Bivariate Spline Method for Numerical Solution of Time Evolution Navier-Stokes Equations over Polygons in Stream Function Formulation

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We use a bivariate spline method to solve the time evolution Navier-Stokes equations numerically. The bivariate splines we use in this paper are in the spline space of smoothness r and degree 3r over triangulated quadrangulations. The stream function formulation for the Navier-Stokes equations is employed. Galerkin's method is applied to discretize the space variables of the nonlinear fourth order equation, Crank-Nicholson's method is applied to discretize the time variable, and Newton's iterative method is then used to solve the resulting nonlinear system. We show the existence and uniqueness of the weak solution in $L_2(0,T;H^2(\Omega))\cap L_\infty(0,T;H^1(\Omega))$ of the 2D nonlinear fourth order problem and give an estimate of how fast the numerical solution converges to the weak solution. The C^1 cubic splines are implemented in MATLAB for solving the Navier-Stokes equations numerically. Our numerical experiments show that the method is effective and efficient. © 2003 John Wiley & Sons, Inc.

I. INTRODUCTION

We are interested in using bivariate spline functions to numerically solve the time evolution Navier-Stokes' equations over a planar polygon Ω. The aim of this research is to provide an efficient numerical tool for fluid simulation. This is a continuation of our

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effort [12] where we studied the bivariate spline method for numerical solution of the steady-state Navier-Stokes equations.

The numerical solutions to Navier-Stokes equations has been extensively studied for many years. See monographs [17], [5], [18], [6], [1], [8] and the references therein. The study has been carried out mainly on the velocity-pressure function formulation and the stream function–vorticity formulation. The stream function formulation of Navier-Stokes equations has not been emphasized in the literature. One of the main reasons is the difficulty of the implementation of smooth finite elements of higher order than linear finite elements. However, the stream function formulation does have some advantages over the two traditional formulations. Indeed, for the velocity-pressure function formulation, there is no boundary conditions for pressure functions and the imcompressibility can only be satisfied approximately. For the stream function–vorticity formulation, there is no boundary conditions for the vorticity functions. One has to check the well-known Babuska-Brezzi condition for the finite element spaces to be able to approximate the solutions. Beside that the stream function formulation does not have those difficiencies, the nonlinear system arising from the stream function formulation has a much smaller size than that from the traditional formulations. (This will be elaborated in more detail later.) Recently we have succeeded in implementing C^1 cubic bivariate spline functions (cf. [10]). The implementation makes numerically solving Navier-Stokes equations in the stream function formulation possible. In this paper, we will demonstrate the advantages and efficiency of the bivariate spline method for Navier-Stokes equations in the stream function formulation. For the completeness and easy accessibility to applied scientists and graduate students, we include a theory of the stream function formulation for Navier-Stokes equations and its numerical approximation.

Let $\Omega \subseteq \mathbf{R}^2$ be a simply connected polygonal domain and $\mathbf{u} = (u_1, u_2)^T$ be the planar velocity of a fluid flow over Ω. Also, let p be the pressure function, $\mathbf{f} = (f_1, f_2)^T$ be the external body force of the fluid, $\mathbf{h} = (h_1, h_2)^T$ the velocity of the fluid flow on the boundary $\partial\Omega$ and $\mathbf{g} = (g_1, g_2)^T$ an initial velocity. Then the time evolution Navier-Stokes equations are

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, & (x,y) \in \Omega, t \in (0,T) \\ \text{div}\mathbf{u} = 0, & (x,y) \in \Omega, t \in (0,T) \\ \mathbf{u} = \mathbf{h}, & (x,y) \in \partial\Omega, t \in (0,T) \\ \mathbf{u}(x,y,0) = \mathbf{g}(x,y), & (x,y) \in \Omega \end{cases} \tag{1.1}$$

where $\mathbf{g}|_{\partial\Omega} = \mathbf{h}|_{t=0}$, $\text{div}(\mathbf{g}) = 0$, $\mathbf{f}$ is in $L_2(0,T;L_2(\Omega))^2$, $\mathbf{h}$ is in $L_2(0,T,\partial\Omega)^2$, and $\mathbf{g} = (g_1, g_2)^T \in H^1(\Omega)^2$. Here, Δ denotes the usual Laplacian operator and ∇ the gradient operator. To motivate our study, we shall first consider the numerical solution of Stokes' equations which are a linearized version of the Navier-Stokes equations. That is, after omitting the nonlinear terms, we have the time evolution Stokes' equations:

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \times (0,T) \\ \text{div}\mathbf{u} = 0 & \text{in } \Omega \times (0,T) \\ \mathbf{u} = \mathbf{h} & \text{on } \partial\Omega \times (0,T) \\ \mathbf{u}(x,y,0) = \mathbf{g}, & \text{in } \Omega \end{cases} \tag{1.2}$$

We are going to use the stream function formulation to solve the Navier-Stokes equations. One of the advantages of the stream function formulation is that the size of the system derived from the stream function formulation by Galerkin's method is much less than that of the system from the velocity-pressure formulation and the vorticity-stream function formulation. Indeed, let N, M, and L be the dimension of the C^0 quadratic finite element space, the bivariate spline space to be introduced below, and the linear finite element space, respectively. In the velocity-pressure formulation (cf. [17]), one needs $2N$ unknowns to approximate the velocity vector and L unknowns for the pressure function and thus, the system of nonlinear equations is of size $(2N + L) \times (2N + L)$. In the stream function-vorticity formulation, one needs N unknowns to approximate the stream function and N unknowns to approximate the vorticity function and hence the size of the system is $2N \times 2N$. In the stream function formulation the size of the system is $M \times M$. In our case, $M < N$. Indeed, let $\Diamond$ be a quadrangulation of Ω which conists of nondegenerate convex quadrilaterals. By adding the two diagonals of each quadrilateral, we obtain a triangulated quadrangulation $\diamondplus$. Let

$$S_3^1(\diamondplus) = \{s \in C^1(\Omega) : s|_t \in \mathbf{P}_3, \forall t \in \diamondplus\}$$

be the bivariate spline space of smoothness 1 and degree 3 which is the well-known Fraejis en Veubeke and Sander's finite element space (cf. [4] and [15]). Let $S_1^0(\diamondplus)$ and $S_2^0(\diamondplus)$ be the continuous linear and quadratic finite element spaces, respectively. Then it is easy to see (cf. Remark 5.1.) that $M = \dim(S_3^1(\diamondplus)) = 5V_i + V_b/2 - 2$, $N = \dim(S_2^0(\diamondplus)) = 8V_i + 4V_b - 7$ and $L = \dim(S_1^0(\diamondplus)) = 2V_i + 3V_b/2 - 1$, where V_i and V_b denotes the interior and boundary vertices of $\Diamond$. Thus, $M < N$.

Another advantage of the stream function formulation is that if the exact solution is very smooth, the numerical solutions obtained from the high order finite elements approximate the exact solution much better than that from the other two formulations which uses continuous linear and quadratic finite elements. For the same tolerance, the stream function formulation will result in fewer refinements of the underlying triangulation than the other two formulations and hence increase the efficiency. We shall also demonstrate by examples that if a solution of the Navier-Stokes equations is singular (less smooth), then the standard local refinement technique can be applied so that numerical solutions from the stream function formulation approximate the singular solution very well.

The only disadvantage of the stream function formulation is the difficulty of implementing high order finite elements, e.g., C^1 quintic Argyris' finite elements or C^1 quadratic Powell-Sabin's finite element. The difficulty arises mainly from the degree of polynomials, e.g. Argyris' element, or from the complicated refinement of the underlying triangulation, e.g., Powell-Sabin's element. However, $S_3^1(\diamondplus)$ has much smaller dimension than the space of Argyris' elements over a triangulation $\triangle$ which uses the same vertices of the quadrangulation $\Diamond$. $S_3^1(\diamondplus)$ has a higher approximation order than the C^1 quadratic finite elements over Powell-Sabin's refinement of $\triangle$ and $\diamondplus$ contains many fewer triangles than Powell-Sabin's refinement of $\triangle$. Also the C^1 cubic finite element space $S_3^1(\triangle_{CT})$, with $\triangle_{CT}$ the Clough-Tocher refinement of $\triangle$, has larger dimension than $S_3^1(\diamondplus)$ (cf. Remark 5.2). Hence these comparisons of the dimensions, approximation powers and the numbers of triangles of underlying triangulations lead us to the conclusion that $S_3^1(\diamondplus)$ is an ideal choice among all C^1 finite element schemes.

We have successfully implemented $S_3^1(\diamondplus)$ in MATLAB (cf. [10]) and applied $S_3^1(\diamondplus)$ to numerically solve the steady state Navier-Stokes equations (cf. [12]). In this paper we shall use spatial approximations in $S_3^1(\diamondplus)$ to numerically solve the time evolution Navier-Stokes equations. More generally, we consider the following bivariate spline space

$$S_{3r}^r(\diamondplus) = \{s \in C^r(\Omega) : s|_t \in \mathbf{P}_{3r}, \forall t \in \diamondplus\}$$

of smoothness r and degree $3r$. Such a spline space was introduced in [11]. We shall study the convergence of the numerical solutions using this general spline space to the exact solutions after we show the existence and uniqueness of the weak solution of the Stokes' and Navier-Stokes equations.

We shall also discuss how to approximate the pressure function in this paper. The pressure function satisfies a second order Poisson equation. We used the Neumann boundary condition which comes directly from the original equations to derive a weak formulation for the pressure function. We will prove the existence of the solution and use the linear finite element method to solve the Poisson equation associated with the pressure.

We have implemented $S_3^1(\diamondsuit)$ in MATLAB to numerically solve the Stokes and Navier-Stokes equations over any general polygonal domains. Our experiments show that the numerical approximations converge to the solution very well. We also implemented the linear finite element method in MATLAB to solve the pressure functions numerically.

In the following two sections, we first introduce the weak formulation of the Stokes equations and then the Navier-Stokes equations. Then we show the existence and uniqueness of the weak solution. After that, we discuss the convergence of numerical solutions to the weak solutions. We address the time discretization problem and provide numerical experiments at the end of each of two sections. Finally, in Section 4, we discuss the numerical approximation of the pressure functions. Our numerical experiments for the pressure approximation are reported at the end of the section.

II. THE STOKES' EQUATIONS

In general, the linearized Navier-Stokes equations are the time evolution Stokes' equations

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega\times(0,T) \\ \text{div}\mathbf{u} = 0 & \text{in } \Omega\times(0,T) \\ \mathbf{u} = \mathbf{h} & \text{on } \partial\Omega\times(0,T) \\ \mathbf{u}(x,y,0) = \mathbf{g}, & \text{in } \Omega \end{cases} \tag{2.1}$$

where $\mathbf{f} = (f_1, f_2)^T$ is in $L_2(0,T;L_2(\Omega))^2$, $\mathbf{h} = (h_1,h_2)$ is in $L_2(0,T;L_2(\partial\Omega))^2$, and $\mathbf{g} = (g_1,g_2)^T \in H^1(\Omega)^2$. Here, $\mathbf{g}|_{\partial\Omega} = \mathbf{h}$ when $t = 0$ and $\text{div}(\mathbf{g}) = 0$.

Since $\text{div}\mathbf{u} = 0$ for each $t \in (0,T)$, there exists a stream function $\varphi(x,y,t)$ such that

$$\mathbf{u} = \mathbf{curl}(\varphi) = \left(\frac{\partial}{\partial y}\varphi, -\frac{\partial}{\partial x}\varphi\right)^T.$$

We derive the initial condition for φ by solving

$$\begin{cases} \Delta\varphi = \frac{\partial}{\partial y}g_1 - \frac{\partial}{\partial x}g_2 & \text{in } \Omega \\ \nabla\varphi\cdot\mathbf{n} = (-\mathbf{g_2},\mathbf{g_1})^\mathbf{T}\cdot\mathbf{n} & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Let $\varphi_\mathbf{g}$ be a solution, unique up to a constant, of the above Neumann problem of Poisson equation. By a simple calculation, the Stokes equations (2.1) may be rewritten as the fourth order differential equation

$$\begin{cases} \frac{\partial}{\partial t}\Delta\varphi - \nu\Delta^2\varphi = \mathrm{curl}(\mathbf{f}) & \text{in } \Omega\times(0,T) \\ \frac{\partial}{\partial n}\varphi = h^{(1)} & \text{on } \partial\Omega\times(0,T) \\ \varphi = h^{(2)} & \text{on } \partial\Omega\times(0,T) \\ \varphi(x,y,0) = \varphi_{\mathbf{g}}(x,y) & \text{in } \Omega, \end{cases} \tag{2.3}$$

where $h^{(1)}$ and $h^{(2)}$ are dependent on $\mathbf{h}$ and will be given later.

Let

$$\begin{aligned} a_1(\varphi,\psi) &= \int_\Omega \nabla\varphi(x,y,t)\nabla\psi(x,y)dxdy \\ a_2(\varphi,\psi) &= \int_\Omega \Delta\varphi(x,y,t)\Delta\psi(x,y)dxdy \\ \langle f,\psi\rangle &= \int_\Omega f(x,y,t)\psi(x,y)dxdy. \end{aligned}$$

To solve (2.3), we introduce a weak formulation as follows: find $\varphi \in L_2(0,T,H^2(\Omega)) \cap L_\infty(0,T;H^1(\Omega))$ such that

$$\begin{cases} a_1(\dot\varphi,\psi) + \nu a_2(\varphi,\psi) = \langle \mathbf{f}, \mathbf{curl}(\psi)\rangle, & \forall\ \psi\in H_0^2(\Omega) \text{ and } t\in(0,T) \\ \frac{\partial\varphi}{\partial n} = h^{(1)},\ \varphi = h^{(2)}, & \text{on } \partial\Omega\times(0,T) \\ \varphi(x,y,0) = \varphi_{\mathbf{g}}(x,y), & \text{in } \Omega \end{cases} \tag{2.4}$$

where $\dot\varphi$ denotes $\dfrac{\partial}{\partial t}\varphi$. We note that $a_1(\dot\varphi,\psi)$ should be understood in the distribution sense in the time variable. Moreover, we will show that $\dot\varphi \in L_2(0,T;H^2(\Omega))$.

Let us take time to explain $h^{(1)}$ and $h^{(2)}$. Since Ω is a polygon with piecewise linear segment boundary, let us write $\partial\Omega = \bigcup_{i=1}^{J}\Gamma_i$ with $\Gamma_i = [w_i, w_{i+1}]$ being a line segment between w_i and w_{i+1}, $i = 1,2,\ldots,J$ and $w_{J+1} = w_1$. We may assume that vertices $w_1,\ldots,w_J$ are so arranged that they are in the counter clockwise direction. Let n_i, τ_i be the outward normal and tangential directions of Γ_i, $i = 1,\ldots,J$, and let n,τ be the normal and tangential directions of $\partial\Omega$ so that $n|_{\Gamma_i} = n_i$ and $\tau|_{\Gamma_i} = \tau_i$. Thus, n,τ are well defined on $\partial\Omega$ except for $w_1,\ldots,w_J$. Since $\mathbf{u} = \mathbf{curl}(\varphi) = \mathbf{h}$ on $\partial\Omega$, we have $\frac{\partial}{\partial n}\varphi = n\cdot\left(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y}\right) = \tau\cdot\mathbf{h}$. Thus, $h^{(1)} = \tau\cdot\mathbf{h}$. Since $\frac{\partial}{\partial\tau}\varphi = -n\cdot\mathbf{h}$, for $(x,y)\in\Gamma_1$, we have

$$\varphi(x,y) = \int_{[w_1,(x,y)]}\frac{\partial}{\partial\tau}\varphi ds + \varphi(w_1) = -\int_{[w_1,(x,y)]} n\cdot\mathbf{h}ds + \varphi(w_1).$$

We may fix φ by choosing $\varphi(w_1) = 0$. Thus,

$$h^{(2)}(x,y) = -\int_{\bigcup_{j=1}^{i-1}\Gamma_j\cup[w_i,(x,y)]} n\cdot\mathbf{h}ds$$

for $(x,y)\in\Gamma_i$, $i = 1,\ldots,J$.

We now resume the discussion of the solution of (2.4). For convenience, let us consider the Stokes equations with homogeneous boundary conditions: find $\varphi\in L_2(0,T,H_0^2(\Omega))$ such that

$$\begin{cases} a_1(\dot\varphi,\psi) + \nu a_2(\varphi,\psi) = \langle\mathbf{f},\mathbf{curl}(\psi)\rangle, & \forall\ \psi\in H_0^2(\Omega) \\ \varphi(x,y,0) = \varphi_{\mathbf{g}}(x,y) & \text{in } \Omega \end{cases} \tag{2.5}$$

which corresponds to the case $\mathbf{h} = 0$ in (2.1).

Theorem 2.1. *Let* $\mathbf{f} \in L_2(0,T,L_2(\Omega))^2$ *and* $\varphi_{\mathbf{g}} \in H_0^2(\Omega)$. *Then there exists a unique solution* $\varphi \in L_2(0,T,H_0^2(\Omega))$ *satisfying (2.5). Furthermore,* $\varphi \in L_\infty(0,T,H^1(\Omega))$ *and* $\dot\varphi \in L_2(0,T,H_0^2(\Omega))$.

