Volume Data Interpolation using Tensor Products of Spherical and Radial Splines

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Abstract. Trivariate splines solve a special case of scattered data interpolation problem in the volume bounded by two concentric spheres. A triangulation Δ of the unit sphere S^2 is constructed based on the vertex set $\mathcal{V}$. Given a partition P of the interval $[1, R]$, let $S^{\tau\times\rho}_{\sigma\times\delta}$ be the space of the spherical splines of degree σ and smoothness τ over Δ tensored with the univariate radial splines of degree δ and smoothness ρ over P. We use a minimal energy method to find a unique smooth spline $s \in S^{\tau\times\rho}_{\sigma\times\delta}$ interpolating given data values at the points $\mathcal{V} \times P$. Numerical investigation is conducted on polynomial reproduction and convergence of the interpolating splines.

§1. Introduction

Assume that the data collected is being bounded by two concentric spheres with radii 1 and R. The data values are sampled at scattered sites $\mathcal{V} = \{v_1, \cdots, v_m\}$ on the unit sphere $\mathbf{S}^2$ along the radial direction at levels $1 = r_1 < \cdots < r_n = R$. This is the case, for example, when the atmospheric data values such as temperature, moisture, and wind velocity vector field are obtained from observation stations located around the world. In Fig. 1., we demonstrate the structure of data sites outside the unit sphere.

One practical problem is to construct a smooth function which interpolates the data values. In this paper we shall use tensor products of spherical splines with radial splines to study interpolation of volume data. Although we could use the method in [4] to construct a 3D interpolatory spline using a tetrahedral partition of the given data sites, we use tensor

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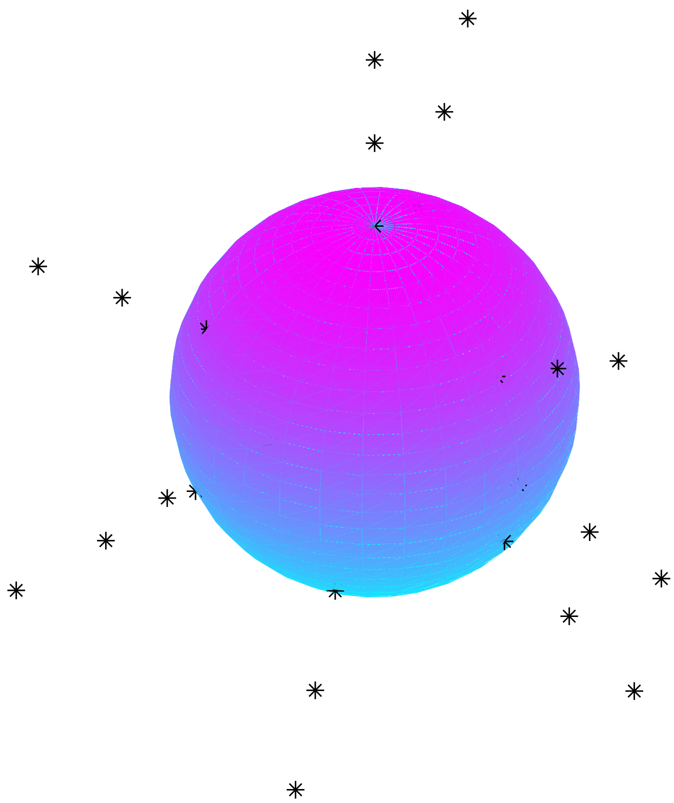

Fig. 1. Schematic illustration of volume data sites

product splines due to the simplicity of the tensor structure of the data sites. Spherical splines are studied in [1], [2], [7] and they have many potential applications in geodesic and atmospherical sciences.

For the data sites $v_j, j = 1, \cdots, m$, we find a spherical triangulation $\triangle$. That is, the data sites v_j are vertices of $\triangle$. For the partition $P := 1 = r_1 < \cdots < r_n = R$ of the interval $[1, R]$, suppose that the set $U := \{r_i v_j, i = 1, \cdots, n, j = 1, \cdots, m\}$ contains all the data sites. This corresponds to a real situation that the data values from observation stations around the world are sampled at the same fixed heights $r_1, \cdots, r_n$. Let f_{ij} denote the value recorded at $r_i v_j$.

