx\right) \le \left(⨍_\Omega f(u) dx\right),$$

where $⨍_\Omega u dx$ denotes the mean value of u over Ω.

Minkowski's inequality: Assume $1 \le p < \infty$ and $u, v \in L^p(\Omega)$. Then

$$\|u + v\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

Young's inequality: Let $1 < p, q < \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then:

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \ (a, b > 0).$$

Chapter 3

Bivariate Spline Approximation for the ROF Model

In this chapter we the bivariate splines approach to approximate the minimizer of the well-known ROF model:

$$\min_{u \in BV(\Omega)} |u|_{BV} + \frac{1}{2\lambda} \int_{\Omega} |u - f|^2, \tag{3.1}$$

where $BV(\Omega)$ stands for the space of all functions of bounded variation over Ω, $|u|_{BV}$ denotes the semi-norm in $BV(\Omega)$, and f is a given function. As a by-product, we propose a new spline method for scattered data fitting by approximating the minimizer above based on discrete image values.

The minimization in (4.1) has been studied for about twenty years. See [44], [1], [12], [5], [10], [23], and many references in [47]. Many numerical methods have been proposed to approximate the minimizer. Typically, one first regularizes the minimization by considering the following ϵ-version of the ROF model:

$$\min_{u \in BV(\Omega)} \int_{\Omega} \sqrt{\epsilon + |\nabla u|^2} + \frac{1}{2\lambda} \int_{\Omega} |u - f|^2, \tag{3.2}$$

where ∇u is the standard gradient of u. Here the first integral is well defined for $u \in W^{1,1}(\Omega)$ which is dense in $BV(\Omega)$ with respect to L^1 topology. But for $u \in BV(\Omega)\backslash W^{1,1}(\Omega)$, we use Acar and Vogel's equivalent formula (2.16). In addition to the prime and dual algorithm (cf. [10]) and projected gradient algorithm (cf. [23]) to numerically solve the minimizer of (4.1) directly, finite difference and finite element methods have been used for numerical solution of (3.2) by solving its Euler-Lagrange equation

$$\text{div}\ \left(\frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}}\right) - \frac{1}{\lambda}(u - f) = 0 \tag{3.3}$$

or its time dependent version

$$u_t = \text{div}\ \left(\frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}}\right) - \frac{1}{\lambda}(u - f) \tag{3.4}$$

starting with $u(x, y, 0) = u_0$ together with Dirichlet or Neumann boundary condition. We refer to [50], [19], [26], and [27] for theoretical studies of finite difference and finite element methods.

To the best knowledge of Dr. Lai and myself, bivariate splines have not been used to solve the nonlinear PDE (3.3) nor time dependent PDE (3.4) in the literature so far. For convenience, let $\epsilon = 1$. It is known(cf. [**?**]) that the following minimization

$$\min\{E(u), \quad u \in BV(\Omega)\}, \tag{3.5}$$

where the energy functional $E(u)$ is defined by

$$E(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx + \frac{1}{2\lambda}\frac{1}{A_\Omega}\int_\Omega |u - f|^2 dx \tag{3.6}$$

has a unique solution, where A_Ω is the area of polygonal domain Ω. We use u_f to denote the minimizer. The discussion of the existence and uniqueness of the minimizer of (3.5) can be found in [1] and [12]. Similarly, we know that

$$\min\{E(u), \quad u \in S_d^r(\triangle)\}, \tag{3.7}$$

has a unique solution. We shall denote by S_f the minimizer of (3.7). Finally, in practice, we are given discrete noised image values over Ω. That is, we have $\{(x_i, f_i), i = 1, \cdots, n\}$ with $x_i \in \Omega$ and noised function values f_i for $i = 1, \cdots, n$. Let

$$E_d(u) = \int_\Omega \sqrt{1 + |\nabla u(x)|^2} dx + \frac{1}{2\lambda}\frac{1}{n}\sum_{i=1}^{n} |u(x_i) - f_i|^2 \tag{3.8}$$

be an energy functional based on the discrete noised image values. We use bivariate splines to solve the following minimization problem:

$$\min\{E_d(u), \quad u \in S_d^r(\Omega)\}, \tag{3.9}$$

and denote the solution by s_f.

One of our main results is in this chapter to show

Theorem 3.0.1 *Suppose Ω is a bounded domain with Lipschitz boundary and $\triangle$ is a β-quasi-uniform triangulation of Ω. If $u_f \in W^{1,1}(\Omega)$, then S_f converges to u_f in $L^2(\Omega)$ norm as the size $|\triangle|$ of triangulation $\triangle$ goes to zero. More precisely,*

$$\|S_f - u_f\|_{L^2(\Omega)} \to 0, \quad \textit{when } |\triangle| \to 0.$$

When $u_f \in W^{1,2}(\Omega)$,

$$\|S_f - u_f\|_{L^2(\Omega)} \le C\sqrt{|\triangle|}$$

for a positive constant C independent of $\triangle$.

Next we shall discuss how to compute the minimizer S_f of (3.7). First the minimizer S_f satisfies the following nonlinear equations: letting $\{\phi_1, \cdots, \phi_N\}$ be a basis for $\mathcal{S}$,

$$\frac{\partial}{\partial t} E(S_f + t\phi_j, f)\big|_{t=0} = \int_\Omega \frac{\nabla S_f \cdot \nabla \phi_j}{\sqrt{1 + |\nabla S_f|^2}} dx + \frac{1}{\lambda}\frac{1}{A_\Omega}\int_\Omega (S_f - f)\phi_j dx = 0 \tag{3.10}$$

for all basis functions $\phi_j, j = 1, \cdots, N$. As these are nonlinear equations, we will use a fixed point iterative algorithm as in [19]. Our next result is to show the convergence of the iterative algorithm. Our analysis is completely different from the one given in [19].

3.1 Existence, Uniqueness and Stability of Solutions S_f in $\mathcal{S}$

In this section we will talk about the existence, uniqueness and the stablity of the bivariate spline solution S_f to (3.7). We begin with the following

Lemma 3.1.1 *Assume that the triangulation $\triangle$ is β-quasi-uniform. The minimization problem (3.7) has one and only one solution over the finite dimensional space $S^r_d(\Omega)$.*

Proof. Let $s_0 \in \mathcal{S}$ be a spline approximation of f, e.g., s_0 satisfies

$$\int_\Omega |s_0 - f|^2 = \min_{s \in \mathcal{S}} \int_\Omega |s - f|^2 \tag{3.11}$$

Let $\mathcal{U}$ be a subset of $\mathcal{S}$ defined by

$$\mathcal{U} = \{u \in \mathcal{S}, \quad I(u) \leq I(s_0) + 1\} \tag{3.12}$$

Clearly, $s_0 \in \mathcal{U}$, so $\mathcal{U}$ is not an empty set. We now show that $\mathcal{U}$ is bounded in $L^2(\Omega)$ norm. For convenience, let

$$\|f\|_2 := \left(\frac{1}{A_\Omega}\int_\Omega |f(x)|^2 dx\right)^{1/2}$$

denote the norm for $L^2(\Omega)$. Indeed, for any $u \in \mathcal{U}$,

$$\begin{aligned}\|u-f\|_2^2 &\leq 2\lambda\left(\frac{1}{2\lambda}\|u-f\|_2^2 + \int_\Omega \sqrt{1+|\nabla u|^2}dx\right) \\ &= 2\lambda I(u) \leq 2\lambda(I(s_0)+1).\end{aligned} \tag{3.13}$$

Thus we have

$$\|u\|_2 \leq \|u-f\|_2 + \|f\|_2 \leq (2\lambda(I(s_0)+1))^{1/2} + \|f\|_2. \tag{3.14}$$

That is, $\mathcal{U}$ is a a bounded set. Since $\mathcal{U}$ is a subset of a finite dimensional space, $\mathcal{U}$ is compact. Let $\{u_j\} \in \mathcal{U}$ be a sequence such that,

$$\lim_{j\to+\infty} E(u_j) = \inf_{u\in\mathcal{S}} E(u).$$

By the compactness for $\mathcal{U}$, there exists a subsequence, say u_j again and $\tilde{u}$ s.t. u_j converges to $\tilde{u}$ in $L^2(\Omega)$ norm.

Next we claim $I(\tilde{u}) = \inf_{u\in\mathcal{S}} I(u)$. It is sufficient to prove that

$$\lim_{j\to\infty}\int_\Omega \sqrt{1+|\nabla u_j|^2}dx = \int_\Omega \sqrt{1+|\nabla \tilde{u}|^2}dx \tag{3.15}$$

Indeed, for any w and fixed $v \in \mathcal{S}$, we have

$$\begin{aligned}\left|\int_\Omega \sqrt{1+|\nabla w|^2}dx - \int_\Omega \sqrt{1+|\nabla v|^2}dx\right| &\leq \frac{1}{2}\int_\Omega |\nabla w - \nabla v|\,|\nabla w + \nabla v|dx \\ &\leq \frac{1}{2}\left(\int_\Omega |\nabla(w-v)|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla(w+v)|^2 dx\right)^{\frac{1}{2}} \\ &= \frac{A_\Omega}{2}\|\nabla(w-v)\|_2\,\|\nabla(w+v)\|_2.\end{aligned}$$

That is, we have

$$\left|\int_\Omega \sqrt{1+|\nabla w|^2}dx - \int_\Omega \sqrt{1+|\nabla v|^2}dx\right| \leq \frac{A_\Omega}{2}\|\nabla(w-v)\|_2\,(\|\nabla w\|_2 + \|\nabla v\|_2). \tag{3.16}$$

Now we need to use Markov's inequality (cf. Theorem 2.1.5) to show that for any spline $s = w, v, w-v \in \mathcal{S}$,

$$\int_\Omega |\nabla s|^2 dx \leq \frac{C\beta^2}{|\triangle|^2}\int_\Omega |s|^2 dx.$$

It follows from (3.16) that

$$\left|\int_\Omega \sqrt{1+|\nabla u_j|^2}dx - \int_\Omega \sqrt{1+|\nabla \tilde{u}|^2}dx\right| \leq \frac{C}{2}\frac{\beta^2}{|\triangle|^2}\|u_j - \tilde{u}\|_2\,(\|u_j\|_2 + \|\tilde{u}\|_2).$$

The convergence of u_j to $\tilde{u}$ in L_2 norm implies the claim (3.15).

The uniqueness follows directly from the strict convexity of the functional $E(u)$. These complete the proof. ■

Lemma 3.1.2 *Assume that the triangulation $\triangle$ is β-quasi-uniform and suppose that the data sites $x_i, i = 1, \cdots, n$ satisfy the condition (2.1). Then the minimization problem (3.9) has one and only one solution.*

Proof. The proof is almost the same as in the proof of Lemma 3.1.1, where $\frac{1}{A_\Omega}\int_\Omega |u-f|^2dx$ has to be replaced by $\frac{1}{n}\sum_{i=1}^{n}|u(x_i)-f_i|^2$. But we are not able to show that u is bounded in L^2 norm unless we use the condition in (2.1). Indeed, similar to the proof in Lemma 3.1.1, we have

$$\sum_{i=1}^{n}|u(x_i)|^2$$

is bounded. Since $u \in \mathcal{S}$, we use (2.1) to have

$$\|u\|_{\infty,\Omega} \leq \frac{1}{F_1}\left(\sum_{i=1}^{n}|u(x_i)|^2\right)^{1/2}$$

and hence, $\int_\Omega |u|^2dx$ is bounded. The remaining the proof is again similar to the corresponding part of the proof of Lemma 3.1.1. We may leave the detail to the interested reader.

Once we have the existence, the solution is unique due to the strictly convexity of $I_d(u)$.

■

3.2 Properties of Minimizer of Continuous Functional

In this section we discuss the properties of the minimizer of the continuous functional (3.2). Since $\int_\Omega \sqrt{1+|\nabla u|^2}dx$ is a convex functional of u, by Theorem 2.5.1, we have the well-known contraction property for the minimizer of a convex functional.

Lemma 3.2.1 *(Contraction).Suppose we have another function $g \in L^2(\Omega)$. Let $S_g \in \mathcal{S}$ be the minimizer of (3.7) associated with g. Then the norm of the difference of S_f and S_g is bounded by the norm of the difference of f and g, i.e.*

$$\|S_f - S_g\|_2 \leq \|f-g\|_2. \tag{3.17}$$

Also we can give an exact value for the energy functionals $I(S_f)$ in the following lemma.

Lemma 3.2.2 *Let S_f be the minimizer of (3.7). Then we have the following equation for $I(S_f)$,*

$$E(S_f) = \frac{1}{2\lambda A_\Omega}(\|f\|_2^2 - \|S_f\|_2^2) + \int_\Omega \frac{1}{\sqrt{1+|\nabla S_f|^2}}dx. \quad (3.18)$$

Proof. By using (3.10), S_f satisfies

$$\int_\Omega \frac{\nabla S_f \cdot \nabla S_f}{\sqrt{1+|\nabla S_f|^2}}dx = \frac{1}{\lambda}\frac{1}{A_\Omega}\int_\Omega (f - S_f)S_f dx$$

Adding $\frac{1}{2\lambda A_\Omega}\|f - S_f\|_2^2$ to both sides above, we get

$$\begin{aligned} I(S_f) - \int_\Omega \frac{1}{\sqrt{1+|\nabla S_f|^2}}dx &= \frac{1}{2\lambda A_\Omega}\|f - S_f\|_2^2 + \frac{1}{\lambda}\frac{1}{A_\Omega}\int_\Omega (f - S_f)S_f dx \\ &= \frac{1}{2\lambda A_\Omega}(\|f\|_2^2 - \|S_f\|_2^2). \end{aligned}$$

This proves the result in the lemma. ■

3.3 Approximation of S_f and s_f to u_f

In this section, we show that S_f and s_f approximate u_f. We first extend the arguments in [17] to show the convergence of S_f and s_f to u_f, the analysis will require $u_f \in W^{2,\infty}(\Omega)$. As an image function may not have such high regularity, next we use the ideas from [53] to show the convergence which only require $u_f \in W^{1,1}$(or in $BV(\Omega)$, if Ω is a rectangular region, which will be discussed in Chapter 4).

Let $\mathcal{S}$ be an N-dimensional space, s_N be an approximation of u_f in $\mathcal{S}$, e.g., s_N is the discrete least squares spline of given data, and S_f be the minimizer of (3.7) in $\mathcal{S}$. Define the error term between u_f and S_f by

$$E_N = \int_\Omega \frac{|\nabla(u_f - S_f)|^2}{\sqrt{1+|\nabla S_f|^2}}dx + \frac{1}{2\lambda}\frac{1}{A_\Omega}\int_\Omega |u_f - S_f|^2 dx. \tag{3.19}$$

then we have the following approximation property of S_f to u_f.

Lemma 3.3.1 *Suppose that u_f can be approximated by s_N well in the following sense that*

$$\int_\Omega |\nabla(u_f - s_N)|^2 dx + \frac{1}{\lambda}\frac{1}{A_\Omega}\int_\Omega |u_f - s_N|^2 dx \to 0, \quad N \to \infty. \tag{3.20}$$

Then S_f can approximates u_f also very well in the sense that $E_N \to 0$, as $N \to \infty$.

Proof. Recall the minimizer S_f of (3.7) satisfies the following equations:

$$\int_\Omega \frac{\nabla S_f \cdot \nabla \phi_j}{\sqrt{1+|\nabla S_f|^2}}dx + \frac{1}{\lambda}\frac{1}{A_\Omega}\int_\Omega (S_f - f)\phi_j dx = 0,\ j = 1, ..., N. \tag{3.21}$$

for every $\phi_j,\ j = 1, ..., N$. Also from (3.5) we have

$$\int_\Omega \frac{\nabla u_f}{\sqrt{1+|\nabla u_f|^2}}\nabla\phi_j dx + \frac{1}{\lambda}\frac{1}{A_\Omega}\int_\Omega u_f\phi_j dx = \frac{1}{\lambda}\frac{1}{A_\Omega}\int_\Omega f\phi_j dx, \quad j = 1, ..., N. \tag{3.22}$$

The subtraction of (3.21) and (3.22) yields

$$\int_\Omega \left(\frac{\nabla u_f}{\sqrt{1+|\nabla u_f|^2}} - \frac{\nabla S_f}{\sqrt{1+|\nabla S_f|^2}} \right) \cdot \nabla\phi_j + \frac{1}{\lambda}\frac{1}{A_\Omega}\int_\Omega (u_f - S_f)\phi_j = 0. \tag{3.23}$$

For convenience, we introduce the following notation for the first term of E_N:

$$\widetilde{E}_N = \int_\Omega \frac{|\nabla(u_f - S_f)|^2}{\sqrt{1+|S_f|^2}}dx \tag{3.24}$$

It is easy to see

$$\widetilde{E}_N = \int_\Omega \frac{\nabla(u_f - S_f)\cdot\nabla(u_f - s_N)}{\sqrt{1+|\nabla S_f|^2}}dx + \int_\Omega \frac{\nabla(u_f - S_f)\cdot\nabla(s_N - S_f)}{\sqrt{1+|\nabla S_f|^2}}dx, \tag{3.25}$$

where s_N is an approximation of u_f in $\mathcal{S}$. The first term gives

$$\int_\Omega \frac{1}{\sqrt{1+|\nabla S_f|^2}}\nabla(u_f - S_f)\cdot\nabla(u_f - s_N)dx \le \left(\int_\Omega \frac{|\nabla(u_f - s_N)|^2}{\sqrt{1+|\nabla S_f|^2}}\right)^{\frac{1}{2}} \widetilde{E}_N^{\frac{1}{2}}. \tag{3.26}$$

The second term of (3.25) gives the following two terms:

$$\begin{aligned} & \int_\Omega \frac{\nabla(u_f - S_f)\cdot\nabla(s_N - S_f)}{\sqrt{1+|\nabla S_f|^2}}dx \\ = \quad & \int_\Omega \left(\frac{\nabla u_f}{\sqrt{1+|\nabla u_f|^2}} - \frac{\nabla S_f}{\sqrt{1+|\nabla S_f|^2}}\right)\cdot\nabla(s_N - S_f)dx \\ & + \int_\Omega \left(\frac{\nabla u_f}{\sqrt{1+|\nabla S_f|^2}} - \frac{\nabla u_f}{\sqrt{1+|\nabla u_f|^2}}\right)\cdot\nabla(s_N - S_f)dx. \end{aligned} \tag{3.27}$$

By equation in (3.23) with replace ϕ_j by $s_N - S_f$, the first term in (3.27) satisfies

$$\begin{aligned} & \int_\Omega \left(\frac{\nabla u_f}{\sqrt{1+|\nabla u_f|^2}} - \frac{\nabla S_f}{\sqrt{1+|\nabla S_f|^2}}\right)\cdot\nabla(s_N - S_f)dx \qquad (3.28) \\ = \quad & -\frac{1}{\lambda}\int_\Omega (u_f - S_f)(s_N - S_f)dx \\ = \quad & -\frac{1}{\lambda}\int_\Omega (u_f - S_f)^2dx - \frac{1}{\lambda}\int_\Omega (u_f - S_f)(s_N - u_f)dx \\ \le \quad & -\frac{1}{\lambda}\int_\Omega (u_f - S_f)^2dx + \frac{1}{\lambda}\|u_f - S_f\|_2\|u_f - s_N\|_2. \end{aligned}$$

The second term in (3.27) satisfies

$$\begin{aligned}
&\int_\Omega \left(\frac{\nabla u_f}{\sqrt{1+|\nabla S_f|^2}} - \frac{\nabla u_f}{\sqrt{1+|\nabla u_f|^2}} \right) \nabla (s_N - S_f) dx \qquad (3.29)\\
\leq & \int_\Omega \frac{|\nabla u_f|\,|\nabla (u_f - S_f)|}{\sqrt{1+|\nabla u_f|^2}\sqrt{1+|\nabla S_f|^2}} |\nabla (s_N - S_f)| dx.
\end{aligned}$$

Letting

$$\gamma(u) = \max_{x\in\Omega} \frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}} < 1,$$

the inequality in (3.29) can be rewritten as

$$\begin{aligned}
&\int_\Omega \left(\frac{\nabla u_f}{\sqrt{1+|\nabla S_f|^2}} - \frac{\nabla u_f}{\sqrt{1+|\nabla u_f|^2}} \right) \nabla (s_N - S_f) dx \qquad (3.30)\\
\leq & \ \gamma(u_f) \int_\Omega \frac{|\nabla (u_f - S_f)||\nabla (s_N - S_f)|}{\sqrt{1+|\nabla S_f|^2}} dx \\
\leq & \ \gamma(u_f) \left(\int_\Omega \frac{|\nabla (u_f - S_f)|^2}{\sqrt{1+|\nabla S_f|^2}} dx + \int_\Omega \frac{|\nabla (u_f - S_f)||\nabla (u_f - s_N)|}{\sqrt{1+|\nabla S_f|^2}} \right) \\
\leq & \ \gamma(u_f) \left(\widetilde{E}_N + \widetilde{E}_N^{1/2} \left(\int_\Omega \frac{|\nabla (u_f - s_N)|^2}{\sqrt{1+|\nabla S_f|^2}} \right)^{1/2} \right).
\end{aligned}$$

Now let us consider the whole error term (3.19). By the discussion above, we have

$$\begin{aligned}
E_N &= \widetilde{E}_N + \frac{1}{2\lambda}\|u_f - S_f\|_2^2 \\
&\le \widetilde{E}_N^{1/2}\left(\int_\Omega \frac{|\nabla(u_f - s_N)|^2}{\sqrt{1+|\nabla S_f|^2}}dx\right)^{1/2} + \frac{1}{2\lambda}\|u_f - S_f\|_2\|u_f - s_N\|_2 \\
&+ \gamma(u_f)\left(\widetilde{E}_N + \widetilde{E}_N^{1/2}\left(\int_\Omega \frac{|\nabla(u_f - s_N)|^2}{\sqrt{1+|\nabla S_f|^2}}dx\right)^{1/2}\right) \\
&\le (1+\gamma(u_f))\widetilde{E}_N^{1/2}\left(\int_\Omega \frac{|\nabla(u_f - s_N)|^2}{\sqrt{1+|\nabla S_f|^2}}dx\right)^{1/2} \\
&+ \frac{1}{2\lambda}\|u_f - S_f\|_{2,\Omega}\|u_f - s_N\|_{2,\Omega} + \gamma(u_f)\widetilde{E}_N \\
&\le 2(\widetilde{E}_N + \frac{1}{2\lambda}\|u_f - S_f\|_2^2)^{\frac{1}{2}}\left(\int_\Omega \frac{|\nabla(u_f - s_N)|}{\sqrt{1+|\nabla S_f|^2}}dx + \frac{1}{2\lambda}\|u_f - s_N\|^2\right)^{\frac{1}{2}} + \gamma(u_f)\widetilde{E}_N \\
&= 2E_N^{\frac{1}{2}}\left(\int_\Omega \frac{|\nabla(u_f - s_N)|}{\sqrt{1+|\nabla S_f|^2}}dx + \frac{1}{2\lambda}\|u_f - s_N\|^2\right)^{\frac{1}{2}} + \gamma(u_f)\widetilde{E}_N.
\end{aligned}$$

Since $\widetilde{E}_N \le E_N$, we can rewrite the above inequality as follows.

$$(1-\gamma(u_f))E_N \le E_N^{\frac{1}{2}}\left(4\int_\Omega |\nabla(u_f - s_N)|^2 dx + \frac{2}{\lambda}\|u_f - s_N\|_2^2\right)^{\frac{1}{2}}.$$

That is, we have

$$E_N^{\frac{1}{2}} \le \frac{2}{1-\gamma(u_f)}\left(\int_\Omega |\nabla(u_f - s_N)|^2 dx + \frac{1}{2\lambda}\frac{1}{A_\Omega}\int_\Omega (u_f - s_N)^2\right)^{\frac{1}{2}}.$$

Therefore, if u_f can be approximated by s_N very well in the sense of (3.20), then $E_N \to 0$. That is, S_f approximates u_f. These complete the proof. ■

Similarly, we can prove

Lemma 3.3.2 *Suppose that u_f can be approximated by s_N well in the following sense that*

$$\int_\Omega |\nabla(u_f - s_N)|^2 dx + \frac{1}{\lambda}\frac{1}{n}\sum_{i=1}^{n}(u_f(x_i) - s_N(x_i))^2 \to 0, \quad n, N \to \infty. \tag{3.31}$$

Then the minimizer s_f of (3.9) can approximates u_f also very well in the following sense:

$$\int_\Omega \frac{|\nabla(u_f - s_f)|^2}{\sqrt{1 + |\nabla s_f|^2}} dx + \frac{1}{2\lambda}\frac{1}{n}\sum_{i=1}^{n}|u_f(x_i) - s_f(x_i)|^2 \to 0 \tag{3.32}$$

as n and N go to ∞.