Proof. We first consider the uniqueness of the solution. Suppose that there exist φ_1 and φ_2 satisfying (2.5). Then $\varphi = \varphi_1 - \varphi_2 \in L_2(0,T,H_0^2(\Omega))$ satisfies

$$\begin{cases} a_1(\dot\varphi,\psi) + \nu a_2(\varphi,\psi) = 0, & \forall\ \psi \in H_0^2(\Omega) \\ \varphi(x,y,0) = 0, & (x,y) \in \Omega. \end{cases}$$

Since $\varphi \in H_0^2(\Omega)$ for each $t \in (0,T)$, we have

$$a_1(\dot\varphi,\varphi) + \nu a_2(\varphi,\varphi) = 0, \text{ or } \frac{1}{2}\frac{d}{dt}|\varphi|_{1,\Omega}^2 + \nu|\varphi|_{2,\Omega}^2 = 0.$$

That is,

$$\frac{d}{dt}|\varphi|_{1,\Omega}^2 \le 0, \quad t \in (0,T)$$

which implies that

$$|\varphi(x,y,t)|_{1,\Omega}^2 - |\varphi(x,y,0)|_{1,\Omega}^2 \le 0$$

Thus we conclude that $|\varphi(x,y,t)|_{1,\Omega}^2 = 0$ which implies that $\varphi(x,y,t)$ is a constant. Since $\varphi(x,y,t)|_{\partial\Omega} = 0$, we see that $\varphi(x,y,t) \equiv 0$. That is, $\varphi_1 \equiv \varphi_2$.

To show the existence of the weak solution, we let $\diamondsuit$ be a triangulated quadrangulation of Ω, $\diamondsuit_n$ be the nth refinement of $\diamondsuit$, and $V_n = S_{3r}^r(\diamondsuit_n) \cap H_0^2(\Omega)$ be the spline space of smoothness r and degree $3r$ over the triangulation $\diamondsuit_n$ (cf. [11]). Let $\{\psi_{n,i}\}_{i=1}^{N_n}$ be a locally supported basis for V_n. Thus, for each $s_n \in V_n$, we may write

$$s_n(x,y,t) = \sum_{i=1}^{N_n} c_{n,i}(t)\psi_{n,i}(x,y).$$

Since $\varphi_{\mathbf{g}} \in H_0^2(\Omega)$, we let

$$S_{\mathbf{g},n} = \sum_{i=1}^{N_n} c_{n,i}(\mathbf{g})\psi_{n,i} \in V_n.$$

be the quasi-interpolant, constructed in [11], of $\varphi_{\mathbf{g}}$ which satisfies

$$|\varphi_{\mathbf{g}} - S_{\mathbf{g},n}|_{k,\Omega} \le C|\diamondsuit_n|^{m-k}|\varphi_{\mathbf{g}}|_{m,\Omega}$$

for $k = 0,1,2$ and $2 \le m \le 3r+1$. Also, let $K_n = [a_1(\psi_{n,i},\psi_{n,j})]_{1\le i,j\le N_n}$ and $B_n = [a_2(\psi_{n,i},\psi_{n,j})]_{1\le i,j\le N_n}$ be the stiffness and bending matrices, respectively. Let $F_n = [\langle \mathbf{f}, \mathbf{curl}(\psi_{n,j})\rangle]_{1\le j\le N_n}$ be a vector of size N_n. It is easy to see that (2.5) with $H_0^2(\Omega)$ replaced by V_n is equivalent to the following system of ordinary differential equations

$$\begin{cases} \frac{d}{dt}\mathbf{c}_n(t) + \nu K_n^{-1}B_n\mathbf{c}_n(t) = K_n^{-1}F \\ \mathbf{c}_n(0) = \mathbf{c}_n(\mathbf{g}) \end{cases}$$

with $\mathbf{c}_n(\mathbf{g}) = (c_{n,1}(\mathbf{g}), \ldots, c_{n,N_n}(\mathbf{g}))^T$, and $\mathbf{c}_n(t) = (c_{n,1}(t), \ldots, c_{n,N_n}(t))^T$. The existence of the unique solution of $\mathbf{c}_n(t)$ follows from the standard theory of ordinary differential equations. Writing

$$\varphi_n(x,y,t) = \sum_{i=1}^{N_n} c_{n,i}(t)\psi_{n,i}(x,y) \in V_n,$$

we know that $\varphi_n \in V_n$ is the weak solution of (2.5) with $H_0^2(\Omega)$ replaced by V_n. We now show that $\{\varphi_n\}_{n=1}^{\infty}$ has a convergent subsequence in $L_2(0,T,H_0^2(\Omega))$.

Clearly, $\varphi_n(x,y,0) = S_{\mathbf{g},n}(x,y)$ converges to $\varphi_{\mathbf{g}}$ in L_2 norm as $n \to \infty$. Since φ_n satisfies

$$a_1(\dot{\varphi_n}, \varphi_n) + \nu a_2(\varphi_n, \varphi_n) = \langle \mathbf{f}, \mathbf{curl}(\varphi_n)\rangle,$$

we have

$$\frac{1}{2}\frac{d}{dt}|\varphi_n|_{1,\Omega}^2 + \nu|\varphi_n|_{2,\Omega}^2 = \langle \mathbf{f}, \mathbf{curl}(\varphi_n)\rangle.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}|\varphi_n|_{1,\Omega}^2 + \nu|\varphi_n|_{2,\Omega}^2 \le \|\mathbf{f}\|_{0,\Omega} C|\varphi_n|_{2,\Omega} \le \frac{C^2}{2\nu}\|\mathbf{f}\|_{0,\Omega}^2 + \frac{\nu}{2}|\varphi_n|_{2,\Omega}^2.$$

Thus, we have

$$\frac{d}{dt}|\varphi_n|_{1,\Omega}^2 + \nu|\varphi_n|_{2,\Omega}^2 \le \frac{C^2}{\nu}\|\mathbf{f}\|_{0,\Omega}^2.$$

Futhermore, we have

$$\begin{aligned}
&|\varphi_n(\cdot,t)|_{1,\Omega}^2 + \nu\int_0^t |\varphi_n(\cdot,s)|_{2,\Omega}^2 ds \\
\le\ & |\varphi_n(\cdot,0)|_{1,\Omega}^2 + \frac{C^2}{\nu}\int_0^t \|\mathbf{f}\|_{0,\Omega}^2 ds \\
\le\ & |S_{\mathbf{g},n}|_{1,\Omega}^2 + \frac{C^2}{\nu}\int_0^T \|\mathbf{f}\|_{0,\Omega}^2 ds \\
\le\ & 2|S_{\mathbf{g},n} - \varphi_{\mathbf{g}}|_{1,\Omega}^2 + 2|\varphi_{\mathbf{g}}|_{1,\Omega}^2 + \frac{8}{\nu}C^2\int_0^T \|\mathbf{f}\|_{0,\Omega}^2 ds \\
\le\ & M < +\infty
\end{aligned}$$

for a constant M independent of n and t. It follows that

$$\int_0^T |\varphi_n|_{2,\Omega}^2 ds \le M/\nu, \text{ and } |\varphi_n(\cdot,t)|_{1,\Omega}^2 \le M \quad \text{for any} \quad t \in (0,T).$$

Thus, $\{\varphi_n : n = 1, \cdots,\}$ is a bounded sequence in $L_\infty(0,T,H^1(\Omega))$. Hence, there exists a subsequence converging weak-star to φ_0 in $L_\infty(0,T,H^1(\Omega))$. On the other hand, $\varphi_n \in L_2(0,T,H_0^2(\Omega))$ is bounded for all n. Therefore, there exists a subsequence which converges weakly to $\varphi_0 \in L_2(0,T,H_0^2(\Omega))$.

We now show that φ_0 satisfies (2.5). Letting $\psi(x,y,t) = \psi_1(x,y)\psi_2(t) \in L_2(\Omega\times(0,T))$ with $\psi_1(x,y) \in H_0^2(\Omega)$ and $\psi_2 \in C_0^1(0,T)$, we have

$$\int_0^T a_1(\dot{\varphi_n}, \psi)dt = -\nu\int_0^T a_2(\varphi_n, \psi)dt + \int_0^T \langle \mathbf{f}, \mathbf{curl}(\psi)\rangle dt$$

$$\longrightarrow -\nu \int_0^T a_2(\varphi_0, \psi)dt + \int_0^T \langle \mathbf{f}, \mathbf{curl}(\psi)\rangle dt.$$

On the other hand, we have

$$\begin{aligned}\int_0^T a_1(\dot\varphi_n, \psi)dt &= -\int_0^T a_1(\varphi_n, \psi_1)\dot\psi_2(t)dt \\ &\longrightarrow -\int_0^T a_1(\varphi_0, \psi_1)\dot\psi_2(t)dt = \int_0^T a_1(\dot\varphi_0, \psi_1)\psi_2(t)dt.\end{aligned}$$

Combining the two right-hand sides above, we have

$$\int_0^T \left[a_1(\dot\varphi_0, \psi_1) + \nu a_2(\varphi_0, \psi_1) - \langle \mathbf{f}, \mathbf{curl}(\psi_1)\rangle\right] \psi_2(t)dt = 0$$

for any $\psi_2 \in C_0^1(0,T)$. It follows that φ_0 satisfies (2.5) for every $t \in (0,T)$ and for any $\psi \in H_0^2(\Omega)$.

Furthermore, letting $\psi = \dot\varphi_n \in V_n$, we have

$$a_1\left(\dot\varphi_n, \dot\varphi_n\right) + \nu a_2\left(\varphi_n, \dot\varphi_n\right) = \langle \mathbf{f}, \mathbf{curl}(\dot\varphi_n)\rangle.$$

The above immediately yields the inequality

$$|\dot\varphi_n|_{1,\Omega}^2 + \frac{\nu}{2}\frac{d}{dt}a_2(\varphi_n,\varphi_n) \le |\mathbf{f}|_{0,\Omega}\,|\dot\varphi_n|_{1,\Omega} \le \frac{1}{2}|\mathbf{f}|_{0,\Omega}^2 + \frac{1}{2}\,|\dot\varphi_n|_{1,\Omega}^2\,.$$

That is, we have

$$|\dot\varphi_n|_{1,\Omega}^2 + \nu\frac{d}{dt}|\varphi_n|_{2,\Omega}^2 \le |\mathbf{f}|_{0,\Omega}^2.$$

It follows that

$$\begin{aligned}\int_0^T |\dot\varphi_n|_{1,\Omega}^2 + \nu|\varphi_n(\cdot,T)|_{2,\Omega}^2 &\le \int_0^T |\mathbf{f}|_{0,\Omega}^2 dt + \nu|\varphi_n(\cdot,0)|_{2,\Omega}^2 \\ &= \int_0^T |\mathbf{f}|_{0,\Omega}^2 dt + \nu|S_{\mathbf{g},n}|_{2,\Omega}^2 \\ &\le \int_0^T |\mathbf{f}|_{0,\Omega}^2 dt + 2\nu|S_{\mathbf{g},n} - \varphi_{\mathbf{g}}|_{2,\Omega}^2 + 2\nu|\varphi_{\mathbf{g}}|_{2,\Omega}^2\end{aligned}$$

which is bounded independent of n since $\varphi_{\mathbf{g}} \in H^2(\Omega)$. Hence,

$$\int_0^T |\dot\varphi_0|_{1,\Omega}^2\, dt < +\infty$$

or $\dot\varphi_0 \in L_2(0,T,H^1(\Omega))$. Thus, we complete the proof of Theorem 2.1. ∎

To consider (2.4) with non-homogeneous boundary conditions, we have to assume that $h^{(1)}$ and $h^{(2)}$ are compatible with the boundary $\partial\Omega$ in the sense that there exists a $\varphi_b \in L_2(0,T,H^2(\Omega))$ such that

$$\varphi_b = h^{(2)} \quad \text{and} \quad \frac{\partial}{\partial n}\varphi_b = h^{(1)} \quad \text{on} \quad \partial\Omega \times (0,T). \tag{2.6}$$

Then the problem (2.4) is equivalent to finding $\varphi_0 \in L_2(0,T,H_0^2(\Omega))$ such that

$$\begin{cases} a_1(\dot\varphi_0,\psi)+\nu a_2(\varphi_0,\psi)=\langle \mathbf{f},\mathbf{curl}(\psi)\rangle - a_1(\dot\varphi_b,\psi)-\nu a_2(\varphi_b,\psi), & \forall\ \psi\in H_0^2(\Omega) \\ \varphi_0(x,y,0)=\varphi_{\mathbf{g}}(x,y)-\varphi_b(x,y,0). \end{cases} \tag{2.7}$$

By Theorem 2.1, there exists a unique $\varphi_0 \in L_2(0,T;H_0^2(\Omega))$ satisfying (2.7). Then $\varphi := \varphi_0+\varphi_b$ satisfies (2.4). The solution φ satisfying (2.4) is unique by the argument in the proof of Theorem 2.1. Hence, we have

Theorem 2.2. *Suppose that $h^{(1)}$ and $h^{(2)}$ are compatible with boundary $\partial\Omega$, In addition, suppose that φ_b satisfying (2.6) is in $\in C^1(0,T,H^2(\Omega))$, i.e., $|\varphi_b|_{2,\Omega}\in C^1[0,T]$. Suppose further that $\varphi_{\mathbf{g}}\in H^2(\Omega)$. Then there exists a unique function φ in $L_2(0,T,H^2(\Omega))$ and also in $L_\infty(0,T;H^1(\Omega))$ satisfying (2.4). Furthermore, $\dot\varphi\in L_2(0,T;H^1(\Omega))$.*

Next we study how to numerically solve (2.3) using bivariate spline space $S_{3r}^r(\Diamond)$. First of all, we note that in general we may not be able to find a spline function $s\in S_{3r}^r(\Diamond)$ satisfying the boundary conditions exactly. Let us find a spline $S_b\in S_{3r}^r(\Diamond)$ approximating the boundary conditions in the following sense: letting $v_1,\cdots,v_b$ be the boundary vertices of the triangulation $\Diamond$ arranged in the counter-clockwise direction,

$$\begin{aligned} S_b(v_i+j(v_{i+1}-v_i)/(3r)) &= h_2(v_i+j(v_{i+1}-v_i)/(3r)), j=0,\cdots,3r \\ \frac{\partial S_b}{\partial n_i}\left(v_i+\frac{j}{3r-1}(v_{i+1}-v_i)\right) &= h_1\left(v_i+\frac{j}{3r-1}(v_{i+1}-v_i)\right), j=0,\cdots,3r-1 \end{aligned}$$

if v_i is a corner or $v_i=v_1$; Otherwise, assuming that $S_b\big|_{[v_{i-1},v_i]}$ and $\frac{\partial S_b}{\partial n_i}\big|_{[v_{i-1},v_i]}$ are determined, we use the C^r smoothness condition at v_i and $2r$ interpolation conditions to determine $S_b\big|_{[v_i,v_{i+1}]}$ and the C^{r-1} smoothness condition and $2r$ interpolation conditions to determine $\frac{\partial S_b}{\partial n_i}\big|_{[v_i,v_{i+1}]}$. These requirements determine the coefficients of the polynomial pieces of S_b on $\partial\Omega$. For convenience let $\tilde h^{(1)} := \frac{\partial}{\partial n}S_b|_{\partial\Omega}$ and $\tilde h^{(2)} := S_b|_{\partial\Omega}$. Let

$$V_0 := S_{3r}^r(\Diamond)\cap H_0^2(\Omega).$$

Our numerical method is to find $S_\varphi\in S_{3r}^r(\Diamond)$ to be the solution of the following weak formulation of the Stokes equations:

$$\begin{cases} a_1(\dot S_\varphi,\psi)+\nu a_2(S_\varphi,\psi)=\langle \mathbf{f},\mathbf{curl}(\psi)\rangle, & \forall\ \psi\in V_0 \text{ and } t\in(0,T) \\ \frac{\partial S_\varphi}{\partial n}=\tilde h^{(1)}, S_\varphi=\tilde h^{(2)}, & \text{on } \partial\Omega\times(0,T) \\ S_\varphi(x,y,0)=S_{\mathbf{g}}(x,y), & \text{in } \Omega \end{cases} \tag{2.8}$$

where $S_{\mathbf{g}}$ is the best approximation of $\varphi_{\mathbf{g}}$ in $S_{3r}^r(\Diamond)$. By using the similar arguments as the proof of Theorem 2.2, we know that the weak solution S_φ exists and is unique. We now show that S_φ converges to φ as the size of triangulation $|\Diamond|$ goes to zero. That is, we need to estimate the error term $S_\varphi-\varphi$. To this end we need to define two norms on any pair of compatible boundary conditions $(h^{(1)},h^{(2)})$ by

$$\|(h^{(1)},h^{(2)})\|_2=\inf\left\{|\varphi_b-s|_{2,\Omega} : s\in H_0^2(\Omega)\right\}$$

and

$$\|(h^{(1)},h^{(2)})\|_1=\inf\left\{|\varphi_b-s|_{2,\Omega}+|\varphi_b-s|_{1,\Omega} : s\in H_0^2(\Omega)\right\},$$

where φ_b is in $H^2(\Omega)$ satisfying boundary conditions as defined in (2.6). We also need the following

Lemma 2.3. *Suppose that $h^{(1)}$ and $h^{(2)}$ are compatible. Then there exists a unique weak solution $\varphi_b \in H^2(\Omega)$ of the biharmonic equation; that is,*

$$a_2(\varphi_b, \psi) = 0, \quad \forall \psi \in H_0^2(\Omega)$$

with $\frac{\partial}{\partial n}\varphi_b| = h^{(1)}$ and $\varphi_b = h^{(2)}$ on $\partial\Omega$. Furthermore, φ_b satisfies the following

$$|\varphi_b|_{2,\Omega} \le 2\|(h^{(1)}, h^{(2)})\|_2$$

and

$$|\varphi_b|_{1,\Omega} \le C\|(h^{(1)}, h^{(2)})\|_1.$$

for a constant C independent of $(h^{(1)}, h^{(2)})$.

