Parameterizations of Univariate Orthogonal Wavelets with Short Support

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Abstract. In this note, we give a complete and simple parameterization for the length-six one-dimensional filters which satisfy the necessary conditions for orthogonality. This parameterization includes all compactly supported univariate scaling functions contained within the interval $[0,5]$ using dilation factor 2. These formulae are a convenient way to generate a continuum of wavelets with varying approximation properties ranging from scaling functions which only reproduce constants to ones which reproduce linears and quadratics. In addition, solutions for the length-eight and length-ten filters are given where the parameters have transcendental constraints. We conclude this note with some interesting numerical experiments comparing the parameterized wavelets of length six with the standard Haar, D4, and D6 wavelets in an image compression scheme.

§1. Introduction

The use of compactly supported orthonormal wavelets is common in various applications. The Daubechies orthonormal wavelets are an important class which have been used extensively for applications such as image compression. The construction of wavelets usually begins with a scaling function ϕ which satisfies various properties such as refinability and orthogonality. Numerous necessary and sufficient conditions have been developed to aid in the construction of wavelets (see [1,6], and many others). One method for constructing compactly supported scaling functions is to begin with the dilation equation

$$\phi(x) = \sum_k h_k \phi(2x - k)$$

Approximation Theory X
Charles K. Chui, Larry L. Schumaker, and Joachim Stöckler (eds.), pp. 1–16.

ISBN 0-8265-xxxx-x.

with a finite number of coefficients. In order for a function of this type to be orthonormal to its integer shifts, the coefficients in the dilation equation must satisfy

$$m(1) = 1$$
$$|m(z)|^2 + |m(-z)|^2 = 1, \quad z = e^{i\omega},$$

where $m(z) = \frac{1}{2}\sum_k h_k z^k$ is the associated trigonometric polynomial. For scaling functions with support contained in $[0,5]$ these necessary conditions suggest a simple technique for parameterizing their solution. A different approach in [8] gives the explicit parameterization of the length-four filters and a constructive technique for longer filters. The method shown here is the same technique we used to parameterize compactly supported bivariate scaling functions(see [2,3,4,5]). Another approach for dealing with the bivariate case can be seen in [7].

As will be shown, the necessary conditions for orthogonality yield a set of linear and nonlinear equations which necessarily imply that various sums of dilation coefficients form perfect squares. This leads to the introduction of the free parameters. In Section 2, we give the parameterization of the length-four filters for completeness. In Section 3, we derive the explicit parameterization for the length-six filters. Section 4 gives the constrained solutions for the length-eight and length-ten filters. Finally, we conclude in Section 5 with an interesting numerical experiment comparing the standard Haar, D4, and D6 wavelets with the parameterized wavelets of length-six in an image compression scheme.

§2. Length-Four Solution

The formulas for the parameterization of the length-four filters were given in [8], but we include them here for completeness. Let $H_4(z) = a_0 + b_0 z + a_1 z^2 + b_1 z^3$.

Lemma 1. *$H_4(z)$ satisfies $H_4(1) = 1$ and*

$$|H_4(z)|^2 + |H_4(-z)|^2 = 1, \qquad \forall z = e^{i\omega} \text{ and } \omega \in \mathbb{R}$$

if and only if

$$a_0 = \frac{1}{4} + \frac{1}{2\sqrt{2}}\cos\alpha, \quad b_0 = \frac{1}{4} + \frac{1}{2\sqrt{2}}\sin\alpha,$$
$$a_1 = \frac{1}{4} - \frac{1}{2\sqrt{2}}\cos\alpha, \quad b_1 = \frac{1}{4} - \frac{1}{2\sqrt{2}}\sin\alpha,$$

for any $\alpha \in \mathbb{R}$.

Proof: First, we have

$$\begin{aligned} |H_4(z)|^2 =& a_0^2 + b_0^2 + a_1^2 + b_1^2 \\ &+ (a_0b_0 + b_0a_1 + a_1b_1)(z + z^{-1}) \\ &+ (a_0a_1 + b_0b_1)(z^2 + z^{-2}) + a_0b_1(z^3 + z^{-3}). \end{aligned}$$

Then $|H_4(z)|^2 + |H_4(-z)|^2 = 1$ implies that

$$\begin{aligned} a_0^2 + a_1^2 + b_0^2 + b_1^2 &= \tfrac{1}{2} \\ a_0a_1 + b_0b_1 &= 0. \end{aligned} \tag{1}$$

It follows from $H_4(1) = 1$ that $a_0 + a_1 + b_0 + b_1 = 1$. Also, $H(-1) = 0$ implies that $a_0 + a_1 - b_0 - b_1 = 0$. Thus, we have

$$a_0 + a_1 = b_0 + b_1 = \frac{1}{2}. \tag{2}$$

Moreover, the two nonlinear equations from (1) imply

$$(a_0 - a_1)^2 + (b_0 - b_1)^2 = \frac{1}{2}$$

or

$$a_0 - a_1 = \frac{1}{\sqrt{2}} \cos\alpha, \quad b_0 - b_1 = \frac{1}{\sqrt{2}} \sin\alpha.$$

Thus, after combining these equations with equation (2), we have the formulas as stated in the lemma.