Let $S^\tau_\sigma(\triangle)$ be the spherical spline space of degree σ and smoothness τ (cf. [2]) and $S^\rho_\delta(P)$ be the univariate spline space of degree δ and smoothness ρ. (The details of these two spaces will be given later.) We use the tensor product $S^{\tau\times\rho}_{\sigma\times\delta}$ of $S^\tau_\sigma(\triangle)$ and $S^\rho_\delta(P)$. Our problem is to find $s \in S^{\tau\times\rho}_{\sigma\times\delta}$ satisfying

$$s(r_i v_j) = f_{ij}, \qquad i = 1, \cdots, n, j = 1, \cdots, m. \tag{1}$$

In general, when σ and δ are large enough, $S^{\tau\times\rho}_{\sigma\times\delta}$ has many solutions for (1). The problem is how to find a reasonable one. In this paper we use a method which minimizes an energy functional subject to the interpolation conditions and the smoothness requirement. We present this method in detail in Section 2.

Our first mathematical problem is to prove that the constrained minimization has a unique solution. Next question is how to compute such a solution. Then we need to demonstrate that solutions produced by our algorithm give good approximations of unknown functions. We have implemented our algorithm in MATLAB and performed several numerical experiments in which data values are collected from some known functions. The values of the spline solutions are then compared with the values of the original functions at many locations other than the data sites. Our numerical experiments show the algorithm works reasonably well. The main difficulty we encountered was the number of the data sites: we were unable to handle more than 160 sites for the spline spaces of mixed degree higher than 4×1 due to the large size of the correspoding systems of linear equations. In practice, the size of the data set is often larger. To overcome this difficulty, we plan to adapt a divide-and-conquer method or a domain decomposition method. The details of the method will be outlined in Section 6.

§2. Preliminaries

Recall that $\triangle$ denotes a triangulation of the sphere $\mathbf{S}^2$ based on vertices in $\mathcal{V}$. We carry $\triangle$ onto all spherical surfaces $\mathbf{S}^2_{r_p}$ with radius r_p centered at the origin by radially projecting each vertex of $\triangle$ onto $\mathbf{S}^2_{r_p}, p = 1, \cdots, n$. The spherical prisms $P_{p,T} := \{u \in \mathbb{R}^3 : \frac{u}{|u|} \in T,\ r_p \leq |u| \leq r_{p+1}\}$, for $T \in \triangle, 1 \leq p \leq n-1$ form a partition $\mathbf{P}$ of the volume W in between $\mathbf{S}^2$ and $\mathbf{S}^2_R$. See Figure 2. We construct basis functions relative to each spherical prism as follows.

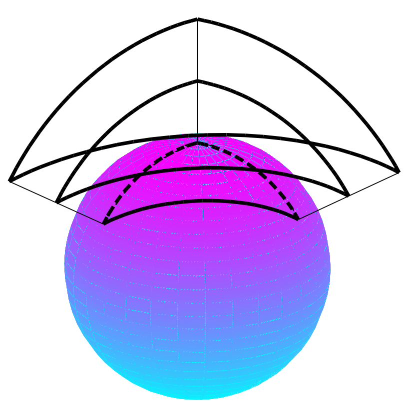

Fig. 2. An example of two spherical prisms stacked together

Fix a spherical prism $P_{p,T}$. For $u \in P_{p,T}$, define $r := |u|$ and let

$$R_l^\delta(r) = \frac{\delta!}{l!(\delta-l)!}\Big(\frac{r_{p+1}-r}{r_{p+1}-r_p}\Big)^{\delta-l}\Big(\frac{r-r_p}{r_{p+1}-r_p}\Big)^l, \quad l = 0, \cdots, \delta, \tag{2}$$

be a set of Bernstein-Bézier polynomial basis functions of degree δ defined on the interval $[r_p,\ r_{p+1}]$.

Lemma 1. *The spherical Bernstein-Bézier basis polynomials with respect to spherical triangle rT are independent of r.*

Proof: Let

$$\bar{B}_{ijk}^\sigma(u) = \frac{\sigma!}{i!j!k!}\bar{b}_1(u)^i\bar{b}_2(u)^j\bar{b}_3(u)^k \tag{3}$$

be the spherical Bernstein-Bézier basis polynomial of degree σ related to the spherical triangle rT as in [1], where $\bar{b}_1(u),\ \bar{b}_2(u),\ \bar{b}_3(u)$ denote spherical barycentric coordinates of u relative to rT defined by, if $T = \langle v_i, v_j, v_k\rangle$,