See [12] for a proof of Lemma 2.3. By using the spline approximation property, we are able to prove the following

Lemma 2.4. *Suppose that $\varphi \in L_\infty(0, T, H^m(\Omega))$ with $2 \le m \le 3r + 1$. Then we have*

$$\begin{aligned} \|(h^{(1)} - \tilde{h}^{(1)}, h^{(2)} - \tilde{h}^{(2)})\|_2 &\le C|\diamondsuit|^{m-2}|\varphi|_{m,\Omega}, \\ \|(h^{(1)} - \tilde{h}^{(1)}, h^{(2)} - \tilde{h}^{(2)})\|_1 &\le C|\diamondsuit|^{m-2}|\varphi|_{m,\Omega}. \end{aligned}$$

Proof. By the definition, we have

$$\begin{aligned} \|(h^{(1)} - \tilde{h}^{(1)}, h^{(2)} - \tilde{h}^{(2)})\|_2 &= \inf\{\|\varphi - S_b - s\|_{2,\Omega} : s \in H_0^2(\Omega)\} \\ &\le \inf\{\|\varphi - S_b - s\|_{2,\Omega} : s \in V_0\} \\ &\le C|\diamondsuit|^{m-2}|\varphi|_{m,\Omega} \end{aligned}$$

by the spline approximation property(cf. [11]). Similarly, we can show the second inequality of Lemma 2.4. ∎

When $h^{(1)}$ and $h^{(2)}$ depend on t, we introduce two auxiliary functions: Let $\tilde{\varphi} \in L_2(0, T, H^2(\Omega))$ satisfy

$$\begin{cases} a_2(\tilde{\varphi}, \psi) = a_2(\varphi, \psi), & \forall \psi \in H_0^2(\Omega), t \in (0, T) \\ \frac{\partial}{\partial n}\tilde{\varphi} = \tilde{h}^{(1)}, & \text{on } \partial\Omega \times (0, T) \\ \tilde{\varphi} = \tilde{h}^{(2)}, & \text{on } \partial\Omega \times (0, T) \end{cases} \tag{2.9}$$

and let $S_{\tilde{\varphi}} \in S_{3r}^r(\diamondsuit)$ satisfy

$$\begin{cases} a_2(S_{\tilde{\varphi}}, \psi) = a_2(\tilde{\varphi}, \psi), & \forall \psi \in V_0, \quad t \in (0, T) \\ \frac{\partial}{\partial n}S_{\tilde{\varphi}} = \tilde{h}^{(1)}, & \text{on } \partial\Omega \times (0, T) \\ S_{\tilde{\varphi}} = \tilde{h}^{(2)}, & \text{on } \partial\Omega \times (0, T) \end{cases} \tag{2.10}$$

The existence of such $\tilde{\varphi} \in L_2(0, T, H^2(\Omega))$ may be justified as follows. Let $\hat{\varphi} = \tilde{\varphi} - \varphi_b \in L_2(0, T; H_0^2(\Omega))$. Then $\hat{\varphi}$ satisfies

$$a_2(\hat{\varphi}, \psi) = a_2(\varphi, \psi) - a_2(\varphi_b, \psi), \quad \forall \psi \in H_0^2(\Omega)$$

and $|\hat\varphi|_{2,\Omega} \le |\varphi|_{2,\Omega} + |\varphi_b|_{2,\Omega}$. Thus, we have

$$\|\hat\varphi\|_{2,\Omega} \le C|\hat\varphi|_{2,\Omega} \le C|\varphi|_{2,\Omega} + |\varphi_b|_{2,\Omega}$$

for any $t \in [0,T]$. Thus, it follows that $\|\tilde\varphi\|_{2,\Omega} \le C|\varphi|_{2,\Omega} + |\varphi_b|_{2,\Omega} + \|\varphi_b\|_{2,\Omega}$. That is, $\tilde\varphi \in L_2(0,T,H^2(\Omega))$.

By Lemma 2.3 and (2.9), we have

$$\begin{aligned} |\varphi - \tilde\varphi|_{2,\Omega} &\le 2\|(h^{(1)} - \tilde h^{(1)}, h^{(2)} - \tilde h^{(2)})\|_2, \quad t \in (0,T) \\ |\varphi - \tilde\varphi|_{1,\Omega} &\le C\|(h^{(1)} - \tilde h^{(1)}, h^{(2)} - \tilde h^{(2)})\|_1, \quad t \in (0,T) \end{aligned}$$

where C is independent of t. Lemma 2.4 now implies the following:

Lemma 2.5. *Suppose that $\varphi \in L_\infty(0,T,H^m(\Omega))$ for $2 \le m \le 3r+1$. Then*

$$|\varphi - \tilde\varphi|_{2,\Omega} \le C|\diamondsuit|^{m-2}|\varphi|_{m,\Omega}, \text{ and } |\varphi - \tilde\varphi|_{1,\Omega} \le C|\diamondsuit|^{m-2}|\varphi|_{m,\Omega}$$

for all $t \in (0,T)$ and for a constant $C > 0$ independent of t.

Suppose that $\varphi \in C^1(0,T,H^2(\Omega))$ and that $\mathbf{h}$ is differentiable with respect to t. Since the right-hand of (2.9) is differentiable in t, we have, by differentiating (2.9) with respect to t,

$$\begin{cases} a_2(\dot{\tilde\varphi}, \psi) = a_2(\dot\varphi, \psi), & \forall \psi \in H_0^2(\Omega), t \in (0,T) \\ \frac{\partial \dot{\tilde\varphi}}{\partial n} = \dot{\tilde h}^{(1)}, & \text{on } \partial\Omega \times (0,T) \\ \dot{\tilde\varphi} = \dot{\tilde h}^{(2)} & \text{on } \partial\Omega \times (0,T). \end{cases}$$

It follows from Lemma 2.3 that

$$\begin{aligned} |\dot\varphi - \dot{\tilde\varphi}|_{2,\Omega} &\le 2\|(\dot h^{(1)} - \dot{\tilde h}^{(1)}, \dot h^{(2)} - \dot{\tilde h}^{(2)})\|_2 \\ |\dot\varphi - \dot{\tilde\varphi}|_{1,\Omega} &\le C\|(\dot h^{(1)} - \dot{\tilde h}^{(1)}, \dot h^{(2)} - \dot{\tilde h}^{(2)})\|_1 \end{aligned}$$

Lemma 2.4 now yields the following:

Lemma 2.6. *Suppose that $\varphi \in C^1(0,T,H^m(\Omega))$ for $2 \le m \le 3r+1$. Then*

$$|\dot\varphi - \dot{\tilde\varphi}|_{2,\Omega} \le C|\diamondsuit|^{m-2}|\dot\varphi|_{m,\Omega} \text{ and } |\dot\varphi - \dot{\tilde\varphi}|_{1,\Omega} \le C|\diamondsuit|^{m-2}|\dot\varphi|_{m,\Omega}$$

for all $t \in (0,T)$ and for a constant $C > 0$ independent of t.

On the other hand, since $\tilde\varphi - S_{\tilde\varphi} \in H_0^2(\Omega)$ for $t \in (0,T)$, we have

$$\begin{aligned} |\tilde\varphi - S_{\tilde\varphi}|_{2,\Omega}^2 &= a_2(\tilde\varphi - S_{\tilde\varphi}, \tilde\varphi - S_{\tilde\varphi}) \\ &= a_2(\tilde\varphi - S_{\tilde\varphi}, \tilde\varphi - S_{\tilde\varphi} - \psi) \\ &\le |\tilde\varphi - S_{\tilde\varphi}|_{2,\Omega}|\tilde\varphi - S_{\tilde\varphi} - \psi|_{2,\Omega} \end{aligned}$$

for any $\psi \in V_0$ by (2.10). That is,

$$\begin{aligned} |\tilde\varphi - S_{\tilde\varphi}|_{2,\Omega} &\le \inf_{\psi \in V_0} |\tilde\varphi - S_{\tilde\varphi} - \psi|_{2,\Omega} \\ &\le |\varphi - \tilde\varphi|_{2,\Omega} + \inf_{\psi \in V_0} |\varphi - S_{\tilde\varphi} - \psi|_{2,\Omega}. \end{aligned}$$

By Lemma 2.5 and the spline approximation property we have

Lemma 2.7. *Suppose that* $\varphi \in L_\infty(0,T,H^m(\Omega))$ *with* $2 \le m \le 3r+1$. *Then*

$$|\tilde{\varphi} - S_{\tilde{\varphi}}|_{2,\Omega} \le C|\diamondsuit|^{m-2}|\varphi|_{m,\Omega} \text{ and } |\tilde{\varphi} - S_{\tilde{\varphi}}|_{1,\Omega} \le C|\diamondsuit|^{m-2}|\varphi|_{m,\Omega}.$$

for all $t \in (0,T)$ *and for constant* $C > 0$ *independent of* t.

Similarly to the derivation of Lemma 2.6 we have

Lemma 2.8. *Suppose that* $\varphi \in C^1(0,T,H^m(\Omega))$ *with* $2 \le m \le 3r+1$. *Then*

$$|\dot{\tilde{\varphi}} - \dot{S}_{\tilde{\varphi}}|_{2,\Omega} \le C|\diamondsuit|^{m-2}|\dot{\varphi}|_{m,\Omega} \text{ and } |\dot{\tilde{\varphi}} - \dot{S}_{\tilde{\varphi}}|_{1,\Omega} \le C|\diamondsuit|^{m-2}|\dot{\varphi}|_{m,\Omega}$$

for all $t \in (0,T)$ *and for a constant* $C > 0$ *independent of* t.

Now we estimate $S_{\tilde{\varphi}} - S_\varphi$. For any $\psi \in V_0$ we have, by (2.8), (2.10), (2.9) and (2.4)

$$\begin{aligned} a_1(\dot{S}_{\tilde{\varphi}} - \dot{S}_\varphi, \psi) + \nu a_2(S_{\tilde{\varphi}} - S_\varphi, \psi) &= a_1(\dot{S}_{\tilde{\varphi}}, \psi) + \nu a_2(S_{\tilde{\varphi}}, \psi) - \langle \mathbf{f}, \mathbf{curl}(\psi)\rangle \\ &= a_1(\dot{S}_{\tilde{\varphi}}, \psi) + \nu a_2(\tilde{\varphi}, \psi) - \langle \mathbf{f}, \mathbf{curl}(\psi)\rangle \\ &= a_1(\dot{S}_{\tilde{\varphi}}, \psi) - a_1(\dot{\varphi}, \psi). \end{aligned}$$

That is, we have

$$a_1(\dot{S}_{\tilde{\varphi}} - \dot{S}_\varphi, \psi) + \nu a_2(S_{\tilde{\varphi}} - S_\varphi, \psi) = a_1(\dot{S}_{\tilde{\varphi}} - \dot{\varphi}, \psi). \tag{2.11}$$

Putting $\psi = S_{\tilde{\varphi}} - S_\varphi \in V_0$ into (2.11), we have

$$\frac{1}{2}\frac{d}{dt}a_1(S_{\tilde{\varphi}} - S_\varphi, S_{\tilde{\varphi}} - S_\varphi) + \nu|S_{\tilde{\varphi}} - S_\varphi|^2_{2,\Omega} \le a_1(\dot{S}_{\tilde{\varphi}} - \dot{\varphi}, S_{\tilde{\varphi}} - S_\varphi).$$

It follows that

$$\frac{1}{2}\frac{d}{dt}|S_{\tilde{\varphi}} - S_\varphi|^2_{1,\Omega} \le |\dot{S}_{\tilde{\varphi}} - \dot{\varphi}|_{1,\Omega}|S_{\tilde{\varphi}} - S_\varphi|_{1,\Omega}$$

which implies that

$$\frac{d}{dt}|S_{\tilde{\varphi}} - S_\varphi|_{1,\Omega} \le |\dot{S}_{\tilde{\varphi}} - \dot{\varphi}|_{1,\Omega}.$$

The triangle inequality and integration of the above inequality with respect to t leads to

$$\begin{aligned} |S_{\tilde{\varphi}} - S_\varphi|_{1,\Omega} &\le |(S_{\tilde{\varphi}} - S_\varphi)\big|_{t=0^+}|_{1,\Omega} + \int_0^t |\dot{S}_{\tilde{\varphi}} - \dot{\tilde{\varphi}}|_{1,\Omega}ds + \int_0^t |\dot{\tilde{\varphi}} - \dot{\varphi}|_{1,\Omega}ds \\ &\le C|(S_{\tilde{\varphi}} - S_\varphi)\big|_{t=0^+}|_{2,\Omega} + \int_0^T |\dot{S}_{\tilde{\varphi}} - \dot{\tilde{\varphi}}|_{1,\Omega}ds + \int_0^T |\dot{\tilde{\varphi}} - \dot{\varphi}|_{1,\Omega}ds. \end{aligned}$$

Note that by (2.9) and (2.10), when $t = 0^+$ we have that

$$a_2(S_{\tilde{\varphi}}, \psi) = a_2(\tilde{\varphi}, \psi) = a_2(\varphi, \psi) = a_2(\varphi_{\mathbf{g}}, \psi)$$

and

$$a_2(S_\varphi, \psi) = a_2(S_{\mathbf{g}}, \psi).$$

for any $\psi \in V_0$. It follows that $a_2(S_{\tilde{\varphi}} - S_\varphi, \psi) = a_2(\varphi_{\mathbf{g}} - S_{\mathbf{g}}, \psi)$ for any $\psi \in V_0$ when $t = 0^+$. Thus, letting $\psi = S_{\tilde{\varphi}} - S_\varphi$ we obtain

$$|(S_{\tilde{\varphi}} - S_\varphi)\big|_{t=0^+}|_{2,\Omega} \le |\varphi_{\mathbf{g}} - S_{\mathbf{g}}|_{2,\Omega}.$$

We note that

$$|\varphi_{\mathbf{g}} - S_{\mathbf{g}}|_{k,\Omega} \le C|\diamondsuit|^{m-k}|\varphi_{\mathbf{g}}|_{m,\Omega}$$

for $k = 0, 1, 2$ with $2 \le m \le 3r+1$ (cf. [11]). Lemmas 2.6 and 2.8 now imply that

$$|S_{\tilde\varphi} - S_\varphi|_{1,\Omega} \le C|\diamondsuit|^{m-2}\left(|\varphi_{\mathbf{g}}|_{m,\Omega} + \left(\int_0^T |\dot\varphi|^2_{m,\Omega}dt\right)^{1/2}\right) \tag{2.12}$$

for all $\varphi \in H^1(0,T,H^m(\Omega))$ with $2 \le m \le 3r+1$.

Putting $\psi = \dot S_{\tilde\varphi} - \dot S_\varphi \in V_0$, into (2.11), we have

$$a_1(\dot S_{\tilde\varphi} - \dot S_\varphi, \dot S_{\tilde\varphi} - \dot S_\varphi) + \nu a_2(S_{\tilde\varphi} - S_\varphi, \dot S_{\tilde\varphi} - \dot S_\varphi) = a_1(\dot S_{\tilde\varphi} - \dot\varphi, \dot S_{\tilde\varphi} - \dot S_\varphi),$$

which together with Poincaré's inequality implies that

$$\begin{aligned}\frac{1}{2}\nu\frac{d}{dt}|S_{\tilde\varphi} - S_\varphi|^2_{2,\Omega} &\le |\dot S_{\tilde\varphi} - \dot\varphi|_{1,\Omega}|\dot S_{\tilde\varphi} - \dot S_\varphi|_{1,\Omega}\\ &\le C|\dot S_{\tilde\varphi} - \dot\varphi|_{1,\Omega}|\dot S_{\tilde\varphi} - \dot S_\varphi|_{2,\Omega}.\end{aligned}$$

Thus we get

$$\frac{d}{dt}|S_{\tilde\varphi} - S_\varphi|_{2,\Omega} \le C_1|\dot S_{\tilde\varphi} - \dot\varphi|_{1,\Omega}$$

which implies that

$$\begin{aligned}|S_{\tilde\varphi} - S_\varphi|_{2,\Omega} &\le |(S_{\tilde\varphi} - S_\varphi)\big|_{t=0^+}|_{2,\Omega} + C\int_0^t |\dot S_{\tilde\varphi} - \dot{\tilde\varphi}|_{1,\Omega}ds + \int_0^t |\dot{\tilde\varphi} - \dot\varphi|_{1,\Omega}ds\\ &\le C|\diamondsuit|^{m-2}|\varphi_{\mathbf{g}}|_{m,\Omega} + C|\diamondsuit|^{m-2}\left(\int_0^T |\dot\varphi|^2_{m,\Omega}dt\right)^{1/2}\end{aligned} \tag{2.13}$$

for $\varphi \in H^1(0,T,H^m(\Omega))$ and $\varphi_{\mathbf{g}} \in H^m(\Omega)$, where C is independent of t. Therefore we can conclude from $\varphi - S_\varphi = \varphi - \tilde\varphi + \tilde\varphi - S_{\tilde\varphi} + S_{\tilde\varphi} - S_\varphi$ the following:

Theorem 2.9. *Suppose that $\varphi \in H^1(0,T,H^m(\Omega))$ and $\varphi_{\mathbf{g}} \in H^m(\Omega)$ with $2 \le m \le 3r+1$. Then*

$$|\varphi - S_\varphi|_{2,\Omega} \le C|\diamondsuit|^{m-2}(\left(\int_0^T |\dot\varphi|^2_{m,\Omega}dt\right)^{1/2} + |\varphi|_{m,\Omega} + |\varphi_{\mathbf{g}}|_{m,\Omega})$$

and

$$|\varphi - S_\varphi|_{1,\Omega} \le C|\diamondsuit|^{m-2}(\left(\int_0^T |\dot\varphi|^2_{m,\Omega}dt\right)^{1/2} + |\varphi|_{m,\Omega} + |\varphi_{\mathbf{g}}|_{m,\Omega})$$

for a constant $C > 0$ independent of t.

Based on Theorem 2.9, we know that S_φ is an approximation of the weak solution of the Stokes equations (2.3). Furthermore, let us give the following estimate which will be used later for the estimate of the pressure function approximation.