On the other hand, using the expressions for the a_i's and b_i's with a straightforward calculation shows that in fact

$$|H_4(z)|^2 + |H_4(-z)|^2 = 1.$$

This completes the proof. □

Example 1. *When $\alpha = \frac{\pi}{4}$, we get $H_4(z) = \frac{1+z}{2}$, which is associated with the Haar wavelet.*

Example 2. *When $\alpha = \frac{5\pi}{12}$, we get $a_0 = \frac{1+\sqrt{3}}{8}$ and $a_1 = \frac{3-\sqrt{3}}{8}$ as well as $b_0 = \frac{3+\sqrt{3}}{8}$, $b_1 = \frac{1-\sqrt{3}}{8}$. Then $H_4(z)$ is associated with the Daubechies D4 wavelet.*

Next, we look for choices of α where $H_4(z)$ has a second-order vanishing moment. That is, $H_4(z) = \left(\frac{1+z}{2}\right)^2 p(z)$ where $p(z)$ is some trigonometric polynomial.

In this case $\frac{d}{dz}H(z)|_{z=-1} = 0$ which is equivalent to $b_0 - 2a_1 + 3b_1 = 0$. That is,

$$\begin{aligned} 0 &= b_0 - 2a_1 + 3b_1 \\ &= \frac{1}{4} + \frac{1}{2\sqrt{2}} \sin\alpha - \frac{1}{2} + \frac{1}{\sqrt{2}} \cos\alpha + \frac{3}{4} - \frac{3}{2\sqrt{2}} \sin\alpha \\ &= \frac{1}{2} + \frac{1}{\sqrt{2}} (\cos\alpha - \sin\alpha) \\ &= \frac{1}{2} + \cos\left(\alpha + \frac{\pi}{4}\right), \end{aligned}$$

which is only satisfied by $\alpha = \frac{5\pi}{12}$ or $\alpha = \frac{13\pi}{12}$. Both are associated with Daubechies' D4 wavelet. Thus, D4 is the only member of the family with two vanishing moments, which is a well-known fact.

§3. Length-Six Solution

Let $H_6(z) = a_0 + b_0 z + a_1 z^2 + b_1 z^3 + a_2 z^4 + b_2 z^5$ be a trigonometric polynomial of $z = e^{i\omega}$.

Lemma 2. *$H_6(z)$ satisfies $H_6(1) = 1$ and*

$$|H_6(z)|^2 + |H_6(-z)|^2 = 1, \forall z = e^{i\omega}, \qquad \omega \in \mathbb{R}$$

if and only if

$$\begin{aligned}
a_0 &= \frac{1}{8} + \frac{1}{4\sqrt{2}}\cos\alpha + \frac{p}{2}\cos\beta \\
a_1 &= \frac{1}{4} - \frac{1}{2\sqrt{2}}\cos\alpha \\
a_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}}\cos\alpha - \frac{p}{2}\cos\beta \\
b_0 &= \frac{1}{8} + \frac{1}{4\sqrt{2}}\sin\alpha + \frac{p}{2}\sin\beta \\
b_1 &= \frac{1}{4} - \frac{1}{2\sqrt{2}}\sin\alpha \\
b_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}}\sin\alpha - \frac{p}{2}\sin\beta,
\end{aligned}$$

where

$$p = \frac{1}{2}\sqrt{1 + \sin(\alpha + \frac{\pi}{4})}$$

for any $\alpha, \beta \in \mathbb{R}$.

Proof: First of all, $H_6(1) = 1$ implies that $\sum_{i=0}^{2} a_i + b_i = 1$. Also, $H_6(-1) = 0$ implies that

$$\sum_{i=0}^{2} a_i - \sum_{i=0}^{2} b_i = 0.$$

It follows that

$$a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = \frac{1}{2}. \tag{3}$$

Next, $|H_6(z)|^2 + |H_6(-z)|^2 = 1$ implies that

$$\begin{aligned}
a_0^2 + a_1^2 + a_2^2 + b_0^2 + b_1^2 + b_2^2 &= \frac{1}{2} \\
a_0 a_1 + a_1 a_2 + b_0 b_1 + b_1 b_2 &= 0 \\
a_0 a_2 + b_0 b_2 &= 0.
\end{aligned} \tag{4}$$

¿From these nonlinear equations, it follows that

$$(a_0 - a_1 + a_2)^2 + (b_0 - b_1 + b_2)^2 = \frac{1}{2},$$

and

$$a_0 - a_1 + a_2 = \frac{1}{\sqrt{2}} \cos\alpha, \quad b_0 - b_1 + b_2 = \frac{1}{\sqrt{2}} \sin\alpha. \tag{5}$$