We now discuss when the conditions (3.20) and (3.31) will hold. We mainly use Theorems 2.1.6 and 2.1.7. When $u_f \in H^2(\Omega)$, the approximation power in Theorem 2.1.6 shows that (3.20) holds if s_N is the quasi-interpolant of f. For the condition (3.31), we may choose s_N to be the discrete least squares spline approximation l_f of u_f in Theorem 2.1.7. Hence, we conclude the following

Theorem 3.3.1 *Suppose that $u_f \in H^2(\Omega)$. Then S_f converges to u_f with respect to L^2 topology as $|\triangle|$ goes to 0. Furthermore, suppose that the data locations satisfy the conditions in (2.1) and (2.6). Then s_f converges to u_f as $|\triangle|$ goes to zero in the sense (3.32).*

Proof. We just need to prove that the conditions (3.20) and (3.31) under our hypothesis. First take $s_N = Qf$ in Theorem 2.1.6, then

$$\|\nabla(s_N - u_f)\|_{\Omega,2} \leq C|\triangle||f|_{\Omega,2,2},$$

and

$$\|s_N - u_f\|_{\Omega,2} \leq C|\triangle|^2|f|_{\Omega,2,2}.$$

By embedding of Lebesgue Spaces, we have $\|\nabla(s_N - u_f)\|_{\Omega,1} \leq C\|\nabla(s_N - u_f)\|_{\Omega,2}$. Therefore, the condition (3.20) holds. Next take s_N the least square spline of u_f in Theorem 2.1.7, then we have

$$\|u_f - s_N\|_{\Omega,\infty} \leq C\frac{F_1}{F_2}|\triangle||u_f|_{\Omega,1,\infty}.$$

Then by Kondrachov embedding theorem, $H^2(\Omega) = W^{2,2}(\Omega) \subset W^{1,\infty}(\Omega)$ when $\Omega \in \mathbb{R}^2$, and hence we have $|u_f|_{\Omega,1,\infty} \leq C\|u_f\|_{H^2(\Omega)}$. Therefore, $\|u_f - s_N\|_{\Omega,\infty} \to 0$ as $|\triangle| \to 0$, and consequently $\frac{1}{n}\sum_{i=1}^{n}(u_f(x_i) - s_N(x_i))^2 \to 0$, as $n, N \to \infty$. ■

In general, u_f may not be in $H^2(\Omega)$. When f is a noised image, we can only assume that $u_f \in H^{1+\alpha}(\Omega) = \mathrm{Lip}(\alpha, L^2(\Omega))$, a fractional Sobolev space for some $\alpha > 0$. In this case, we need a result like the one in Theorem 2.1.6 for functions in a fractional Sobolev space, e.g. $H^{m+\alpha}(\Omega)$, for $\alpha > 0$. The study of the approximation order of bivariate splines using a fractional Sobolve space norm is beyond the scope of this paper. We leave the problem to the interested reader.

Next we shall show that S_f converges to u_f, the minimizer of (3.5) for the same λ, when $u_f \in W^{1,1}(\Omega)$, which has a lower regularity than the space $W^{2,\infty}(\Omega)$ as the Theorem 3.3.1 requires.

Lemma 3.3.3 *Let u_f be the solution to (3.5). For any $u \in BV(\Omega)$,*

$$\|u - u_f\|_2^2 \leq 2\lambda A_\Omega(E(u) - E(u_f)). \tag{3.33}$$

In particular, we have

$$\|S_f - u_f\|_2^2 \leq 2\lambda A_\Omega(E(S_f) - E(u_f)). \tag{3.34}$$

Proof. Using the concept of sub-differentiation and its basic property (see, e.g. [24]), we have

$$0 = \partial E(u_f) = \partial J(u_f) - \frac{1}{\lambda A_\Omega}(f - u_f) \text{ and } \langle \partial J(u_f), u - u_f\rangle \leq J(u) - J(u_f),$$

where $J(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx$. From the above equations, it follows

$$\frac{1}{\lambda A_\Omega} \int_\Omega (f - u_f)(u - u_f) dx \leq \int_\Omega \sqrt{1 + |\nabla u|^2} - \sqrt{1 + |\nabla u_f|^2} dx. \tag{3.35}$$

We can write

$$\begin{aligned} & E(u) - E(u_f) \\ = & \int_\Omega \sqrt{1 + |\nabla u|^2} - \sqrt{1 + |\nabla u_f|^2} dx + \frac{1}{2\lambda |A_\Omega|} \left(\int_\Omega |u - f|^2 dx - \int_\Omega |u_f - f|^2 dx \right) \\ = & \int_\Omega \sqrt{1 + |\nabla u|^2} - \sqrt{1 + |\nabla u_f|^2} dx + \frac{1}{2\lambda |A_\Omega|} \left(\int_\Omega (u - u_f + u_f - f)^2 dx - \int_\Omega |u_f - f|^2 dx \right) \\ = & \int_\Omega \sqrt{1 + |\nabla u|^2} - \sqrt{1 + |\nabla u_f|^2} dx + \frac{1}{\lambda |A_\Omega|} \int_\Omega (u - u_f)(u_f - f) dx + \frac{1}{2\lambda |A_\Omega|} \int_\Omega |u - u_f|^2 dx \\ \geq & \frac{1}{2\lambda |A_\Omega|} \int_\Omega |u - u_f|^2 dx \end{aligned}$$

by (3.35). Therefore the inequality (3.33) holds. Hence, we have (3.34). ■

Next we need to show that $E(S_f) - E(u_f) \to 0$. To this end, we recall two standard concepts. Since $\Omega \subset \mathbb{R}^2$ is an region with piecewise smooth boundary $\partial\Omega$ and u_f is assumed to be in $W^{1,1}(\Omega)$, using the extension theorem in [45], there exists a linear operator $\mathfrak{E} : W^{1,1}(\Omega) \to W^{1,1}(\mathbb{R}^2)$ such that,

(i)$\mathfrak{E}(u_f)|_\Omega = u_f$

(ii)$\mathfrak{E}$ maps $W^{1,1}(\Omega)$ continuously into $W^{1,1}(\mathbb{R}^2)$:

$$\|\mathfrak{E}(u_f)\|_{W^{1,1}(\mathbb{R}^2)} \leq C \|u_f\|_{W^{1,1}(\Omega)}. \tag{3.36}$$

Note that $\mathfrak{E}(u_f)$ is a compactly supported function in $W^{1,1}(\mathbb{R}^2)$. Thus, without loss of generality we may assume $u_f \in W^{1,1}(\mathbb{R}^2)$.

Let u_f^ϵ be the mollification of u_f defined by

$$u_f^\epsilon(x) = \int_{\Omega_\epsilon} \eta_\epsilon(x-y)u_f(y)dy = \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u_f(y)dy,$$

where $\Omega_\epsilon := \{x \in \mathbb{R}^2 \mid \operatorname{dist}(x,\Omega) < \epsilon\}$. It is known that $\|u_f^\epsilon - u_f\|_2 \to 0$ as $\epsilon \to 0$ and $u_f^\epsilon \in C_0^\infty(\Omega_\epsilon)$. See, e.g. [25].

Lemma 3.3.4 *If $u \in W^{1,2}(\Omega)$,*

$$\|u^\epsilon - u\| \leq C|u|_{W^{1,2}(\Omega)}\epsilon \tag{3.37}$$

for a positive constant C independent of ϵ and f.

Proof.

$$\begin{aligned} |u^\epsilon - u| &= \left| \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)\,dy - \int_{B(x,\epsilon)} \eta_\epsilon(x-y)\,dy u(x) \right| \\ &\leq \left| \int_{B(x,\epsilon)} \eta_\epsilon(x-y)(u(y)-u(x))\,dy \right| \end{aligned} \tag{3.38}$$

Rewrite $u(y) - u(x)$ in its integral form

$$u(y) - u(x) = \int_0^1 \nabla u(x + t(y-x)) \cdot (y-x)\,dt.$$

Note that $|y - x| \leq \epsilon$, then

$$|u(y) - u(x)| \leq \int_0^1 |\nabla u(x + t(y-x)|\,dt \cdot \epsilon.$$

Apply the above inequality to (3.38),

$$
\begin{aligned}
|u^\epsilon - u| &\leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y)-u(x)|\,dy \\
&\leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y) \int_0^1 |\nabla u(x+t(y-x))|\,dt \cdot \epsilon\,dy
\end{aligned}
$$

Now square both sides, and use the convexity of square function(twice)

$$
\begin{aligned}
|u^\epsilon - u|^2 &\leq \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y) \int_0^1 |\nabla u(x+t(y-x))|\,dt \cdot \epsilon\,dy \right)^2 \\
&\leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y) \left(\int_0^1 |\nabla u(x+t(y-x))|\,dt \right)^2 \epsilon^2\,dy \\
&\leq \int_{B(x,\epsilon)} \epsilon^2 \eta_\epsilon(x-y) \int_0^1 |\nabla u(x+t(y-x))|^2\,dt\,dy
\end{aligned}
$$

Integrate both sides over Ω and use the fact that η_ϵ is bounded by

$$
\eta_\epsilon(x) \leq \frac{C}{\epsilon^2},
$$

then

$$
\begin{aligned}
\int_\Omega |u^\epsilon - u|^2\,dx &\leq \int_\Omega \int_{B(x,\epsilon)} \epsilon^2 \eta_\epsilon(x-y) \int_0^1 |\nabla u(x+t(y-x))|^2\,dt\,dy\,dx \\
&\leq C \int_\Omega \int_{B(0,\epsilon)} \int_0^1 |\nabla u(x+tz))|^2\,dt\,dz\,dx \qquad (\text{let } z = y-x) \\
&\leq C \int_0^1 \int_{B(0,\epsilon)} \int_\Omega |\nabla u(x+tz)|^2\,dx\,dz\,dt \\
&\leq C|u|^2_{W^{1,2}(\Omega_\epsilon))}\epsilon^2 \\
&\leq C|u|^2_{W^{1,2}(\Omega))}\epsilon^2.
\end{aligned}
$$

Taking square root on both sides gives the result. ■

Our general plan to show $E(S_f) - E(u_f) \to 0$ is to establish the following sequence of inequalities:

$$E(u_f) \leq E(S_f) \leq E(Qu_f^\epsilon) \leq E(u_f^\epsilon) + \text{err}(|\triangle|, \epsilon) \leq E(u_f) + \text{err}_\epsilon + \text{err}(|\triangle|, \epsilon),$$

where Qu_f^ϵ is a spline approximation of u_f^ϵ as in Theorem 2.2 and $\text{err}(|\triangle|, \epsilon)$ and err_ϵ are error terms which will go to zero when ϵ and $|\triangle|$ go to zero.

We first show $E(u_f^\epsilon)$ approximates $E(u_f)$.

Lemma 3.3.5 *Let u_f^ϵ be the mollification of u_f defined above. Then $E(u_f^\epsilon)$ approximates $E(u_f)$, when $\epsilon \to 0$. In particular, when $u_f \in W^{1,2}(\Omega), E(u_f^\epsilon) - E(u_f) \leq C\epsilon$ for a positive constant dependent on u_f.*

Proof. First we claim that

$$E(u_f^\epsilon) \leq E(u_f) + err_\epsilon$$

for an error term err_ϵ

$$err_\epsilon = \int_{\Omega_\epsilon \setminus \Omega} \sqrt{1 + |\nabla u_f(y)|^2} dy + \frac{1}{2\lambda} \left(\|u_f^\epsilon - u_f\|_2^2 + 2\|u_f^\epsilon - u_f\|_2 \, \|u_f - f\|_2 \right) \qquad (3.39)$$

which will be shown to go to zero when $\epsilon \to 0$ below.

By the convexity of $\sqrt{1 + |t|^2}$ and the property of the mollifier, we have

$$\sqrt{1 + |\nabla u_f^\epsilon(x)|^2} = \sqrt{1 + \left| \int_{B(0,\epsilon)} \eta_\epsilon(x-y) \nabla u_f(y) dy \right|^2} \leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y) \sqrt{1 + |\nabla u_f(y)|^2} dy.$$

It follows that

$$
\begin{aligned}
\int_\Omega \sqrt{1+|u_f^\epsilon(x)|^2}dx &\le \int_\Omega \int_{B(x,\epsilon)} \eta_\epsilon(x-y)\sqrt{1+|\nabla u_f(y)|^2}dydx. \\
&\le \int_{\Omega_\epsilon} \sqrt{1+|\nabla u_f(y)|^2} \int_{B(y,\epsilon)} \eta_\epsilon(x-y)dxdy \\
&= \int_{\Omega_\epsilon} \sqrt{1+|\nabla u_f(y)|^2}dy. \\
&= \int_\Omega \sqrt{1+|\nabla u_f(y)|^2}dy + \int_{\Omega_\epsilon\setminus\Omega} \sqrt{1+|\nabla u_f(y)|^2}dy.
\end{aligned}
$$

By $u_f \in W^{1,1}(\mathbb{R}^2)$ and (3.36), it follows that

$$
\int_{\Omega_\epsilon\setminus\Omega} \sqrt{1+|\nabla u_f(y)|^2}dy \le \int_{\Omega_\epsilon\setminus\Omega} \left(1+|\nabla u_f(y)|\right)dy \to 0, \quad as\ \epsilon \to 0.
$$

Next we have

$$
\frac{1}{2\lambda}\|u_f^\epsilon - f\|_2^2 \le \frac{1}{2\lambda}\left(\|u_f^\epsilon - u_f\|_2^2 + 2\|u_f^\epsilon - u_f\|_2\|u_f - f\|_2 + \|u_f - f\|_2^2\right).
$$

Since $\|u_f^\epsilon - u_f\|_2 \to 0$ as explained above and $\|u_f - f\|_2$ is bounded because $\frac{1}{2\lambda}\|u_f - f\|^2 \le E(0)$,

$$
\frac{1}{2\lambda}\left(\|u_f^\epsilon - u_f\|_2^2 + 2\|u_f^\epsilon - u_f\|_2\|u_f - f\|_2\right) \to 0, \quad as\ \epsilon \to 0.
$$

This finishes the proof of our claim.

Clearly, $u_f^\epsilon \in W^{1,1}(\Omega) \subset BV(\Omega)$. As u_f is the minimizer in $BV(\Omega)$, it follows that

$$
E(u_f) \le E(u_f^\epsilon) \le E(u_f) + err_\epsilon,
$$

which implies $E(u_f^\epsilon)$ approximates $E(u_f)$ when $\epsilon \to 0$.

When $u_f \in W^{1,2}(\Omega)$, the above analysis applies. Together with (**??**), we use (3.39) to conclude

$$err_\epsilon \leq C|u_f|_{W^{1,2}(\Omega)}\epsilon.$$

■

We next estimate $E(Qu_f^\epsilon) - E(u_f^\epsilon)$. To do so, we need semi-norm $|u_f^\epsilon|_{W^{2,1}(\Omega_\epsilon)}$.

Lemma 3.3.6 *For any fixed $\epsilon > 0$, $u_f^\epsilon \in W^{2,1}(\Omega_\epsilon)$ and*

$$|u_f^\epsilon|_{W^{2,1}(\Omega_\epsilon)} \leq \frac{C}{\epsilon}|u_f|_{W^{1,1}(\Omega)} \tag{3.40}$$

for a constant $C > 0$.

Proof. Due to the mollification, $u_f^\epsilon \in W^{2,1}(\Omega_\epsilon)$. Letting D_1 denote the partial derivative with respect to the first variable, we consider $\|D_1D_1u_f^\epsilon\|_{L^1(\Omega_\epsilon)}$. Recall that $u_f^\epsilon(x) = \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u_f(y)dx$. We have

$$D_1D_1u_f^\epsilon = -\int_{B(x,\epsilon)} D_1u_f(y)D_1\eta_\epsilon(x-y)dy.$$

It follows that

$$\begin{aligned}
|D_1D_1u_f^\epsilon|_{L^1(\Omega_\epsilon)} &= \int_{\Omega_\epsilon}\left|\int_{B(x,\epsilon)} D_1u_f(y)D_1\eta_\epsilon(x-y)dy\right|dx \\
&\leq \int_{\Omega}\int_{B(x,\epsilon)} |D_1u_f(y)||D_1\eta_\epsilon(x-y)|dydx \\
&\leq \int_{\Omega_\epsilon} |D_1u_f(y)|\int_{B(y,\epsilon)} |D_1\eta_\epsilon(x-y)|dxdy \\
&= \int_{\Omega_\epsilon} |D_1u_f(y)|dy\int_{B(0,\epsilon)} |D_1\eta_\epsilon(x)|dx.
\end{aligned}$$

Since

$$|D_1\eta_\epsilon(x)| \leq \frac{8C\exp(-2)}{\epsilon^3},$$

we have

$$|D_1D_1u_f^\epsilon|_{L^1(\Omega_\epsilon)} \leq \|D_1u_f\|_{L^1(\Omega)}\frac{C\exp(-2)}{\epsilon^3}\int_{B(0,\epsilon)}dx \leq |u_f|_{W^{1,1}(\Omega)}\frac{C'}{\epsilon} \tag{3.41}$$

for a positive constant C'. Using the similar arguments, we can show that $\|D_1D_2u_f^\epsilon\|_{L^1(\Omega_\epsilon)}$ and $\|D_2D_2u_f^\epsilon\|_{L^1(\Omega_\epsilon)}$ have the same upper bound as in (3.41). And thus, we prove that

$$|u_f^\epsilon|_{W^{2,1}(\Omega_\epsilon)} \leq \frac{C}{\epsilon}|u_f|_{W^{1,1}(\Omega)}$$

for another positive constant $C > 0$. ■

Recall $\triangle$ is a triangulation of Ω. Let $\triangle' = \{t_i\}$ be a new triangulation of Ω_ϵ with $\triangle \subset \triangle'$ and $|\triangle'| = |\triangle|$. Using Theorem 2.1.6, we can choose $Qu_f^\epsilon \in \mathcal{S}(\triangle')$ such that

$$\|D_1^\alpha D_2^\beta(Q(u_f^\epsilon) - u_f^\epsilon)\|_{L^1(\Omega_\epsilon)} \leq C|\triangle|^{2-\alpha-\beta}|u_f^\epsilon|_{W^{2,1}(\Omega_\epsilon)} \tag{3.42}$$

for all $\alpha + \beta = 1$.

Lemma 3.3.7 *Let $\tilde{s} := Qu_f^\epsilon|_\Omega$ be the restriction of Qu_f^ϵ on Ω which is a spline in $\mathcal{S}$. Then $E(\tilde{s})$ approximates $E(u_f^\epsilon)$, when $\frac{|\triangle|}{\epsilon} \to 0$.*

Proof. We first estimate the difference between $|E(\tilde{s}) - E(u_f^\epsilon)|$ by

$$\begin{aligned}
& |E(\tilde{s}) - E(u_f^\epsilon)| \\
\leq & \left|\int_\Omega \sqrt{1+|\nabla\tilde{s}|^2} - \sqrt{1+|\nabla u_f^\epsilon|^2}dx\right| + \frac{1}{2\lambda}\left|\|\tilde{s}-f\|_2^2 - \|u_f^\epsilon - f\|_2^2\right| \\
\leq & \left|\int_\Omega \sqrt{1+|\nabla\tilde{s}|^2} - \sqrt{1+|\nabla u_f^\epsilon|^2}dx\right| + \frac{1}{2\lambda}\left(\|\tilde{s}-u_f^\epsilon\|_2^2 + 2\|\tilde{s}-u_f^\epsilon\|_2\,\|u_f^\epsilon - f\|_2\right).
\end{aligned}$$

Let $err(|\triangle|, \epsilon)$ be the term on the right-hand side of the inequality above. Let us show that $err(|\triangle|, \epsilon) \to 0$, as $|\triangle|/\epsilon \to 0$. Note that $Q(u_f^\epsilon)$ is supported over Ω_ϵ by the construction of quasi-interpolatory operator Q.

$$\|\nabla(\tilde{s}-u_f^\epsilon)\|_{L^1(\Omega)} \leq \|\nabla(Q(u_f^\epsilon)-u_f^\epsilon)\|_{L^1(\Omega_\epsilon)} \leq C|\triangle'||u_f^\epsilon|_{W^{2,1}(\Omega_\epsilon)} \leq \frac{C|\triangle|}{\epsilon}|u_f|_{W^{1,1}(\Omega)} \quad (3.43)$$

by using the inequality in (3.42) with $\alpha+\beta=1$. Hence, we have

$$\begin{aligned}
\left|\int_\Omega \sqrt{1+|\nabla\tilde{s}|^2}-\sqrt{1+|\nabla u_f^\epsilon|^2}dx\right| &= \left|\int_\Omega \frac{|\nabla\tilde{s}|^2-|\nabla u_f^\epsilon|^2}{\sqrt{1+|\nabla\tilde{s}|^2}\sqrt{1+|\nabla u^{\epsilon_f}|^2}}dx\right| \\
\leq & \int_\Omega \frac{|\nabla\tilde{s}-\nabla u_f^\epsilon||\nabla\tilde{s}+\nabla u_f^\epsilon|}{\sqrt{1+|\nabla\tilde{s}|^2}\sqrt{1+|\nabla u_f^\epsilon|^2}}dx \\
\leq & \|\nabla(\tilde{s}-u_f^\epsilon)\|_{L^1(\Omega)} \\
\leq & C'\frac{|\triangle|}{\epsilon}|u_f|_{W^{1,1}(\Omega)}.
\end{aligned}$$

Here both C and C' are positive constants independent of $\epsilon, \triangle, u_f$.