Theorem 2.10. *Assume that the triangulation* $\diamondsuit$ *is quasi-uniform. Suppose that* $\varphi \in H^1(0,T,H^m(\Omega))$ *and* $\varphi_{\mathbf{g}} \in H^m(\Omega)$ *with* $3 \le m \le 3r+1$. *Then*

$$|\dot\varphi - \dot S_\varphi|_{1,\Omega} \le C|\diamondsuit|^{m-3}\Big(\left(\int_0^T |\dot\varphi|^2_{m,\Omega} dt\right)^{1/2} + |\varphi|_{m,\Omega} + |\dot\varphi|_{m,\Omega} + |\varphi_{\mathbf{g}}|_{m,\Omega}\Big).$$

Proof. Letting $\psi = \dot S_{\tilde\varphi} - \dot S_\varphi$ in (2.11), we have

$$|\dot S_{\tilde\varphi} - \dot S_\varphi|^2_{1,\Omega} = a_1(\dot S_{\tilde\varphi} - \dot\varphi, \dot S_{\tilde\varphi} - \dot S_\varphi) - \nu a_2(S_{\tilde\varphi} - S_\varphi, \dot S_{\tilde\varphi} - \dot S_\varphi).$$

By Cauchy-Schwarz's inequality and the inequality $|\dot S_{\tilde\varphi} - \dot S_\varphi|_{2,\Omega} \le \frac{C}{|\diamondsuit|}|\dot S_{\tilde\varphi} - \dot S_\varphi|_{1,\Omega}$ which follows from the quasi-uniform assumption on the triangulation, we have

$$\begin{aligned}|\dot S_{\tilde\varphi} - \dot S_\varphi|_{1,\Omega} &\le |\dot S_{\tilde\varphi} - \dot\varphi|_{1,\Omega} + \frac{\nu C}{|\diamondsuit|}|S_{\tilde\varphi} - S_\varphi|_{2,\Omega} \\ &\le |\dot S_{\tilde\varphi} - \dot{\tilde\varphi}|_{1,\Omega} + |\dot{\tilde\varphi} - \dot\varphi|_{1,\Omega} + \frac{\nu C}{|\diamondsuit|}|S_{\tilde\varphi} - S_\varphi|_{2,\Omega}.\end{aligned}$$

By Lemmas 2.6 and 2.8 and (2.13). we get

$$|\dot S_{\tilde\varphi} - \dot S_\varphi|_{1,\Omega} \le C|\diamondsuit|^{m-3}\Big(\left(\int_0^T |\dot\varphi|^2_{m,\Omega} dt\right)^{1/2} + |\varphi|_{m,\Omega} + |\dot\varphi|_{m,\Omega} + |\varphi_{\mathbf{g}}|_{m,\Omega}\Big).$$

The conclusion of this theorem follows from Lemmas 2.6 and 2.8 and the following inequality

$$|\dot\varphi - \dot S_\varphi|_{1,\Omega} \le |\dot\varphi - \dot{\tilde\varphi}|_{1,\Omega} + |\dot{\tilde\varphi} - \dot S_{\tilde\varphi}|_{1,\Omega} + |\dot S_{\tilde\varphi} - \dot S_\varphi|_{1,\Omega}.$$

This completes the proof. ∎

Next we consider the error in the computation of S_φ that results from the discretization of the time variable. Writing

$$S_\varphi(x,y,t) = \sum_{j=1}^{N_1} c_j(t)\psi_j(x,y),$$

we let $\mathbf{c}(t) = (c_1(t), \ldots, c_{N_1}(t))^T$. Then we know that $\mathbf{c}$ satisfies

$$K\frac{d}{dt}\mathbf{c}(t) + \nu B\mathbf{c}(t) = F$$

with $K := K_n, B := B_n, F := F_n$ for a fixed n. To solve this system of ordinary differential equations, we discretize the time variable into $0 = t_0 < t_1 < \ldots < t_n = T$ and note that

$$\frac{\mathbf{c}(t_{i+1}) - \mathbf{c}(t_i)}{t_{i+1} - t_i} = \frac{1}{2}\left(\frac{d}{dt}\mathbf{c}(t_i) + \frac{d}{dt}\mathbf{c}(t_{i+1})\right) + O(|t_{i+1} - t_i|^2).$$

Thus writing $\Delta t_i = t_{i+1} - t_i$ we have

$$K(\mathbf{c}(t_{i+1}) \;-\; \mathbf{c}(t_i)) + \frac{\Delta t_i}{2}\nu B(\mathbf{c}(t_{i+1}) + \mathbf{c}(t_i))$$

$$= \frac{\Delta t_i}{2}(F(t_{i+1} + F(t_i)) + O(K(\Delta t_i)^3).$$

Solving the above equation for $\mathbf{c}(t_{i+1})$ yields

$$\begin{aligned}\mathbf{c}(t_{i+1}) &= (K + \frac{\Delta t_i}{2}\nu B)^{-1}(K - \frac{\Delta t_i}{2}\nu B)\mathbf{c}(t_i) \\ &+\frac{\Delta t_i}{2}(K + \frac{\Delta t_i}{2}\nu B)^{-1}(F(t_i) + F(t_{i+1})) \\ &+(K + \frac{\Delta t_i}{2}\nu B)^{-1}KO((\Delta t_i)^3). \qquad (2.14)\end{aligned}$$

This leads to the classical Crank-Nicholson method for Stokes' equation:

$$\begin{aligned}\mathbf{c}_{i+1} &= (K + \frac{\Delta t_i}{2}\nu B)^{-1}(K - \frac{\Delta t_i}{2}\nu B)\mathbf{c}_i \\ &+\frac{\Delta t_i}{2}(K + \frac{\Delta t_i}{2}\nu B)^{-1}(F(t_i) + F(t_{i+1})) \qquad (2.15)\end{aligned}$$

for $i = 0, 1, 2, \ldots, n-1$ with $\mathbf{c}_0 = \mathbf{c}(\mathbf{g})$ which is the coefficient vector of $S_{\mathbf{g}}$. We now give an estimate for $\mathbf{c}(t_i) - \mathbf{c}_i$.

Lemma 2.11. *Suppose that* $\max\{\Delta t_i, 0 \le i \le n-1\} \le M \min\{\Delta t_i, 0 \le i \le n-1\}$. *Letting* $\mathbf{e}_i = \mathbf{c}(t_i) - \mathbf{c}_i$, *we have*

$$\|\mathbf{e}_i\|_{\ell^2} = O(|\Delta t|^2),$$

where $\Delta t = \max_{0\le i\le n-1} \Delta t_i$. *Letting* $S_{\varphi,i}(x,y) = \sum_{j=1}^{N_1} c_{ij}\psi_j(x,y)$ *with* $\mathbf{c}_i = (c_{i,1}, \cdots, c_{i,N_1})^T$ *being the solution of (2.14) for* $i = 0, \cdots, n-1$, *we have*

$$\|S_\varphi(\cdot, t_i) - S_{\varphi,i}(\cdot)\|_{0,\Omega} \le C|\Delta t|^2.$$

Proof. Let $A_i = \left(K + \frac{\Delta t_i}{2}B\right)^{-1}\left(K - \frac{\Delta t_i}{2}B\right)$. We observe that the positivity of the eigenvalues λ_i of K and the eigenvalues μ_i of B implies that

$$\|A_i\|_2 = \max_{1\le j\le N_1} \frac{|1 - \lambda_j^{-1}\mu_j\frac{\Delta t_i}{2}|}{1 + \lambda_j^{-1}\mu_j\frac{\Delta t_i}{2}} < 1, \quad i = 0, \cdots, n-1.$$

Next let

$$E_i = \left(K + \frac{\Delta t_i}{2}B\right)^{-1} KO(\Delta t_i)^3 = \left(I + \frac{\Delta t_i}{2}K^{-1}B\right)^{-1} O(|\Delta t_i|^3).$$

Then we have from (2.14) and (2.15)

$$\mathbf{e}_{i+1} = A_i\mathbf{e}_i + E_i = \prod_{j=0}^{i} A_j\mathbf{e}_0 + \sum_{j=1}^{i} A_{j+1}\ldots A_iE_j = \sum_{j=1}^{i} A_{j+1}\ldots A_iE_j.$$

Now as the matrices K and B are positive definite it follows that $\|E_i\|_{\ell^2} = O(|\Delta t_i|^3)$ and hence $\|\mathbf{e}_{i+1}\|_{\ell^2} \le nO((\Delta t)^3) \le O(|\Delta t|^2)$. Furthermore,

$$\|S_\varphi(\cdot, t_i) - S_{\varphi,i}(x,y)\|_{0,\Omega} \le \|\mathbf{e}_i\|_{\ell^2}\sqrt{\sum_{j=1}^{N_1}\int_\Omega |\psi_j|^2 dxdy} \le O(|\Delta t|^2)$$

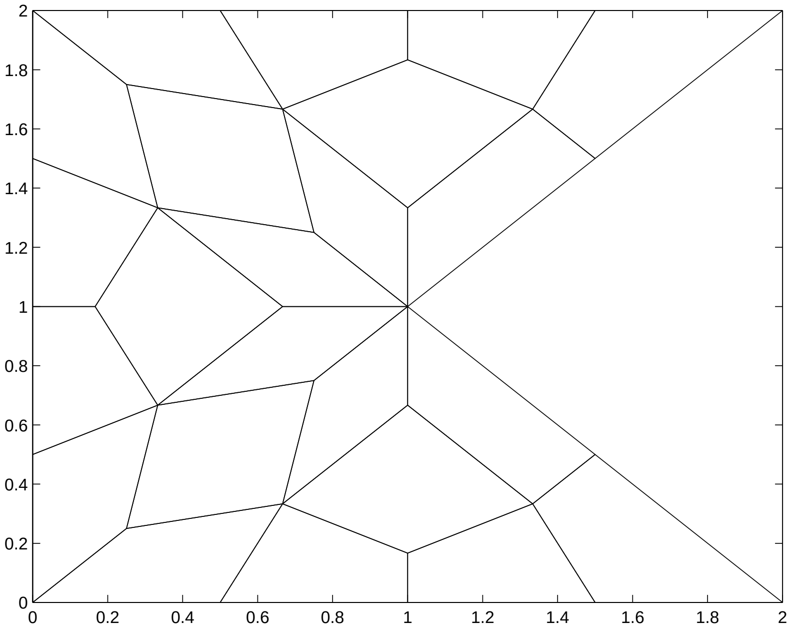

FIG. 1. A Quadrangulation of K-shape Domain

since all ψ_j are locally supported and uniformly bounded by a constant which is dependent only on the smallest angle of the underlying triangulation (cf. [11]). ■

We have implemented in MATLAB the above numerical method for solving Stokes' equation using the bivariate C^1 cubic spline space $S_3^1(\diamondplus)$. We tested our programs for many known exact solutions and compared with the numerical solutions over many different polygonal domains. In the following example, we consider a K-shape domain. We start with a triangulated quadrangulation as shown in Fig. 1 and then refine it twice in Fig. 2. The time interval $[0,1]$ is first divided into 10 subintervals and then 20 and 40 subintervals. We tested many known exact solutions by first computing the corresponding right-hand side functions and boundary functions using MATHEMATICA and then feeding the resulting functions into our programs to find numerical solutions. We evaluate the numerical solutions over 201×201 equally spaced points over $[0,2] \times [0,2]$ restricted to the K-shaped domain. Table I is a list of the maximum errors of the numerical solutions against the exact solutions.

We can see that the convergence is very fast and appears to be fourth order. Let us also show the convergence of the derivatives of the numerical solutions to the corresponding derivatives of the exact solutions. Table II lists the maximum errors of the derivative with respect to x.

Next we consider the following Stokes equation:

$$\begin{cases} \frac{\partial}{\partial t}\Delta\varphi - \nu\Delta^2\varphi = 1 & \text{in } \Omega \times (0,T) \\ \frac{\partial}{\partial n}\varphi = 0, \varphi = 0, & \text{on } \partial\Omega \times (0,T) \\ \varphi(x,y,0) = 0 & \text{in } \Omega, \end{cases}$$

with Ω the same K-shape domain. We start with a quadrangulation as in Fig. 1. Then we refine it four times uniformly. We use the numerical solution from the fourth refinement

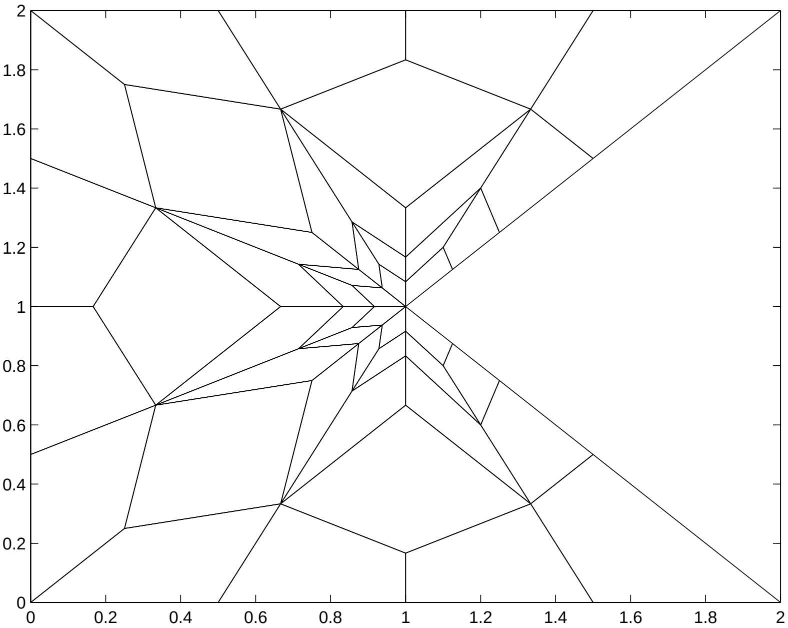

FIG. 2. The First Irregular Refinement of the K-shape Domain

TABLE I. Maximum Errors of Some Stream Functions

Matrix Sizes	150×150	527×527	1971×1971
$\sin(1+t(x+y))$	2.1307×10^{-3}	1.3624×10^{-4}	8.4965×10^{-6}
$(x^2+y^2)^{5/2}\sin(t\pi/2)$	3.7349×10^{-1}	2.4737×10^{-2}	1.5924×10^{-3}
$\exp(1+t(x+y))$	2.1053×10^{-1}	1.6672×10^{-2}	1.1765×10^{-3}
$\sin(t\pi/2)\sin(x+y)$	1.6503×10^{-3}	1.2829×10^{-4}	8.6280×10^{-6}
$(1+t^2)(x^2+y^2)^{5/2}$	3.7349×10^{-1}	2.4737×10^{-2}	1.5924×10^{-3}
$(1+\sin(t*\pi/2))(x^4+y^4)$	1.5625×10^{-2}	9.7366×10^{-4}	6.1191×10^{-5}

as an approximate exact solution and compute the maximum error against the numerical solutions from the initial quadrangulation and its three refinements. We list them below as well as the associated matrix size. On the other hand, we use a local refinement technique (cf. [12]) to refine the initial quadrangulation at and nearby the singular point $(1,1)$ three times as shown in Figures 3 and 4. We list the maximum errors of the numerical solutions based on those irregular refinements against the approximate exact solution. Also, we list the matrix sizes associated with the three irregular refinements.

From Table III above we can see that when a local refinement technique is applied, the bivariate spline method is able to approximate the singular solution very well.

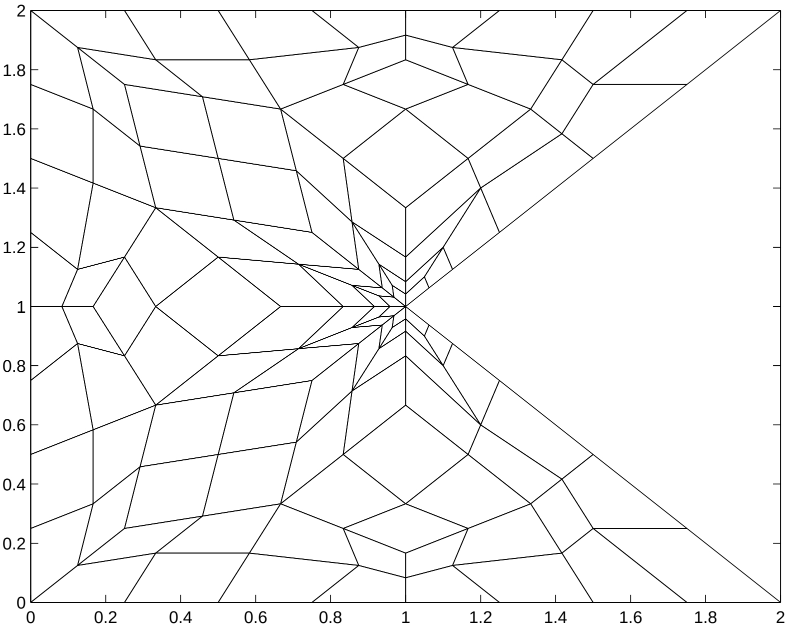

FIG. 3. Another Irregular Refinement of the K-shape Domain

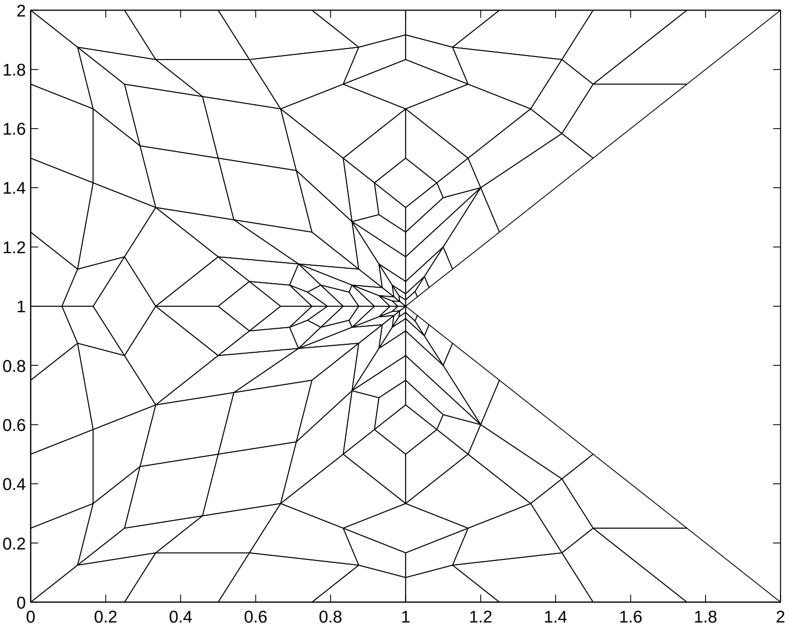

FIG. 4. One More Irregular Refinement of the K-shape Domain

TABLE II. The Maximum Errors of the x-Derivatives of the Stream Functions

Matrix Sizes	150×150	527×527	1971×1971
$\sin(1+t(x+y))$	9.6261×10^{-3}	1.1997×10^{-3}	1.5073×10^{-4}
$(x^2+y^2)^{5/2}\sin(t\pi/2)$	1.5600	1.8135×10^{-1}	2.7082×10^{-2}
$\exp(1+t(x+y))$	9.4434×10^{-1}	1.3327×10^{-1}	1.6454×10^{-2}
$\sin(t\pi/2)\sin(x+y)$	6.1440×10^{-3}	1.0793×10^{-3}	1.7177×10^{-4}
$(1+t^2)(x^2+y^2)^{5/2}$	1.5596	1.8125×10^{-1}	2.7080×10^{-2}
$(1+\sin(t*\pi/2))(x^4+y^4)$	6.8278×10^{-2}	9.5971×10^{-3}	1.5428×10^{-3}

TABLE III. Comparison of the Maximum Errors using Regular and Irregular Refinements

	Regular Refinements			
Matrix Sizes	150×150	527×527	1971×1971	7619×7619
Maximum Errors	4.1383×10^{-4}	1.1350×10^{-4}	3.9227×10^{-5}	1.2130×10^{-5}
	Irregular Refinements			
Matrix Sizes	150×150	278×278	655×655	849×849
Maximum Errors	4.1383×10^{-4}	9.5104×10^{-5}	2.1512×10^{-5}	8.4526×10^{-6}

III. THE NAVIER-STOKES EQUATIONS

In this section, we consider the time evolution Navier-Stokes equations:

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega\times(0,T) \\ \text{div}\mathbf{u} = 0, & \text{in } \Omega\times(0,T) \\ \mathbf{u} = \mathbf{h}, & \text{in } \partial\Omega\times(0,T) \\ \mathbf{u}(x,y,0) = \mathbf{g}(x,y), & \text{in } \Omega \end{cases} \tag{3.1}$$

where $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ are given in $L_2(0,T,L_2(\Omega))^2$, $H^1(\Omega)^2$, and $L_2(0,T,L_2(\partial\Omega))^2$, respectively. Here, $\mathbf{g}|_{\partial\Omega} = \mathbf{h}$ when $t=0$ and $\text{div}(\mathbf{g}) = 0$.