Combining (5) with (3), we have

$$\begin{aligned} a_1 &= \frac{1}{4} - \frac{1}{2\sqrt{2}} \cos\alpha, & b_1 &= \frac{1}{4} - \frac{1}{2\sqrt{2}} \sin\alpha \\ a_0 + a_2 &= \frac{1}{4} + \frac{1}{2\sqrt{2}} \cos\alpha, & b_0 + b_2 &= \frac{1}{4} + \frac{1}{2\sqrt{2}} \cos\alpha. \end{aligned} \tag{6}$$

Moreover,

$$\begin{aligned} (a_0 - a_2)^2 + (b_0 - b_2)^2 &= \frac{1}{2} - a_1^2 - b_1^2 \\ &= \frac{1}{2} - (\frac{1}{4} - \frac{1}{4\sqrt{2}}(\cos\alpha + \sin\alpha)) \\ &= \frac{1}{4} + \frac{1}{4\sqrt{2}}(\cos\alpha + \sin\alpha) \\ &= \frac{1}{4} + \frac{1}{4}\sin(\alpha + \frac{\pi}{4}), \end{aligned}$$

which implies that

$$\begin{aligned} a_0 - a_2 &= \frac{1}{2}\sqrt{1 + \sin(\alpha + \frac{\pi}{4})} \cos\beta \\ b_0 - b_2 &= \frac{1}{2}\sqrt{1 + \sin(\alpha + \frac{\pi}{4})} \sin\beta. \end{aligned}$$

The result follows by adding these equations to those from (6).

On the other hand, a routine calculation establishes that $H_6(z)$ with these coefficients satisfies $H(1) = 1$ and $|H_6(z)|^2 + |H_6(-z)|^2 = 1$. This completes the proof. □

Example 3. *If $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{4}$, then $H_6(z)$ is associated with the Haar wavelet.*

Example 4. *If $\alpha = \frac{5\pi}{12}$ and $\beta = \frac{\pi}{3}$, then $H_6(z)$ is associated with the Daubechies D4 wavelet.*

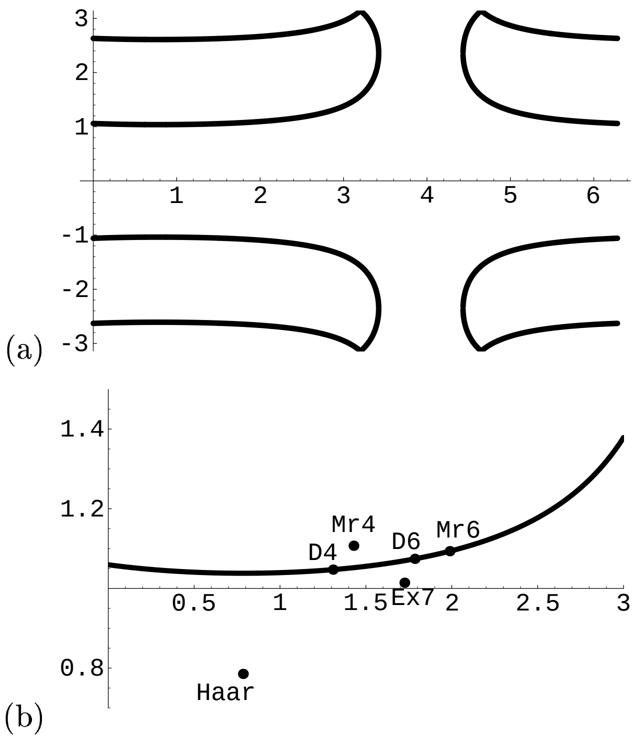

Fig. 1. (a) A plot of the ordered pairs (α, β) for which $H_6(z)$ has two vanishing moments, (b) a zoomed in plot of the above graph showing the positions of Haar, D4, D6, Example 7 (Ex7), and the most-regular length-four (Mr4) and length-six (Mr6) scaling functions.

Example 5. *If*

$$\cos\alpha = -\frac{1}{4}\sqrt{8+\sqrt{15+12\sqrt{10}}}, \quad \sin\alpha = \frac{1}{4}\sqrt{8-\sqrt{15+12\sqrt{10}}},$$
$$\cos\beta = \frac{1}{4}\sqrt{8-8\sqrt{-25+8\sqrt{10}}}, \quad \sin\beta = \frac{1}{4}\sqrt{8+8\sqrt{-25+8\sqrt{10}}},$$

then $H_6(z)$ is the filter associated with Daubechies D6 wavelet.