$$rv_i\bar{b}_1(u) + rv_j\bar{b}_2(u) + rv_k\bar{b}_3(u) = u,$$

or

$$v_i\bar{b}_1(u) + v_j\bar{b}_2(u) + v_k\bar{b}_3(u) = \frac{u}{|u|}.$$

By the uniqueness of the barycentric coordinates

$$\bar{b}_i(u) = b_i(v), \quad i = 1, 2, 3,$$

where $b_i(v),\ i = 1, 2, 3$ are the spherical barycentric coordinates related to T, and the point $v := \frac{u}{|u|}$ is the radial projection of u onto the surface of the unit sphere $\mathbf{S}^2$. Thus

$$\bar{B}_{ijk}^\sigma(u) = \frac{\sigma!}{i!j!k!}b_1(v)^i b_2(v)^j b_3(v)^k =: B_{ijk}^\sigma(v),$$

with B_{ijk}^σ being defined with respect to T. That is, the value of $\bar{B}_{ijk}^\sigma$ is independent of r. □

Thus, we simply use B_{ijk}^σ for every spherical prism $P_{p,T}$, $1 \le p \le n-1$. With R_l^δ and B_{ijk}^σ, we let

$$q_{p,T}(u) = \sum_{l=0}^{\delta}\sum_{i+j+k=\sigma} C_{ijk}^l(p,T)B_{ijk}^\sigma(v)R_l^\delta(r)$$

be a polynomial of mixed degree $\sigma \times \delta$ defined with respect to $P_{p,T}$. Now we define our spline function s on W to be a piecewise polynomial of mixed degree $\sigma \times \delta$. That is, let

$$S_{\sigma\times\delta}(\mathbf{P}) := \{s, s|_{P_{p,T}} = q_{p,T},\ P_{p,T} \in \mathbf{P}\}$$

be the discontinuous spline space of mixed degree $\sigma \times \delta$. For computational purpose, we let C denote the vector of all coefficients $C^l_{ijk}(p,T),\ i+j+k = \sigma,\ l = 0,\cdots,\delta,\ T \in \triangle,\ p = 1,\cdots,n-1$.

Next, we explain the smoothness conditions between adjacent prisms in the same layer of $\mathbf{P}$. Let $P_{p,T}$ and $P_{p,\hat{T}}$ be two spherical prisms sharing a common face $\langle r_p v_2, r_p v_3,\ r_{p+1}v_2, r_{p+1}v_3\rangle$, say. Let $q(u)$ and $\hat{q}(u)$ be the two polynomials that coincide with the spline in $P_{p,T}$ and $P_{p,\hat{T}}$. We have

$$\begin{aligned} q(u) &= \sum_{l=0}^{\delta} R^\delta_l(r) \sum_{i+j+k=\sigma} C^l_{ijk}(p,T) B^\sigma_{ijk}(v), \\ \hat{q}(u) &= \sum_{l=0}^{\delta} R^\delta_l(r) \sum_{i+j+k=\sigma} C^l_{ijk}(p,\hat{T}) \hat{B}^\sigma_{ijk}(v). \end{aligned} \tag{4}$$

Note that the radial functions are defined on the same interval $[r_p, r_{p+1}]$ and therefore are the same for both polynomials. Consider the pairs of polynomial pieces corresponding to the same radial basis functions. For a fixed l between 0 to δ, we let

$$\begin{aligned} p^l(v) &:= \sum_{i+j+k=\sigma} C^l_{ijk}(p,T) B^\sigma_{ijk}(v), \\ \hat{p}^l(v) &:= \sum_{i+j+k=\sigma} C^l_{ijk}(p,\hat{T}) \hat{B}^\sigma_{ijk}(v). \end{aligned}$$

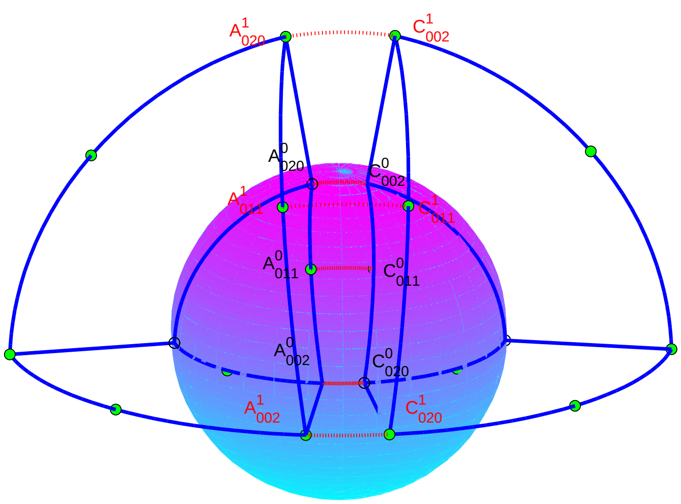

Fig. 3. Schematic illustration of Bernstein-Bézier coefficients along the common face of two neighboring spherical prisms for $\sigma = 2, \delta = 1$.