It is not hard to see the quantity $\|u_f^\epsilon - f\|_2$ is bounded because

$$\|u_f^\epsilon - f\|_2 \leq \|u_f^\epsilon - u_f\|_2 + \|u_f - f\|_2 \leq 1 + \sqrt{2\lambda A_\Omega}\|f\|_2$$

as $\|u_f^\epsilon - u_f\|_2 \leq 1$ if ϵ small enough and by using the property of the minimizer u_f. By using the well-known Sobolev inequality: for any function $g \in W^{1,1}(\Omega_\epsilon)$,

$$\|g\|_{L^2(\Omega_\epsilon)} \leq C|\nabla g|_{L^1(\Omega_\epsilon)}$$

for Ω_ϵ with C^1 boundary (cf. [25]), we have,

$$\|\tilde{s}-u_f^\epsilon\|_{L^2(\Omega)} \leq \|\tilde{s}-u_f^\epsilon\|_{L^2(\Omega_\epsilon)} \leq C\|\nabla(\tilde{s}-u_f^\epsilon)\|_{L^1(\Omega_\epsilon)} \leq \frac{C|\triangle|}{\epsilon}|u_f|_{W^{1,1}(\Omega)}$$

by (3.43). Therefore, we conclude that $err(|\triangle|,\epsilon) \to 0$, as $|\triangle|/\epsilon \to 0$, and thus $E(\tilde{s})$

approximates $E(u_f^\epsilon)$. ■

Summarizing the discussion above, we have

Theorem 3.3.2 *Suppose that* $u_f \in W^{1,1}(\Omega)$. *Then* S_f *approximates* u_f *in* $L^2(\Omega)$ *when* $|\triangle| \to 0$. *In particular, when* $u_f \in W^{1,2}(\Omega)$,

$$\|S_f - u_f\|_2 \leq C|u_f|_{W^{1,2}(\Omega)}\sqrt{|\triangle|}$$

for a positive constant C *independent of* $|\triangle|$.

Proof. Since $\mathcal{S} \subset W^{1,1}(\Omega)$, we have $E(u_f) \leq E(S_f)$. Also $\tilde{s} \in \mathcal{S}$ implies $E(S_f) \leq E(\tilde{s})$. By Lemmas 3.3.5 and 3.3.7, we have

$$E(u_f) \leq E(S_f) \leq E(\tilde{s}) \leq E(u_f^\epsilon) + err(|\triangle|, \epsilon) \leq E(u_f) + err_\epsilon + err(|\triangle|, \epsilon).$$

For each triangulation $\triangle$, we choose $\epsilon = \sqrt{|\triangle|}$ which ensures $|\triangle|/\epsilon \to 0$. Thus, the above error terms go to zero as $|\triangle| \to 0$. By Lemma 3.3.3, S_f converges to u_f in L^2 norm.

When $u_f \in W^{1,2}(\Omega)$, we have $err_\epsilon \leq \epsilon C|u_f|_{W^{1,2}(\Omega)}$ and $err(|\triangle|, \epsilon) \leq C\sqrt{|\triangle|}|u_f|_{W^{1,2}(\Omega)}$ as we trivially have $|u_f|_{W^{1,1}(\Omega)} \leq C|u_f|_{W^{1,2}(\Omega)}$ with positive constant C dependent only on A_Ω. These complete the proof. ■

3.4 A Fixed Point Algorithm and Its Convergence

The following iterations will be used to approximate S_f.

Algorithm 3.4.1 *Given* $u^{(k)}$, *we find* $u^{(k+1)} \in \mathcal{S}$ *such that*

$$\int_\Omega \frac{\nabla u^{(k+1)} \cdot \nabla \phi_j}{\sqrt{1 + |\nabla u^{(k)}|^2}} dx + \frac{1}{\lambda A_\Omega} \int_\Omega u^{(k+1)} \phi_j dx = \frac{1}{\lambda A_\Omega} \int_\Omega f \phi_j dx, \quad \text{for all } j = 1, ..., n. \quad (3.44)$$

We first show that the above iteration is well defined. Since $u^{(k+1)} \in \mathcal{S}$, it can be written as $u^{(k+1)} = \sum_i^n c_i^{(k+1)} \phi_i$. Plugging it in (4.31), we have

$$\sum_i^n c_i^{(k+1)} \left(\int_\Omega \frac{\nabla \phi_i \cdot \nabla \phi_j}{\sqrt{1 + |\nabla u^{(k)}|^2}} dx + \frac{1}{\lambda A_\Omega} \int_\Omega \phi_i \phi_j dx \right) = \frac{1}{\lambda A_\Omega} \int_\Omega f \phi_j dx, \quad j = 1, ..., n. \tag{3.45}$$

Denote by

$$\begin{aligned} D^{(k)} &:= (d_{i,j}^{(k)})_{N \times N} \text{ with } d_{i,j}^{(k)} = \lambda \int_\Omega \frac{\nabla \phi_i \cdot \nabla \phi_j}{\sqrt{1 + |\nabla u^{(k)}|^2}} dx, \\ M &:= (m_{i,j})_{N \times N} \text{ with } m_{i,j} = \frac{1}{A_\Omega} \int_\Omega \phi_i \phi_j dx, \\ \mathbf{v} &:= (v_j, j = 1, \cdots, N) \text{ with } v_j = \frac{1}{A_\Omega} \int_\Omega f \phi_j dx. \end{aligned}$$

Then to solve equation of (3.45) is equivalent to solving the equation

$$(D^{(k)} + M)\mathbf{c}^{(k+1)} = \mathbf{v}, \tag{3.46}$$

where $\mathbf{c}^{(k+1)} = [c_1^{(k+1)}, c_2^{(k+1)}, ..., c_n^{(k+1)}]^T$. Furthermore, if imposing the smoothness conditions $H\mathbf{c} = 0$ in Theorem 2.1.2, we need to solve the following linear systems

$$\begin{pmatrix} H' & D^{(k)} + M \\ 0 & H \end{pmatrix} \begin{pmatrix} \alpha \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ 0 \end{pmatrix}. \tag{3.47}$$

Lemma 3.4.1 *(3.46) has a unique solution* $\mathbf{c}^{(k+1)}$.

Proof. It is easy to prove $D^{(k)}$ is semi-positive definite and M is positive definite because, for any nonzero $\mathbf{c} = (c_i)_n$,

$$\mathbf{c}^T D^{(k)} \mathbf{c} = \lambda \int_\Omega \frac{|\sum_i^n c_i \nabla \phi_i|^2}{\sqrt{1 + |\nabla u^{(k)}|^2}} dx \geq 0,$$

and

$$\mathbf{c}^T\ M\ \mathbf{c} = \frac{1}{A_\Omega}\int_\Omega |\sum_i^n c_i\phi_i|^2 dx > 0.$$

Moreover $(D^{(k)} + M)$ is also positive definite, and hence invertible. So (3.46) has a unique solution. ■

Lemma 3.4.2 *$\{u^{(k)}, k = 1, 2, \cdots\}$ are bounded in L^2 norm by $\|f\|_2$ for all $k > 0$. That is,*

$$\|u^{(k+1)}\|_2 \leq \|f\|_2. \tag{3.48}$$

Also, there exists a positive constant C dependent on β and $|\triangle|$ such that

$$\|\nabla u^{(k+1)}\|_2 \leq C\|f\|_2.$$

Proof. Multiply $(\mathbf{c}^{(k+1)})^T$ to both hand-side of (3.46), we have

$$\lambda\int_\Omega \frac{|\nabla u^{(k+1)}|^2}{\sqrt{1+|\nabla u^{(k)}|^2}}dx + \frac{1}{A_\Omega}\int_\Omega |u^{(k+1)}|^2 dx = \frac{1}{A_\Omega}\int_\Omega f u^{(k+1)} dx. \tag{3.49}$$

Since the first term of (3.49) are nonnegative, we have

$$\|u^{(k+1)}\|_2^2 \leq \frac{1}{A_\Omega}\int_\Omega f u^{(k+1)} \leq \|f\|_2 \|u^{(k+1)}\|_2 \tag{3.50}$$

which yields

$$\|u^{(k+1)}\|_2 \leq \|f\|_2,$$

and hence $\|u^{(k+1)}\|_2$ is bounded if $f \in L^2(\Omega)$.

Let λ_{min} is the smallest eigenvalue of M, then $\lambda_{min} > 0$ by the positivity of M. Since

$$\|u^{(k)}\|_2^2 = (c^{(k)})^t * M * c^{(k)},$$

then

$$\lambda_{min}\|c^{(k)}\|_2^2 \leq \|u^{(k)}\|_2^2 \leq \|f\|_2^2,$$

which implies

$$\|c^{(k)}\|_2^2 \leq \frac{1}{\lambda_{min}}\|f\|_2^2$$

Consider the matrix $P = (p_{i,j})_{n\times n}$, where $p_{i,j} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j dx$. Then $\|\nabla u^{(k)}\|_2^2 = (c^{(k)})^t * P * c^{(k)}$. It is easy to see that P is also semi-positive definite . Suppose π_{max} is the largest eigenvalue of P, then $\pi_{max} \geq 0$, and

$$\|\nabla u^{(k)}\|_2^2 = (c^{(k)})^t * P * c^{(k)} \leq \pi_{max}\|c^{(k)}\|_2^2 \leq \frac{\pi_{max}}{\lambda_{min}}\|f\|_2^2$$

or

$$\|\nabla u^{(k)}\|_2 \leq \sqrt{\frac{\pi_{max}}{\lambda_{min}}}\|f\|_2 \tag{3.51}$$

which finishes the proof. ■

Next we need to show that the iterative algorithm above converges. We need the following inequality. Note that the proof of the inequality is different from the one in Lemma 3.3.3. The reason is that $u^{(k+1)}$ is not a minimizer of $E(u)$ in $\mathcal{S}$. Thus the technique of the sub-differentiation can not be applied here. We have to give a different proof.

Lemma 3.4.3 *If $u^{(k+1)}$ is the solution of our Algorithm 3.4.1, then the following inequality holds*

$$\|u^{(k)} - u^{(k+1)}\|^2 \leq 2\lambda A_\Omega(E(u^{(k)}) - E(u^{(k+1)})). \tag{3.52}$$

Proof. First of all we use (4.31) to have

$$\frac{1}{\lambda A_\Omega}\int_\Omega (f - u^{(k+1)})(u^{(k)} - u^{(k+1)})dx = \int_\Omega \frac{\nabla u^{(k)} \cdot \nabla u^{(k+1)}}{\sqrt{1 + |\nabla u^{(k)}|^2}}dx - \int_\Omega \frac{|\nabla u^{(k+1)}|^2}{\sqrt{1 + |\nabla u^{(k)}|^2}}dx$$

since $u^{(k)} - u^{(k+1)}$ is a linear combination of $\phi_j, j = 1, \cdots, n$. Then the following inequality follows.

$$\frac{1}{\lambda A_\Omega}\int_\Omega (f-u^{(k+1)})(u^{(k)}-u^{(k+1)})dx \le \int_\Omega \frac{|\nabla u^{(k)}|^2}{2\sqrt{1+|\nabla u^{(k)}|^2}}dx - \int_\Omega \frac{|\nabla u^{(k+1)}|^2}{2\sqrt{1+|\nabla u^{(k)}|^2}}dx. \quad (3.53)$$

Now we are ready to prove (3.52). The difference between $E(u^{(k)})$ and $E(u^{(k+1)})$ is

$$\begin{aligned}
& E(u^{(k)}) - E(u^{(k+1)}) \\
= & \int_\Omega \sqrt{1+|\nabla u^{(k)}|} - \sqrt{1+|\nabla u^{(k+1)}|^2}dx + \frac{1}{2\lambda A_\Omega}\int_\Omega |u^{(k)}-f|^2 - |u^{(k+1)}-f|^2 dx \\
= & \int_\Omega \sqrt{1+|\nabla u^{(k)}|} - \sqrt{1+|\nabla u^{(k+1)}|^2}dx + \frac{1}{2\lambda A_\Omega}\int_\Omega (u^{(k)}-u^{(k+1)})(u^{(k)}+u^{(k+1)}-2f)dx \\
= & \int_\Omega \sqrt{1+|\nabla u^{(k)}|} - \sqrt{1+|\nabla u^{(k+1)}|^2}dx + \int_\Omega \frac{1}{\lambda A_\Omega}(u^{(k+1)}-f)(u^{(k)}-u^{(k+1)}) \\
& + \frac{1}{2\lambda A_\Omega}\int_\Omega |u^{(k)}-u^{(k+1)}|^2 dx
\end{aligned}$$

which yields the result of this lemma since the first two terms in the last equation above is not negative. Indeed, by applying (4.35), we have

$$\begin{aligned}
& \int_\Omega \sqrt{1+|\nabla u^{(k)}|} - \sqrt{1+|\nabla u^{(k+1)}|^2}dx + \frac{1}{\lambda A_\Omega}\int_\Omega (u^{(k+1)}-f)(u^{(k)}-u^{(k+1)}) \\
\ge & \int_\Omega \sqrt{1+|\nabla u^{(k)}|^2} - \sqrt{1+|\nabla u^{(k+1)}|^2}dx - \int_\Omega \frac{|\nabla u^{(k)}|^2}{2\sqrt{1+|\nabla u^{(k)}|^2}}dx + \int_\Omega \frac{|\nabla u^{(k+1)}|^2}{2\sqrt{1+|\nabla u^{(k)}|^2}}dx \\
= & \int_\Omega \frac{2+|\nabla u^{(k)}|^2+|\nabla u^{(k+1)}|^2}{2\sqrt{1+|\nabla u^{(k)}|^2}} - \sqrt{1+|\nabla u^{(k+1)}|^2}dx \\
\ge & \int_\Omega \frac{\sqrt{1+|\nabla u^{(k)}|^2}\sqrt{1+|\nabla u^{(k+1)}|^2}}{\sqrt{1+|\nabla u^{(k)}|^2}} - \sqrt{1+|\nabla u^{(k+1)}|^2}dx = 0.
\end{aligned}$$

We have thus established the proof. ■

We are ready to show the convergence of $u^{(k)}$ to the minimizer S_f.

Theorem 3.4.1 *The sequence* $\{u^{(k)}, k = 1, 2, \cdots, \}$ *obtained from Algorithm 3.4.1 converges to the true minimizer* S_f.

Proof. By Lemma 3.4.2, the sequence $\{u^{(k)}, k = 1, \cdots, \}$ is bounded. Actually, we know $\|u^{(k)}\|_2 \leq \|f\|_2$. So there must be a convergent subsequence $\{u^{(n_j)}, n_1 < n_2 < \cdots <\}$. Suppose $u^{(n_j)} \to \bar{u}$. By Lemma 4.4.2, we see $\{E(u^{(k)}), k = 1, 2, \cdots, \}$ is a decreasing sequence and bounded below, so $\{E(u^{(k)})\}$ is convergent as well as any subsequence of it. We use Lemma 4.4.2 to have

$$\begin{aligned} \|u^{(n_j+1)} - \bar{u}\|_2^2 &\leq 2\|u^{(n_j+1)} - u^{(n_j)}\|_2^2 + 2\|u^{(n_j)} - \bar{u}\|_2^2 \\ &\leq 4\lambda A_\Omega (E(u^{(n_j)}) - E(u^{(n_j+1)})) + 2\|u^{(n_j)} - \bar{u}\|_2^2 \to 0, \end{aligned}$$

which implies $u^{(n_j+1)} \to \bar{u}$.

According to Markov's inequality, i.e. Theorem 2.1.5, we have

$$\int_\Omega |\nabla u^{(n_j)} - \nabla \bar{u}|^2 dx \leq \frac{\beta^2}{|\triangle|^2} \int_\Omega |u^{(n_j)} - \bar{u}|^2 dx. \tag{3.54}$$

It follows from the convergence of $u^{(n_j)} \to \bar{u}$ that $\nabla u^{(n_j)} \to \nabla \bar{u}$ in L^2 norm as well. Replacing $u^{(n_j)}$ by $u^{(n_j+1)}$ above, we have $\nabla u^{(n_j+1)} \to \nabla \bar{u}$ too by the convergence of $u^{(n_j+1)} \to \bar{u}$. As $u^{(n_j)}$, $u^{(n_j+1)}$ and $\bar{u}$ are spline functions in $\mathcal{S}$. The convergence of $u^{(n_j)}$ and $u^{(n_j+1)}$ to $\bar{u}$, respectively implies the coefficients of $u^{(n_j)}$ and $u^{(n_j+1)}$ in terms of the basis functions $\phi_j, j = 1, \cdots, N$ are convergent to the coefficients of $\bar{u}$, respectively.

Since $u^{(n_j+1)}$ solves the equations (4.31), we have

$$\int_\Omega \frac{\nabla u^{(n_j+1)} \cdot \nabla \phi_j}{\sqrt{1 + |\nabla u^{(n_j)}|^2}} dx = \int_\Omega \frac{f - u^{(n_j+1)}}{\lambda A_\Omega} \phi_i dx \tag{3.55}$$

for all $\phi_i, i = 1, \cdots, N$. Letting $j \to \infty$, we obtain

$$\int_\Omega \frac{\nabla \bar{u} \cdot \nabla \phi_i}{\sqrt{1 + |\nabla \bar{u}|^2}} dx = \int_\Omega \frac{f - \bar{u}}{\lambda A_\Omega} \phi_i dx \tag{3.56}$$

for all $i = 1, \cdots, N$. That is, $\bar{u}$ is a local minimizer. Since the functional is convex, a local minimizer is the global minimizer and hence, $\bar{u} = S_f$. Thus all convergent subsequences of $\{u^{(k)}\}$ converge to S_f. ■

3.5 Numerical Results

We have implemented our bivariate spline approach in MATLAB and performed several image enhancement experiments: image inpainting, image rescaling and wrinkle removal. We shall briefly explain how to choose a polygonal domain, how to triangulate a polygonal domain, how to use a bivariate spline space in following subsections. After these, we report our numerical results.

We mainly use the standard Delaunay triangulation algorithm to find a triangulation of a polygon. A key ingredient is to choose boundary points as equally-spaced as possible and points inside the polygon as evenly-distributed as possible. If some points are clustered near the boundary of the polygon, we have to thin a few point off. In addition, we check the triangles from the Delaunay triangulation method to see which one is outside of the domain and delete such triangles. Triangles in triangulations from our MATLAB code have almost uniform in size and in area. In our computation we mainly use Bivariate Splines $S_3^1(\Omega)$, $S_5^1(\Omega)$ and $S_7^1(\Omega)$.

Example 3.5.1 (Image Inpainting) *In Fig. 3.1, we show a damaged image and the recovered image by using the minimal surface area fitting splines. Assume that we can find the locations of damaged parts, we use the known image values to recover the loss image* *data values. In the computation, we used the triangulations in Fig. 3.2. Fig 3.3 and Fig 3.4 demonstrate two examples of our approach to recover specified regions of damaged images which have* 50% *lost and* 60% *data lost respectively. However, if there is a large number of triangles in our triangulations, such as the case in Fig.3.2, then the size of linear system we need to solve in 3.47 will be huge, because the size of* $D^{(k)}$ *is* $R\binom{d+2}{2}$*-by-*$R\binom{d+2}{2}$*, where* R *is the number of triangles, and* d *is the degree of polynomial of the Bivariate splines we use. This might cause the "Out of Memory" problem in some computers. To solve this problem, we use the domain decomposition technique developed in [37], which can transfer the problem into solving a set of linear systems of small size.*

Figure 3.1: Inpainting domain are marked by two blacked words. $d = 3$ and $r = 1$ are used in our computation.

Example 3.5.2 (Image Scaling) *Due to the accuracy of evaluation of spline funtions, we can apply the minimum surface area fitting splines to image rescaling and compared our method to the bicubic and bilinear interpolation methods provided by MATLAB function "imresize". In Fig. 3.5, the two images showed in the first column(not in actual size) are*

Figure 3.2: These two triangulations are used in our computation.

scaled by 10. One can notice that both the bicubic and bilinear interpolation methods lead to the gibbs discontinuity effect at the edges of the two images, while our approach barely show any such effect. In Fig. 3.6 and Fig. 3.7, our bivariate splines method is compared to the bilinear method. Since the bilinear method only works on rectangular domain, the whole rectangular region, including the dark area is enlarged. A shortcoming of this method is the discontinuity across the boundary between the dark area and image leads to the jagged edges on the boundary of the image. For our bivariate splines, due to the capability to approximate functions on a flexible polygon domain, only the image itself is enlarged, which avoids the jagged edges problem.

Example 3.5.3 (Wrinkle Removal) *Finally we present a wrinkle removal experiment. We are interested in reducing some wrinkles from a human face. We identify a couple of regions of interest near eyes and cheeks and apply our bivariate spline approach over each*

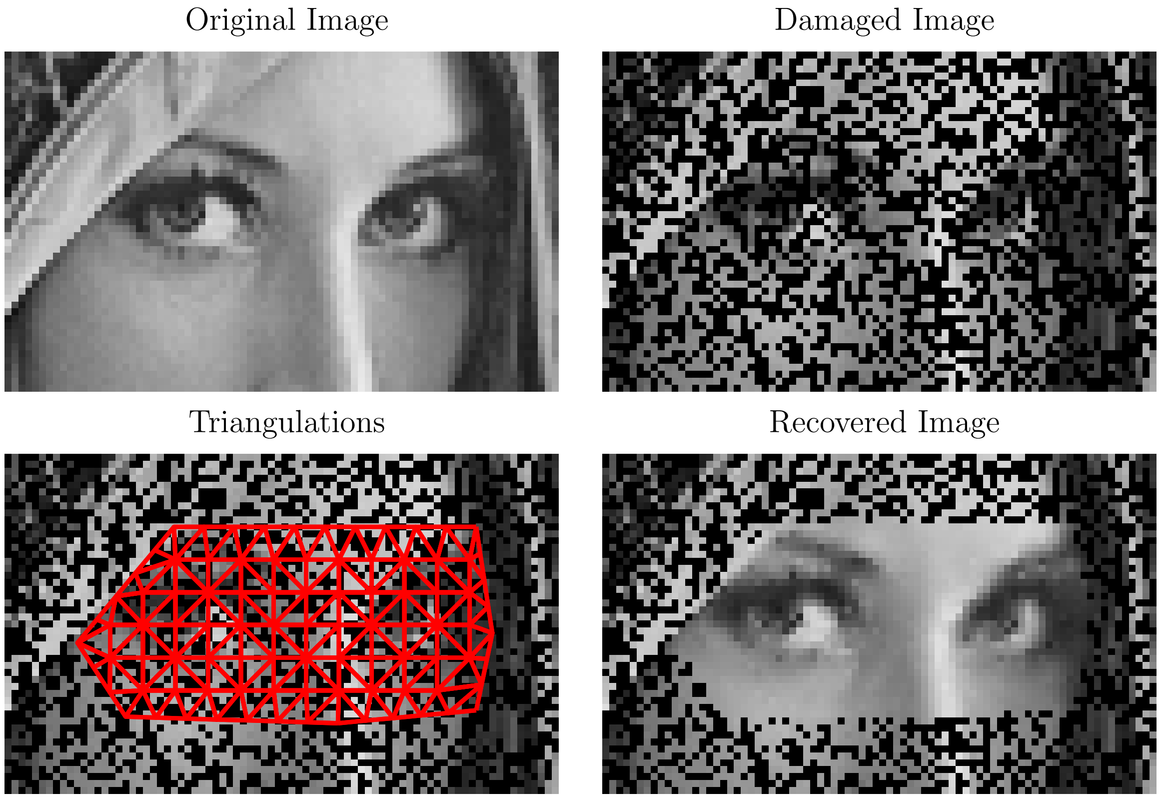


Figure 3.3: Inpainting: Recovering specified region of an image of 50% data lost.

region. In Fig. 3.8, two images are shown. The human face on the right is clearly enhanced in the areas near eyes and cheeks.