Example 6. *To see when $H_6(z)$ has two vanishing moments, we require $H'(-1) = 0$ which implies*

$$\begin{aligned} 0 =& H'(-1) = b_0 - 2a_1 + 3b_1 - 4a_2 + 5b_2 \\ =& 1/2 + 2b_1 + 4b_2 - 2a_1 - 4a_2 \\ =& 1/2 - \sqrt{2(1+\sin(\alpha+\pi/4))}\sin(\beta-\pi/4) \end{aligned}$$

or

$$\sin(\alpha+\pi) = \frac{1}{8\sin^2(\beta-\pi/4)} - 1.$$

When the ordered pair (α, β) lies along the curves given in Figure 1(a), $H_6(z)$ has two vanishing moments.

Example 7. *When $\alpha = 1.725080699801023$ and $\beta = 1.01424683724616$, the scaling function is as shown in Figure 2(b). This scaling function performed better than Haar, D4, and D6 in our numerical experiments as shown in the last Section. It should be noted that this scaling function does not have a second-order vanishing moment as can be seen in Figure 1(b).*

Example 8. *With $\alpha = 1.4288992721907328$, $\beta = 1.1071487177940904$, $H_6(z)$ is associated with the most-regular length-four filter as given in [1]. When $\alpha = 1.9886461158096038$ and $\beta = 1.0934936891036087$, $H_6(z)$ is associated with the most-regular length-six filter as given in [1]. Each of these scaling functions are shown in Figure 2(c) and 2(d).*

§4. Constrained Solutions of Lengths Eight and Ten

The length-eight solution has 4 parameters and a transcendental constraint, and the length-ten solution has 5 parameters and a transcendental constraint as well. We will need the following lemma in the proofs of Lemmas 4 and 5.

Lemma 3. *Suppose a, b, c, and $d \in \mathbb{R}$. Then $a^2 + b^2 + c^2 + d^2 = 1$ if and only if*

$$a = \cos\beta\cos\gamma, \quad b = \cos\beta\sin\gamma, \quad c = \sin\beta\cos\theta, \quad d = \sin\beta\sin\theta$$

for β, γ, and $\theta \in \mathbb{R}$.

Let $H_8(z) = a_0 + b_0 z + a_1 z^2 + b_1 z^3 + a_2 z^4 + b_2 z^5 + a_3 z^6 + b_3 z^7$ be a trigonometric polynomial of $z = e^{i\omega}$.

Lemma 4. *$H_8(z)$ satisfies $H_8(1) = 1$ and*

$$|H_8(z)|^2 + |H_8(-z)|^2 = 1, \forall z = e^{i\omega}, \qquad \omega \in \mathbb{R}$$

if and only if

$$\begin{aligned}
a_0 &= \frac{1}{8} + \frac{1}{4\sqrt{2}}\cos\alpha + \frac{1}{2\sqrt{2}}\cos\beta\cos\gamma \\
a_1 &= \frac{1}{8} - \frac{1}{4\sqrt{2}}\cos\alpha + \frac{1}{2\sqrt{2}}\cos\beta\sin\gamma \\
a_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}}\cos\alpha - \frac{1}{2\sqrt{2}}\cos\beta\cos\gamma \\
a_3 &= \frac{1}{8} - \frac{1}{4\sqrt{2}}\cos\alpha - \frac{1}{2\sqrt{2}}\cos\beta\sin\gamma
\end{aligned}$$