We need to consider the derivatives across the common face independent of r and thus we deal with continuity of the spherical functions $p^l(v)$

and $\hat{p}^l(v)$ defined on triangles $T := \langle v_1,\ v_2,\ v_3\rangle$ and $\hat{T} := \langle v_2,\ v_3,\ v_4\rangle$, correspondingly. From the well-known smoothness conditions for $p^l(v)$ and $\hat{p}^l(v)$ over T and $\hat{T}$ (cf. [1]), we obtain the following

Lemma 2. *Let q and $\hat{q}$ be two polynomials of degree $\sigma \times \delta$ defined on spherical prisms $P_{p,T}$ and $P_{p,\hat{T}}$, respectively. That is, q and $\hat{q}$ are given in (4). Then $q(u)$ and $\hat{q}(u)$ join with C^s continuity across a common face $\langle r_p v_2, r_p v_3,\ r_{p+1}v_2, r_{p+1}v_3\rangle$ if and only if*

$$C^l_{ijk}(p,\hat{T}) = \sum_{r+s+t=i} C^l_{r,j+s,k+t}(p,T)B^i_{rst}(v_4),$$

for all $i = 1,\cdots,s,\ j,k$ such that $i+j+k=\sigma$ and $l = 0,\cdots,\delta$.

Next, consider two spherical prisms $P_{p,T}$ and $P_{p+1,T}$ having the same triangular face $T_{p+1} = r_{p+1}T,\ T := \langle v_1,\ v_2,\ v_3\rangle$, say. Let

$$\begin{aligned} q(u) &= \sum_{i+j+k=\sigma} B^\sigma_{ijk}(v)\sum_{l=0}^{\delta} C^l_{ijk}(p,T)R^\delta_l(r), \\ \hat{q}(u) &= \sum_{i+j+k=\sigma} B^\sigma_{ijk}(v)\sum_{l=0}^{\delta} C^l_{ijk}(p+1,T)\hat{R}^\delta_l(r) \end{aligned} \tag{5}$$

be the corresponding polynomials on $P_{p,T}$ and $P_{p+1,T}$. Note that the functions $B^\sigma_{ijk}(v)$ are defined with respect to the same triangle, and thus are the same for q and $\hat{q}$ by Lemma 1. Thus q and $\hat{q}$ will join smoothly across T_{p+1} if and only if

$$p_{ijk}(u) = \sum_{l=0}^{\delta} C^l_{ijk}(p,T)R^\delta_l(r) \quad \text{and} \quad \hat{p}_{ijk}(u) = \sum_{l=0}^{\delta} C^l_{ijk}(p+1,T)\hat{R}^\delta_l(r)$$

join smoothly at T_{p+1} for each $i+j+k=\sigma$. Since these are univariate polynomials, smoothness conditions follow from setting

$$\begin{aligned} p_{ijk}(r_{p+1}) &= \hat{p}_{ijk}(r_{p+1}), \\ \frac{\partial}{\partial r}p_{ijk}(r_{p+1}) &= \frac{\partial}{\partial r}\hat{p}_{ijk}(r_{p+1}) \end{aligned}$$

and so on for each triple $i+j+k=\sigma$.

Fix any $\tau \geq 0$ and $\rho \geq 0$. Assemble two kinds of smoothness conditions for all common faces (spherical triangular faces as well as curved trapezoidal faces) into a matrix M so that a spline s with coefficients C satisfying $MC = 0$ has a required continuity $\tau \times \rho$ on W. In this paper, we shall consider the tensor product spline space

$$S^{\tau\times\rho}_{\sigma\times\delta}(\mathbf{P}) = \{s \in S_{\sigma\times\delta}(\mathbf{P}), s \in \mathcal{C}^\tau(\triangle)\times\mathcal{C}^\rho([1,R])\}.$$

We shall identify the spline space $S^{\tau\times\rho}_{\sigma\times\delta}(\mathbf{P})$ with all vectors $C \in \mathbb{R}^N$ such that $MC = 0$. Here $N = \binom{\sigma+2}{2} n_\triangle(\delta+1)(n-1)$ and $n_\triangle$ denotes the number of triangles in $\triangle$.