Example 3.5.4

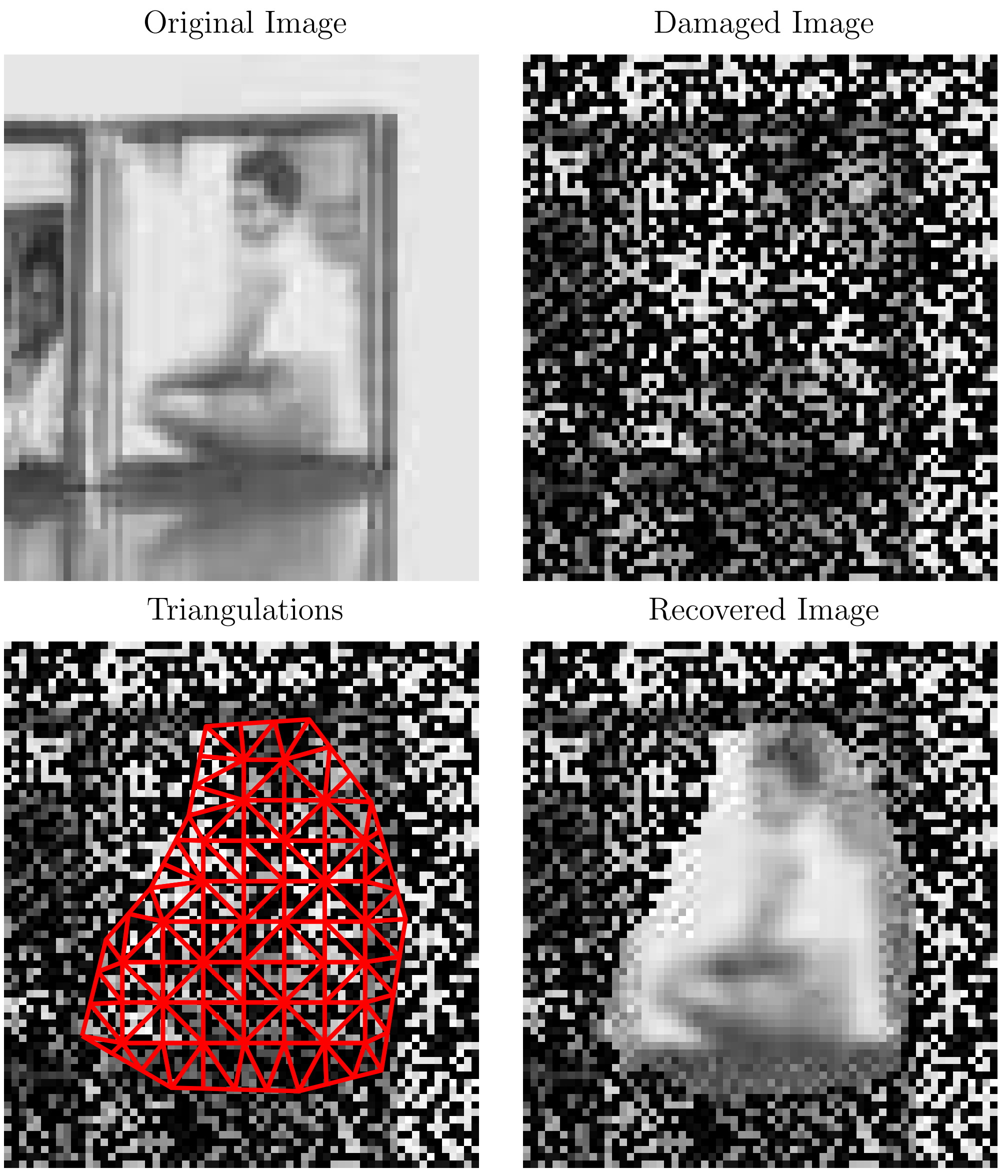


Figure 3.4: Inpainting: Recovering specified region of an image of 60% data lost.

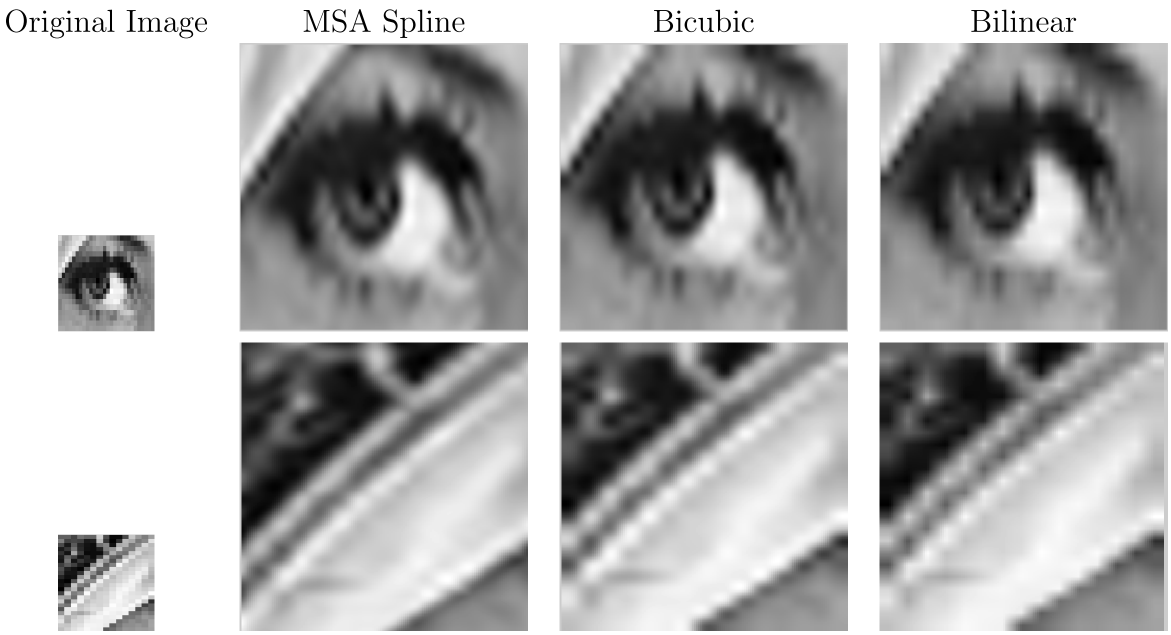


Figure 3.5: Images are scaled by 10 by using our spline method($d = 7, r = 1$), Bicubic and Bilinear interpolation respectively.

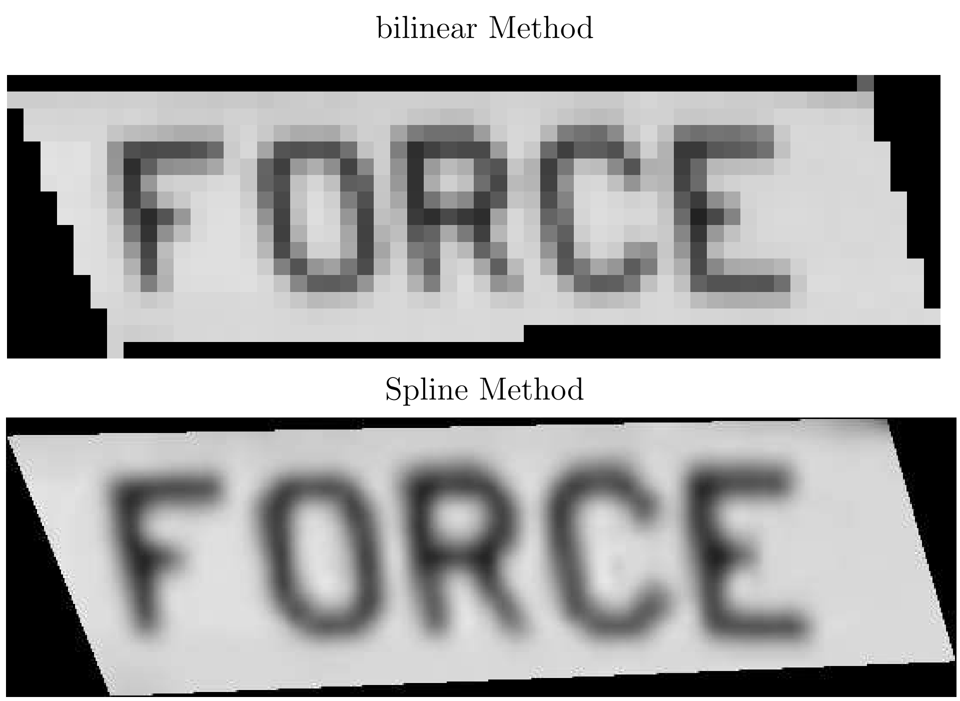


Figure 3.6: Bilinear vs. Bivariate Splines

bilinear Method Spline Method

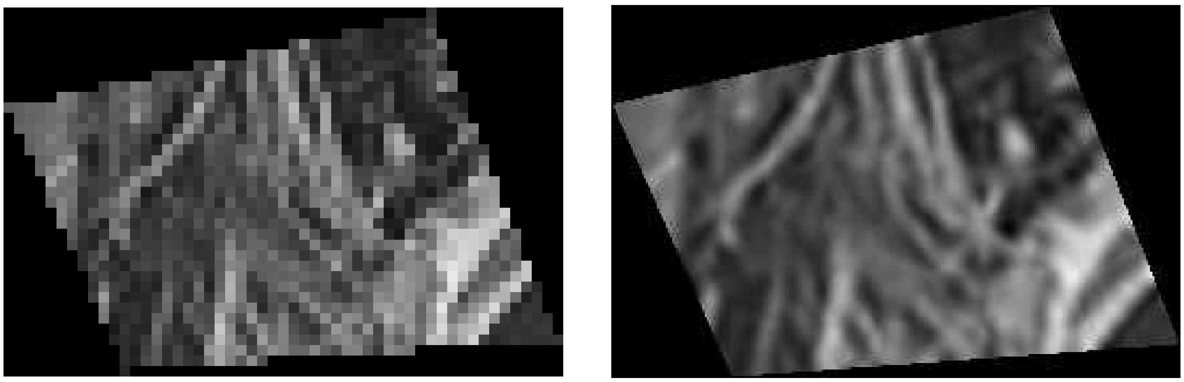


Figure 3.7: Bilinear vs. Bivariate Splines

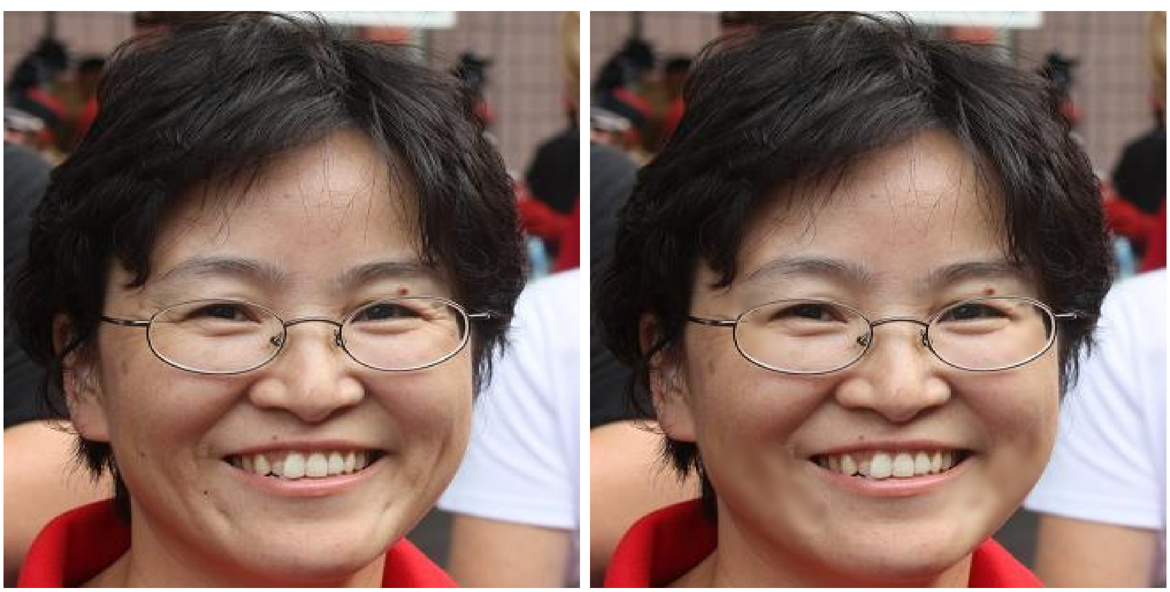


Figure 3.8: A face with wrinkles on the left and the face with reduced wrinkles on the right. We use $d = 3$ and $r = 1$ in our computation.

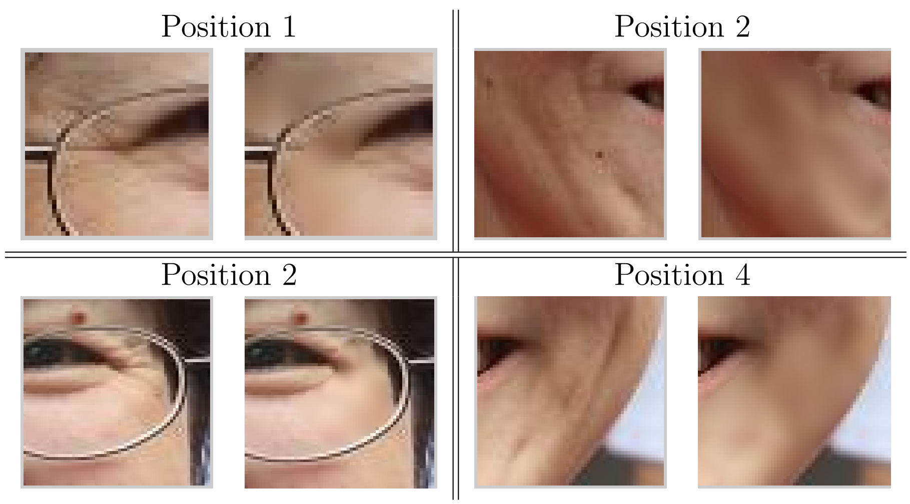


Figure 3.9: Enlarged pictures of 4 positions where wrinkles are removed.

Chapter 4

Bivariate Splines Approximation to a TV-L^p Model

Recall that the ROF model can be given in the following equivalent form:

$$\min_{u\in \mathrm{BV}(\Omega)} |u|_{BV} + \frac{1}{2\lambda}\int_\Omega |u-f|^2 dx. \tag{4.1}$$

Many extensions of the ROF model can be found in the literature [2], [3], and [4], [38], [39], [14], [54], [55], and [22]. For example, the following TV-L^1 model is closely related to the original ROF model:

$$\min_{u\in \mathrm{BV}(\Omega)} |u|_{BV} + \frac{1}{\lambda}\int_\Omega |u-f| dx. \tag{4.2}$$

The above TV-L^1 model was studied in the context of image denoising and deblurring by many researchers as mentioned in the literatures above. As the energy functional

$$\mathcal{E}(u) = |u|_{BV} + \frac{1}{\lambda}\int_\Omega |u-f| dx$$

is convex and lower semi-continuous, it is easy to see the existence of the minimizers. However, due the fact that the energy functional is not strictly convex, the minimizers are not unique. Many results related to the set of minimizers were obtained. For example, properties of maximum principle, monotonicity, commutation with constants, affine invariance and contract invariance are studied. See [14], [55], and [22].

In addition to the TV-L^1 model, the following TV-L^p model has been proposed in the literature:

$$\min_{u\in \mathrm{BV}(\Omega)} |u|_{BV} + \frac{1}{p\lambda}\int_\Omega |u-f|^p dx \tag{4.3}$$

for $p \geq 1$. Besides $p = 1$ and $p = 2$, it is not clear in the literature that if this TV-L^p model makes sense. We shall present a statistical explanation together with numerical examples to show when the noises of an image are from random variables of p-exponential distribution with $p \neq 2$, the TV-L^p model will be useful. Thus we shall discuss some properties of the minimizers of the TV-L^p model. To compute the minimizers, we shall use the following (ϵ, η)-version of TV-L^p model for $p \geq 1$.

$$\min_{u\in \mathrm{BV}(\Omega)} \int_\Omega \sqrt{\epsilon + |\nabla u|^2} dx + \frac{1}{p\lambda}\int_\Omega (\eta + |u-f|^2)^{p/2} dx. \tag{4.4}$$

for $\epsilon, \eta > 0$.

It is shown that when $\epsilon \to 0$,

$$J_\epsilon(u) := \int_\Omega \sqrt{\epsilon + |\nabla u|^2} dx.$$

converges to $|u|_{BV}$. It is easy to see that when $\eta \to 0$, the second term in the minimization (4.4) converges to $\|f-u\|_p^p$. Hence, the (ϵ, η) version of TV-L^p model gives an approximation of the standard TV-L^p model (4.3).

When $\epsilon, \eta > 0$, the energy functional $E_{\epsilon,\eta}$ associated with (4.4) is strictly convex and hence, the minimizer $u_f := u_{f,\epsilon,\eta}$ is unique in $\mathrm{BV}(\Omega)$. We shall show that the minimizers of the (ϵ, η)-version of the TV-L^p has the stable and continuous properties. It is clear that we can find the minimizers of the TV-L^p model in a finite dimensional space, e.g. finite element space, or more generally bivariate spline space, $S_d^r(\triangle)$ which is the spline space of degree d and smoothness r over triangulation $\triangle$. The key point is that the minimizers in the finite dimensional space converge to the minimizers in the BV(Ω). The main result in this paper is the following

Theorem 4.0.1 *Fix $p \geq 1$ $\eta > 0$ and $\epsilon > 0$. Suppose f is bounded and u_f is the minimizer of (4.4). Suppose that a spline space $\mathcal{S}$ contains $S_d^r(\triangle)$ for a degree $d \geq 3r+2$ as a subspace. Let S_f be the minimizer of*

$$\min_{u \in \mathcal{S}} \int_\Omega \sqrt{\epsilon + |\nabla u|^2} dx + \frac{1}{p\lambda} \int_\Omega |u - f|^p dx \tag{4.5}$$

in $\mathcal{S}$. Then

$$\|S_f - u_f\|_2 \leq C\lambda\sqrt{|\triangle|}, \tag{4.6}$$

where C is a positive constant independent of $|\triangle|$, the size of triangulation $\triangle$.

This study leads to a numerical approach to compute an approximation of the minimizer of the TV-L^p model for $p \geq 1$. An iterative algorithm will be derived to solve the associated nonlinear problem in the finite dimensional space. We shall show that the numerical algorithm converges. Several numerical examples will be shown to demonstrate the effectiveness and convenience for image denoising.

This chapter is organized as follows. First, we shall give a statistical explanation of the TV-L^p model in §4.1. We then study the properties of the minimizers in §4.2. With these preparation, we discuss how to approximate the minimizer of the TV-L^p model using spline spaces in §4.3 and present an iterative algorithm and show that the iterative algorithm is convergent in §5. Finally we show two examples to demonstrate that the TV-L^p model is indeed useful to denoise images whose contains some non-Gaussian noises.

4.1 A Statistical Explanation of the TV-L^p Model

Recall that the classic Rudin-Osher-Fetami (ROF) model for image denoising is the following minimization:

$$\min_{u\in BV(\Omega)}\{|u|_{BV}, \quad \text{subject to } var(u-f)\le \sigma_0^2\} \tag{4.7}$$

where $f = u_0 + \xi$ is a given noised image and u_0 is the original image, and $var(\cdot)$ stands for the standard variance. The minimizer u_f is supposed to be the clean image which is expected to be close to the original image u_0. Thus it is expected that $u_f - f$ is very close to ξ, i.e.,

$$u_f - f \simeq \xi. \tag{4.8}$$

In the discrete setting, suppose that $f = \{f_i, \cdots, f_n\}$ is a given image with $f_i = u_i + \xi_i$, where u_i are pixel values of grayscale and ξ_i are noise values. Suppose that ξ_i are from independent identically distributed random variables whose mean is zero and variance σ^2. A standard method to estimate σ^2 is

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\xi_i - \bar{\xi})^2,$$

where $\bar{\xi} = \dfrac{1}{n}\sum_{i=1}^n \xi_i$. It is known that s^2 is an unbiased estimator. That is, $E(s^2) = \sigma^2$. Note that $\bar{\xi} \approx 0$. Thus, by (4.8),

$$s^2 \simeq \frac{1}{n-1}\sum_{i=1}^{n}(\xi_i)^2 = \frac{1}{n-1}\sum_{i=1}^{n}(u_i - f_i)^2 \simeq \frac{1}{A_\Omega}\int_\Omega |u-f|^2 dx$$

where A_Ω stands for the area of Ω. Hence, the ROF model can be rewritten as

$$\min_{u\in BV(\Omega)}\{|u|_{BV}, \quad \text{subject to } \frac{1}{A_\Omega}\int_\Omega |u-f|^2dx \le \sigma_0^2\} \tag{4.9}$$

By the Chambolle-Lions theorem [12], the minimization is equivalent to the following unconstrained problems

$$\min_{u\in BV(\Omega)} |u|_{BV} + \frac{1}{2\lambda}\int_\Omega |u-f|^2dx \tag{4.10}$$

for some λ dependent on σ_0 if $\sigma_0 \in (0, \frac{1}{A_\Omega}\int_\Omega |u_0|^2dx]$.

Furthermore, suppose that we have an *a prior* knowledge of the distribution of ξ_i. For example, ξ_i are independent identically distributed random variables with the same probability density function:

$$g_p(x) = \frac{C_p}{b}\exp\left(-\left|\frac{x}{b}\right|^p\right), p \ge 1 \tag{4.11}$$

with $C_p = \dfrac{p}{2\Gamma(1/p)}$ and b being an fixed parameter. It is easy to see that the mean $E(\xi_i) = 0$ and the variance

$$\sigma^2 = var(\xi_i) = \frac{\Gamma(3/p)}{\Gamma(1/p)}b^2. \tag{4.12}$$

One of the standard methods to estimate the parameter b in statistics is the maximum likelihood method. Let us use the discrete setting to explain it again. For the given image $f = \{f_1, \cdots, f_n\}$, with the random variables $\xi_i, i = 1, \cdots, n$ which are iid with the same probability density function g_p, the event $\xi_i, i = 1, \cdots, n$ happens when the joint probability

$$L(\xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n|\, b) = \left(\frac{C_p}{b}\right)^n \exp\left(-\frac{\sum_i^n |\xi_i|^p}{b^p}\right)$$

is maximized. To compute the maximum, it is equivalent to find the maximum of the logarithm of the above maximal likelihood function

$$\log L = n\log(C_p) - n\log(b) - \frac{\sum_i^n |\xi_i|^p}{b^p}.$$

Thus, we solve

$$\frac{d\log(L)}{db} = -\frac{n}{b} + \frac{p\sum_i^n |\xi_i|^p}{b^{(p+1)}} = 0.$$

It follows that the maximum likelihood estimator $\hat{b}$ for b is $\hat{b} = \left(\frac{p}{n}\sum_{i=1}^{n}|\xi_i|^p\right)^{1/p}$.

Therefore, the variance

$$\begin{aligned} var(u-f) &= \sigma^2 = \frac{\Gamma(3/p)}{\Gamma(1/p)}b^2 \simeq \frac{\Gamma(3/p)}{\Gamma(1/p)}\left(\frac{p}{n}\sum_i^n |\xi_i|^p\right)^{2/p} \\ &= \frac{\Gamma(3/p)}{\Gamma(1/p)}\left(\frac{p}{n}\sum_i^n |u_i - f_i|^p\right)^{2/p} \approx \frac{\Gamma(3/p)}{\Gamma(1/p)}\left(\frac{p}{A_\Omega}\int_\Omega |u-f|^p\right)^{2/p} \end{aligned}$$

by (4.12) and (4.8) when $n \to \infty$. The ROF model in this case is

$$\min_{u\in BV(\Omega)} \{|u|_{BV}, \quad \text{subject to } \frac{\Gamma(3/p)}{\Gamma(1/p)}\left(\frac{p}{A_\Omega}\int_\Omega |u-f|^p\right)^{2/p} \le \sigma_0^2\} \tag{4.13}$$

Similar to the proof of the Chambolle-Lions theorem [12], the minimization is equivalent to the following

$$\min_{u\in BV(\Omega)} |u|_{BV} + \frac{p}{2\lambda}\left(\int_\Omega |u-f|^p\right)^{2/p} \tag{4.14}$$

for some λ dependent on σ_0 if σ_0 is not too big.