Since $\text{div}\mathbf{u} = 0$, there exists a stream function $\varphi(x,y,t)$ such that

$$\mathbf{u} = \mathbf{curl}(\varphi) = \left(\frac{\partial}{\partial y}\varphi, -\frac{\partial}{\partial x}\varphi\right)^T, \quad (x,y,t)\in\Omega\times(0,T).$$

In particular, when $t=0$, we have $\mathbf{curl}(\varphi) = \mathbf{g}$. Thus, φ at $t=0$ satisfies (2.2). As in §2, we let $\varphi_{\mathbf{g}} \in H^1(\Omega)$ be the unique solution up to a constant of the Poisson equation (2.2) with nonhomogeneous Neumann boundary condition. As in §2, the Navier-Stokes equations (3.1) may be given as a fourth order nonlinear biharmonic equation

$$\begin{cases} \frac{\partial}{\partial t}\Delta\varphi - \nu\Delta^2\varphi - \frac{\partial}{\partial x}\left(\Delta\varphi\frac{\partial}{\partial y}\varphi\right) + \frac{\partial}{\partial y}\left(\Delta\varphi\frac{\partial}{\partial x}\varphi\right) = \text{curl}(\mathbf{f}) & \text{in } \Omega\times(0,T) \\ \frac{\partial}{\partial n}\varphi = h^{(1)}, \varphi = h^{(2)}, & \text{on } \partial\Omega\times(0,T) \\ \varphi(x,y,0) = \varphi_{\mathbf{g}}(x,y), & \text{in } \Omega \end{cases} \tag{3.2}$$

where $h^{(1)}$ and $h^{(2)}$ are the same as given in §2. The weak formulation of (3.2) is as follows: find $\varphi \in L_2(0,T,H^2(\Omega)) \cap L_\infty(0,T;H^1(\Omega))$ such that

$$\begin{cases} a_1(\dot{\varphi},\psi) + \nu a_2(\varphi,\psi) + b(\varphi;\varphi,\psi) = \langle \mathbf{f}, \mathbf{curl}(\psi)\rangle & \forall \psi \in H_0^2(\Omega) \text{ and } t \in (0,T) \\ \frac{\partial \varphi}{\partial n} = h^{(1)}, \varphi = h^{(2)}, & \text{on } \partial\Omega \times (0,T) \\ \varphi(x,y,0) = \varphi_{\mathbf{g}}(x,y), & \text{in } \Omega, \end{cases} \tag{3.3}$$

where $b(\varphi;\varphi,\psi)$ is the trilinear form defined by

$$b(\theta;\varphi,\psi) = \int_\Omega \Delta\theta \left(\frac{\partial}{\partial x}\varphi \frac{\partial}{\partial y}\psi - \frac{\partial}{\partial y}\varphi \frac{\partial}{\partial x}\psi \right) dxdy.$$

For convenience, let us first consider the Navier-Stokes equations with homogeneous boundary conditions. Find $\varphi \in L_2(0,T,H_0^2(\Omega)) \cap L_\infty(0,T;H^1(\Omega))$ such that

$$\begin{cases} a_1(\dot{\varphi},\psi) + \nu a_2(\varphi,\psi) + b(\varphi;\varphi,\psi) = \langle \mathbf{f}, \mathbf{curl}(\psi)\rangle, & \forall \psi \in H_0^2(\Omega) \\ \varphi(x,y,0) = \varphi_{\mathbf{g}}(x,y) & \text{in } \Omega. \end{cases} \tag{3.4}$$

We make sense of $a_1(\dot{\varphi},\psi)$ in (3.3) and (3.4) as in Section 2.

Theorem 3.1. *Suppose that $\varphi_{\mathbf{g}} \in H^2(\Omega)$ and that $\mathbf{f} \in L_\infty(0,T;H^1(\Omega))$. Then there exists a unique solution $\varphi_0 \in L_2(0,T,H_0^2(\Omega)) \cap L_\infty(0,T;H^1(\Omega))$ satisfying (3.4). Furthermore, $\dot{\varphi} \in L_2(0,T;L_2(\Omega))$.*

Before giving a proof of Theorem 3.1, we need a lemma used in the proof of the uniqueness of the solution.

Lemma 3.2. *For $\theta,\varphi,\psi \in H_0^2(\Omega)$,*

$$|b(\theta;\varphi,\psi)| \le C|\theta|_{2,\Omega}|\varphi|_{1,\Omega}^{1/2}|\varphi|_{2,\Omega}^{1/2}|\psi|_{1,\Omega}^{1/2}|\psi|_{2,\Omega}^{1/2}.$$

Lemma 3.2 is a straightforward consequence of the following inequality

$$\int_\Omega |f(x,y)|^4 dxdy \le 2\int_\Omega |f(x,y)|^2 dxdy \int_\Omega |\nabla f(x,y)|^2 dxdy$$

for any $f \in H_0^1(\Omega)$ (cf. [7]). Next we need Lemma 3.4 which will be used in passing to the limit in the trilinear form. To prove Lemma 3.4, we recall the following well-known result:

Lemma 3.3. *Let $Y = \{\varphi \in L_{\alpha_0}(0,T;X_0), \dot{\varphi} \in L_{\alpha_1}(0,T;X_1)\}$ be a vector space with norm*

$$\|\varphi\|_Y = \|\varphi\|_{L_{\alpha_0}(0,T;X_0)} + \|\dot{\varphi}\|_{L_{\alpha_1}(0,T;X_1)}$$

where $X_0 \subset X_1$ are Hilbert spaces. Let X be a Hilbert space satisfying $X_0 \subset X \subset X_1$. If the injections are continuous and the one from X_0 to X is compact, then the injection from Y to $L_{\alpha_0}(0,T;X)$ is compact.

We refer [14] for a proof. We are now in a position to prove the following

Lemma 3.4. *Suppose that φ_n converges to φ_0 weakly in $L_2(0,T;H_0^2(\Omega))$ and that $\dot{\varphi}_n$ is uniformly bounded in $L_2(0,T;L_2(\Omega))$. Then $\forall\ \psi \in H_0^2(\Omega)$*

$$\int_0^T b(\varphi_n;\varphi_n,\psi)dt \longrightarrow \int_0^T b(\varphi_0;\varphi_0,\psi)dt, \quad \text{as} \quad n \to +\infty.$$

Proof. We begin by noting that the uniform boundedness of $\dot{\varphi}_n$ implies that this sequence has a subsequence weakly convergent in $L_2(0,T;L_2(\Omega))$. It is easy to see that the subsequence converges to $\dot{\varphi}_0$, and that in fact $\dot{\varphi}_n$ converges to $\dot{\varphi}_0$. We next claim that $\int_0^T b(\varphi_n,\varphi_n,\psi)dt$ is convergent to $\int_0^T b(\varphi_0,\varphi_0,\psi)dt$ for any $\psi \in C^1(\overline{\Omega}) \cap H_0^2(\Omega)$ which is dense in $H_0^2(\Omega)$. Indeed, we note that by Lemma 3.3 $\{\varphi_n\}$ converges strongly in $L_2(0,T;H_0^1(\Omega))$. We note that for any $\psi \in C^1(\overline{\Omega}) \cap H_0^2(\Omega)$ we have the inequalities

$$\begin{aligned}
&\left|\int_0^T b(\varphi_n,\varphi_n,\psi)dt - \int_0^T b(\varphi_0,\varphi_0,\psi)dt\right| \\
\leq\ & |\int_0^T b(\varphi_n,\varphi_n-\varphi_0,\psi)dt| + |\int_0^T b(\varphi_n-\varphi_0,\varphi_0,\psi)dt| \\
\leq\ & \left(\int_0^T |\varphi_n|_{2,\Omega}^2 dt\right)^{1/2} \left(\|\frac{\partial\psi}{\partial x}\|_{L^\infty(\Omega)} + \|\frac{\partial\psi}{\partial y}\|_{L^\infty(\Omega)}\right) \left(\int_0^T |\varphi_n-\varphi_0|_{1,\Omega}^2 dt\right)^{1/2} \\
& + |\int_0^T b(\varphi_n-\varphi_0,\varphi_0,\psi)dt|.
\end{aligned}$$

Since $\{\varphi_n : n = 1,2,\cdots\} \subset L_2(0,T;H^2(\Omega))$ is a bounded sequence, the first term above is convergent to zero. Since $\psi \in C^1(\overline{\Omega})$, $\frac{\partial}{\partial x}\varphi_0\frac{\partial}{\partial y}\psi - \frac{\partial}{\partial y}\varphi_0\frac{\partial}{\partial x}\psi \in L_2(0,T,L_2(\Omega))$. The second term converges to zero by the weak convergence of $\{\varphi_n, n = 1,2,\cdots,\}$ in $L_2(0,T;H^2(\Omega))$. The claim follows and the conclusion of this lemma follows from the previously mentioned boundedness of the sequence φ_n. ∎

Proof. (Proof of Theorem 3.1.) We first discuss the uniqueness formally. Let φ_1 and φ_2 in $L_2(0,T;H_0^2(\Omega))$ satisfy (3.4). Then $\varphi = \varphi_1 - \varphi_2$ satisfies

$$\begin{cases} a_1(\dot{\varphi},\psi) + \nu a_2(\varphi,\psi) + b(\varphi_1;\varphi,\psi) + b(\varphi;\varphi_2,\psi) = 0, & \forall\ \psi \in H_0^2(\Omega) \\ \varphi(x,y,0) = 0, & (x,y) \in \Omega. \end{cases}$$

Putting $\psi = \varphi$ in the above equation, we have, by Lemma 3.2,

$$\begin{aligned}
\frac{d}{dt}|\varphi|_{1,\Omega}^2 + \nu|\varphi|_{2,\Omega}^2 &= -b(\varphi;\varphi_2,\varphi) \leq C|\varphi|_{2,\Omega}^{3/2}|\varphi|_{1,\Omega}^{1/2}|\varphi_2|_{2,\Omega} \\
&\leq \frac{(\alpha|\varphi|_{2,\Omega}^{3/2})^{4/3}}{\frac{4}{3}} + \frac{\left(\frac{C}{\alpha}|\varphi|_{1,\Omega}^{1/2}|\varphi_2|_{2,\Omega}\right)^4}{4} \\
&= \frac{3}{4}\alpha^{4/3}|\varphi|_{2,\Omega}^2 + \frac{1}{4}\left(\frac{C}{\alpha}\right)^4 |\varphi|_{1,\Omega}^2|\varphi_2|_{2,\Omega}^4.
\end{aligned}$$

By choosing $\alpha = \left(\frac{4}{3}\nu\right)^{3/4}$, we have

$$\frac{d}{dt}|\varphi|_{1,\Omega}^2 \leq \frac{1}{4}\left(\frac{C}{\alpha}\right)^4 |\varphi_2|_{2,\Omega}^4|\varphi|_{1,\Omega}^2.$$

For simplicity, let $\beta = \frac{1}{4}\left(\frac{C}{\alpha}\right)^4$. Assume that $|\varphi_2|_{2,\Omega} \in L_4(0,T)$ which will be proved in Lemma 3.6. Then it follows that

$$\exp\left(-\beta\int_0^t |\varphi_2|_{2,\Omega}^4 ds\right)\frac{d}{dt}|\varphi|_{1,\Omega}^2 \le \exp\left(-\beta\int_0^t |\varphi_2|_{2,\Omega}^4 ds\right)\beta|\varphi_2|_{2,\Omega}^4|\varphi|_{1,\Omega}^2$$

or

$$\frac{d}{dt}\left[\exp\left(-\beta\int_0^t |\varphi_2|_{2,\Omega}^4 ds\right)|\varphi|_{1,\Omega}^2\right] \le 0.$$

Since $|\varphi(\cdot,0)|_{1,\Omega}^2 = 0$, we integrate the above inequality over $(0,t)$ to get

$$\exp\left(-\beta\int_0^t |\varphi_2|_{2,\Omega}^4 ds\right)|\varphi(\cdot,t)|_{1,\Omega}^2 \le 0$$

or $|\varphi(\cdot,t)|_{1,\Omega}^2 = 0$ which implies that $\varphi(\cdot,t) = 0$ since $\varphi \in H_0^2(\Omega)$. That is, $\varphi_1 = \varphi_2$ and hence the solution must be unique.

Next we discuss the existence of the solution. Recall that $\diamondsuit_n$ is the nth refinement of $\diamondsuit$ and that

$$V_n = S_{3r}^r(\diamondsuit_n) \cap H_0^2(\Omega).$$

For any $s \in V_n$ we may write

$$s_n(x,y,t) = \sum_{j=1}^{N_n} c_{n,j}(t)\psi_{nj}(x,y)$$

with $\mathbf{c}_n(t) = (c_{n1}(t),\ldots,c_{n,N_n}(t))^T$ as in §2.

Recall also from §2 that $S_{\mathbf{g},n}$ is the quasi-interpolant of $\varphi_{\mathbf{g}}$ which satisfies

$$|\varphi_{\mathbf{g}} - S_{\mathbf{g},n}|_{k,\Omega} \le C|\diamondsuit_n|^{m-k}$$

for $k = 0,1,2$ with $2 \le m \le 3r+1$. Recall the stiff and bending matrices K_n and B_n as well as the right-hand side F_n from §2. The weak formulation of (3.4) for V_n may be rewritten as

$$\mathbf{c}_n'(t) + \nu K_n^{-1}B_n\mathbf{c}_n(t) + K_n^{-1}\mathbf{c}_n(t)^T\mathbf{B}\mathbf{c}_n(t) = K_n^{-1}F_n, \tag{3.5}$$

where $\mathbf{B}$ is the 3 dimensional matrix associated with the trilinear form $b(\varphi;\varphi,\psi)$. That is,

$$\mathbf{B} = [b(\psi_{n,i},\psi_{nj},\psi_{nk})]_{1\le i,j,k\le N_n}.$$

The nonlinear system of ordinary differential equations in (3.5) with the initial condition $\mathbf{c}_n(0) = (c_{n,i}(\mathbf{g}),\ldots,c_{n,N_n}(\mathbf{g}))^T$ has a maximal solution defined on an interval $(0,T_n)$, where $\mathbf{c}_n(\mathbf{g}) := (c_{n,1}(\mathbf{g}),\ldots,c_{n,N_n}(\mathbf{g}))^T$ is the coefficient vector of $S_{\mathbf{g},n}$. If $T_n < T$, then $|\varphi_n(\cdot,t)|_{1,\Omega}$ must be unbounded as $t \to T_n$, but we claim that $\varphi_n(\cdot,t)$ is bounded independent of t. Thus $T_n > T$. Let us prove the claim.