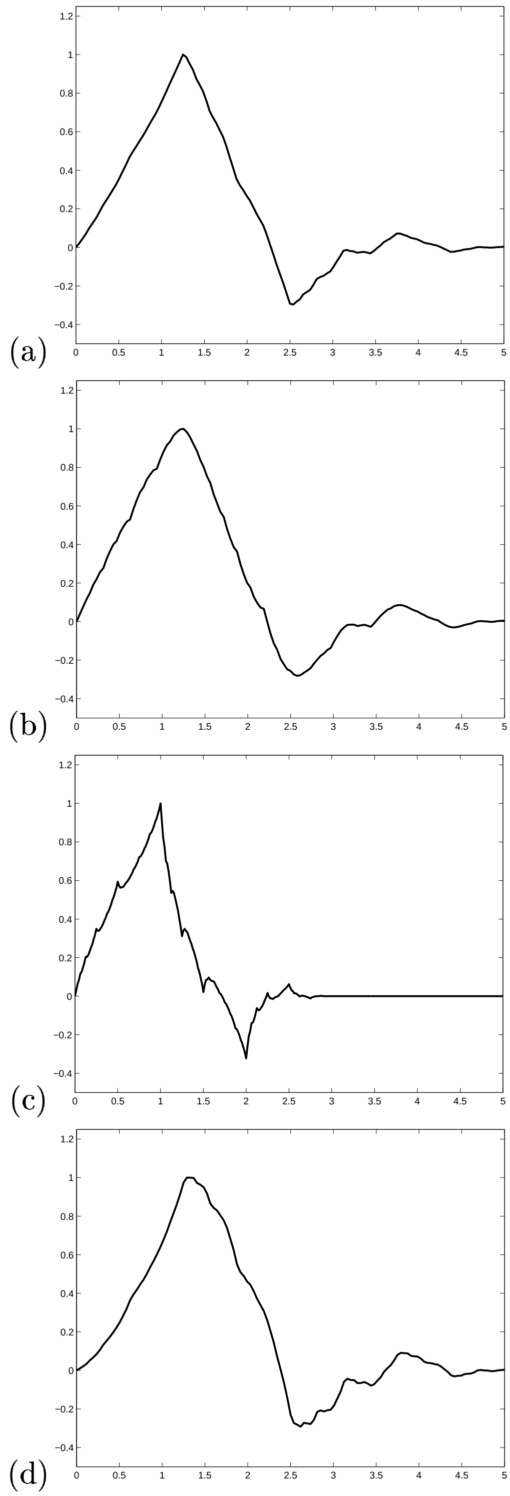

Fig. 2. Graphs of four different scaling functions: (a) D6 scaling function with three vanishing moments from Example 5, (b) H6 scaling function from Example 7, (c) most-regular length-four scaling function from Example 8, (d) most-regular length-six scaling function from Example 8.

$$
\begin{aligned}
b_0 &= \frac{1}{8} + \frac{1}{4\sqrt{2}}\sin\alpha + \frac{1}{2\sqrt{2}}\sin\beta\cos\theta \\
b_1 &= \frac{1}{8} - \frac{1}{4\sqrt{2}}\sin\alpha + \frac{1}{2\sqrt{2}}\sin\beta\sin\theta \\
b_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}}\sin\alpha - \frac{1}{2\sqrt{2}}\sin\beta\cos\theta \\
b_3 &= \frac{1}{8} - \frac{1}{4\sqrt{2}}\sin\alpha - \frac{1}{2\sqrt{2}}\sin\beta\sin\theta,
\end{aligned}
$$

where α, β, θ, *and* $\gamma \in \mathbb{R}$ *satisfy*

$$
\begin{aligned}
&\sqrt{2}\cos\theta\sin\beta - 2\cos\theta\sin\alpha\sin\beta + \sqrt{2}\cos\beta(\cos\gamma - \sin\gamma) \\
&- 4\cos^2\beta\cos\gamma\sin\gamma - 2\cos\alpha\cos\beta(\cos\gamma + \sin\gamma) \\
&- \sqrt{2}\sin\beta\sin\theta - 2\sin\alpha\sin\beta\sin\theta - 4\cos\theta\sin^2\beta\sin\theta = 0.
\end{aligned}
$$

Proof: As before, $H_8(1) = 1$ implies that $\sum_{i=0}^{3} a_i + b_i = 1$ and $H_8(-1) = 0$ implies that

$$\sum_{i=0}^{3} a_i - \sum_{i=0}^{3} b_i = 0.$$

It follows that

$$a_0 + a_1 + a_2 + a_3 = b_0 + b_1 + b_2 + b_3 = \frac{1}{2}. \tag{7}$$

Again, $|H_8(z)|^2 + |H_8(-z)|^2 = 1$ implies the following set of nonlinear equations:

$$
\begin{aligned}
a_0^2 + a_1^2 + a_2^2 + a_3^2 + b_0^2 + b_1^2 + b_2^2 + b_3^2 &= \frac{1}{2} \\
a_0a_1 + a_1a_2 + a_2a_3 + b_0b_1 + b_1b_2 + b_2b_3 &= 0 \\
a_0a_2 + a_1a_3 + b_0b_2 + b_1b_3 &= 0 \\
a_0a_3 + b_0b_3 &= 0.
\end{aligned} \tag{8}
$$

By adding all the equations together from (8), we see that

$$(a_0 - a_1 + a_2 - a_3)^2 + (b_0 - b_1 + b_2 - b_3)^2 = \frac{1}{2}.$$

So,

$$a_0 - a_1 + a_2 - a_3 = \frac{1}{\sqrt{2}}\cos\alpha, \quad b_0 - b_1 + b_2 - b_3 = \frac{1}{\sqrt{2}}\sin\alpha,$$

which implies that

$$
\begin{aligned}
a_0 + a_2 &= \frac{1}{4} + \frac{1}{2\sqrt{2}}\cos\alpha, \quad a_1 + a_3 = \frac{1}{4} - \frac{1}{2\sqrt{2}}\cos\alpha \\
b_0 + b_2 &= \frac{1}{4} + \frac{1}{2\sqrt{2}}\sin\alpha, \quad b_1 + b_3 = \frac{1}{4} - \frac{1}{2\sqrt{2}}\sin\alpha.
\end{aligned}
$$

Continuing along this line, by subtracting the third equation from the first equation of (8), we have

$$(a_0 - a_2)^2 + (a_1 - a_3)^2 + (b_0 - b_2)^2 + (b_1 - b_3)^2 = \frac{1}{2}.$$

Using Lemma 3,

$$a_0 - a_2 = \frac{1}{\sqrt{2}} \cos\beta \cos\gamma, \quad a_1 - a_3 = \frac{1}{\sqrt{2}} \cos\beta \sin\gamma$$
$$b_0 - b_3 = \frac{1}{\sqrt{2}} \sin\beta \cos\theta, \quad b_1 - b_3 = \frac{1}{\sqrt{2}} \sin\beta \sin\theta.$$

Combining these four equations with the previous group of four, we arrive at the necessary formulae as listed in the lemma.