Lemma 4.1.1 *The estimate* $\widehat{b^p} := \frac{p}{n}\sum_i^n |\xi_i|^p$ *is a unbiased estimator for* b^p. *However, when* $p < 2$, $\hat{b}^2 = \left(\frac{p}{n}\sum_i^n |\xi_i|^p\right)^{2/p}$ *is a biased estimator of* b^2 *and hence the* $\widehat{var}(u-f) := \frac{\Gamma(3/p)}{\Gamma(1/p)}\hat{b}^2$ *is a biased estimator of* $var(u-f)$.

Proof. Since ξ_i are i.i.d.,

$$\begin{aligned} E(\widehat{b^p}) &= E\left(\frac{p}{n}\sum_i^n |\xi_i|^p\right) = pE(|\epsilon_1|^p) = 2pC_p\int_0^\infty \frac{|x|^p}{b}\exp\left(-\left|\frac{x}{b}\right|^p\right)dx \\ &= 2C_pb^pp\int_0^\infty y^p\exp(-y^p)dy = b^p \end{aligned}$$

by using integration by parts.

On the other hand, as $y = x^{2/p}$ is a strict convex function when $p < 2$, by Jensen's inequality,

$$E(\hat{b}^2) = E((\hat{b}^p)^{2/p}) > (E(\hat{b}^p))^{2/p} = (b^p)^{2/p} = b^2.$$

That is, $\hat{b}^2$ is a biased estimator of b^2. Since $var(u - f) = \dfrac{\Gamma(3/p)}{\Gamma(1/p)}b^2$, $\widehat{var}(u - f)$ is also a biased estimator of $var(u - f)$. ■

Next let us point out that $\widehat{b^p}$ is a sufficient statistics in the sense that it summarizes all information the entire random sample can provide about b^p.

Lemma 4.1.2 *The estimator $\widehat{b^p}$ in Lemma 4.1.1 is a sufficient estimator of b^p.*

Proof. Let $u = \sum_i^n |\xi_i|^p$ be a random variable. Then the likelihood function of b is

$$L(\xi_1, ..., \xi_n|b) = \left(\frac{C_p}{b}\right)^n \exp\left(-\frac{u}{b^p}\right).$$

By Fisher-Neyman factorization theorem (cf. [51]), u is a sufficient statistic of b. Therefore $\widehat{b^p}$ is a sufficient estimator of b^p. ■

Finally, since $\widehat{b^p}$ is unbiased and sufficient, by Rao-Blackwell theorem (cf. [51]), it is the minimum-variance unbiased estimator. That is,

$$E((\widehat{b^p} - b^p)^2) = \min_{\xi,\text{unbiased r.v.}} E((\xi - b^p)^2).$$

Such a minimizer is unique.

Therefore it is better to use the unbiased estimator $\widehat{b^p}$ than $\hat{b}^2$ and hence, we consider the model

$$\min_{u\in BV(\Omega)} |u|_{BV} + \frac{1}{p\lambda}\int_\Omega |u-f|^p dx.$$

This justifies the TV-L^p model in this paper.

We shall explain how to generate such a random variable subject to probability density function g_p in the section on computational results together with numerical examples to demonstrate the performance of TV-L^p model for p-exponential distributions.

4.2 Basic Properties of the (ϵ, η)-version TV-L^p Model

We first rewrite the (ϵ, η)-version of the TV-L^p model as follows:

$$\min_{u\in \mathrm{BV}(\Omega)} E_{\epsilon,\eta}(u) \tag{4.15}$$

with energy functional

$$E_{\epsilon,\eta}(u) := J_\epsilon(u) + \frac{1}{p\lambda}\int_\Omega (\eta + (u-f)^2)^{p/2} \tag{4.16}$$

It is easy to see that when $p \geq 1$ and $\eta > 0$, the minimization functional $E_{\epsilon,\eta}$ is strictly convex , weakly lower semi-continuous, BV-coercive and hence according to Theorem 2.3.7, there exists a unique minimizer $u_f \in \mathrm{BV}(\Omega)$ for any $\epsilon \geq 0$. Similarly, when $p > 1$, $\eta = 0$ and $\epsilon \geq 0$ or when $p = 1$, $\eta = 0$ and $\epsilon > 0$, $E_{\epsilon,\eta}$ is also strictly convex and lower semi-continuous. Thus, the minimizer u_f is unique. Only when $p = 1$ and $\eta = 0$ and $\epsilon = 0$, the functional $E_{\epsilon,\eta}(u)$ is not strictly convex. The minimizers are not unique for certain $\lambda > 0$. See the discussion in [14].

We shall further study other properties of u_f in this section. We begin with the following

Lemma 4.2.1 *Let u_f be the minimizer of problem (4.15) for input f. Consider the setting (p, ϵ, η) such that the minimizer u_f is unique. If f is bounded, then u_f is also bounded and*

$$\inf_{x\in\Omega} f(x) \le u_f(x) \le \sup_{x\in\Omega} f(x).$$

Proof. Let $\bar{u}_f$ be the truncation of u_f by $\sup_{x\in\Omega} f(x)$ and $\inf_{x\in\Omega} f(x)$, i.e.,

$$\bar{u}_f(y) = \begin{cases} \sup_{x\in\Omega} f(x), & \text{if } u_f(y) > \sup_{x\in\Omega} f(x) \\ u_f(y), & \text{if } \inf_{x\in\Omega} f(x) < u_f(y) < \sup_{x\in\Omega} f(x) \\ \inf_{x\in\Omega} f(x), & \text{if } \inf_{x\in\Omega} f(x) > u_f(y). \end{cases}$$

It is easy to verify if $u_f \in W^{1,1}(\Omega)$,

$$|\bar{u}_f(y) - f(y)| \le |u(y) - f(y)|, \forall y \in \Omega$$

and

$$\int_\Omega \sqrt{\epsilon + |\nabla \bar{u}_f|^2} \le \int_\Omega \sqrt{\epsilon + |\nabla u_f|^2}.$$

According to Theorem 2.3.1, for $u_f \in \mathrm{BV}(\Omega)$, we can find a sequence $\{u_n\}$ in $W^{1,1}(\Omega)$ that converges to u_f in $L^1(\Omega)$ and

$$\lim_{n\to\infty} \int_\Omega \sqrt{\epsilon + |\nabla u_n|^2} = \int_\Omega \sqrt{\epsilon + |\nabla u_f|^2}.$$

We also have $\bar{u}_n$ converge to $\bar{u}_f$ in $L^1(\Omega)$ as n tends to infinity, where $\bar{u}_n$ is the truncation of u_n. Note that the functional $J_\epsilon(u)$ is lower semi-continuous (cf. [Acart-Vogel'94]). Then

$$\int_\Omega \sqrt{\epsilon + |\nabla \bar{u}_f|^2} \leq \varliminf_{n\to\infty} \int_\Omega \sqrt{\epsilon + |\nabla \bar{u}_n|^2}$$
$$\leq \lim_{n\to\infty} \int_\Omega \sqrt{\epsilon + |\nabla u_n|^2} = \int_\Omega \sqrt{\epsilon + |\nabla u_f|^2}.$$

Then

$$E_{\epsilon,\eta}(\bar{u}_f) \leq E_{\epsilon,\eta}(u_f)$$

which implies $\bar{u}_f = u_f$ because of the uniqueness of the solution. ■

We next present the continuous property of the minimizers.

Theorem 4.2.1 *Fix $p \geq 1$ and $\eta > 0$. Suppose u_f is the solution of problem (4.4) with f bounded. Then for any bounded function u with* $\sup|u| \leq \sup|f|$,

$$\|u - u_f\|_2 \leq C\lambda(E_{\epsilon,\eta}(u) - E_{\epsilon,\eta}(u_f))$$

for all $\epsilon \geq 0$, where C depends on $\sup|f|$ *and η.*

Proof. We first give the Euler-Lagrange equation for the minimizer u_f of (4.15)

$$\partial J_\epsilon(u) + \frac{1}{\lambda}\frac{u_f - f}{(\epsilon + (u_f - f)^2)^{1-p/2}} = 0 \tag{4.17}$$

Its equivalent inequality is

$$J_\epsilon(u) - J_\epsilon(u_f) \geq \left\langle -\frac{1}{\lambda}\frac{u_f - f}{(\eta + (u_f - f)^2)^{1-p/2}}, u - u_f \right\rangle, \tag{4.18}$$

where $\langle f, g\rangle$ is the standard inner product in $L^2(\Omega)$.

Following this calculation, we continue to compute the difference between two energies.

Assume u_f is the minimizer for an input function f, u is a $L^p(\Omega)$ function

$$
\begin{aligned}
& E_{\epsilon,\eta}(u) - E_{\epsilon,\eta}(u_f) \\
= & \frac{1}{\lambda}\int_\Omega (\eta + (u-f)^2)^{p/2} - (\eta + (u_f - f)^2)^{p/2}\,dx + J_\epsilon(u) - J_\epsilon(u_f).
\end{aligned}
$$

Using (4.18), we have

$$
\begin{aligned}
& E_{\epsilon,\eta}(u) - E_{\epsilon,\eta}(u_f) \\
\geq & \frac{1}{\lambda}\int_\Omega (\eta + (u-f)^2)^{p/2} - (\eta + (u_f - f)^2)^{p/2}\,dx \\
& + \left\langle -\frac{1}{\lambda}\frac{u_f - f}{(\eta + (u_f - f)^2)^{1-p/2}}, u - u_f \right\rangle
\end{aligned}
$$

For simplicity we let

$$
\phi_\eta(x) = \frac{1}{p}(\eta + x^2)^{p/2} \tag{4.19}
$$

which is convex and infinitely differentiable for all x when $\eta > 0$. Then

$$
\begin{aligned}
& \lambda(E_{\epsilon,\eta}(u) - E_{\epsilon,\eta}(u_f)) \\
\geq & \int_\Omega \phi_\eta(u-f) - \phi_\eta(u_f - f)\,dx - \langle \phi'_\eta(u_f - f), u - u_f \rangle \\
= & \int_\Omega \phi''_\eta(\zeta)(u - u_f)^2\,dx
\end{aligned} \tag{4.20}
$$

where ζ is a function between $u - f$ and $u_f - f$. Direct calculation shows

$$
\phi''_\eta(y) = \frac{(\eta + (p-1)y^2)}{(\eta + y^2)^{2-p/2}} \tag{4.21}
$$

is a positive decreasing function which goes to zero when $y \to \infty$.

Lemma 4.2.1 and the assumption imply that both $u - f$ and $u_f - f$ area bounded by

$2 \sup |f|$ and hence,

$$\phi''(\zeta) \geq C$$

with C depending on $\sup |f|$ and η. Hence, the result follows. ■

With similar argument, we can show that

Theorem 4.2.2 *Fix $p \geq 1$ and $\eta > 0$. Suppose that f and g are bounded. Let u_f and u_g be the minimizers of TV-L^p model (4.15) associated with images f and g, respectively. Then*

$$\|u_f - u_g\|_2 \leq C\|f - g\|_2$$

for all $\epsilon \geq 0$, where $C > 0$ depends on the bound of f and g as well as η.

Proof. Assume u_f is the minimizer for f and u_g is the minimizer for g, by the definition, we have

$$\langle \partial J_\epsilon(u_f), u_g - u_f \rangle \leq J(u_g) - J(u_f)$$
$$\langle \partial J_\epsilon(u_g), u_f - u_g \rangle \leq J(u_f) - J(u_g)$$

Adding these two inequalities, and using (4.18) we have

$$\langle \phi'_\eta(u_g - g) - \phi'_\eta(u_f - f), u_g - u_f \rangle \leq 0.$$

Note that

$$\phi'_\eta(u_g - g) - \phi'_\eta(u_f - f) = \phi''_\eta(\xi)(u_g - u_f + f - g). \tag{4.22}$$

It is easy to see

$$
\begin{aligned}
&\langle \phi''_\eta(\xi)(u_g - u_f), u_g - u_f \rangle \\
\le\ &\langle \phi''_\eta(\xi)(g - f), u_g - u_f \rangle \\
\le\ &(\langle \phi''_\eta(\xi)(g - f), g - f \rangle)^{1/2} (\langle \phi''_\eta(\xi)(u_g - u_f), u_g - u_f \rangle)^{1/2}
\end{aligned}
$$

by Cauchy-Schwarz inequality. It thus follows

$$\langle \phi''_\eta(\xi)(u_g - u_f), u_g - u_f \rangle \le \langle \phi''_\eta(\xi)(g - f), g - f \rangle. \tag{4.23}$$

As $\phi''_\eta(\xi)$ is bounded from below and from above when u_g and u_f are bounded, the desired inequality follows. ■

4.3 Bivariate Splines Approximation of the TV-L^p Model

To find the minimizers of the TV-L^p minimization, we approximate them by considering the minimization problem in a finite dimensional space. For example, we can use continuous piecewise linear finite element space $S^r_d(\triangle)$.

The minimization problem in the spline space is formulated as

$$\min\{E_{\epsilon,\delta}(u), \quad u \in \mathcal{S}\}. \tag{4.24}$$

We shall study the relationship between the minimizer of (4.24) and the minimizer of the original problem (4.15). Assuming S_f is the minimizer of (4.24), we shall prove that $E_{\epsilon,\eta}(S_f) - E_{\epsilon,\eta}(u_f) \to 0$ as the size $|\triangle|$ of the triangulation tends to zero, where $|\triangle|$ is the largest of the lengths of edges in $\triangle$.

We first introduce the notation of extension of functions on $\Omega = [0, 1] \times [0, 1]$: for any function u defined on Ω, let $\operatorname{Ext} u$ be the extension of u defined on $\mathbb{R}^2$ by first reflecting u about the boundary of Ω and then periodically extending the resulting function to the whole plane $\mathbb{R}^2$. For detail, see [53]. Let ψ be a standard symmetric non-negative mollifier and define

$$u^\delta(x) = \int_{R^2} \operatorname{Ext} u(x-y)\psi(\frac{y}{\delta})\frac{dy}{\delta^2}.$$

Lemma 4.3.1 *Suppose f is bounded. Then*

$$E_{\epsilon,\eta}(u_f^\delta) \leq E_{\epsilon,\eta}(u_f) + C\delta|u_f|_{\mathrm{BV}}$$

where $C > 0$ is a constant dependent on $\|f\|_\infty$ and p.

Proof. First we claim that

$$\int_\Omega \sqrt{\epsilon + |\nabla u_f^\delta|^2} \leq \int_\Omega \sqrt{\epsilon + |\nabla u_f|^2}$$

for δ small enough. Indeed, it is straightforward to verify that this inequality holds for $u_f \in W^{1,1}(\Omega)$ by using the convexity of $J_\epsilon(u)$ and the property of the mollifier for any $\delta > 0$. For $u_f \in \mathrm{BV}(\Omega)$, the inequality still holds due to the fact that any BV function can be approximated by $W^{1,1}$ functions in the sense of Theorem 2.3.1.

Next we have

$$\left| \int_\Omega (\eta + (u_f^\delta - f)^2)^{p/2} - (\eta + (u_f - f)^2)^{p/2} \right| \leq \int_\Omega \left| \phi_\eta'(\xi)(u_f^\delta - u_f) \right|,$$

where $\phi_\eta(x)$ is the function defined in (4.19). Since f is bounded, we have u_f is also bounded by Lemma 4.2.1, then

$$\left|\int_\Omega (\epsilon + (u_f^\delta - f)^2)^{p/2} - (\epsilon + (u_f - f)^2)^{p/2}\right|$$
$$\leq C\|u_f^\delta - u_f\|_{L^1} \leq C\delta |u_f|_{\mathrm{BV}},$$

where $\omega(u_f, \delta)_{L^1}$ is the modulus of smoothness in $L^1(\Omega)$. The last inequality follows from the fact that BV space is identical to the Lipschitz space $\mathrm{Lip}(1, L^1)$, see [18]. ■

We need to use the following standard result.

Lemma 4.3.2 *If* $u \in \mathrm{BV}(\Omega)$,

$$|u^\delta|_{W^{2,1}} \leq \frac{C}{\delta}|u|_{\mathrm{BV}}$$

Proof. We have shown in Lemma 3.3.6 that the inequality holds for $u \in W^{1,1}(\Omega)$. The inequality still holds by the facts in Theorem 2.3.1. ■

We now prove that the energy of the spline approximation is close to the energy of the smoothed function.

Lemma 4.3.3 *Suppose* f *is bounded. Suppose that* $\mathcal{S}$ *contains a spline space* $S_d^r(\triangle)$ *for a degree* $d \geq 3r + 2$ *as a subspace. Let* $Qu_f^\delta \in \mathcal{S}$ *be the quasi-interpolatory spline mentioned in Theorem 2.1.6. Then*

$$E_{\epsilon,\eta}(Qu_f^\delta) \leq E_{\epsilon,\eta}(u_f^\delta) + C(\frac{|\Delta|}{\delta} + |\Delta|)|u_f|_{\mathrm{BV}}.$$

Proof. We estimate the difference between $|E_{\epsilon,\eta}(Qu_f^\delta) - E_{\epsilon,\eta}(u_f^\delta)|$ by

$$
\begin{aligned}
&|E_{\epsilon,\eta}(Qu_f^\delta) - E_{\epsilon,\eta}(u_f^\delta)| \\
\le &\left| \int_\Omega \sqrt{1 + |\nabla Qu_f^\delta|^2} - \sqrt{1 + |\nabla u_f^\epsilon|^2} dx \right| \\
&+ \frac{1}{\lambda} \left| \int_\Omega (\epsilon + (Qu_f^\delta - f)^2)^{p/2} - (\epsilon + (u_f^\delta - f)^2)^{p/2} \right| .
\end{aligned}
$$

For the first term on the right side of the inequality

$$
\begin{aligned}
&\left| \int_\Omega \sqrt{\epsilon + |\nabla Qu_f^\delta|^2} - \sqrt{\epsilon + |\nabla u_f^\delta|^2} dx \right| \\
= &\left| \int_\Omega \frac{|\nabla Qu_f^\delta|^2 - |\nabla u_f^\delta|^2}{\sqrt{1 + |\nabla Qu_f^\delta|^2} + \sqrt{\epsilon + |\nabla \tilde{u_f^\delta}|^2}} dx \right| \\
\le &\int_\Omega \frac{|\nabla Qu_f^\delta - \nabla u_f^\delta||\nabla Qu_f^\delta + \nabla u_f^\delta|}{\sqrt{\epsilon + |\nabla Qu_f^\delta|^2} + \sqrt{\epsilon + |\nabla \tilde{u_f^\delta}|^2}} dx \\
\le &\|\nabla (Qu_f^\delta - u_f^\delta)\|_{L^1}
\end{aligned}
$$

By Theorem 2.1.6(with $m = 1$),

$$
\|\nabla (Qu_f^\delta - u_f^\delta)\|_{L^1} \le C|\Delta||u_f^\delta|_{W^{2,1}} \le C\frac{|\Delta|}{\delta}|u_f|_{\mathrm{BV}}
$$

Here we have used Lemma 4.3.2.

For the second term, since f is bounded, u_f, u_f^δ and Qu_f^δ are also bounded. We apply

Theorem 2.1.6

$$\begin{aligned}
&\left|\int_\Omega (\eta + (Qu_f^\delta - f)^2)^{p/2} - (\eta + (u_f^\delta - f)^2)^{p/2}\right| \\
&\le \int_\Omega \left|\phi_\eta(\xi)(Qu_f^\delta - u_f^\delta)\right| \\
&\le C\int_\Omega |(Qu_f^\delta - u_f^\delta)|\, dx \\
&\le C|\Delta||u_f^\delta|_{W^{1,1}} \le C|\Delta||u_f|_{\mathrm{BV}}
\end{aligned}$$

The last inequality uses Theorem 2.4.1 that for any function $u \in \mathrm{BV}(\Omega)$

$$|u^\delta|_{W^{1,1}} \le |u|_{\mathrm{BV}}.$$

Summarized the discussion above, we have completed the proof. ■

Finally we are ready to prove the main result in this section

Theorem 4.3.1 *Fix $p \ge 1$ and $\eta > 0$. Suppose f is bounded. Suppose that $\mathcal{S}$ contains a spline space $S_d^r(\triangle)$ for a degree $d \ge 3r + 2$ as a subspace. Let $\delta = \sqrt{|\triangle|}$. Then*

$$E_{\epsilon,\eta}(S_f) - E_{\epsilon,\eta}(u_f) \le C\sqrt{|\Delta|}\, E_{\epsilon,\eta}(0) \tag{4.25}$$

for all $\epsilon \ge 0$, where C is a constant dependent on f, p, and the smallest angle $\theta_\triangle$ of triangulation $\triangle$. By Theorem 4.2.1,

$$\|S_f - u_f\|_2 \le C\lambda\sqrt{|\triangle|}, \tag{4.26}$$

where C is another positive constant independent of $|\triangle|$.

Proof. Combine Lemma 4.3.1 and Lemma 4.3.3. We have

$$E_{\epsilon,\eta}(u_f) \le E_{\epsilon,\eta}(S_f) \le E_{\epsilon,\eta}(Qu_f^\delta) \le E_{\epsilon,\eta}(u_f) + C(\frac{|\Delta|}{\delta} + |\Delta| + \delta)|u_f|_{\mathrm{BV}}.$$

For $\delta = \sqrt{|\Delta|}$, we have

$$E_{\epsilon,\eta}(S_f) - E_{\epsilon,\eta}(u_f) \le C\sqrt{|\Delta|}\,|u_f|_{\rm BV} \le C\sqrt{|\Delta|}E_{\epsilon,\eta}(0).$$

The inequality (4.26) follows from (4.25) and Theorem 4.2.1 directly. ■

4.4 A Numerical Algorithm

In this section, we shall derive an iterative algorithm to compute the minimizers of the (ϵ, η) TV-L^p model and show that the iterative algorithm converges. The discussion is similar to the one in [33] where the minimizer of the TV-L_2 model was considered. The proof has to be carefully modified when $p < 2$. Thus, we present a detail discussion here. Recall

$$E_{\epsilon,\eta}(u) := \int_\Omega \sqrt{\epsilon + |\nabla u|^2}dx + \frac{1}{p\lambda}\int_\Omega (\eta + |u-f|^2)^{p/2}dx \tag{4.27}$$

for $\epsilon \ge 0$ and $\eta \ge 0$. Note that it is also defined for $p > 0$. We first have

Theorem 4.4.1 *Fix $p > 0, \epsilon \ge 0$ and $\eta > 0$. There exists a solution of the following minimization problem*

$$\min\{E_{\epsilon,\eta}(s),\ s \in \mathcal{S}\}, \tag{4.28}$$

where $\mathcal{S}$ is a spline space explained in the previous section. When $p > 1$ or when $p = 1$ and $\eta > 0$, the minimizer is unique.

Proof. Let $D := \{E_{\epsilon,\eta}(s) \le E_{\epsilon,\eta}(0), s \in \mathcal{S}\}$. It is not hard to see $D \subset \mathcal{S}$ is bounded and hence, is compact. Note that $E_{\epsilon,\eta}(u)$ is a continuous functional for any fixed $p > 0, \eta \ge 0, \eta > 0$. Therefore there exists a spline $S_f \in D$ which achieves the minimum, i.e., $E_{\epsilon,\eta}(S_f) \le E_k(s), s \in D$.