In general, the interpolation and smoothness conditions are not sufficient to uniquely determine the spline. Therefore we find the spline function $s \in S^{\tau\times\rho}_{\sigma\times\delta}(\mathbf{P})$ which interpolates the given volume data and minimizes the following energy functional:

$$\mathcal{E}(h) = \int_W (\frac{\partial^2 h}{\partial r^2})^2 + (\diamond h)^2 dxdydz, \tag{6}$$

where

$$(\diamond h)^2 = \sum_{\mu,\nu\in\{x,y,z\}} (D^2_{\mu\nu} h_\alpha)^2 \tag{7}$$

is a sum of squares of second order spherical derivatives of h. Neamtu and Schumaker, introduced in [7] Sobolev-type semi-norms for spherical functions which annihilate homogeneous polynomials. This work motivates the definition of the energy functional in terms of (7) (cf. [8]), which annihilates constant and linear homogeneous polynomials. A function h defined on the unit sphere is first extended homogeneously to all of $\mathbb{R}^3$ by

$$h_\alpha(u) = |u|^\alpha h(\frac{u}{|u|}).$$

In (7) α takes on value of 0 if the degree σ of spherical basis polynomials is even, and $\alpha = 1$ if σ is odd. Next, second order partial derivatives of h_α are taken with respect to cartesian directions x, y, z.

For a general function $h(u) = h(x, y, z)$ to compute (6) we first change variables by $u = rv$, with $r = |u|$ and v being a unit direction of u. We differentiate $h(r, \cdot)$ with respect to r to compute $\frac{\partial^2 h}{\partial r^2}$, and then apply (7) to $h(\cdot, v)$, as explained above, treating h as a function defined on the unit sphere.

To implement the minimization of (6) over $S^{\tau\times\rho}_{\sigma\times\delta}(\mathbf{P})$, we need to be able to compute the energy contributions of basis functions. We denote the sub-vector of coefficients C of s associated with a given prism $P_{p,T}$ by $C_{p,T}$. That is, $C_{p,T}$ consists of $C^l_{ijk}(p, T), i + j + k = \sigma, 0 \le l \le \delta$. Re-indexing (i, j, k, l) by J, $\Upsilon := \{J\}$, and denoting the basis functions by $B_J(v)R_J(r) = B^\sigma_{ijk}(v)R^\delta_l(r)$, we let

$$\begin{aligned} &E_{JK}(p, T) \\ &= \int_{r_p}^{r_{p+1}} r^2 (\frac{\partial^2 R_J}{\partial r^2})(\frac{\partial^2 R_K}{\partial r^2}) dr \int_T B_J B_K dv \\ &+ \int_{r_p}^{r_{p+1}} r^2 R_J R_K dr \int_T \diamond B_J \diamond B_K dv, \end{aligned}$$

with dr and dv denoting radial and spherical differentials respectively.

Therefore, for any tensor-product spline s with a coefficient vector C, we have

$$\mathcal{E}(s) = C^T E C, \tag{8}$$

where $E = \text{ diag } ([E_{JK}(p,T)]_{J,K\in\Upsilon}, T \in \triangle)$ is a block-diagonal matrix.

§3. Existence and Uniqueness

In this section we assume that the tensor product spline space $S^{\tau\times\rho}_{\sigma\times\delta}(\mathbf{P})$ contains at least one spline that interpolates the given data values f_{ij} at the corresponding points $r_i v_j$. Let

$$\Gamma(f) := \{s \in S^{\tau\times\rho}_{\sigma\times\delta}(\mathbf{P}), \quad s(r_i v_j) = f_{ij}, \quad i = 1, \cdots, n, j = 1, \cdots, m\} \neq \emptyset.$$

Our goal is to uniquely determine $s \in \Gamma(f)$ that minimizes the energy functional (6). This is equivalent to finding a coefficient vector C that minimizes the quadratic form (8) while satisfying the linear constraints $KC = F$ (interpolation) and $MC = 0$ (smoothness). Here K is a matrix describing interpolation conditions which will be defined in §4 together with the vector F of data values f_{ij} re-indexed from 1 to nm.

We will prove that such spline exists and is unique. We will need the following Lemmas.

Lemma 3. *Let $\mathcal{P}$ be the space of spherical homogeneous polynomials of degree d. If d is even, $\diamond p = 0$ if and only if $p \equiv C$ for some constant C. If d is odd, $\diamond p = 0$ if and only if p is a linear homogeneous polynomial.*

Lemma 4. *Let p be a polynomial of degree d. Suppose*

$$\frac{d^2}{dx^2} p(x) = 0, \quad \forall x \in \mathbb{R}\backslash\{0\}.$$

Then p is a linear polynomial.

Proof of Lemma 3 can be found in [8], Lemma 4 is elementary.