Plugging these formulae into the nonlinear system of equations reveals that the first and third equations from (8) are satisfied, but the second and fourth equations produce a single transcendental constraint as stated in the lemma. □

Example 9. *Let* $\alpha = 2.2400754386946162, \beta = 0.7535419996522459$, $\gamma = 0.9614024467911164$, *and* $\theta = -0.025413001147374 89$. *Then* $H_8(z)$ *is associated with Daubechies scaling function D8.*

Example 10. *In order for* $H_8(z)$ *to have a second order vanishing moment, we require* $H'(-1) = 0$. *It follows that*

$$a_1 + 2a_2 + 3a_3 = b_1 + 2b_2 + 3b_3 + \frac{1}{4}.$$

Using the solution formulae in Lemma 4, we have

$$\frac{1}{2} - \sin(\alpha - \frac{\pi}{4}) + 2\cos\beta \sin(\gamma + \frac{\pi}{4}) - 2\sin\beta \sin(\theta + \frac{\pi}{4}) = 0.$$

Combining this with the transcendental constraint in Lemma 4, we can find many solutions for $H_8(z)$ *with the second order vanishing moment. For example,*

$$\alpha = \frac{5\pi}{12}, \beta = -\frac{\pi}{2}, \gamma = -\frac{\pi}{4}, \text{ and } \theta = -\frac{\pi}{4},$$

and

$$\alpha = \frac{5\pi}{12}, \beta = \pi, \gamma = -\frac{\pi}{4}, \text{ and } \theta = -\frac{\pi}{4}.$$

Let $H_{10}(z) = a_0 + b_0 z + a_1 z^2 + b_1 z^3 + a_2 z^4 + b_2 z^5 + a_3 z^6 + b_3 z^7 + a_4 z^8 + b_4 z^9$ be a trigonometric polynomial of $z = e^{i\omega}$.

Lemma 5. *$H_{10}(z)$ satisfies $H_{10}(1) = 1$ and*

$$|H_{10}(z)|^2 + |H_{10}(-z)|^2 = 1, \forall z = e^{i\omega}, \qquad \omega \in \mathbb{R}$$

if and only if

$$\begin{aligned}
a_0 &= \frac{1}{16} + \frac{1}{8\sqrt{2}}\cos\alpha + \frac{1}{4\sqrt{2}}\cos\beta\cos\gamma + \frac{r}{2}\cos\delta \\
a_1 &= \frac{1}{8} - \frac{1}{4\sqrt{2}}\cos\alpha + \frac{1}{2\sqrt{2}}\cos\beta\sin\gamma \\
a_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}}\cos\alpha - \frac{1}{2\sqrt{2}}\cos\beta\cos\gamma \\
a_3 &= \frac{1}{8} - \frac{1}{4\sqrt{2}}\cos\alpha - \frac{1}{2\sqrt{2}}\cos\beta\sin\gamma \\
a_4 &= \frac{1}{16} + \frac{1}{8\sqrt{2}}\cos\alpha + \frac{1}{4\sqrt{2}}\cos\beta\cos\gamma - \frac{r}{2}\cos\delta \\
b_0 &= \frac{1}{16} + \frac{1}{8\sqrt{2}}\sin\alpha + \frac{1}{4\sqrt{2}}\sin\beta\cos\theta + \frac{r}{2}\sin\delta \\
b_1 &= \frac{1}{8} - \frac{1}{4\sqrt{2}}\sin\alpha + \frac{1}{2\sqrt{2}}\sin\beta\sin\theta \\
b_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}}\sin\alpha - \frac{1}{2\sqrt{2}}\sin\beta\cos\theta \\
b_3 &= \frac{1}{8} - \frac{1}{4\sqrt{2}}\sin\alpha - \frac{1}{2\sqrt{2}}\sin\beta\sin\theta \\
b_4 &= \frac{1}{16} + \frac{1}{8\sqrt{2}}\sin\alpha + \frac{1}{4\sqrt{2}}\sin\beta\cos\theta - \frac{r}{2}\sin\delta,
\end{aligned}$$

where

$$r = \sqrt{\frac{1}{2} - a_1^2 - a_2^2 - a_3^2 - b_1^2 - b_2^2 - b_3^2}$$

and $\alpha, \beta, \theta, \gamma$, and $\delta \in \mathbb{R}$ satisfy

$$\begin{aligned}
&\cos\beta\left(\cos\gamma(\sqrt{2} - 2\cos\alpha) - 8\sqrt{2}r\cos\delta\sin\gamma\right) + \\
&\quad \sin\beta\left(\cos\theta(\sqrt{2} - 2\sin\alpha) - 8\sqrt{2}r\sin\delta\sin\theta\right) = 0.
\end{aligned}$$