When $p > 1$ or when $p = 1$ and $\eta > 0$, the minimization functional $E_{\epsilon,\eta}$ is strictly convex and hence, the minimizer is unique. ■

Let us use S_f to be a minimizer of the minimization problem (4.28). Since $\eta > 0$, $E_{\epsilon,\eta}(u)$ is Gâteau differentiable. A minimizer S_f must happen at one of the stationary points. Let $\phi_1, ..., \phi_n$ be a basis for spline space $\mathcal{S}$. The following iteration algorithm will be used to approximate S_f

Algorithm 4.4.1 *Given $u^{(k)}$, we find $u^{(k+1)} \in \mathcal{S}$ such that*

$$\int_\Omega \frac{\nabla u^{(k+1)} \cdot \nabla \phi_j}{\sqrt{\epsilon + |\nabla u^{(k)}|^2}} dx + \frac{1}{\lambda} \int_\Omega \frac{u^{(k+1)} \phi_j}{(\eta + |u^{(k)} - f|^2)^{1-p/2}} dx$$
$$= \frac{1}{\lambda} \int_\Omega \frac{f \phi_j}{(\eta + |u^{(k)} - f|^2)^{1-p/2}} dx, \quad \forall j = 1, ..., n. \tag{4.29}$$

We first show that the above iteration is well defined. Since $u^{(k+1)} \in \mathcal{S}$, it can be written as $u^{(k+1)} = \sum_i^n c_i^{(k+1)} \phi_i$. Plugging it in (4.29), we have

$$\sum_i^n c_i^{(k+1)} \left(\int_\Omega \frac{\nabla \phi_i \cdot \nabla \phi_j}{\sqrt{\epsilon + |\nabla u^{(k)}|^2}} dx + \frac{1}{\lambda} \int_\Omega \frac{\phi_i \phi_j}{(\eta + |u^{(k)} - f|^2)^{1-p/2}} dx \right)$$
$$= \frac{1}{\lambda} \int_\Omega \frac{f \phi_j}{\eta + |u^{(k)} - f|^2)^{1-p/2}} dx. \tag{4.30}$$

Denote by

$$\begin{aligned}
D^{(k)} &:= (d_{i,j}^{(k)})_{N \times N} \text{ with } d_{i,j}^{(k)} = \lambda \int_\Omega \frac{\nabla \phi_i \cdot \nabla \phi_j}{\sqrt{\epsilon + |\nabla u^{(k)}|^2}} dx \\
M^{(k)} &:= (m_{i,j}^{(k)})_{N \times N} \text{ with } m_{i,j} = \int_\Omega \frac{\phi_i \phi_j}{(\eta + |u^{(k)} - f|^2)^{1-p/2}} dx, \\
\mathbf{v}^{(k)} &:= (v_j^{(k)}, j = 1, \cdots, N) \text{ with } v_j = \int_\Omega \frac{f \phi_j}{(\eta + |u^{(k)} - f|^2)^{1-p/2}} dx.
\end{aligned}$$

Then to solve (4.30) is equivalent to solving the equation

$$(D^{(k)} + M^{(k)})c^{(k+1)} = \mathbf{v^{(k)}}, \tag{4.31}$$

where $c^{(k+1)} = [c_1^{(k+1)}, c_2^{(k+1)}, ..., c_n^{(k+1)}]^T$.

Theorem 4.4.2 *The algorithm 4.4.1 has a unique solution in $\mathcal{S}$.*

Proof. It is easy to see $D^{(k)}$ is semi-positive definite and M is positive defined because, for any nonzero $\mathbf{c} = (c_i)_n$, because

$$\mathbf{c}^T D^{(k)} \mathbf{c} = \lambda \int_\Omega \frac{|\sum_i^n c_i \nabla \phi_i|^2}{\sqrt{\epsilon + |\nabla u^{(k)}|^2}} dx \geq 0,$$

and

$$\mathbf{c}^T \ M^{(k)} \ \mathbf{c} = \int_\Omega \frac{|\sum_i^n c_i \phi_i|^2}{(\eta + |u^{(k)} - f|^2)^{1-p/2}} dx > 0.$$

Therefore $(D^{(k)} + M^{(k)})$ is also positive definite, and hence invertible. So (4.31) has a unique solution and so does (4.29). ■

Next we show the iterative solution converges to the minimizer S_f in $\mathcal{S}$. We need the following few lemmas.

Lemma 4.4.1 *For $0 < p \leq 2$, the following inequality holds*

$$\frac{1}{p}\left((\eta + x^2)^{p/2} - (\eta + y^2)^{p/2}\right) \geq \frac{(x-y)^2}{2(\eta + x^2)^{1-p/2}} + \frac{y(x-y)}{(\eta + x^2)^{1-p/2}} \tag{4.32}$$

for all $x, y \geq 0$.

Proof. Since

$$\begin{aligned}
& (\eta + x^2)^{p/2} - (\eta + y^2)^{p/2} \\
= \quad & \frac{(\eta + x^2) - (\eta + x^2)^{1-p/2}(\eta + y^2)^{p/2}}{(\eta + x^2)^{1-p/2}} \\
= \quad & \frac{\eta + x^2 + py^2 - pxy - (\eta + x^2)^{1-p/2}(\eta + y^2)^{p/2}}{(\eta + x^2)^{1-p/2}} + \frac{py(x-y)}{(\eta + x^2)^{1-p/2}}
\end{aligned}$$

By Young's inequality which states that if $0 \leq a, b \leq 1$ and $a + b = 1$, then

$$ax + by \geq x^a y^b, \quad \text{where } x, y \geq 0,$$

we have

$$\begin{aligned}
& \eta + x^2 + py^2 - pxy - (\eta + x^2)^{1-p/2}(\eta + y^2)^{p/2} - \frac{p}{2}(x-y)^2 \\
= \quad & (1 - \frac{p}{2})(\eta + x^2) + \frac{p}{2}(\eta + y^2) - (\eta + x^2)^{1-p/2}(\eta + y^2)^{p/2} \geq 0.
\end{aligned}$$

It follows that

$$\frac{1}{p}\left((\eta + x^2)^{p/2} - (\eta + y^2)^{p/2}\right) \geq \frac{(x-y)^2}{2(\eta + x^2)^{1-p/2}} + \frac{y(x-y)}{(\eta + x^2)^{1-p/2}}$$

■

Lemma 4.4.2 *Let $u^{(k+1)}$ be the solution of our Algorithm 4.4.1. Suppose that $0 < p \leq 2$. Then the following inequality holds*

$$E_{\epsilon,\eta}(u^{(k)}) - E_{\epsilon,\eta}(u^{(k+1)}) \geq \frac{1}{\lambda} \int_\Omega \frac{(u^{(k)} - u^{(k+1)})^2}{2(\eta + |u^{(k)} - f|^2)^{1-p/2}} dx \tag{4.33}$$

where $E_{\epsilon,\eta}(\cdot)$ is the energy functional given in (4.15).

Proof. Since $\int_\Omega \frac{|\nabla u|^2}{2\sqrt{\epsilon+|\nabla u^{(k)}|^2}}dx$ is a convex functional of u, by (4.29) and convexity, the following inequality holds,

$$\frac{1}{\lambda}\int_\Omega \frac{(f-u^{(k+1)})(u^{(k)}-u^{(k+1)})}{(\eta+|u^{(k)}-f|^2)^{1-p/2}}dx \le \int_\Omega \frac{|\nabla u^{(k)}|^2}{2\sqrt{\epsilon+|\nabla u^{(k)}|^2}}dx - \int_\Omega \frac{|\nabla u^{(k+1)}|^2}{2\sqrt{\epsilon+|\nabla u^{(k)}|^2}}dx.$$

With $w^{(k)} = u^{(k)} - f$, the above inequality can be rewritten as follows.

$$\frac{1}{\lambda}\int_\Omega \frac{(-w^{(k+1)})(w^{(k)}-w^{(k+1)})}{(\eta+|w^{(k)}|^2)^{1-p/2}}dx \le \int_\Omega \frac{|\nabla u^{(k)}|^2}{2\sqrt{\epsilon+|\nabla u^{(k)}|^2}}dx - \int_\Omega \frac{|\nabla u^{(k+1)}|^2}{2\sqrt{\epsilon+|\nabla u^{(k)}|^2}}dx. \quad (4.34)$$

Note that with (4.34), we have

$$\begin{aligned}
& \int_\Omega \sqrt{\epsilon+|\nabla u^{(k)}|} - \sqrt{\epsilon+|\nabla u^{(k+1)}|^2}dx + \frac{1}{\lambda}\int_\Omega \frac{(w^{(k+1)})(w^{(k)}-w^{(k+1)})}{(\eta+|w^{(k)}|^2)^{1-p/2}}dx \\
\ge\ & \int_\Omega \sqrt{\epsilon+|\nabla u^{(k)}|^2} - \sqrt{\epsilon+|\nabla u^{(k+1)}|^2}dx \\
& -\int_\Omega \frac{|\nabla u^{(k)}|^2}{2\sqrt{\epsilon+|\nabla u^{(k)}|^2}}dx + \int_\Omega \frac{|\nabla u^{(k+1)}|^2}{2\sqrt{\epsilon+|\nabla u^{(k)}|^2}}dx \\
=\ & \int_\Omega \frac{2\epsilon+|\nabla u^{(k)}|^2+|\nabla u^{(k+1)}|^2}{2\sqrt{\epsilon+|\nabla u^{(k)}|^2}} - \sqrt{\epsilon+|\nabla u^{(k+1)}|^2}dx \\
\ge\ & \int_\Omega \frac{\sqrt{\epsilon+|\nabla u^{(k)}|^2}\sqrt{\epsilon+|\nabla u^{(k+1)}|^2}}{\sqrt{\epsilon+|\nabla u^{(k)}|^2}} - \sqrt{\epsilon+|\nabla u^{(k+1)}|^2}dx = 0.
\end{aligned}$$

Letting $x = w^{(k)}$ and $y = w^{(k+1)}$ in (4.32) in Lemma 4.4.1, we have

$$\begin{aligned}
& \frac{1}{p}\left((\eta+(w^{(k)})^2)^{p/2} - (\eta+(w^{(k+1)})^2)^{p/2}\right) \\
\ge & \frac{(w^{(k)}-w^{(k+1)})^2}{2(\eta+(w^{(k)})^2)^{1-p/2}} + \frac{w^{(k+1)}(w^{(k)}-w^{(k+1)})}{(\eta+(w^{(k)})^2)^{1-p/2}}
\end{aligned} \quad (4.35)$$

Now we are ready to prove (4.33). By using (4.35), the difference between $E_{\epsilon,\eta}(u^{(k)})$ and $E_{\epsilon,\eta}(u^{(k+1)})$ is

$$
\begin{aligned}
& E_{\epsilon,\eta}(u^{(k)}) - E_{\epsilon,\eta}(u^{(k+1)}) = \int_\Omega \left(\sqrt{\epsilon + |\nabla u^{(k)}|} - \sqrt{\epsilon + |\nabla u^{(k+1)}|^2} \right) dx \\
& + \frac{1}{p\lambda} \int_\Omega \left((\eta + |w^{(k)}|^2)^{p/2} - (\eta + |w^{(k+1)}|^2)^{p/2} \right) dx \\
\geq \; & \underbrace{\int_\Omega \sqrt{\epsilon + |\nabla u^{(k)}|} - \sqrt{\epsilon + |\nabla u^{(k+1)}|^2} dx + \frac{1}{\lambda} \int_\Omega \frac{w^{(k+1)}(w^{(k)} - w^{(k+1)})}{(\eta + |w^{(k)}|^2)^{1-p/2}} dx}_{\geq 0} \\
& + \frac{1}{\lambda} \int_\Omega \frac{(w^{(k)} - w^{(k)})^2}{2(\eta + |w^{(k)}|^2)^{1-p/2}} dx
\end{aligned}
$$

Since $w^{(k)} - w^{(k)} = (u^{(k)} - u^{(k)})$, we have thus established the proof. ■

Lemma 4.4.3 *Suppose that $p \in (0, 2]$ and $\eta \geq 0$ and $\epsilon \geq 0$. If $\|f\|_\infty$ is bounded, then $\|u^{(k)}\|_\infty \leq C$ for some constant C independent of k.*

Proof. From Theorem 4.4.2, we see that $\{E_{\epsilon,\eta}(u^{(k)})\}$ is a decreasing sequence. Thus,

$$\lambda E_{\epsilon,\eta}(u^{(0)}) \geq \lambda E_{\epsilon,\eta}(u^{(k)}) \geq \int_\Omega (\eta + |u^{(k)} - f|^2)^{p/2} dx \geq \int_\Omega |u^{(k)} - f|^p dx.$$

For $p \geq 1$, we have

$$\|u^{(k)}\|_p \leq \|f\|_p + \|u^{(k)} - f\|_p \leq \left(\lambda E_{\epsilon,\eta}(u^{(0)}) \right)^{1/p} + A_\Omega^{1/p} \|f\|_\infty$$

where A_Ω is the area of Ω. It follows that $\|u^{(k)}\|_p$ is bounded. Similarly, when $p \in (0, 1)$, we have

$$\|u^{(k)}\|_p^p \leq \|f\|_p^p + \|u^{(k)} - f\|_p^p \leq \lambda E_{\epsilon,\eta}(u^{(0)}) + A_\Omega \|f\|_\infty^p.$$

In finite dimensional space, $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are equivalent. Therefore $\|u^{(k)}\|_\infty$ is also bounded.

■

Lemma 4.4.4 *If $\|f\|_\infty$ is bounded, then $\|u^{(k+1)} - u^{(k)}\|_2 \to 0$ as $k \to \infty$.*

Proof. By Lemma 4.4.3, $\|f\|_\infty$ is bounded implies $\|u^{(k)}\|_\infty$ is bounded. Therefore

$$|u^{(k)} - f|^2 \le (\|u^{(k)}\|_\infty + \|f\|_\infty)^2 < \infty$$

Let $\|(\eta + |u^{(k)} - f|^2)^{1-p/2}\|_\infty = M$. Then $M < \infty$. By Lemma 4.4.2, we have

$$\begin{aligned} E_{\epsilon,\eta}(u^{(k)}) - E_{\epsilon,\eta}(u^{(k+1)}) &\ge \frac{1}{\lambda}\int_\Omega \frac{(u^{(k)} - u^{(k+1)})^2}{2(\eta + |u^{(k)} - f|^2)^{1-p/2}} dx \\ &\ge \frac{1}{\lambda M}\int_\Omega (u^{(k)} - u^{(k+1)})^2 dx, \end{aligned}$$

that is,

$$\|u^{(k+1)} - u^{(k)}\|_2 \le \lambda M \left(E_{\epsilon,\eta}(u^{(k)}) - E_{\epsilon,\eta}(u^{(k+1)})\right).$$

So $\{E_{\epsilon,\eta}(u^{(k)})\}$ is a decreasing sequence and bounded below, and hence converges, which implies $E_{\epsilon,\eta}(u^{(k)}) - E_{\epsilon,\eta}(u^{(k+1)}) \to 0$ and therefore $\|u^{(k+1)} - u^{(k)}\|_2 \to 0$ as $k \to \infty$. ■

Theorem 4.4.3 *Fix $p > 1$ or $p = 1$ with $\eta > 0$. The sequence $u^{(k)}$ obtained from Algorithm 4.4.1 converges to the true minimizer S_f.*

Proof. By Lemma 4.4.3, the sequence $\{u^{(k)}\}$ is bounded. So there must be a convergent subsequence $\{u^{(n_j)}\}$. Suppose $u^{(n_j)} \to \bar{u}$. We use Lemma 4.4.4 to have

$$\|u^{(n_j+1)} - \bar{u}\|_2 \le \|u^{(n_j+1)} - u^{(n_j)}\|_2 + \|u^{(n_j)} - \bar{u}\|_2 \to 0$$

which implies $u^{(n_j+1)} \to \bar{u}$.

According to Markov's inequality (cf. [36]), we have

$$\int_\Omega |\nabla u^{(n_j)} - \nabla \bar{u}|^2 dx \le \frac{\beta^2}{|\triangle|^2}\int_\Omega |u^{(n_j)} - \bar{u}|^2 dx, \tag{4.36}$$

where $\beta > 0$ is a constant dependent on the smallest angle $\theta_\triangle$ of $\triangle$. It follows from the convergence of $u^{(n_j)} \to \bar{u}$ that $\nabla u^{(n_j)} \to \nabla \bar{u}$ in L^2 norm as well. Replace $u^{(n_j)}$ by $u^{(n_j+1)}$ above, we have $\nabla u^{(n_j+1)} \to \nabla \bar{u}$ too by the convergence of $u^{(n_j+1)} \to \bar{u}$. As $u^{(n_j)}$, $u^{(n_j+1)}$ and $\bar{u}$ are spline functions in $S_d^r(\triangle)$. The convergence of $u^{(n_j)}$ and $u^{(n_j+1)}$ to $\bar{u}$, respectively implies the coefficients of $u^{(n_j)}$ and $u^{(n_j+1)}$ in terms of the basis functions $\phi_j, j = 1, \cdots, N$ are convergent to the coefficients of $\bar{u}$, respectively.

Since $u^{(n_j+1)}$ solves the equations (4.29), we have

$$\begin{aligned}\int_\Omega \frac{\nabla u^{(n_j+1)} \cdot \nabla \phi_j}{\sqrt{\epsilon + |\nabla u^{(n_j)}|^2}} dx + \frac{1}{\lambda} \int_\Omega \frac{u^{(n_j+1)} \phi_j}{(\eta + |u^{(k)} - f|^2)^{1-p/2}} dx \\ = \frac{1}{\lambda} \int_\Omega \frac{f \phi_j}{(\eta + |u^{(n_j)} - f|^2)^{1-p/2}} dx,\end{aligned}$$

for all $\phi_i, i = 1, \cdots, N$. Letting $j \to \infty$, we obtain

$$\int_\Omega \frac{\nabla \bar{u} \cdot \nabla \phi_j}{\sqrt{\epsilon + |\nabla \bar{u}|^2}} dx + \frac{1}{\lambda} \int_\Omega \frac{\bar{u} \phi_j}{(\eta + |\bar{u} - f|^2)^{1-p/2}} dx = \frac{1}{\lambda} \int_\Omega \frac{f \phi_j}{(\eta + |\bar{u} - f|^2)^{1-p/2}} dx,$$

for all $i = 1, \cdots, N$. That is, $\bar{u}$ is a local minimizer. Since the functional is convex when $p > 1$ or when $p = 1$ with $\eta > 0$, a local minimizer is the global minimizer and hence, $\bar{u} = S_f$. Thus all convergent subsequences of $\{u^{(k)}\}$ converge to S_f, which implies $\{u^{(k)}\}$ itself converges to S_f. ■

4.5 Numerical Examples

4.5.1 The Rejection Method

In this subsection we will discuss a method of generating noises following the distribution in (4.11). It is called the rejection sampling, which can be implemented without the knowledge of cumulated density function. It is also commonly called the acceptance-rejection method or "accept-reject algorithm".

Algorithm 4.5.1 (The Rejection Sampling Algorithm) *Let $M > 1$ be a real number and h be an envelope distribution.*

1. *Sample x from $h(x)$, and u from uniform distribution over the unit interval $U(0,1)$.*
2. *Check weather or not $u \leq \dfrac{g(x)}{Mh(x)}$.*
    - *if this holds, accept x as a realization of $g(x)$.*
    - *if not, reject the value x.*

This algorithm generates random numbers from a probability distribution function $g(x)$ by using an envelope distribution $Mh(x)$ for some $M > 1$. In this way, we generate random noises subject to the p-exponential distribution. The distributions of Laplacian noises of $p = 1$, $p = 1.5$ and $p = 2$ are shown in Fig. 4.1.

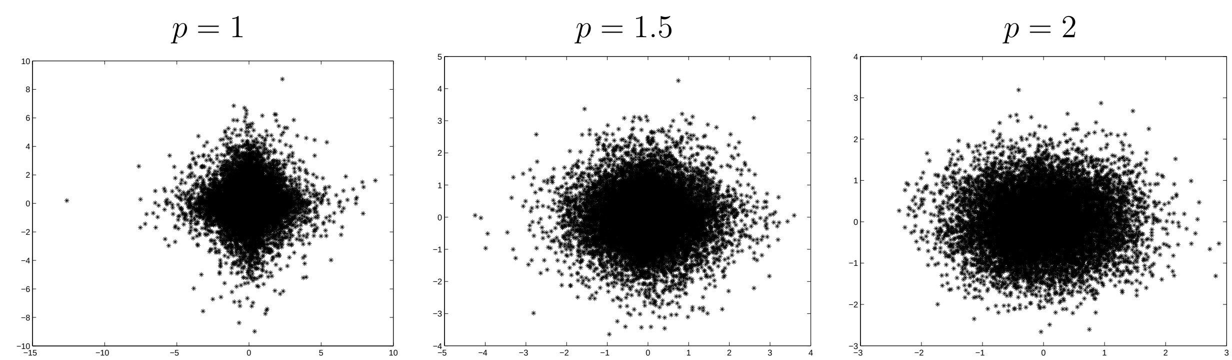


Figure 4.1: Distribution of Laplacian noises of $p = 1$, $p = 1.5$ and $p = 2$.

4.5.2 Numerical Results

Example 4.5.1 *In this example we examine the convergency of our algorithm. That is, for fixed triangulation size $|\triangle|$, $\|u^{(k+1)} - u^{(k)}\|_2 \to 0$, as $k \to \infty$.*

(a) First we try to illuminate that our algorithm converges regardless the shape of region. To make our experiment more convincing, we use a figure from a natural environment. We test our algorithm on seven triangulations of the following figure. The input function f is a noised image with white noise of $\sigma = 20$. For each patch, we get a convergent sequence from our numerical algorithm as shown in Fig. 4.3.

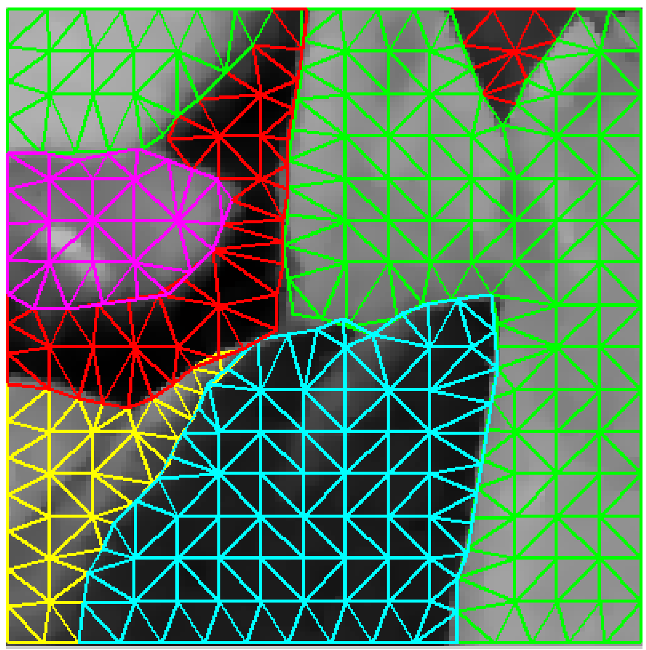


Figure 4.2: Triangulations used to test convergence of the algorithm.