Lemma 5. *There exists a unique spline $s_0 \in S^{\tau\times\rho}_{\sigma\times\delta}(\mathbf{P})$ such that*

$$\mathcal{E}(s_0) = min\{\mathcal{E}(s), \quad s \in \Gamma(0)\}.$$

Proof: Clearly, s_0 is in $\Gamma(0)$, and $\mathcal{E}(s_0) = 0$ which is the absolute minimum of $\mathcal{E}$. We need to show that the null function $s_0(u) = 0$ is the only spline $s \in S^{\tau\times\rho}_{\sigma\times\delta}(\mathbf{P})$ such that $\mathcal{E}(s) = 0$. Therefore, suppose

$$\int_W (\frac{\partial^2 s}{\partial r^2})^2 + (\diamond s)^2 dw = 0.$$

Then we must have

$$\diamond s = 0$$

and

$$\frac{\partial^2 s}{\partial r^2} = 0$$

on each spherical prism. Lemma 3 implies that s is linear over any sphere with radius $r \in [r_0, r_n]$ (if σ is odd) and constant (if σ is even). Lemma 4 implies that s is linear along any radial segment $\{ru : r \in [r_0, r_u]\}$, where $u \in \mathbf{S}^2$. Since s is required to have the value 0 at the sampling points, it follows that $s = 0$ everywhere. □

We are now in the position to proof our main result.

Theorem 6. *Assume that $\Gamma(f) \neq \emptyset$. Then there exists a unique spline function $s_f \in S^{\tau\times\rho}_{\sigma\times\delta}(\mathbf{P})$ satisfying*

$$\mathcal{E}(s_f) = \min\{\mathcal{E}(s), s \in \Gamma(f)\}. \tag{9}$$

Proof: Since $\Gamma(f)$ is a nonempty closed convex set, s_f exists. The uniqueness follows from Lemma 5. □

§4. Computational Algorithm

In this section we explain how to compute the minimal energy spline s_f interpolating the given data. We first explain the interpolation conditions.

It follows directly from the properties of spherical and univariate Bernstein-Bézier splines that

$$\begin{aligned} q_{p,T}(r_p v_1) &= C^0_{\sigma 00}(p,T) & \qquad q_{p,T}(r_{p+1} v_1) &= C^\delta_{\sigma 00}(p,T) \\ q_{p,T}(r_p v_2) &= C^0_{0\sigma 0}(p,T) & \qquad q_{p,T}(r_{p+1} v_2) &= C^\delta_{0\sigma 0}(p,T) \\ q_{p,T}(r_p v_3) &= C^0_{00\sigma}(p,T) & \qquad q_{p,T}(r_{p+1} v_3) &= C^\delta_{00\sigma}(p,T) \end{aligned}$$

with $T = \langle v_1, v_2, v_3 \rangle$ and similarly for other triangles. We assemble a matrix K and a vector F based on interpolation conditions, e.g.,

$$\begin{aligned} f(r_p v_1) &= C^0_{\sigma 00}(p,T) & \qquad f(r_{p+1} v_1) &= C^\delta_{\sigma 00}(p,T) \\ f(r_p v_2) &= C^0_{0\sigma 0}(p,T) & \qquad f(r_{p+1} v_2) &= C^\delta_{0\sigma 0}(p,T) \\ f(r_p v_3) &= C^0_{00\sigma}(p,T) & \qquad f(r_{p+1} v_3) &= C^\delta_{00\sigma}(p,T) \end{aligned}$$

so that if $KC = F$ for the coefficient vector C of a spline s, then s interpolates the given function values stored in F.

Now to minimize (8) subject to the linear constraints $KC = F$ and $SC = 0$, we use the Lagrange multiplier method. Namely, we look for a coefficient vector C that satisfies the constraints and such that the gradient of (8) is a linear combination of the rows of K and M. These conditions are expressed by the linear system

$$\begin{bmatrix} E & M^T & K^T \\ M & 0 & 0 \\ K & 0 & 0 \end{bmatrix} \begin{bmatrix} C \\ \lambda \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F \end{bmatrix}.$$

Here λ and η are Lagrange multiplier vectors.