By choosing $\psi = \varphi_n \in V_n$ in (3.4), we have

$$a_1(\dot{\varphi}_n,\varphi_n) + \nu a_2(\varphi_n,\varphi_n) = \langle \mathbf{f}, \mathbf{curl}(\varphi_n)\rangle$$

which implies that

$$\frac{1}{2}\frac{d}{dt}|\varphi_n|_{1,\Omega}^2 + \nu|\varphi_n|_{2,\Omega}^2 \le C\|\mathbf{f}\|_{L_2(\Omega)^2}|\varphi_n|_{2,\Omega} \le \frac{\nu}{2}|\varphi_n|_{2,\Omega}^2 + \frac{1}{2\nu}C^2\|\mathbf{f}\|_{L_2(\Omega)^2}^2.$$

It follows that

$$\frac{d}{dt}|\varphi_n|_{1,\Omega}^2 + \nu|\varphi_n|_{2,\Omega}^2 \le \frac{1}{\nu}C^2\|\mathbf{f}\|_{L_2(\Omega)^2}^2.$$

Integrating the above inequality over $(0, t)$, we have

$$\begin{aligned} |\varphi_n(\cdot,t)|_{1,\Omega}^2 + \int_0^t \nu|\varphi_n|_{2,\Omega}^2 ds &\le |S_{\mathbf{g},n}(\cdot)|_{1,\Omega}^2 + \int_0^t \frac{1}{\nu}C^2\|\mathbf{f}\|_{L_2(\Omega)^2}^2 ds \\ &\le |\varphi_{\mathbf{g}} - S_{\mathbf{g},n}|_{1,\Omega}^2 + |\varphi_{\mathbf{g}}|_{1,\Omega}^2 + \frac{C^2}{\nu}\int_0^T \|\mathbf{f}\|_{L_2(\Omega)^2}^2 ds \end{aligned}$$

for any $t \in (0,T)$ and any n which implies that the sequence $\{\varphi_n\}$ is a bounded set in $L_\infty(0,T;H_0^1(\Omega))$. Thus, we have established the claim.

The above inequality also implies that $\int_0^t |\varphi_n|_{2,\Omega}^2 ds$ is bounded independent of t and n. Thus $\{\varphi_n\}$ is a bounded sequence in $L_2(0,T,H_0^2(\Omega))$ and hence has a weakly convergent subsequence. Without loss of generality, we may assume that whole sequence $\{\varphi_n\}$ converges to φ_0 weakly in $L_2(0,T,H_0^2(\Omega))$. Since $\{\varphi_n, n = 1,2,\cdots,\}$ is bounded in $L_\infty(0,T;H^1(\Omega))$, $\{\varphi_n\}$ has a weak* convergent subsequence. Without loss, we may assume that whole sequence $\{\varphi_n, n = 1,2,\cdots,\}$ converges in the weak* sense. Thus, $\varphi_0 \in L_\infty(0,T;H^1(\Omega))$.

We now show that φ_0 satisfies (3.4). Letting $\psi(x,y,t) = \psi_1(x,y)\psi_2(t) \in L_2(\Omega \times (0,T))$ with $\psi_1(x,y) \in H_0^2(\Omega)$ and $\psi_2 \in C_0^1(0,T)$, we have by Lemma 3.4 and the weak convergence of $\{\varphi_n\}$ that

$$\begin{aligned} \int_0^T a_1(\dot\varphi_n,\psi)dt &= -\nu\int_0^T a_2(\varphi_n,\psi)dt - \int_0^T b(\varphi_n;\varphi_n,\psi)dt + \int_0^T \langle \mathbf{f}, \mathbf{curl}(\psi)\rangle dt \\ &\longrightarrow -\nu\int_0^T a_2(\varphi_0,\psi)dt - \int_0^T b(\varphi_0,\varphi_0,\psi)dt + \int_0^T \langle \mathbf{f}, \mathbf{curl}(\psi)\rangle dt. \end{aligned}$$

On the other hand, the weak* convergence and integration by parts show that

$$\begin{aligned} \int_0^T a_1(\dot\varphi_n,\psi)dt &= -\int_0^T a_1(\varphi_n,\psi_1)\dot\psi_2(t)dt \\ &\longrightarrow -\int_0^T a_1(\varphi_0,\psi_1)\dot\psi_2(t)dt = \int_0^T a_1(\dot\varphi_0,\psi_1)\psi_2(t)dt. \end{aligned}$$

Combining the two right-hand sides above, we have

$$\int_0^T \left[a_1(\dot\varphi_0,\psi_1) + \nu a_2(\varphi_0,\psi_1) + b(\varphi_0;\varphi_0,\psi_1) - \langle \mathbf{f}, \mathbf{curl}(\psi_1)\rangle\right]\psi_2(t)dt = 0$$

for any $\psi_2 \in C_0^1[0,T]$. It follows that φ_0 satisfies (3.4) for every $t \in (0,T)$ and for any $\psi \in H_0^2(\Omega)$.

Furthermore, we need to show that $\dot\varphi_n$ is uniformly bounded in $L_2(0,T;L_2(\Omega))$ which has been assumed in Lemma 3.4. To this end we have to use another formulation of the trilinear form which is equivalent to the original form. Note that

$$\begin{aligned} -\frac{\partial}{\partial x}\left(\Delta\varphi\frac{\partial}{\partial y}\varphi\right) + \frac{\partial}{\partial y}\left(\Delta\varphi\frac{\partial}{\partial x}\varphi\right) &= -\frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\varphi\frac{\partial}{\partial y}\varphi\right) + \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\varphi\frac{\partial}{\partial x}\varphi\right) \\ &\quad -\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\varphi\frac{\partial}{\partial y}\varphi\right) + \frac{\partial}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\varphi\frac{\partial}{\partial y}\varphi\right). \end{aligned}$$

Thus we have

$$\begin{aligned} b(\varphi;\varphi,\psi) &= \int_\Omega \Big(-\frac{\partial}{\partial y}\varphi \frac{\partial}{\partial x}\varphi \frac{\partial}{\partial x}\frac{\partial}{\partial x}\psi + \frac{\partial}{\partial x}\varphi \frac{\partial}{\partial x}\varphi \frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi \\ &\quad - \frac{\partial}{\partial y}\varphi \frac{\partial}{\partial y}\varphi \frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi + \frac{\partial}{\partial x}\varphi \frac{\partial}{\partial y}\varphi \frac{\partial}{\partial y}\frac{\partial}{\partial y}\psi \Big)\, dxdy. \end{aligned}$$

For any $\psi \in H_0^2(\Omega)$, we have

$$\begin{aligned} \langle \Delta\dot{\varphi}_n, \psi\rangle &= -a_1(\dot{\varphi}_n, \psi) = \nu a_2(\varphi_n, \psi) + b(\varphi_n, \varphi_n, \psi) - \langle \mathbf{f}, \mathbf{curl}(\psi)\rangle \\ &\le \nu|\varphi_n|_{2,\Omega}|\psi|_{2,\Omega} + |\nabla\varphi_n|^2_{L_4(\Omega)}|\psi|_{2,\Omega} + C|\mathbf{f}|_{L_2(\Omega)^2}|\psi|_{2,\Omega}. \end{aligned}$$

If follows that

$$\langle \Delta\dot{\varphi}_n, \psi\rangle \le (\nu|\varphi_n|_{2,\Omega} + C|\varphi_n|_{1,\Omega}|\varphi_n|_{2,\Omega} + |\mathbf{f}|_{L_2(\Omega)^2})|\psi|_{2,\Omega}$$

and hence,

$$\begin{aligned} \int_0^T \langle \Delta\dot{\varphi}_n, \psi\rangle dt &\le C\left(\int_0^T |\varphi_n|^2_{2,\Omega} dt + \max_{0\le t\le T}|\varphi_n|_{1,\Omega}\int_0^T |\varphi_n|^2_{2,\Omega}dt + \int_0^T |\mathbf{f}|^2_{L_2(\Omega)^2}dt\right)^{1/2} \\ &\quad \times \left(\int_0^T |\psi|^2_{2,\Omega}dt\right)^{1/2}. \end{aligned}$$

The uniform boundedness of φ_n in $L_\infty(0,T;H_0^1(\Omega))$ and $L_2(0,T;H_0^2(\Omega))$ now implies that $\Delta\dot{\varphi}_n$ is uniformly bounded in $L_2(0,T,H^{-2}(\Omega))$ and hence that $\dot{\varphi}_n$ is uniformly bounded in $L_2(0,T;L_2(\Omega))$. We have thus completed the proof. ∎

We now take time to prove the following fact which has been used in the uniqueness part of the proof above.

Lemma 3.5. *Suppose that* $\mathbf{f} \in L_\infty(0,T,H^1(\Omega))$. *Then the weak solution* φ *of (3.4) actually belongs to* $L_\infty(0,T;H^2(\Omega))$.

Proof. Note that the weak formulation (3.4) may be given in the following form

$$-\langle \Delta\dot{\varphi}, \psi\rangle - \nu a_1(\Delta\varphi, \psi) + b(\varphi,\varphi,\psi) = -\langle \mathrm{curl}(\mathbf{f}), \psi\rangle.$$

for any $\psi \in H_0^2(\Omega)$. Let $\psi = \Delta\varphi$. Using the previously mentioned inequality of Ladyzhenskaya we have

$$\begin{aligned} |b(\varphi,\varphi,\Delta\varphi) &\le |\nabla(\Delta|\varphi)|_{L_2(\Omega)}|\nabla\varphi|_{L_4(\Omega)}|\Delta\varphi|_{L_4(\Omega)} \\ &\le |\nabla(\Delta\varphi)|^{3/2}_{L_2(\Omega)}|\nabla\varphi|_{L_4(\Omega)}|\Delta\varphi|^{1/2}_{L_2(\Omega)} \\ &\le \nu|\nabla(\Delta\varphi)|^2_{L_2(\Omega)} + C|\nabla\varphi|^4_{L_4(\Omega)}|\Delta\varphi|^2_{L_2(\Omega)} \end{aligned}$$

for a constant $C > 0$. Thus it follows that

$$\begin{aligned} &\frac{d}{dt}|\Delta\varphi|^2_{L_2(\Omega)} = \langle \Delta\dot{\varphi}, \Delta\varphi\rangle \\ = \ &-\nu a_1(\Delta\varphi,\Delta\varphi) - b(\varphi,\varphi,\Delta\varphi) + \langle \mathrm{curl}(\mathbf{f}), \Delta\varphi\rangle \end{aligned}$$

$$\begin{aligned}&\le -\nu|\nabla(\Delta\varphi)|^2_{L_2(\Omega)} + \nu|\nabla(\Delta\varphi)|^2_{L_2(\Omega)} + C|\nabla\varphi|^4_{L_4(\Omega)}|\Delta\varphi|^2_{L_2(\Omega)}\\ &\quad +|\nabla(\mathbf{f})|_{L_2(\Omega)^2}|\Delta\varphi|_{L_2(\Omega)}\\ &\le C|\nabla\varphi|^2_{L_2(\Omega)}|\Delta\varphi|^4_{L_2(\Omega)} + |\nabla(\mathbf{f})|_{L_2(\Omega)^2}|\Delta\varphi|_{L_2(\Omega)}.\end{aligned}$$

Multiplying both sides of the inequality above by $\exp(-C\int_0^t |\nabla\varphi|^2_{L_2(\Omega)}|\Delta\varphi|^2_{L_2(\Omega)}ds)$, we have

$$\begin{aligned}&\frac{d}{dt}\left[|\Delta\varphi|^2_{L_2(\Omega)}\exp(-C\int_0^t |\nabla\varphi|^2_{L_2(\Omega)}|\Delta\varphi|^2_{L_2(\Omega)}ds)\right]\\ &\le |\nabla(\mathbf{f})|_{L_2(\Omega)^2}|\Delta\varphi|_{L_2(\Omega)}\exp(-C\int_0^t |\nabla\varphi|^2_{L_2(\Omega)}|\Delta\varphi|^2_{L_2(\Omega)}ds)\\ &\le |\nabla(\mathbf{f})|_{L_2(\Omega)^2}|\Delta\varphi|_{L_2(\Omega)}.\end{aligned}$$

Since $\varphi \in L_\infty(0,T;H^1(\Omega))$ and $\mathbf{f} \in L_2(0,T,H^1(\Omega)^2)$, we conclude that

$$\begin{aligned}|\Delta\varphi|^2_{L_2(\Omega)} &\le \exp(C\int_0^T |\nabla\varphi|^2_{L_2(\Omega)}|\Delta\varphi|^2_{L_2(\Omega)}dt)\big(|\Delta\varphi\big|_{t=0}|^2_{L_2(\Omega)}\\ &\quad + \int_0^T |\nabla(\mathbf{f})|_{L_2(\Omega)^2}|\Delta\varphi|_{L_2(\Omega)}dt\big)\end{aligned}$$

or

$$|\Delta\varphi|^2_{L_2(\Omega)} \le C_2\exp\left(C_1\int_0^T |\varphi|^2_{2,\Omega}dt\right)$$

for constants C_1 and C_2 which may be dependent on $\mathbf{f}$, $\max\{|\varphi|^2_{1,\Omega} : t \in (0,T)\}$, and $\int_0^T |\varphi|^2_{2,\Omega}dt$. As $\varphi \in L_2(0,T,H^2(\Omega))$ it follows that $\varphi \in L_\infty(0,T;H^2(\Omega))$. Therefore we have completed the proof. ∎

From Lemma 3.5, we see that $\varphi \in L_\infty(0,T;W^{1,4})$ by Sobolev's Embedding Theorem, where $W^{1,4} = \{f \in L_2(\Omega) : |\nabla f|_{L_4(\Omega)} < +\infty\}$. It follows that $\mathbf{u} \in L_\infty(0,T;L_4(\Omega))^2$. Then the following well-known Theorem may be applied to conclude that $\mathbf{u} = \mathbf{curl}(\varphi)$ is in fact the exact solution of the Navier-Stokes equations. See [16] for a proof of the following theorem.

Theorem 3.6. *(Serrin). Let* $\mathbf{u}$ *be a weak solution of Navier-Stokes equations. If the solution* $\mathbf{u} \in L_s(0,T;L_r(\Omega))^2$ *with* $\frac{2}{s} + \frac{2}{r} < 1$ *then* $\mathbf{u}$ *is locally bounded (hence locally regular).*

To consider (3.2) with nonhomogeneous boundary conditions, we have to assume that $h^{(1)}$ and $h^{(2)}$ are compatible with the boundary $\partial\Omega$ in the sense that there exists a $\varphi_b \in L_\infty(0,T,H^2(\Omega))$ such that

$$\varphi_b = h^{(2)} \quad \text{and} \quad \frac{\partial}{\partial n}\varphi_b = h^{(1)} \quad \text{on} \quad \partial\Omega\times(0,T). \tag{3.6}$$

Then the problem (3.2) is equivalent to finding $\varphi_0 \in L_2(0,T,H_0^2(\Omega))$ such that

$$\begin{cases} a_1(\dot\varphi_0,\psi) + \nu a_2(\varphi_0,\psi) + b(\varphi_0;\varphi_0,\psi) + b(\varphi_0;\varphi_b,\psi) + b(\varphi_b;\varphi_0,\psi)\\ = \langle\mathbf{f},\mathbf{curl}(\psi)\rangle - a_1(\dot\varphi_b,\psi) - \nu a_2(\varphi_b,\psi) - b(\varphi_b;\varphi_b,\psi),\\ \varphi_0(x,y,0) = \varphi_{\mathbf{g}}(x,y) - \varphi_b(x,y,0)\end{cases} \tag{3.7}$$

for all $\psi \in H_0^2(\Omega)$. The proof of Theorem 3.1 may be applied to this case with little modifications. The first modification concerns uniqueness. Suppose that there are two solutions ϕ_1 and ϕ_2 satisfying (3.7). Then $\varphi = \phi_1 - \phi_2 \in H_0^2(\Omega)$ for all $t \in (0,T)$ and satisfies

$$a_1(\dot{\varphi},\psi) + \nu a_2(\varphi,\psi) + b(\phi_1;\varphi,\psi) + b(\varphi;\phi_2,\psi) + b(\varphi;\varphi_b,\psi) + b(\varphi_b;\varphi,\psi) = 0.$$

Letting $\psi = \varphi$ in the above equation, we have, by Lemma 3.2,

$$\frac{d}{dt}|\varphi|_{1,\Omega}^2 + \nu|\varphi|_{2,\Omega}^2 = -b(\varphi;\varphi_2,\varphi) - b(\varphi;\varphi_b,\varphi) \le C|\varphi|_{2,\Omega}^{3/2}|\varphi|_{1,\Omega}^{1/2}\left(|\varphi_2|_{2,\Omega} + |\varphi_b|_{2,\Omega}^{1/2}|\varphi_b|_{2,\Omega}^{1/2}\right).$$

Then the same arguments may be used to show the uniqueness of the solution of the weak formulation (3.7). In fact, the argument for uniqueness in the proof of Theorem 3.1 actually works for the original weak formulation (3.3).