(b)Second, we try to illuminate that our algorithm converges for all p value. The input function f is a noised image with Laplacian noise $p = 1$, $\sigma = 30$ defined on a circular region. We test the case when $p = 1$, $p = 1.5$ and $p = 2$. The result is show in

Example 4.5.2 *Next we examine the convergence of the solution in the spline space to the solution in the BV space. That is, for fixed p, when $|\triangle| \to 0$, $u_f(\triangle)$ converges in L^2 norm.*

Example 4.5.3 *It is well-known that for noised image with Gaussian noises, the original ROF model (i.e. TVL_2) is better than the TV-L_1 model. We now present some evidence that when an image polluted by noises subject to a Laplace distribution, the noised image* *is better denoised by the TV-L^p model than by the ROF model for $p < 2$. In this example, we examine the effectiveness of our model. We try to give some evidence that the TV-L^p model performs best to the Laplacian noise of the same p value. The input function is a sine function over a circle domain with $p = 1$, $p = 1.5$ and $p = 2$ noises of $\sigma = 30$ respectively. The results are shown in Table 4.1, 4.2 and 4.3 respectively. The clean figure is shown in Fig. 4.5. And noised images and best recovered image are shown in Fig. 4.6, Fig. 4.7 and Fig. 4.8 respectively.*

Table 4.1: Denoising: Laplacian noises of $p = 1$.

$p = 1$		$p = 1.5$		$p = 2$	
λ	PSNR	λ	PSNR	λ	PSNR
0.05	33.40	0.5	33.17	0.5	32.28
0.1	**33.48**	1	**33.21**	2	32.33
1	31.48	2	33.0591	7	**32.43**
2	27.8108	3	32.7137	10	32.42

Table 4.2: Denoising: Laplacian noises of $p = 1.5$.

$p = 1$		$p = 1.5$		$p = 2$	
λ	PSNR	λ	PSNR	λ	PSNR
0.05	33.91	0.1	34.21	0.5	34.19
0.1	**33.94**	0.5	**34.22**	1	34.20
0.5	33.58	1	34.19	2	**34.21**
1	32.36	2	33.95	3	34.20

Table 4.3: Denoising: Laplacian noises of $p = 2$.

$p = 1$		$p = 1.5$		$p = 2$	
λ	PSNR	λ	PSNR	λ	PSNR
0.05	34.02	0.1	34.45	0.5	34.61
0.1	**34.06**	0.5	**34.50**	1	34.62
0.5	33.91	1	34.49	3	**34.63**
1	32.84	2	34.31	5	34.60

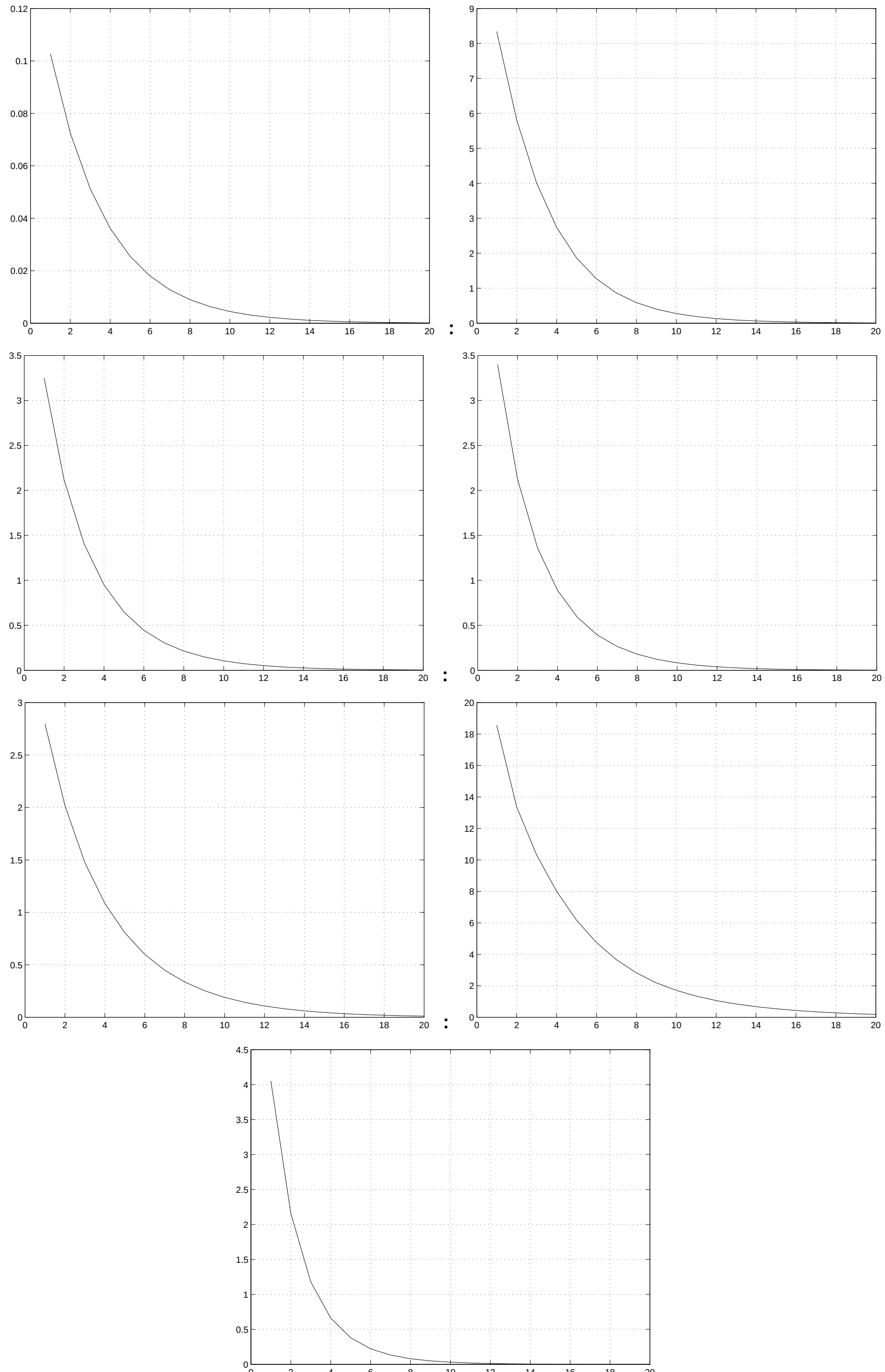


Figure 4.3: Convergency of numerical algorithm on different regions.

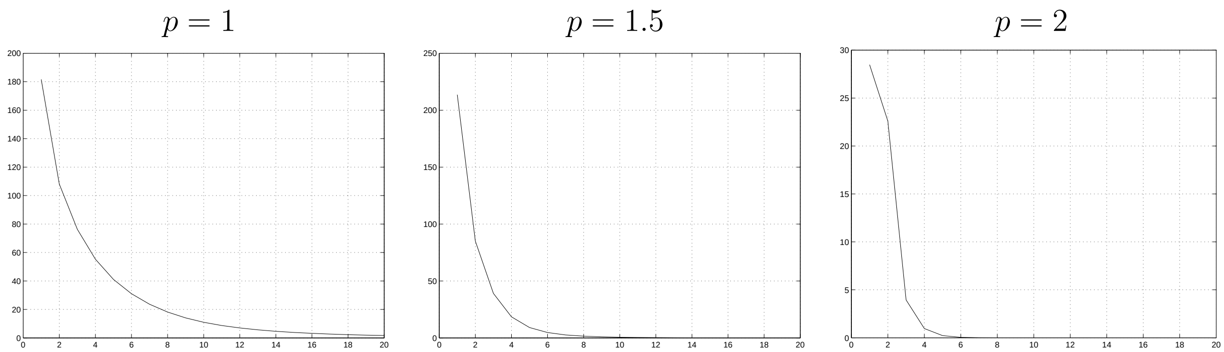


Figure 4.4: Convergency of numerical algorithm in different p values.

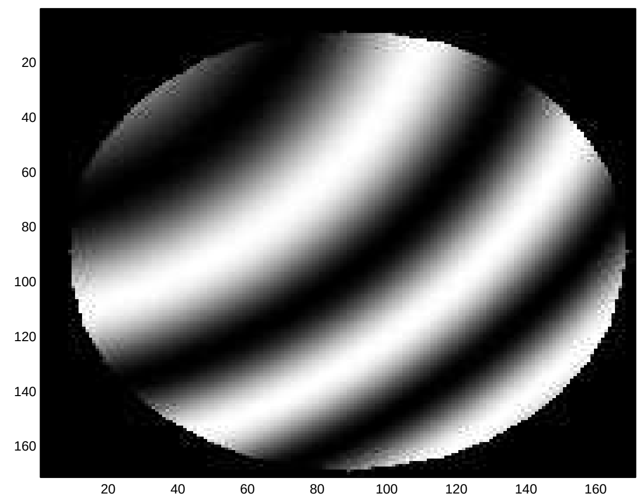


Figure 4.5: Clean image

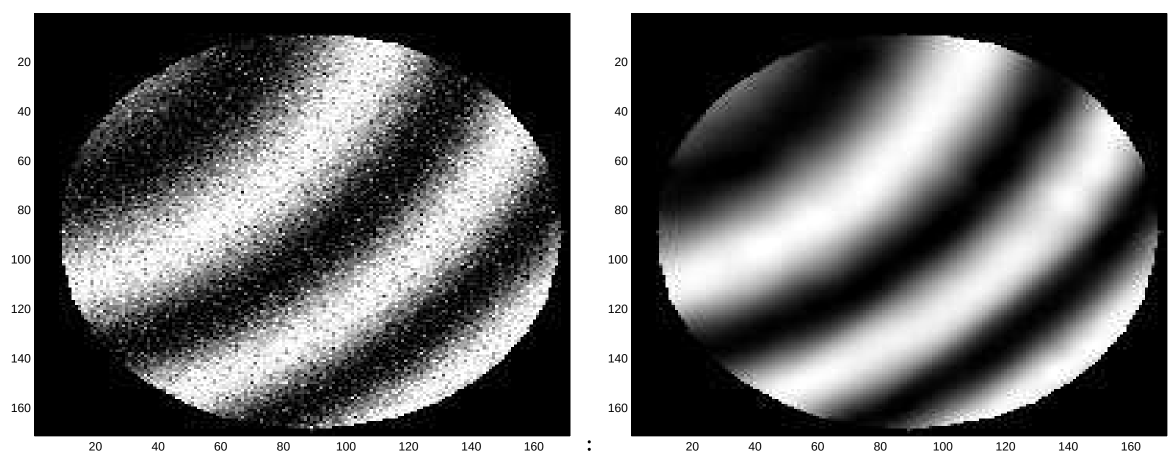


Figure 4.6: Noised image of $p = 1$ and the denoised image with TV-L^p method with $p = 1$

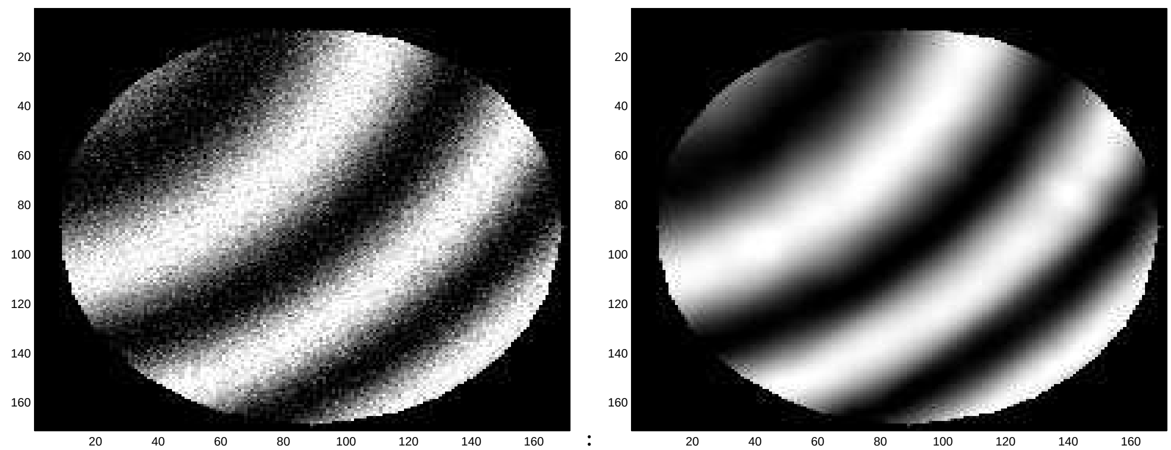


Figure 4.7: Noised image of $p = 1$ and the denoised image with TV-L^p method with $p = 1$

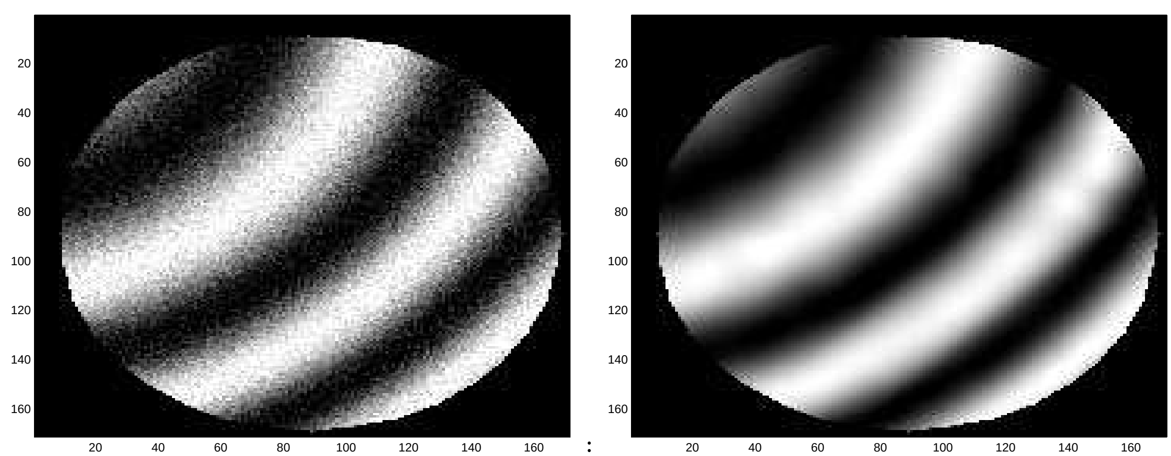


Figure 4.8: Noised image of $p = 1$ and the denoised image with TV-L^p method with $p = 1$

Chapter 5

Image Segmentation and Triangulation

5.1 Level Set Method

In the application of image segmentation, we can use an interface(a curve) to separate one region from another. To modify the interface we can assign a force F–the equation of motion to control how to move each point of the interface. The force F is usually given according to some physic mechanisms, for example, gravity, the ratio of the fluid density, and the surface tension between two regions. In the figure 5.1, the interface in red separate the square domain into regions and it is expanding under a force F.

However things get pretty complicated when the shape of the interface breaks into two or try to cross over itself. Therefore, rather than follow the interface, the level set method introduced by Osher and Sethian [40] consider the interface as the intersection of a surface an the x-y plane, or in another word, the zero-level contour of a surface. Therefore, in two dimensions, the level set method amounts to representing a closed curve Γ in a plane as the

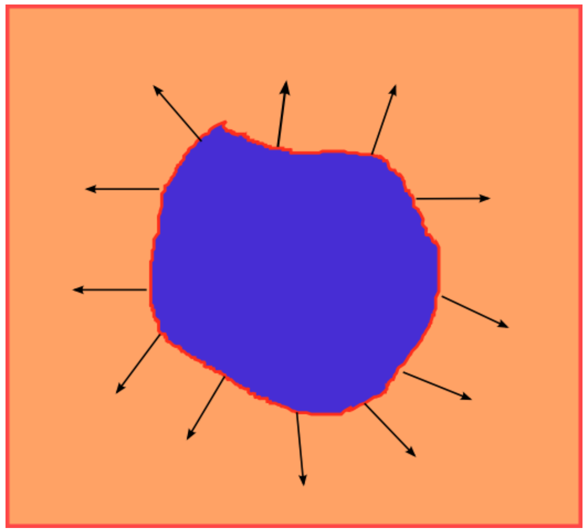


Figure 5.1: Level set method demonstration.

zero level set of a two-dimensional auxiliary function ϕ,

$$\Gamma = \{(x, y)|\phi(x, y|) = 0\}$$

and then manipulating Γ *implicitly* by tracing the surface function $\phi(x, y)$ instead. In figure 5.2, the the intersection of red surface ϕ and the blue x-y plane yield a the cutting planes in the first row. The boundary of the gray region is the interface Γ. One can see that although the interface breaks into two curves, the topology of the surface doesn't change at all. At first glance, it might not be a good idea to trade a problem of a moving curve into a problem of moving surface. Because we need to assign a force to each point on the domain, instead of the curve only. It seems it will add on the cost of calculation. However, the level set method processes some more critical properties – all the complicated problem of breaking and merging curves.

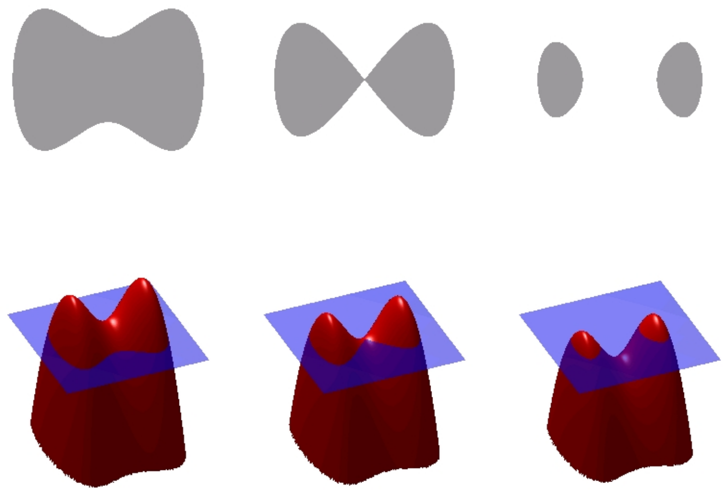


Figure 5.2: Demonstration of the change of the topology of the interface with level set method.

5.2 Active Contour Method

The active contour method proposed in [15] by Tony F. Chan and Luminita A. Vese originates from the idea of the level set method. Its basic idea is to evolve a closed curve to detect objects in an image, subject to the minimization of an energy defined in (5.1) below. For simplicity, let us assume that the image u is formed by two regions of approximatively piecewise-constant intensities of two distinct values u_1 and u_2 and they are separated by a contour $C_0 := \{x : \phi(x) = 0, x \in \Omega\}$. The goal is to find the "fittest" boundary C which best approximates C_0. One numerically computes an approximation C of C_0. Then the image is segmented into two distinguished regions: one is inside C and the other is outside C. In [15] the research considered the following minimization functional on C:

$$F(C) = \mu Length(C) + \nu Area(inside\ C) + \int_{inside(C)} |u(x) - u_1|^2 dxdy + \int_{outside(C)} |u(x) - u_2|^2 dx, \tag{5.1}$$

where C is a variable curve represented by level set $\{x : \phi(x) = 0, x \in \Omega\}$. In our computation, $\mu = \nu = 1/2$ and ϕ is approximated by a piecewise constant function over Ω. Here $u_1 := u_1(C)$ and $u_2 := u_2(C)$ are the average values of the image inside and outside C, respectively, which are defined as

$$u_1 = \int_{inside(C)} u(x,y)dxdy \text{ and } u_2 = \int_{outside(C)} u(x,y)dxdy.$$

Then C_0 is the minimizer of the fitting term

$$\inf_C F(C).$$

Here $\phi(x,y)$ is the surface function associated to the interface C in the level set method, such that

$$\phi(x,y) := \begin{cases} > 0, & (x,y)\ inside\ C \\ = 0, & (x,y)\ on\ C \\ < 0, & (x,y)\ outside\ C. \end{cases}$$

Recall that the Heavisible function H and Dirac Delta function δ are defined by

$$H(z) = \begin{cases} 1, & z \leq 0; \\ 0, & z < 0, \end{cases}$$

and

$$\delta(z) = \frac{d}{dz} H(z).$$

Then we have,

$$H(\phi(x,y)) = \begin{cases} 1, & (x,y) \ inside \ or \ on \ C; \\ 0, & otherwise, \end{cases}$$

$$|\nabla H(\phi(x,y))| = \begin{cases} 1, & (x,y) \ on \ C; \\ 0, & otherwise, \end{cases}$$

and

$$|\nabla H(\phi(x,y))| = \delta(\phi(x,y))|\nabla \phi(x,y)|.$$

Therefore, the length of C, the areas of the regions inside and outside C can be written respectively as

$$Length(\phi = 0) = \int_\Omega \delta(\phi(x,y))|\nabla \phi(x,y)|dxdy,$$

$$Area(\phi \geq 0) = \int_\Omega H(\phi(x,y))dxdy,$$

and

$$Area(\phi < 0) = \int_\Omega 1 - H(\phi(x,y))dxdy.$$

Similarly u_1 and u_2 can be given as:

$$u_1(\phi) = \frac{\int_\Omega u_0(x,y)H(\phi(x,y))dxdy}{\int_\Omega H(\phi(x,y))dxdy}, \tag{5.2}$$

$$u_2(\phi) = \frac{\int_\Omega u_0(x,y)(1 - H(\phi(x,y)))dxdy}{\int_\Omega (1 - H(\phi(x,y)))dxdy}. \tag{5.3}$$

Moreover, we can rewrite the following integrations on the entire domain Ω:

$$\int_{\phi > 0} |u_0(x,y) - u_1|^2 dxdy = \int_\Omega |u_0(x,y) - u_1|^2 H(\phi(x,y))dxdy,$$

$$\int_{\phi<0} |u_0(x,y) - u_2|^2 dxdy = \int_\Omega |u_0(x,y) - u_2|^2 (1 - H(\phi(x,y))| dxdy.$$

However, since the Heavisible function H and Dirac Delta function δ are not differentiable, we use the following regularizations instead,

$$H_\varepsilon(x) = \frac{1}{2}(1 + \frac{2}{\pi} arctan(\frac{x}{\varepsilon})),$$

$$\delta_\varepsilon(x) = \frac{\varepsilon^2}{(\pi(\varepsilon^2 + x^2))}.$$

Now we can rewrite the energy function (5.1) as

$$\begin{aligned} F_\varepsilon(u_1, u_2, \phi) &= \mu \int_\Omega \delta_\varepsilon(\phi)|\nabla\phi| dxdy \\ &+ \nu \int_\Omega H_\varepsilon(\phi) dxdy \\ &+ \lambda_1 \int_\Omega |u_0 - u_1|^2 H_\varepsilon(\phi) dxdy \\ &+ \lambda_2 \int_\Omega |u_0 - u_2|^2 (1 - H_\varepsilon(\phi)) dxdy \end{aligned} \tag{5.4}$$

Since the energy functional (5.4) is strictly convex, we just to need find a local minimum, which is the solution of the Euler-Lagrange equation. Now we use the calculus of variation method to calculate the Euler-Lagrange equation associated to the minimization problem of the energy function (5.4). First, notice that

$$\text{div}\left(\delta_\varepsilon(\phi)\frac{\nabla\phi}{|\nabla\phi|}\right) = \delta'_\varepsilon(\phi)|\nabla\phi| + \delta_\varepsilon(\phi)\text{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right).$$

Let $\vec{n}$ be the normal vector of $\partial\Omega$ and ψ the test function. When $t = 0$, we have

$$
\begin{aligned}
& \frac{d}{dt}\int_\Omega \delta_\varepsilon(\phi + t\psi)|\nabla(\phi + t\psi)|dxdy \\
= & \int_\Omega \delta'_\varepsilon(\phi)\psi|\nabla\phi|dxdy + \int_\Omega \delta_\varepsilon(\phi)\frac{\nabla\phi}{|\nabla\phi|}\cdot\nabla\psi \\
= & \int_\Omega \delta'_\varepsilon(\phi)\psi|\nabla\phi|dxdy + \int_{\partial\Omega} \delta_\varepsilon(\phi)\frac{\nabla\phi}{|\nabla\phi|}\cdot\overrightarrow{n}\psi dxdy - \int_\Omega \text{div}\left(\delta_\varepsilon(\phi)\frac{\nabla\phi}{|\nabla\phi|}\right)\psi dxdy \\
= & \int_\Omega \delta'_\varepsilon(\phi)\psi|\nabla\phi|dxdy - \int_\Omega \text{div}\left(\delta_\varepsilon(\phi)\frac{\nabla\phi}{|\nabla\phi|}\right)\psi dxdy \\
= & -\int_\Omega \delta_\varepsilon \text{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\psi dxdy,
\end{aligned}
$$

and

$$\frac{d}{dt}H_\varepsilon(\phi + t\psi) = \delta_\varepsilon(\phi)\psi.$$

Assuming $\frac{d}{dt}u_1(\phi + t\psi)|_{t=0} = 0$ and $\frac{d}{dt}u_2(\phi + t\psi)|_{t=0} = 0$, then we get the following Euler-Lagrange Equation of (5.4),

$$
\begin{cases}
0 = \delta_\varepsilon(\phi)\left(\mu\text{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) - \nu - \lambda_1|u_0 - u_1|^2 + \lambda_2|u_0 - u_2|^2\right), & \text{in } \Omega; \\
\frac{\partial\phi}{\partial\overrightarrow{n}} = 0, & \text{on } \partial\Omega;
\end{cases}
\tag{5.5}
$$

Using the method of steepest descent, we can approximate the solution of the Euler-Lagrange equation above by evolving the following time-flow partial differential equation

$$
\begin{cases}
\frac{\partial\phi}{\partial t} = \delta_\varepsilon(\phi)\left(\mu\text{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) - \nu - \lambda_1|u_0 - u_1|^2 + \lambda_2|u_0 - u_2|^2\right), & in(0, +\infty) \times \Omega; \\
\frac{\partial\phi}{\partial\overrightarrow{n}} = 0, & on(0, +\infty) \times \partial\Omega; \\
\phi(0, x, y) = \phi_0(x, y), & \text{when } t = 0, (x, y) \in \Omega.
\end{cases}
\tag{5.6}
$$

After the discretization the above time flow PDE, in each iteration of numerical steps, we need to update u_1 and u_2 according to (5.2) and (5.3).