Proof: Along the same lines, we have the following nonlinear equations to solve:

$$\begin{aligned}
a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + b_0^2 + b_1^2 + b_2^2 + b_3^2 + b_4^2 &= \frac{1}{2} \\
a_0a_1 + a_1a_2 + a_2a_3 + a_3a_4 + b_0b_1 + b_1b_2 + b_2b_3 + b_3b_4 &= 0 \\
a_0a_2 + a_1a_3 + a_2a_4b_0b_2 + b_1b_3 + b_2b_4 &= 0 \qquad (9) \\
a_0a_3 + a_1a_4 + b_0b_3 + b_1b_4 &= 0 \\
a_0a_4 + b_0b_4 &= 0,
\end{aligned}$$

as well as the linear equation $\sum_{i=0}^{4} a_i = \sum_{i=0}^{4} b_i$. Using various combinations of these equations as we have before, it can be seen that

$$a_0 + a_1 + a_2 + a_3 + a_4 = \frac{1}{2}, \qquad b_0 + b_1 + b_2 + b_3 + b_4 = \frac{1}{2}$$

$$a_0 - a_1 + a_2 - a_3 + a_4 = \frac{1}{\sqrt{2}}\cos\alpha, \quad b_0 - b_1 + b_2 - b_3 + b_4 = \frac{1}{\sqrt{2}}\sin\alpha$$

$$a_0 - a_2 + a_4 = \frac{1}{\sqrt{2}}\cos\beta\cos\gamma, \qquad b_0 - b_2 + b_4 = \frac{1}{\sqrt{2}}\sin\beta\cos\theta$$

$$a_1 - a_3 = \frac{1}{\sqrt{2}}\cos\beta\sin\gamma, \qquad b_1 - b_3 = \frac{1}{\sqrt{2}}\sin\beta\sin\theta.$$

So, we can solve for $a_1, a_2, a_3, b_1, b_2,$ and b_3 as

$$\begin{aligned}
a_1 &= \frac{1}{8} - \frac{1}{4\sqrt{2}}\cos\alpha + \frac{1}{2\sqrt{2}}\cos\beta\sin\gamma \\
a_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}}\cos\alpha - \frac{1}{2\sqrt{2}}\cos\beta\cos\gamma \\
a_3 &= \frac{1}{8} - \frac{1}{4\sqrt{2}}\cos\alpha - \frac{1}{2\sqrt{2}}\cos\beta\sin\gamma \\
b_1 &= \frac{1}{8} - \frac{1}{4\sqrt{2}}\sin\alpha + \frac{1}{2\sqrt{2}}\sin\beta\sin\theta \\
b_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}}\sin\alpha - \frac{1}{2\sqrt{2}}\sin\beta\cos\theta \\
b_3 &= \frac{1}{8} - \frac{1}{4\sqrt{2}}\sin\alpha - \frac{1}{2\sqrt{2}}\sin\beta\sin\theta.
\end{aligned}$$

Using the first and last equation from (9), we have

$$(a_0 - a_4)^2 + a_1^2 + a_2^2 + a_3^2 + (b_0 - b_4)^2 + b_1^2 + b_2^2 + b_3^2 = \frac{1}{2}.$$

Hence, it follows that

$$\begin{aligned}
a_0 - a_4 &= r\cos\delta \\
b_0 - b_4 &= r\sin\delta \\
r &= \sqrt{\frac{1}{2} - a_1^2 - a_2^2 - a_3^2 - b_1^2 - b_2^2 - b_3^2}.
\end{aligned}$$

Thus, the necessary formulae from the lemma follow. Upon substituting these formulae back into the nonlinear equations, we find that the first, third, and last equations are satisfied where as the second and fourth produce a single transcendental equation. □

Example 11. *Let* $\alpha = 2.6829477415207257$, $\beta = 0.714939482206344$, $\gamma = 1.028362000753886$, $\theta = 0.2225085841811395$, *and suppose* $\delta = 1.2907830472783552$. *Then* $H_{10}(z)$ *is associated with the Daubechies D10 wavelet.*

Example 12. *In order for $H_{10}(z)$ to satisfy the second order vanishing moment condition, we need $H'_{10}(-1) = 0$, i.e.*

$$\begin{aligned} H'_{10}(-1) &= b_0 + 3b_1 + 5b_2 + 7b_3 + 9b_4 - 2a_1 - 4a_2 - 6a_3 - 8a_4 \\ &= \frac{1}{2} + 4r\cos\delta - 4r\sin\delta + \sqrt{2}\cos\beta\sin\gamma - \sqrt{2}\sin\beta\sin\theta \\ &= 0. \end{aligned}$$

There are numerous possibilities. For example, when

$$\alpha = 2.8975168508124955, \beta = \frac{3\pi}{4}, \gamma = \frac{\pi}{6}, \theta = 0, \text{ and } \delta = \frac{\pi}{4},$$

then both the transcendental condition and the second moment condition are satisfied.