The system above is singular, however it can be successfully solved by using, for example, the following method (cf. [3]). For a general linear system

$$\begin{pmatrix} A & L^T \\ L & 0 \end{pmatrix} \begin{pmatrix} c \\ \lambda \end{pmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}, \tag{10}$$

we solve the sequence of linear systems

$$\begin{pmatrix} A & L^T \\ L & -\epsilon I \end{pmatrix} \begin{bmatrix} c^{(l+1)} \\ \lambda^{(l+1)} \end{bmatrix} = \begin{bmatrix} F \\ G - \epsilon\lambda^{(l)} \end{bmatrix}, \tag{11}$$

$l = 0, 1, 2, \cdots$, for an arbitrary $\epsilon > 0$, where $\lambda^0 = 0$ and I is the identity matrix.

It is known that the vectors $c^{(l)}$ converge to the solution c of (10), in the sense that there exists a constant α such that $\|c^{k+1} - c\| \leq \alpha\epsilon\|c^k - c\|$ for all k provided that the symmetric part of A is nonnegative definite with respect to L (cf. [3]). The main advantage of this method as opposed to, for example, the least square approach is using the inverse of the matrix of size A only instead of inverting the entire coefficient matrix. The iterative steps mainly consist of the following

$$(A + \frac{1}{\epsilon} L^T L) c^{(l+1)} = A c^{(l)} + \frac{1}{\epsilon} L^T G.$$

(for details, see [3]).

Theorem 7. *E is symmetric and positive definite with respect to $[S; K]$.*

Proof: Since $C^T E C = \mathcal{E}(s)$ by equation (8), formula (6) implies $\mathcal{E}(s) \geq 0$, it follows that $C^T E C \geq 0$ for all C. By Lemma 5, $\mathcal{E}(s) = 0$ implies that $s = 0$; From the linear independence of the basis functions, we conclude that $C^T E C = 0$ only if $C = 0$. □

Having computed the coefficients of the spline C, we can evaluate the spline as follows. Given a point $u \in W$ we determine its position relative to the partition $\mathbf{P}$ and extract the corresponding set of coefficients $C^l_{ijk}(p, T),\ l = 0, \cdots, \delta, i + j + k = \sigma$ out of C. First evaluate the radial functions $R^\delta_l(r),\ l = 0, \cdots, \delta$ at $|u|$. Then $q_{p,T}(u)$ is a sum of $\delta + 1$ spherical polynomials defined with respect to the same basis triangle and having the coefficients $C^0_{ijk}(p, T),\ C^1_{ijk}(p, T), \cdots, C^\delta_{ijk}(p, T)$ correspondingly. Each spherical piece can be evaluated using the well-known de Casteljau algorithm at $\frac{u}{|u|}$ [1]. Adding up the results we have $q_{p,T}(u)$.

§5. Numerical Experiments

We have implemented the above algorithm in MATLAB and performed several numerical experiments on the convergence of the minimal energy spline interpolation.

As a consistency check, we began our experiments with the interpolation of the constant function $f_1(u) = 1$, i.e. of data values $f_{1,ij} = f_1(r_i v_j) = 1$, with tensor splines of various degree and smoothness orders. It is known that the space of spherical splines of even (resp. odd) degree d is generated by spherical harmonics of even (resp.odd) degrees $\leq d$ [5]. It follows that the function $f_1(u) = 1$ belongs to $S^{\tau\times\rho}_{\sigma\times\delta}(\mathbf{P})$ if and only if the spherical degree σ is even. Since this spline is smooth to all orders and has minimum (zero) energy, it is the correct solution expected by our algorithm. Similarly, linear homogeneous functions are expected to be reproduced if σ is odd. Moreover, if $p^\delta(r)$ is a polynomial of degree δ in the radial direction and q^σ is a spherical homogeneous polynomial of degree σ then $p^0(r)g^1(v)$ is reproduced in spaces of odd spherical degrees and $p^1(r)q^0(v)$ is reproduced in spaces of even spherical degrees. To illustrate all that we also experiment with functions $f_2(x,y,z) = x+y+z$, $f_3(x,y,z) = \sqrt{x^2+y^2+z^2}$ and $f_4(x,y,z) = \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}$.

In the first round of tests, we used a triangulation Δ_1 of the unit sphere based on 6 equally-spaced vertices. The initial radial partition P_1 had two intervals, $[1,\ 1.5]$ and $[1.5,\ 2]$. The approximation error $|s(u) - f(u)|/|f|$ was evaluated at $768 \times 16 = 12288$ points distributed throughout the domain W.

Results on reproduction are reported in Tables 2 and 3.