The second modification concerns the boundedness of a sequence φ_n satisfying (3.7) with V_n in the replace of $H_0^2(\Omega)$, where $V_n = S_{3r}^r(\diamondsuit_n) \cap H_0^2(\Omega)$. By choosing $\psi = \varphi_n$ in (3.7), we have

$$\begin{aligned} a_1(\dot{\varphi}_n,\varphi_n) + \nu a_2(\varphi_n,\varphi_n) &= -b(\varphi_n,\varphi_b,\varphi_n) + < \mathbf{f}, \mathbf{curl}(\varphi_n) > \\ &\quad -a_1(\dot{\varphi}_b,\varphi_n) - \nu a_2(\varphi_b,\varphi_n) + b(\varphi_b,\varphi_b,\varphi_n). \end{aligned}$$

By using Lemma 3.2 and Poincaré's inequality we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\varphi_n|_{1,\Omega}^2 &+ \nu|\dot{\varphi}_n|_{2,\Omega}^2 \le C|\varphi_n|_{2,\Omega}^{3/2}|\varphi_n|_{1,\Omega}^{1/2}|\varphi_b|_{1,\Omega}^{1/2}|\varphi_b|_{2,\Omega}^{1/2} \\ &+ C\left(\|\mathbf{f}\|_{L_2(\Omega)^2} + |\dot{\varphi}_b|_{1,\Omega} + \nu|\varphi_b|_{2,\Omega} + |\varphi_b|_{2,\Omega}^{3/2}|\varphi_b|_{1,\Omega}^{1/2}\right)|\varphi_n|_{2,\Omega} \\ \le &\quad \frac{(\alpha|\varphi_n|_{2,\Omega}^{3/2})^{4/3}}{4/3} + \frac{(C/\alpha|\varphi_n|_{1,\Omega}^{1/2}|\varphi_b|_{1,\Omega}^{1/2}|\varphi_b|_{2,\Omega}^{1/2})^4}{4} \\ &+ \frac{C}{\nu}\left(\|\mathbf{f}\|_{\mathbf{L_2(\Omega)^2}} + |\dot{\varphi}_\mathbf{b}|_{\mathbf{1,\Omega}} + \nu|\varphi_\mathbf{b}|_{\mathbf{2,\Omega}} + |\varphi_\mathbf{b}|_{\mathbf{2,\Omega}}^{\mathbf{3/2}}|\varphi_\mathbf{b}|^{\mathbf{1/2}}\right)^2 + \frac{\nu}{4}|\varphi_n|_{2,\Omega}^2. \end{aligned}$$

By choosing $\alpha = \left(\frac{2\nu}{3}\right)^{3/4}$ and letting

$$F(t) = \frac{2C}{\nu}(\|\mathbf{f}\|_{L_2(\Omega)^2} + |\dot{\varphi}_b|_{1,\Omega} + \nu|\varphi_b|_{2,\Omega} + |\varphi_b|_{2,\Omega}^{3/2}|\varphi_b|_{1,\Omega}^{1/2})^2$$

and $\beta(t) = \frac{1}{2}(C/\alpha)^4|\varphi_b|_{1,\Omega}^2|\varphi_b|_{2,\Omega}^2$, we have the following inequality

$$\frac{d}{dt}|\varphi_n|_{1,\Omega}^2 + \nu|\varphi_n|_{2,\Omega}^2 \le \beta(t)|\varphi_n|_{1,\Omega}^2 + F(t). \tag{3.8}$$

First, the above inequality (3.8) yields

$$\frac{d}{dt}|\varphi_n|_{1,\Omega}^2 \le \beta(t)|\varphi_n|_{1,\Omega}^2 + F(t).$$

Multiplying both sides of the above inequality by $\exp(-\int_0^t \beta(s)ds)$ we obtain

$$\frac{d}{dt}(\exp(-\int_0^t \beta(s)ds)|\varphi_n|_{1,\Omega}^2) \le \exp(-\int_0^t \beta(s)ds)F(t) \le F(t).$$

Integrating with respect to t, we get

$$\exp(-\int_0^t \beta(s)ds)|\varphi_n|_{1,\Omega}^2 \le |\varphi_n(\cdot,0)|_{1,\Omega}^2 + \int_0^t \exp(-\int_0^s \beta(u)du)F(s)ds$$

or

$$\begin{aligned} |\varphi_n|_{1,\Omega}^2 \ &\le 2\ \exp(\int_0^T \beta(s)ds)\left(|S_{\mathbf{g},n} - \varphi_{\mathbf{g}}|_{1,\Omega}^2 + |\varphi_{\mathbf{g}}|_{1,\Omega}^2\right) \\ &\quad + \exp\left(\int_0^T \beta(s)ds\right)\int_0^t F(s)ds. \end{aligned}$$

Thus, $|\varphi_n|_{1,\Omega}^2 \le M < +\infty$ for all n and all t. Hence, $\{\varphi_n : n = 1, 2, \cdots\}$ is a bounded sequence in $L_\infty(0, T, H^1(\Omega))$.

Secondly, the integration of the inequality (3.8) from 0 to T gives

$$\int_0^T \frac{d}{dt}|\varphi_n|_{1,\Omega}^2 dt + \nu\int_0^T |\varphi_n|_{2.\Omega}^2 dt \le \int_0^T \beta(t)|\varphi_n|_{1,\Omega}^2 dt + \int_0^T F(t)dt.$$

Thus, it follows that

$$\begin{aligned} \nu\int_0^T |\varphi_n|_{2,\Omega}^2 dt \ &\le\ |\varphi_n\big|_{t=0}|_{1,\Omega}^2 + \int_0^T F(t)dt + \max_{0\le t\le T}|\varphi_n|_{1,\Omega}^2 \int_0^T \beta(t)dt \\ &=\ 2|S_{\mathbf{g},n} - \varphi_{\mathbf{g}}|_{2,\Omega}^2 + 2|\varphi_{\mathbf{g}}|_{2,\Omega}^2 + \int_0^T F(t)dt + \max_{0\le t\le T}|\varphi_n|_{1,\Omega}^2 \int_0^T \beta(t)dt. \end{aligned}$$

Since we have just seen that $|\varphi_n|_{1,\Omega}^2 \le M < +\infty$ for all n and all t, it follows that $\int_0^T |\varphi_n|_{2,\Omega}^2 dt$ is bounded independent of n. Since $\varphi_n \in H_0^2(\Omega)$, by Poincaré's inequality we conclude that $\{\varphi_n\}$ is a bounded sequence in $L_2(0, T, H_0^2(\Omega))$.

By the argument in the proof of Theorem 3.1, we can show that there exists φ_0 satisfying (3.7). Hence, $\varphi_0 + \varphi_b$ satisfies (3.3). Therefore, we conclude the following

Theorem 3.7. *Suppose that $h^{(1)}$ and $h^{(2)}$ are compatible with boundary $\partial\Omega$. Suppose that there exists a φ_b satisfying the boundary conditions (3.6) such that $\varphi_b \in C^1(0, T, H^2(\Omega))$. Suppose further that $\varphi_{\mathbf{g}} \in H^2(\Omega)$. Then there exists a unique $\varphi \in L_2(0, T, H^2(\Omega)) \cap L_\infty(0, T, H^1(\Omega))$ satisfying (3.3).*

Next we consider the spline approximation of the solution φ. As before, let $\diamondsuit$ be a triangulated quadrangulation of Ω and let $S_{3r}^r(\diamondsuit)$ be the spline space of degree $3r$ and smoothness r with $r \ge 1$. In general, we may not be able to find a spline function $s \in S_{3r}^r(\diamondsuit)$ satisfying the boundary conditions exactly. As in §2, we let $\tilde{h}^{(1)}$ and $\tilde{h}^{(2)}$ be the spline approximations of the boundary functions $h^{(1)}$ and $h^{(2)}$ respectively. Recall that $V_0 = S_{3r}^r(\diamondsuit) \cap H_0^2(\Omega)$. Let S_φ be the spline approximation of the solution of the weak formulation of the Navier-Stokes equation:

$$\begin{cases} a_1(\dot{S_\varphi}, \psi) + \nu a_2(S_\varphi, \psi) + b(S_\varphi; S_\varphi, \psi) = \langle \mathbf{f}, \mathbf{curl}(\psi)\rangle, & \forall\psi \in S_{3r}^r(\diamondsuit), t \in (0, T) \\ \frac{\partial}{\partial n}S_\varphi = \tilde{h}^{(1)}, S_\varphi = \tilde{h}^{(2)}, & \text{on } \partial\Omega \times (0, T) \\ S_\varphi(x, y, 0) = S_{\mathbf{g}}(x, y), & \text{in } \Omega \end{cases} \tag{3.9}$$

where $S_{\mathbf{g}}$ is the best approximation of the initial value $\varphi_{\mathbf{g}}$ in $S_{3r}^r(\diamondsuit)$.

In order to show that S_φ converges to φ we need to introduce two auxiliary functions: Let $\tilde\varphi \in L_2(0,T,H^2(\Omega))$ satisfy

$$\begin{cases} a_2(\tilde\varphi,\psi) = a_2(\varphi,\psi), & \forall\psi \in H_0^2(\Omega), t \in (0,T) \\ \frac{\partial}{\partial n}\tilde\varphi = \tilde h^{(1)}, & \text{on } \partial\Omega \times (0,T) \\ \tilde\varphi = \tilde h^{(2)}, & \text{on } \partial\Omega \times (0,T) \end{cases}$$

and let $S_{\tilde\varphi} \in S_{3r}^r(\diamondsuit)$ satisfy

$$\begin{cases} a_2(S_{\tilde\varphi},\psi) = a_2(\tilde\varphi,\psi), & \forall\psi \in V_0, \quad t \in (0,T) \\ \frac{\partial}{\partial n}S_{\tilde\varphi} = \tilde h^{(1)}, & \text{on } \partial\Omega \times (0,T) \\ S_{\tilde\varphi} = \tilde h^{(2)}, & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{3.10}$$

As in §2, we have estimates of $\varphi - \tilde\varphi$, $\tilde\varphi - S_{\tilde\varphi}$, $\dot\varphi - \dot{\tilde\varphi}$ and $\dot{\tilde\varphi} - \dot S_{\tilde\varphi}$. That is, Lemma 2.5–2.8 may be applied. Now we only need to estimate $S_{\tilde\varphi} - S_\varphi$. Consider

$$\begin{aligned} & a_1(\dot S_{\tilde\varphi} - \dot S_\varphi,\psi) + \nu a_2(S_{\tilde\varphi} - S_\varphi,\psi) + b(S_\varphi, S_{\tilde\varphi} - S_\varphi,\psi) \\ = \;& a_1(\dot S_{\tilde\varphi},\psi) + \nu a_2(S_{\tilde\varphi},\psi) - a_1(\dot S_\varphi,\psi) - \nu a_2(S_\varphi,\psi) - b(S_\varphi,S_\varphi,\psi) + b(S_\varphi,S_{\tilde\varphi},\psi) \\ = \;& a_1(\dot S_{\tilde\varphi},\psi) + \nu a_2(\varphi,\psi) - <\mathbf{f},\mathbf{curl}(\psi)> + b(\varphi,\varphi,\psi) + b(S_\varphi,S_{\tilde\varphi},\psi) - b(\varphi,\varphi,\psi) \\ = \;& a_1(\dot S_{\tilde\varphi} - \dot\varphi,\psi) + b(S_\varphi,S_{\tilde\varphi} - \varphi,\psi) + b(S_\varphi - \varphi,\varphi,\psi) \\ = \;& a_1(\dot S_{\tilde\varphi} - \dot\varphi,\psi) + b(S_\varphi,S_{\tilde\varphi} - \varphi,\psi) + b(S_\varphi - S_{\tilde\varphi},\varphi,\psi) + b(S_{\tilde\varphi} - \varphi,\varphi,\psi). \end{aligned} \tag{3.11}$$

Let $\psi = S_{\tilde\varphi} - S_\varphi \in V_0$ in the above. We get, by Lemma 3.2,

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|S_{\tilde\varphi} - S_\varphi|_{1,\Omega}^2 + \nu|S_{\tilde\varphi} - S_\varphi|_{2,\Omega}^2 \\ \le \;& |\dot S_{\tilde\varphi} - \dot\varphi|_{1,\Omega}|S_{\tilde\varphi} - S_\varphi|_{1,\Omega} \\ & + C|S_\varphi|_{2,\Omega}|\nabla(S_{\tilde\varphi} - \varphi)|_{L_4(\Omega)}|S_{\tilde\varphi} - S_\varphi|_{1,\Omega}^{\frac12}|S_{\tilde\varphi} - S_\varphi|_{2,\Omega}^{\frac12} \\ & + C|S_\varphi - S_{\tilde\varphi}|_{2,\Omega}^{\frac32}|\nabla\varphi|_{L_4(\Omega)}|S_{\tilde\varphi} - S_\varphi|_{1,\Omega}^{\frac12} \\ & + C|S_{\tilde\varphi} - \varphi|_{2,\Omega}|\nabla\varphi|_{L_4(\Omega)}|S_{\tilde\varphi} - S_\varphi|_{1,\Omega}^{\frac12}|S_{\tilde\varphi} - S_\varphi|_{2,\Omega}^{\frac12}, \end{aligned}$$

where C denotes a positive constant independent of t which may vary line by line. We now give an estimate for each of the four terms on the right hand side of the above inequality.

$$\begin{aligned} & |\dot S_{\tilde\varphi} - \dot\varphi|_{1,\Omega}|S_{\tilde\varphi} - S_\varphi|_{1,\Omega} \le C|\dot S_{\tilde\varphi} - \dot\varphi|_{1,\Omega}^2 + \frac{\nu}{8}|S_{\tilde\varphi} - S_\varphi|_{2,\Omega}^2; \\ & C|S_\varphi|_{2,\Omega}|\nabla(S_{\tilde\varphi} - \varphi)|_{L_4(\Omega)}|S_{\tilde\varphi} - S_\varphi|_{1,\Omega}^{\frac12}|S_{\tilde\varphi} - S_\varphi|_{2,\Omega}^{\frac12} \\ \le \;& \frac34\left(\left(\frac{8}{\nu}\right)^{\frac14} C|S_\varphi|_{2,\Omega}|\nabla(S_{\tilde\varphi} - \varphi)|_{L_4(\Omega)}|S_{\tilde\varphi} - S_\varphi|_{1,\Omega}^{\frac12}\right)^{\frac43} + \frac{\nu}{8}\left(|S_{\tilde\varphi} - S_\varphi|_{2,\Omega}^{\frac12}\right)^4 \\ = \;& C_1\left(|S_\varphi|_{2,\Omega}|\nabla(S_{\tilde\varphi} - \varphi)|_{L_4(\Omega)}\right)^{\frac43}|S_{\tilde\varphi} - S_\varphi|_{1,\Omega}^{\frac23} + \frac{\nu}{8}|S_{\tilde\varphi} - S_\varphi|_{2,\Omega}^2 \\ \le \;& \frac13|S_{\tilde\varphi} - S_\varphi|_{1,\Omega}^2 + C_2(|S_\varphi|_{2,\Omega}|\nabla(S_{\tilde\varphi} - \varphi)|_{L_4(\Omega)})^2 + \frac{\nu}{8}|S_{\tilde\varphi} - S_\varphi|_{2,\Omega}^2; \\ & C|S_\varphi - S_{\tilde\varphi}|_{2,\Omega}^{\frac32}|\nabla\varphi|_{L_4(\Omega)}|S_{\tilde\varphi} - S_\varphi|_{1,\Omega}^{\frac12} \\ \le \;& C_1|\nabla\varphi|_{L_4(\Omega)}^{\frac43}|S_{\tilde\varphi} - S_\varphi|_{1,\Omega}^2 + \frac{\nu}{8}|S_\varphi - S_{\tilde\varphi}|_{2,\Omega}^2; \end{aligned}$$

$$\begin{aligned}
& C|S_{\tilde{\varphi}}-\varphi|_{2,\Omega}|\nabla\varphi|_{L_4(\Omega)}|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^{\frac{1}{2}}|S_{\tilde{\varphi}}-S_\varphi|_{2,\Omega}^{\frac{1}{2}} \\
\le\ & C_1(|S_{\tilde{\varphi}}-\varphi|_{2,\Omega}|\nabla\varphi|_{L_4(\Omega)}|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^{\frac{1}{2}})^{\frac{4}{3}}+\frac{\nu}{8}|S_{\tilde{\varphi}}-S_\varphi|_{2,\Omega}^2 \\
=\ & C_1\left(|S_{\tilde{\varphi}}-\varphi|_{2,\Omega}|\nabla\varphi|_{L_4(\Omega)}\right)^{\frac{4}{3}}|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^{\frac{2}{3}}+\frac{\nu}{8}|S_{\tilde{\varphi}}-S_\varphi|_{2,\Omega}^2 \\
\le\ & C_2|S_{\tilde{\varphi}}-\varphi|_{2,\Omega}^2|\nabla\varphi|_{L_4(\Omega)}^2+\frac{1}{3}|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^2+\frac{\nu}{8}|S_{\tilde{\varphi}}-S_\varphi|_{2,\Omega}^2
\end{aligned}$$

for some positive constants C, C_1, C_2 which may vary line by line. Putting these estimates into the previous inequality, we have

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^2+\frac{\nu}{2}|S_{\tilde{\varphi}}-S_\varphi|_{2,\Omega}^2 \\
\le\ & (\frac{2}{3}+C|\nabla\varphi|_{L_4(\Omega)}^2)|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^2+C|\dot{S}_{\tilde{\varphi}}-\dot{\varphi}|_{1,\Omega}^2 \\
& +C|S_\varphi|_{2,\Omega}^2|\nabla(S_{\tilde{\varphi}}-\varphi)|_{L_4(\Omega)}^2+C|S_{\tilde{\varphi}}-\varphi|_{2,\Omega}^2|\nabla\varphi|_{L_4(\Omega)}^2.
\end{aligned}$$

By Sobolev's embedding theorem, $|\nabla\varphi|_{L_4(\Omega)}\le C\|\varphi\|_{2,\Omega}$ and by Lemma 3.6, $\|\varphi\|_{2,\Omega}\le C$ independent of t. Another application of Sobolev's embedding theorem and the arguments of Lemmas 2.5 and 2.7 yield

$$|\nabla(S_{\tilde{\varphi}}-\varphi)|_{L_4(\Omega)}\le C\|S_{\tilde{\varphi}}-\varphi\|_{2,\Omega}\le C|\triangle|^{m-2}|\varphi|_{m,\Omega}.$$

The arguments of Lemmas 2.6 and 2.8 yield $|\dot{S}_{\tilde{\varphi}}-\dot{\varphi}|_{1,\Omega}\le C|\triangle|^{m-2}|\dot{\varphi}|_{m,\Omega}$. Thus the previous inequality may be simplified as follows:

$$\begin{aligned}
& \frac{d}{dt}|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^2+\nu|S_{\tilde{\varphi}}-S_\varphi|_{2,\Omega}^2 \\
\le\ & C_1|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^2+C_2(t)|\triangle|^{2m-4}\left(|\varphi|_{m,\Omega}^2+|\dot{\varphi}|_{m,\Omega}^2\right) \qquad (3.12)
\end{aligned}$$

for a positive bounded constant C_1 independent of t and $C_2(t)=C(1+|S_\varphi|_{2,\Omega})$. From (3.11), we have

$$\frac{d}{dt}|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^2\le C_1|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^2+C_2(t)|\triangle|^{2m-4}\left(|\varphi|_{m,\Omega}^2+|\dot{\varphi}|_{m,\Omega}^2\right).$$

Multiplication of both sides of the above inequality by $\exp(-C_1t)$ yields

$$\begin{aligned}
& \frac{d}{dt}\left(\exp\left(\int_0^t -C_1 s ds\right)|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^2\right) \\
\le\ & \exp\left(-C_1t\right)C_2(t)|\triangle|^{2m-4}\left(|\varphi|_{m,\Omega}^2+|\dot{\varphi}|_{m,\Omega}^2\right) \\
\le\ & C_2(t)|\triangle|^{2m-4}\left(|\varphi|_{m,\Omega}^2+|\dot{\varphi}|_{m,\Omega}^2\right).
\end{aligned}$$

It follows that

$$|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^2\le\exp(C_1T)\int_0^t C_2(t)|\triangle|^{2m-4}|\left(|\varphi|_{m,\Omega}^2+|\dot{\varphi}|_{m,\Omega}^2\right)ds.$$

That is, we have

$$|S_{\tilde{\varphi}}-S_\varphi|_{1,\Omega}^2\le\exp(C_1T)|\triangle|^{2m-4}\left[\max_{0\le t\le T}|\varphi|_{m,\Omega}^2+\max_{0\le t\le T}|\dot{\varphi}|_{m,\Omega}^2\right]\int_0^T C_2(t)dt.$$

By Lemma 3.7 we can conclude that

$$|S_{\tilde{\varphi}} - S_\varphi|_{1,\Omega} \leq C|\Phi|^{m-2}\left[\max_{0\leq t\leq T}|\varphi|_{m,\Omega} + \max_{0\leq t\leq T}|\dot{\varphi}|_{m,\Omega}\right] \tag{3.13}$$

for all $t \in [0,T]$, where C is a positive constant dependent on T.