§5. Numerical Experiment

We have implemented an image compression scheme as a means of comparing the parameterized solutions of length six with the standard wavelets of Haar, D4, and D6. (We are still working on the comparison using H_8 and H_{10}. The results will be reported elsewhere.) The scheme consists of the following:

- Decomposing the gray-scale values of various images with size 512 × 512 to a maximum number of levels.
- Encoding the decomposed image using an embedded zero-tree encoder to a specified file size. All of the images in this experiment have a file size of 262,159 bytes which is approximately one byte per pixel. The actual compressed file sizes are 32,793 bytes (8:1), 16,409 bytes (16:1), 8,217 bytes (32:1), and 4,121 bytes (64:1).
- Decoding the compressed file.
- Reconstructing the image using the wavelet transform and rounding the values to the nearest integer.
- Calculating the peak signal to noise ratio (PSNR) which is a measure of the root mean squared error (RMSE) in the sense that

$$RMSE = \frac{1}{512^2}\sum_{i,j=1}^{512}(p_{i,j} - \hat{p}_{i,j})^2$$

$$PSNR = 20\log_{10}(\frac{255}{RMSE})$$

where $p_{i,j}$ is the original gray-scale value and $\hat{p}_{i,j}$ is the value after reconstruction.

Image: Lena 512 × 512				
Wavelet	8:1	16:1	32:1	64:1
Haar	36.2258	32.6462	29.5685	27.5420
D4	38.4440	34.9209	31.6733	28.8185
D6	38.7819	35.3234	32.0479	29.0727
H6(Ex7)	38.8167	35.4208	32.1585	29.1588

Image: Barbara 512 × 512				
Wavelet	8:1	16:1	32:1	64:1
Haar	30.4954	26.8119	24.6329	22.7409
D4	32.8675	28.6364	25.6853	23.3821
D6	33.4311	29.0735	25.9431	23.4715
H6(Ex7)	33.6441	29.2804	26.1074	23.5789

Image: Boat 512 × 512				
Wavelet	8:1	16:1	32:1	64:1
Haar	34.7720	30.7103	27.5762	25.4130
D4	35.6517	31.5910	28.5165	26.0746
D6	35.8593	31.8088	28.6402	26.1880
H6(Ex7)	35.9301	31.9080	28.7187	26.2733

Image: Finger-print 512 × 512				
Wavelet	8:1	16:1	32:1	64:1
Haar	32.4415	29.9961	28.5828	27.3396
D4	33.3224	30.8632	29.4448	27.9762
D6	33.8077	31.2097	29.8049	28.2562
H6(Ex7)	33.9236	31.3387	29.9687	28.3928

Image: Marmousi 512 × 512				
Wavelet	8:1	16:1	32:1	64:1
Haar	35.7244	31.1309	27.7014	25.0277
D4	41.3471	36.2831	31.8262	27.5415
D6	43.6310	37.7401	33.0496	28.9160
H6(Ex7)	45.5190	38.6440	33.9109	29.5047

Image: Crowd 512 × 512				
Wavelet	8:1	16:1	32:1	64:1
Haar	33.3109	29.2032	26.1346	23.7222
D4	34.7214	30.6307	27.4101	24.8170
D6	35.1740	31.0444	27.7601	25.0348
H6(Ex7)	35.2893	31.1590	27.8690	25.1465

Tab. 1. PSNR comparison of the standard Daubechies wavelets and the parameterized wavelet of length six from Example 7.

Figure 3 shows the images used in the compression scheme, and Tab. 1 gives the PSNR values for the standard wavelets versus a parameterized wavelet for each image at various compression ratios. The parameterized wavelet was selected based upon having the largest PSNR values for the parameter values near D6. The search for the parameterized wavelet was limited, and a more exhaustive search may yield better results. Note that a higher PSNR value is better. In this case, we were able to find a single

Fig. 3. Test images: Lenna, Finger-print, Barbara, Marmousi, Boat, and Crowd.

length six parameterized filter given in Example 7 which out-performed all the others tested on all six test images and at each compression ratio.

Acknowledgments. This project was partially funded by a research enhancement grant from the Kentucky NSF EPSCoR Foundation.

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