Table 1. Reproduction of Tensors of Spherical Constants with Radial Linear Polynomials

Degree $\sigma\times\delta$	Smoothness $r_s \times r_r$	$\frac{\|\|f_1-s\|\|_\infty}{\|\|f_1\|\|_\infty}$	$\frac{\|\|f_4-s\|\|_\infty}{\|\|f_4\|\|_\infty}$
2×1	0×0	10^{-15}	10^{-15}
2×2	0×0	10^{-10}	10^{-10}
2×3	0×1	10^{-9}	10^{-9}
4×1	1×0	10^{-9}	10^{-9}
4×2	1×0	10^{-9}	10^{-9}
4×3	1×1	10^{-9}	10^{-9}

We now report on the numerical experiments on the convergence of the minimal energy interpolatory splines. Consider a general function

$$\begin{aligned} f_5(x,y,z) =& (1 - .2\sin(\pi r))(1 + (x/r)^5 + \exp 0.2(y/r)^3 + \exp 0.1(z/r)^2 \\ &+ 3(x/r)(y/r)(z/r)), \end{aligned}$$

Table 2. Reproduction of Tensors of Spherical Linear Homogeneous Polynomials with Radial Linear Polynomials

Degree $\sigma \times \delta$	Smoothness $r_s \times r_r$	$\frac{\|\|f_2-s\|\|_\infty}{\|\|f_2\|\|_\infty}$	$\frac{\|\|f_3-s\|\|_\infty}{\|\|f_3\|\|_\infty}$
3×1	1×0	10^{-9}	10^{-9}
3×2	1×0	10^{-9}	10^{-9}
3×3	1×1	10^{-8}	10^{-8}
5×1	2×0	10^{-10}	10^{-10}
5×2	2×0	10^{-9}	10^{-9}
5×3	2×1	10^{-8}	10^{-8}

with $r^2 = x^2 + y^2 + z^2$.

For this set of experiments, we first used the triangulation $\triangle_1$, then a triangulation $\triangle_2$ derived from Δ_1 by subdividing each triangle into four parts, and finally a triangulation $\triangle_3$ derived from $\triangle_2$ similar way. With each refinement radial intervals are divided in half. For each space the accuracy of the spline approximation is checked against the exact function values at 12288 points as shown in Table 3.

Table 3. Interpolation of a General Function

Degree $\sigma \times \delta$	Smoothness $r_s \times r_r$	$\frac{\|\|f_5-s\|\|_\infty}{\|\|f_5\|\|_\infty}$ Δ_1, P_1	$\frac{\|\|f_5-s\|\|_\infty}{\|\|f_5\|\|_\infty}$ Δ_2, P_2	$\frac{\|\|f_5-s\|\|_\infty}{\|\|f_5\|\|_\infty}$ Δ_3, P_3
2×1	0×0	0.2274	0.1689	0.0613
2×2	0×0	0.2250	0.1689	0.0613
2×3	0×1	0.3327	0.2586	0.0893
3×1	1×0	0.6133	0.2103	0.0099
3×2	1×0	0.6147	0.2098	0.0100
3×3	1×1	0.6007	0.2046	0.0100
4×1	1×0	0.2543	0.1703	0.0192
4×2	1×0	0.2526	0.1707	···
4×3	1×1	0.3509	0.2271	···
5×1	2×0	0.4130	0.1754	···
5×2	2×0	0.4179	0.1761	···
5×3	2×1	0.4404	0.2210	···

Unfortunately, due to a large size of matrices involved in the computation of minimal energy interpolatory splines, we were not able to obtain error values in spaces with mixed degree higher than 4×1 over the last partition. The values in the last column are extremely valuable for assessment of the spline performance. One way to obtain some estimate of these values is to adapt a domain decomposition technique. It has been shown (cf. [6]) that such a technique provides local splines converging to the global minimal energy splines in the planar case. In next section we outline the procedure for the volume domain.

§6. Domain Decomposition Procedure

We propose to find C^1 interpolatory splines from a large data set, through a domain decomposition technique. We first triangulate the data locations. Then we divide the unit sphere into several non-overlapping sub-domains using the edges of the triangulation $\triangle$. We expand each sub-domain Ω_i by adding all triangles in $\triangle$ that have a vertex in common with Ω_i. Let $\tilde{\Omega}_i$ be the expansion of Ω_i. For a better approximation, we can continue to expand $\tilde{\Omega}_i$ in this way and so on. Now we find the minimal energy interpolatory spline S_1 over $\tilde{\Omega}_i$ which interpolates the data values over the data sites located in $\tilde{\Omega}_i$. Similarly, we handle the rest of sub-domains. We use the Bézier coefficients of the prisms over Ω_i as an approximation of the global minimal energy spline interpolation over Ω_i. We are working on the implementation of the domain decomposition technique and the results will be reported elsewhere.

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