To deal with complicated images with more than two distinguished regions, we have to apply the active contour segmentation method iteratively. We implement this method based on numerical integration with equally-spaced grids. Figure 5.2 gives an example which shows the process of the iterations. Figure (a) is the original image to be segmented. (b) is the resulting image after the first iteration of the active contour method, the original images is divided into two regions(black and white), or five separate regions A,B,C,D and E; In figure (c) we go on to divide region A into three separate region A1,A2 and A4 by one more iteration of the active contour method; (d) shows the combining the results of these two iterations by assigning different colors to each separate region;

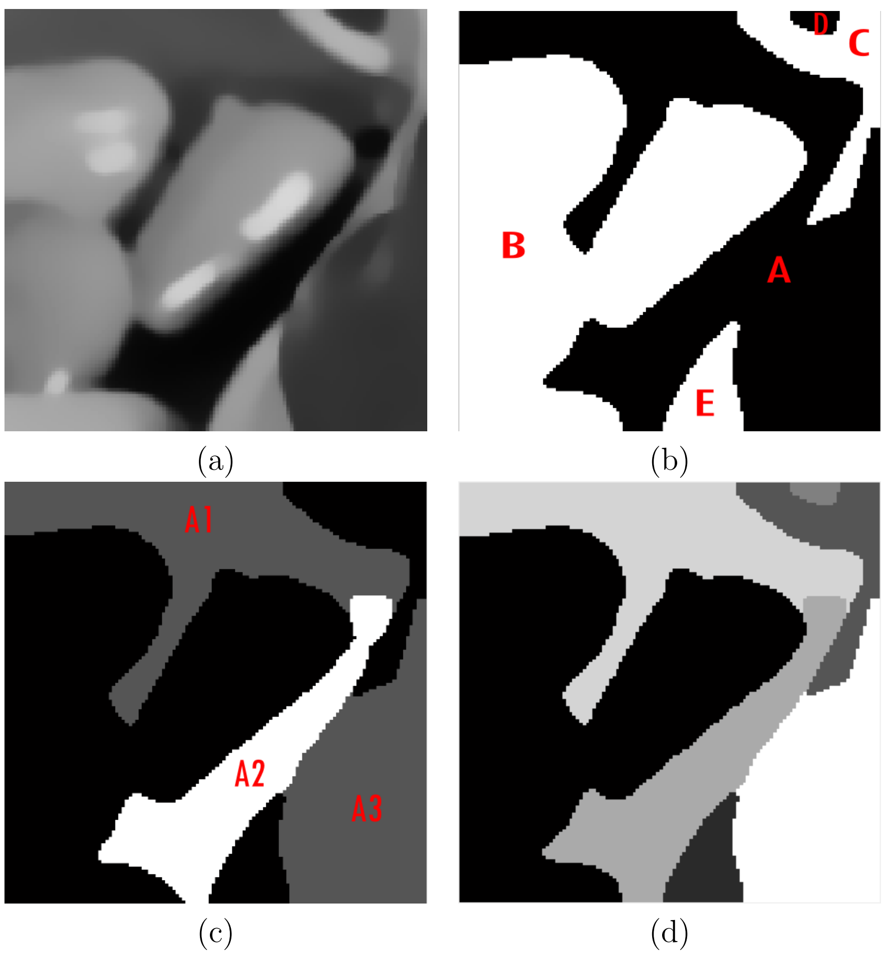


Figure 5.3: Application of iterative active contours in image segmentation.

5.3 Numerical Algorithm – The Finite Difference Method

In this section we propose a finite difference scheme to approximate the partial differential equation (5.6). Given $u_{i,j} = u(ih, jh)$, then the Forward difference, Backward difference, and Central difference are defined by

$$\Delta_x^+ u_{i,j} := u_{i+1,j} - u_{i,j}, \quad \Delta_y^+ u_{i,j} := u_{i,j+1} - u_{i,j};$$

$$\Delta_x^- u_{i,j} := u_{i,j} - u_{i-1,j}, \quad \Delta_y^- u_{i,j} := u_{i,j} - u_{i,j-1};$$

and

$$\Delta_x^c u_{i,j} := u_{i+1,j} - u_{i-1,j}, \quad \Delta_y^c u_{i,j} := u_{i,j+1} - u_{i,j-1}.$$

All the three differences above can be used to approximate the first derivative $\frac{\partial}{\partial x}(\text{or } \frac{\partial}{\partial y})$. It is up to the user to decide which scheme to use. For example we can approximate

$$|\nabla u|^2 \simeq \left(\frac{\Delta_x^+ u_{i,j}}{h}\right)^2 + \left(\frac{\Delta_y^c u_{i,j}}{2h}\right)^2.$$

However when we design our scheme, a thumb of rule to follow is the scheme should treat all grid points "fairly", that is using all grid points equally times. For example, if we use forward difference Δ_x^+ to approximate the first derivative then we should use backward difference Δ_x^- for the second derivative, e.x.,

$$u_{xx} \simeq \frac{\Delta_x^-(\Delta_x^+ u_{i,j})}{h^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}.$$

Or in another word, the grid points we use should be symmetrically located around u_{ij}, e.x.,

$$u_{xy} \simeq \frac{\Delta_x^c \Delta_y^c u_{i,j}}{(2h)^2} = \frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{(2h)^2}.$$

Follow the same rule, we use the following scheme to approximate the Laplacian Operator,

$$\begin{aligned}\Delta u &= \operatorname{div}(\nabla u) = \frac{\Delta_x^-}{h}\left(\frac{\Delta_x^+ u_{ij}}{h}\right) + \frac{\Delta_y^-}{h}\left(\frac{\Delta_y^+ u_{ij}}{h}\right) \\ &= \frac{1}{h^2}(\Delta_x^-\Delta_x^+ u_{ij} + \Delta_y^-\Delta_y^+ u_{ij}) = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}.\end{aligned}$$

There are two basic types of finite difference schemes, the explicit and implicit schemes. An explicit scheme is one which calculate the state of the system at a future time from the state at the current time, i.e., if $u^n = u(x, n\Delta t)$, $u^{n+1} = G(u^n)$ for some operator G, here . In an implicit scheme, we find an approximation for the future state by solving a system of equation involving both the future and current time step, i.e., $G(u^{n+1}) = u^n$. Generally, we can attain higher order of approximation than the explicit scheme. However, for nonlinear operator G, an implicit schemes is hard to implemented.

Since we are using finite difference scheme to approximate (5.6), we have to assume the domain of function Ω is rectangular and the values of u_0 on grid points $\{u_{ij}\}$ are given. For non rectangular domain, we can expand it to a rectangular domain by multiplying a characteristic function. And we use the predictor-corrector method to design our numerical scheme. In numerical analysis, a predictor-corrector is a two-step process which first calculates an roughly approximation of the desired quality, and then refines this initial approximation by another means. Denote r_i the discretion of the nonlinear function $\frac{1}{\sqrt{1+|\nabla\phi|^2}}$ in the diffusion term div $\left(\frac{\nabla\phi}{\sqrt{1+|\nabla\phi|^2}}\right)$. In order to follow the "fair" rule, we use two set of approximations. And since it is to approximate an nonlinear function, we use the explicit scheme here.

$$r_{ij}^x = \frac{1}{\sqrt{1 + ((\triangle_-^x \phi_{i,j}^n)/h)^2 + ((\triangle_c^y \phi_{i,j}^n)/2h)^2}}$$

$$r_{ij}^y = \frac{1}{\sqrt{1 + ((\triangle_-^y \phi_{i,j}^n)/h)^2 + ((\triangle_c^x \phi_{i,j}^n)/2h)^2}}$$

Then we have

$$
\begin{aligned}
& \operatorname{div}\left(\frac{\nabla\phi}{\sqrt{1+|\nabla\phi|^2}}\right) \\
\simeq\ & \frac{1}{h^2}\left(\Delta_x^+(\Delta_x^-\phi_{ij}r_{ij}^x)+\Delta_y^+(\Delta_y^-\phi_{ij}r_{ij}^y)\right) \\
=\ & \frac{1}{h^2}\left(\phi_{i+1j}r_{i+1j}^x+\phi_{i-1j}r_{ij}^x+\phi_{ij+1}r_{ij+1}^y+\phi_{ij-1}r_{ij}^y-\phi_{ij}(r_{i+1j}^x+r_{ij}^x+r_{ij+1}^y+r_{ij}^y)\right).
\end{aligned}
$$

Let

$$
C_{ij}=-\left(\nu+\lambda_1(u_{i,j}-c_{i,j}^1)^2-\lambda_2(u_{i,j}-c_{i,j}^2)^2\right).
$$

In the prediction step we first write down the implicit discrete form of (5.6)

$$
\begin{aligned}
\frac{\phi_{i,j}^{n+1}-\phi_{i,j}^n}{\triangle t} =\ & \frac{\mu\delta(\phi_{i,j}^{n+1})}{h^2}(r_{ij}^x\phi_{i-1,j}^{n+1}+r_{i+1j}^x\phi_{i+1,j}^{n+1}+r_{ij}^y\phi_{i,j-1}^{n+1}+r_{ij+1}^y\phi_{i,j+1}^{n+1}) \\
& -\frac{\mu\delta(\phi_{i,j}^{n+1})}{h^2}(r_{i+1j}^x+r_{ij}^x+r_{ij+1}^y+r_{ij}^y)\phi_{i,j}^{n+1}-\delta(\phi_{i,j}^{n+1})C_{ij} \\
& +\frac{\tau}{h^2}(r_{ij}^x\phi_{i-1,j}^{n+1}+r_{i+1j}^x\phi_{i+1,j}^{n+1}+r_{ij}^y\phi_{i,j-1}^{n+1}+r_{ij+1}^y\phi_{i,j+1}^{n+1}) \\
& -\frac{\tau}{h^2}(r_{i+1j}^x+r_{ij}^x+r_{ij+1}^y+r_{ij}^y)\phi_{i,j}^{n+1}.
\end{aligned} \tag{5.7}
$$

Then move all the linear terms of $\phi_{i,j}^{n+1}$ to the left hand side:

$$
\begin{aligned}
& \left(1+\left(\frac{\mu\triangle t}{h^2}\delta(\phi_{i,j}^{n+1})+\frac{\tau\triangle t}{h^2}\right)(r_{ij}^x+r_{i+1j}^x+r_{ij}^y+r_{ij+1}^y)\right)\phi_{i,j}^{n+1}-\phi_{i,j}^n \qquad (5.8) \\
=\ & \triangle t\delta(\phi_{i,j}^{n+1})\left(\frac{\mu}{h^2}(r_{ij}^x\phi_{i-1,j}^{n+1}+r_{i+1j}^x\phi_{i+1,j}^{n+1}+r_{ij}^y\phi_{i,j-1}^{n+1}+r_{ij+1}^y\phi_{i,j+1}^{n+1})\right)-\triangle t\delta(\phi_{i,j}^{n+1})C_{ij} \\
& +\frac{\tau\triangle t}{h^2}(r_{ij}^x\phi_{i-1,j}^{n+1}+r_{i+1j}^x\phi_{i+1,j}^{n+1}+r_{ij}^y\phi_{i,j-1}^{n+1}+r_{ij+1}^y\phi_{i,j+1}^{n+1}).
\end{aligned}
$$

Then we replace ϕ^{n+1} in all nonlinear terms by ϕ^n:

$$\left(1+\left(\frac{\mu\triangle t}{h^2}\delta(\phi^n_{i,j})+\frac{\tau\triangle t}{h^2}\right)(r^x_{ij}+r^x_{i+1j}+r^y_{ij}+r^y_{ij+1})\right)\phi^{n+1}_{i,j}-\phi^n_{i,j} \tag{5.9}$$
$$= \triangle t\delta(\phi^n_{i,j})\left(\frac{\mu}{h^2}(r^x_{ij}\phi^{n+1}_{i-1,j}+r^x_{i+1j}\phi^{n+1}_{i+1,j}+r^y_{ij}\phi^{n+1}_{i,j-1}+r^y_{ij+1}\phi^{n+1}_{i,j+1})\right)-\triangle t\delta(\phi^n_{i,j})C_{ij}$$
$$+\frac{\tau\triangle t}{h^2}(r^x_{ij}\phi^{n+1}_{i-1,j}+r^x_{i+1j}\phi^{n+1}_{i+1,j}+r^y_{ij}\phi^{n+1}_{i,j-1}+r^y_{ij+1}\phi^{n+1}_{i,j+1}).$$

To predict ϕ^{n+1}, we replace all linear terms of ϕ^{n+1} by ϕ^n on the right hand side of (5.11), and solve it to get the predictor $\bar{\phi}^{n+1}$ as follows

$$\left(1+\left(\frac{\mu\triangle t}{h^2}\delta(\phi^n_{i,j})+\frac{\tau\triangle t}{h^2}\right)(r^x_{ij}+r^x_{i+1j}+r^y_{ij}+r^y_{ij+1})\right)\bar{\phi}^{n+1}_{i,j}-\phi^n_{i,j} \tag{5.10}$$
$$= \triangle t\delta(\phi^n_{i,j})\left(\frac{\mu}{h^2}(r^x_{ij}\phi^{n}_{i-1,j}+r^x_{i+1j}\phi^{n}_{i+1,j}+r^y_{ij}\phi^{n}_{i,j-1}+r^y_{ij+1}\phi^{n}_{i,j+1})\right)-\triangle t\delta(\phi^n_{i,j})C_{ij}$$
$$+\frac{\tau\triangle t}{h^2}(r^x_{ij}\phi^{n}_{i-1,j}+r^x_{i+1j}\phi^{n}_{i+1,j}+r^y_{ij}\phi^{n}_{i,j-1}+r^y_{ij+1}\phi^{n}_{i,j+1}).$$

Finally, in the correction step we replace ϕ^{n+1} in all nonlinear terms, and the right hand side of (5.11) with $\bar{\phi}^{n+1}$,

$$\left(1+\left(\frac{\mu\triangle t}{h^2}\delta(\bar{\phi}^{n+1}_{i,j})+\frac{\tau\triangle t}{h^2}\right)(r^x_{ij}+r^x_{i+1j}+r^y_{ij}+r^y_{ij+1})\right)\phi^{n+1}_{i,j}-\phi^n_{i,j} \tag{5.11}$$
$$= \delta(\bar{\phi}^{n+1}_{i,j})\left(\frac{\mu}{h^2}(r^x_{ij}\bar{\phi}^{n+1}_{i-1,j}+r^x_{i+1j}\bar{\phi}^{n+1}_{i+1,j}+r^y_{ij}\bar{\phi}^{n+1}_{i,j-1}+r^y_{ij+1}\bar{\phi}^{n+1}_{i,j+1})\right)-\delta(\bar{\phi}^{n+1}_{i,j})C_{ij}$$
$$+\frac{\tau}{h^2}(r^x_{ij}\bar{\phi}^{n+1}_{i-1,j}+r^x_{i+1j}\bar{\phi}^{n+1}_{i+1,j}+r^y_{ij}\bar{\phi}^{n+1}_{i,j-1}+r^y_{ij+1}\bar{\phi}^{n+1}_{i,j+1}).$$

Solving the above equation, we get ϕ^{n+1}.

5.4 Numerical Results

Example 5.4.1 *In Fig. 5.4 and Fig. 5.5, we demonstrate using the active contour model to separate distinguished textures. One can see that both zebra and leopard have different* *texture to their background. Once we have extracted the texture of the animals, we can use the active contour method to separate the texture and the background.*

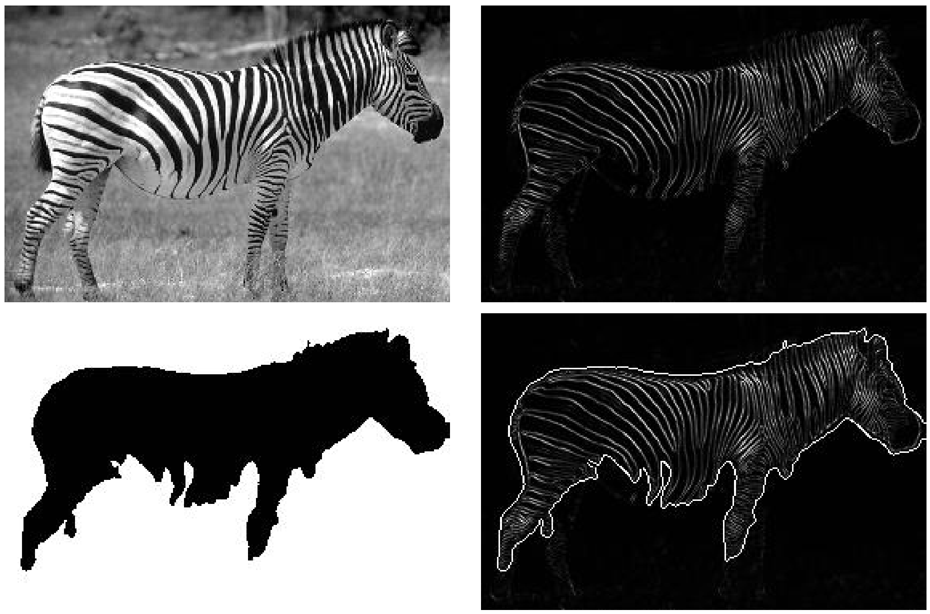


Figure 5.4: Segmentation of Texture by Active Contour Method

Example 5.4.2 *In Fig. 5.6, (b) is the segmentation after applying the Active Contour Method on (a) directly. One can see that in this case, the two coarse patches can not be distinguished, because the segmentation is carried out according to the grey level. After taking the gradient in (c), the two coarse patches are singled out by the Active Contour Method in (d).*

Example 5.4.3 *Fig. 5.7 shows an example of triangulation according to the segmentation result from the Active Contour Method.*

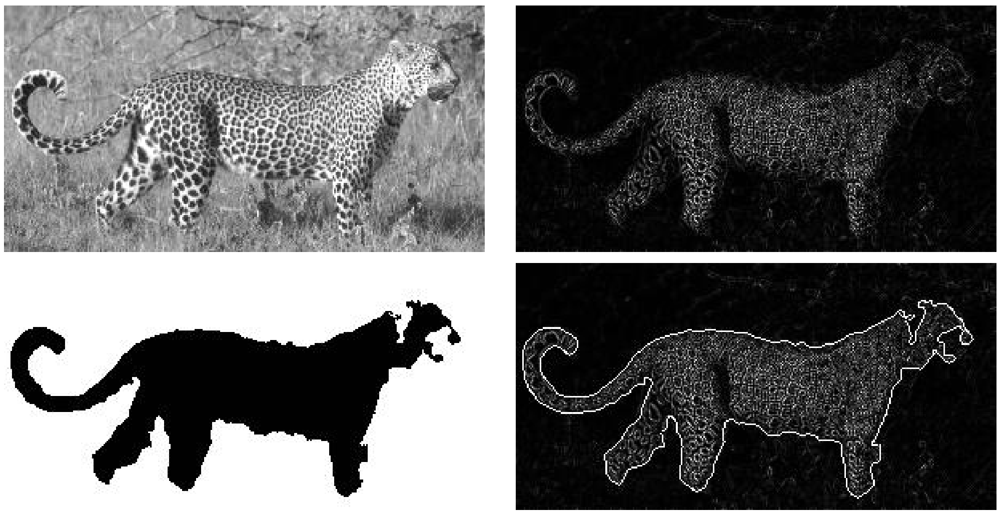
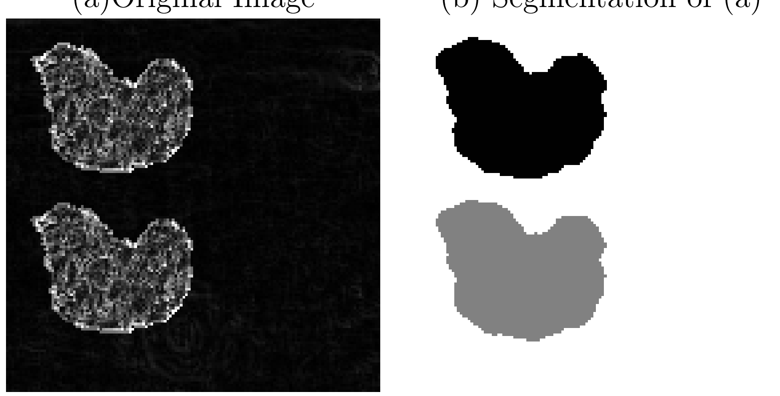


Figure 5.5: Segmentation of Texture by Active Contour Method

(a)Original Image (b) Segmentation of (a)



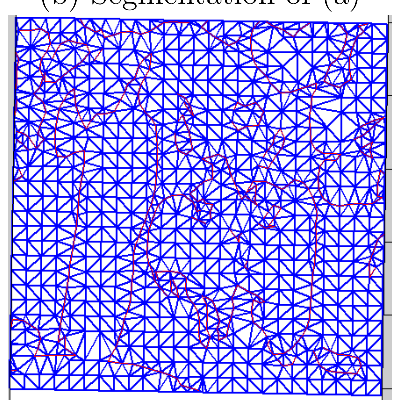
(c) Gradient Image of (a) (d) Segmentation of (c)

Figure 5.6: Segmentation according to grey level and gradient level respectively

(a)Original Image

(b) Segmentation of (a)

(c)Contour of (b)



(d)Triangulation according to (b)

Figure 5.7: Triangulation according to the segmentation from Active Contour Method.

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