Lemma 3.8. *Suppose that $f \in L_\infty(0,T,L_2(\Omega))$ and that $S_b \in C^1(0,T,H^1(\Omega))$ and $S_b \in L_4(0,T,H^2(\Omega))$. Then*

$$\left(\int_0^T |S_\varphi|^2_{2,\Omega}ds\right)^{\frac{1}{2}} \leq C,$$

where C is a positive constant dependent on T.

Proof. Recall that $S_b \in H^2(\Omega)$ satisfies the boundary conditions in (3.9). It follows from (3.9) that

$$\begin{aligned} a_1(\dot{S}_\varphi - \dot{S}_b,\psi) \quad &+\nu a_2(S_\varphi - S_b,\psi) + b(S_\varphi - S_b, S_\varphi - S_b,\psi) \\ &= \langle \mathbf{f}, \mathbf{curl}\psi\rangle - a_1(\dot{S}_b,\psi) - \nu a_2(S_b,\psi) - b(S_b, S_\varphi - S_b,\psi) \\ &\quad -b(S_\varphi - S_b, S_b,\psi) + b(S_b,S_b,\psi) \end{aligned} \tag{3.14}$$

for any $\psi \in V_0$. Let $\psi = S_\varphi - S_b \in V_0$. Lemma 3.2 now yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|S_\varphi - S_b|^2_{1,\Omega} + \nu|S_\varphi - S_b|^2_{2,\Omega} \\ \leq\ & |\mathbf{f}|_{0,\Omega}|S_\varphi - S_b|_{1,\Omega} + |\dot{S}_b|_{1,\Omega}|S_\varphi - S_b|_{1,\Omega} + \nu|S_b|_{2,\Omega}|S_\varphi - S_b|_{2,\Omega} \\ &+C|S_b|^{\frac{3}{2}}_{2,\Omega}|S_b|^{\frac{1}{2}}_{1,\Omega}|S_\varphi - S_b|_{2,\Omega} + C|S_\varphi - S_b|^{\frac{3}{2}}_{2,\Omega}|S_b|^{\frac{1}{2}}_{1,\Omega}|S_b|^{\frac{1}{2}}_{2,\Omega}|S_\varphi - S_b|^{\frac{1}{2}}_{1,\Omega}. \end{aligned}$$

Furthermore, we use the standard inequalities to have

$$\begin{aligned} &\frac{d}{dt}|S_\varphi - S_b|^2_{1,\Omega} + \nu|S_\varphi - S_b|^2_{2,\Omega} \\ \leq\ & C(|\mathbf{f}|^2_{2,\Omega} + |\dot{S}_b|^2_{1,\Omega} + |S_b|^2_{2,\Omega} + |S_b|^4_{2,\Omega}) + C|S_b|^2_{1,\Omega}|S_b|^2_{2,\Omega}|S_\varphi - S_b|^2_{1,\Omega} \\ =\ & C_1(t) + C_2(t)|S_\varphi - S_b|^2_{1,\Omega}. \end{aligned}$$

That is, we have

$$\frac{d}{dt}|S_\varphi - S_b|^2_{1,\Omega} \leq C_2(t)|S\varphi - S_b|^2_{1,\Omega} + C_1(t)$$

which implies that

$$\frac{d}{dt}(\exp(-\int_0^t C_2(s)ds)|S_\varphi - S_b|^2_{1,\Omega}) \leq C_1(t)\exp(-\int_0^t C_2(s)ds).$$

Next, we integrate both sides of the the above inequality with respect to t and then multiply both sides by the term $\exp(\int_0^t C_2(s)ds)$ to get

$$|S_\varphi - S_b|^2_{1,\Omega} \leq \exp\left(\int_0^T C_2(s)ds\right)\left(|S_\varphi - S_b|_{t=0}|^2_{1,\Omega} + \int_0^t C_1(s)ds\right).$$

Note that

$$|(S_\varphi - S_b)|_{t=0}|_{1,\Omega} = |S_{\mathbf{g}} - S_b|_{t=0}|_{1,\Omega} \le C < +\infty.$$

It follows now that

$$|S_\varphi - S_b|_{1,\Omega}^2 \le C < +\infty \text{ for all } t \in [0,T].$$

Furthermore, we have

$$\begin{aligned} &|S_\varphi - S_b|_{1,\Omega}^2 - |(S_\varphi - S_b)|_{t=0}|_{1,\Omega}^2 + \nu \int_0^t |S_\varphi - S_b|_{2,\Omega}^2 ds \\ \le\; &\int_0^T C_1(s)ds + \int_0^T C_2(s)|S_\varphi - S_b|_{1,\Omega}^2 ds. \end{aligned}$$

That is

$$\nu \int_0^T |S_\varphi - S_b|_{2,\Omega}^2 ds \le \int_0^T C_1(s)ds + C\int_0^T C_2(s)ds + |S_\varphi - S_b|_{t=0}|_{1,\Omega}^2 \le C$$

for all $t \in [0,T]$ with C now dependent on T. ■

Next we note that (3.11) and (3.12) imply that

$$\begin{aligned} |S_{\tilde\varphi} \quad &- S_\varphi|_{1,\Omega}^2 + \nu \int_0^t |S_{\tilde\varphi} - S_\varphi|_{2,\Omega}^2 ds \\ &\le \int_0^t C_1 |S_{\tilde\varphi} - S_\varphi|_{1,\Omega}^2 ds + \int_0^t C_2(s)|\diamondsuit|^{2m-4}\left[|\varphi|_{m,\Omega}^2 + |\dot\varphi|_{m,\Omega}^2\right] ds \\ &\le C|\diamondsuit|^{2m-4}\left[\max_{0\le t\le T}|\varphi|_{m,\Omega}^2 + \max_{0\le t\le T}|\dot\varphi|_{m,\Omega}^2\right]. \end{aligned}$$

Thus, we obtain

$$\sqrt{\int_0^T |S_{\tilde\varphi} - S_\varphi|_{2,\Omega}^2 dt} \le C|\diamondsuit|^{m-2}\left[\max_{0\le t\le T}|\varphi|_{m,\Omega} + \max_{0\le t\le T}|\dot\varphi|_{m,\Omega}\right]$$

for a constant $C > 0$ dependent on T. We therefore conclude from $\varphi - S_\varphi = \varphi - \tilde\varphi + \tilde\varphi - S_{\tilde\varphi} + S_{\tilde\varphi} - S_\varphi$ the following

Theorem 3.9. *Suppose that $\varphi \in C^1(0,T,H^m(\Omega))$ and that $\varphi_{\mathbf{g}} \in H^m(\Omega)$ with $2 \le m \le 3r+1$. Then*

$$\sqrt{\int_0^T |\varphi - S_\varphi|_{2,\Omega}^2 dt} \le C|\diamondsuit|^{m-2}\left[\max_{0\le t\le T}|\varphi|_{m,\Omega} + \max_{0\le t\le T}|\dot\varphi|_{m,\Omega} + |\varphi_g|_{m,\Omega}^2\right]$$

and

$$|S_{\tilde\varphi} - S_\varphi|_{1,\Omega} \le C|\diamondsuit|^{m-2}\left[\max_{0\le t\le T}|\varphi|_{m,\Omega} + \max_{0\le t\le T}|\dot\varphi|_{m,\Omega}\right]$$

where C is a constant dependent on T.

Next we need to give an estimate on $\dot\varphi - \dot S_\varphi$. Let $\psi = \dot S_{\tilde\varphi} - \dot S_\varphi \in V_0$ in (3.11). We have

$$|\dot S_{\tilde\varphi} \quad - \dot S_\varphi|_{1,\Omega}^2 + \nu a_2(S_{\tilde\varphi} - S_\varphi, \dot S_{\tilde\varphi} - \dot S_\varphi) + b(S_\varphi, S_{\tilde\varphi} - S_\varphi, \dot S_{\tilde\varphi} - \dot S_\varphi)$$

$$= a_1(\dot S_{\tilde\varphi} - \dot\varphi, \dot S_{\tilde\varphi} - \dot S_\varphi) + b(S_\varphi, S_{\tilde\varphi} - \varphi, \dot S_{\tilde\varphi} - \dot S_\varphi)$$
$$+ b(S_\varphi - S_{\tilde\varphi}, \varphi, \dot S_{\tilde\varphi} - \dot S_\varphi) + b(S_{\tilde\varphi} - \varphi, \varphi, \dot S_{\tilde\varphi} - \dot S_\varphi).$$

Note that by Cauchy-Schwarz's inequality and the inequality in Lemma 3.2 we have

$$\begin{aligned}
\nu a_2(S_{\tilde\varphi} - S_\varphi, \dot S_{\tilde\varphi} - \dot S_\varphi) &\le \nu |S_{\tilde\varphi} - S_\varphi|_{2,\Omega} |\dot S_{\tilde\varphi} - \dot S_\varphi|_{2,\Omega},\\
b(S_\varphi, S_{\tilde\varphi} - S_\varphi, \dot S_{\tilde\varphi} - \dot S_\varphi) &\le C|S_\varphi|_{2,\Omega}|S_{\tilde\varphi} - S_\varphi|_{2,\Omega}|\dot S_{\tilde\varphi} - \dot S_\varphi|_{2,\Omega},\\
b(S_\varphi, S_{\tilde\varphi} - \varphi, \dot S_{\tilde\varphi} - \dot S_\varphi) &\le C|S_\varphi|_{2,\Omega}|S_{\tilde\varphi} - \varphi|_{2,\Omega}|\dot S_{\tilde\varphi} - \dot S_\varphi|_{2,\Omega},\\
b(S_\varphi - S_{\tilde\varphi}, \varphi, \dot S_{\tilde\varphi} - \dot S_\varphi) &\le C|S_\varphi - S_{\tilde\varphi}|_{2,\Omega}|_{2,\Omega}|\varphi|_{2,\Omega}|\dot S_{\tilde\varphi} - \dot S_\varphi|_{2,\Omega},\\
b(S_\varphi - \varphi, \varphi, \dot S_{\tilde\varphi} - \dot S_\varphi) &\le C|S_{\tilde\varphi} - \varphi|_{2,\Omega}|\varphi|_{2,\Omega}|\dot S_\varphi - \dot S_\varphi|_{2,\Omega},\\
\text{and}\qquad a_1(\dot S_{\tilde\varphi} - \dot\varphi, \dot S_{\tilde\varphi} - \dot S_\varphi) &\le |\dot S_{\tilde\varphi} - \dot\varphi|_{1,\Omega}|\dot S_{\tilde\varphi} - \dot S_\varphi|_{1,\Omega}.
\end{aligned}$$

Note also that $|\dot S_{\tilde\varphi} - \dot S_\varphi|_{2,\Omega} \le \dfrac{C}{|\diamondsuit|}|\dot S_{\tilde\varphi} - \dot S_\varphi|_{1,\Omega}$, where we have assumed that $\diamondsuit$ is quasi-uniform. Then we have

$$\begin{aligned}
|\dot S_{\tilde\varphi} - \dot S_\varphi|_{1,\Omega} \;\le\; & |\dot S_{\tilde\varphi} - \dot\varphi|_{1,\Omega} + \frac{C}{|\diamondsuit|}|S_{\tilde\varphi} - S_\varphi|_{2,\Omega}(\nu + |S_\varphi|_{2,\Omega} + |\varphi|_{2,\Omega})\\
&+ \frac{C}{|\diamondsuit|}|S_{\tilde\varphi} - \varphi|_{2,\Omega}(|S_\varphi|_{2,\Omega} + |\varphi|_{2,\Omega}).
\end{aligned}$$

By the same arguments of Lemmas 2.5–2.8 we have

$$|\dot S_{\tilde\varphi} - \dot\varphi|_{1,\Omega} \le C|\diamondsuit|^{m-2}(|\dot\varphi|_{m,\Omega} + |\varphi|_{m,\Omega} + |\varphi_{\mathbf g}|_{m,\Omega})$$
$$|S_{\tilde\varphi} - \varphi|_{2,\Omega} \le C|\diamondsuit|^{m-2}(|\dot\varphi|_{m,\Omega} + |\varphi|_{m,\Omega} + |\varphi_{\mathbf g}|_{m,\Omega}).$$

Recall that in the proof of Theorem 3.3 we showed that

$$\left(\int_0^T |S_{\tilde\varphi} - S_\varphi|^2_{2,\Omega} dt\right)^{\frac12} \le C|\diamondsuit|^{m-2}\left(\max_{0\le t\le T}|\dot\varphi|_{m,\Omega} + \max_{0\le t\le T}|\varphi|_{m,\Omega}\right).$$

By Lemmas 3.6 and 3.7 we have

$$\begin{aligned}
&\int_0^T |\dot S_{\tilde\varphi} - \dot S_\varphi|_{1,\Omega} dt \le CT|\diamondsuit|^{m-2}(|\dot\varphi|_{m,\Omega} + |\varphi|_{m,\Omega} + |\varphi_{\mathbf g}|_{m,\Omega})\\
&+ C\sqrt{T}|\diamondsuit|^{m-3}(|\dot\varphi|_{m,\Omega} + |\varphi|_{m,\Omega} + |\varphi_{\mathbf g}|_{m,\Omega})\left(\left(\int_0^T |S_\varphi|^2_{2,\Omega} dt\right)^{\frac12} + \left(\int_0^T |\varphi|^2_{2,\Omega} dt\right)^{\frac12}\right)\\
&+ C|\diamondsuit|^{m-3}(|\dot\varphi|_{m,\Omega} + |\varphi|_{m,\Omega} + |\varphi_{\mathbf g}|_{m,\Omega})\left(\int_0^T (\nu + |S_\varphi|^2_{2,\Omega} + |\varphi^2_{2,\Omega}) dt)^{\frac12}\right).
\end{aligned}$$

It follows that

$$\int_0^T |\dot S_{\tilde\varphi} - \dot S_\varphi|_{1,\Omega} dt \le C|\diamondsuit|^{m-3}(|\dot\varphi|_{m,\Omega} + |\varphi|_{m,\Omega} + \varphi_{\mathbf g}|_{m,\Omega}).$$

Hence,we have

Theorem 3.10. *Under the assumption of Theorem 2.3. Then we have*

$$\int_0^T |\dot\varphi - \dot S_\varphi|_{1,\Omega} dt \le C|\diamondsuit|^{m-3}(|\dot\varphi|_{m,\Omega} + |\varphi|_{m,\Omega} + |\varphi_{\mathbf{g}}|_{m,\Omega})$$

where C is dependent of T.

Finally, we consider the computation of S_φ by discretizing the time variable $t \in (0,T]$. Writing

$$S_\varphi(x,y,t) = \sum_{j=1}^{N_1} c_j(t)\psi_j(x,y),$$

we let $\mathbf{c}(t) = (c_1(t),\ldots,c_{N_1}(t))^T$. Then we know that $\mathbf{c}$ satisfies

$$K\frac{d}{dt}\mathbf{c}(t) + \nu B\mathbf{c}(t) + \mathbf{c}(t)^T\mathbf{B}\mathbf{c}(t) = F(t)$$

with stiffness matrix K, bending matrix B, 3-dimensional matrix $\mathbf{B}$ and the right-hand size F. To solve this system of ordinary differential equations, let us discretize the time variable into $0 = t_0 < t_1 < \ldots < t_n = T$ and note that

$$\frac{\mathbf{c}(t_{i+1}) - \mathbf{c}(t_i)}{t_{i+1} - t_i} = \frac{1}{2}\left(\frac{d}{dt}\mathbf{c}(t_i) + \frac{d}{dt}\mathbf{c}(t_{i+1})\right) + O(|t_{i+1} - t_i|^2).$$

Thus, recalling that $\Delta t_i = t_{i+1} - t_i$, we have

$$\begin{aligned} K(\mathbf{c}(t_{i+1}) \quad & -\mathbf{c}(t_i)) + \frac{\Delta t_i}{2}\nu B(\mathbf{c}(t_{i+1}) + \mathbf{c}(t_i)) + \frac{\Delta t_i}{2}\mathbf{c}(t_{i+1})^T\mathbf{B}\mathbf{c}(t_{i+1}) + \frac{\Delta t_i}{2}\mathbf{c}(t_i)^T\mathbf{B}\mathbf{c}(t_i) \\ & = \frac{\Delta t_i}{2}(F(t_{i+1}) + F(t_i)) + O(K(\Delta t_i)^3). \end{aligned}$$

In the other word, we have

$$\begin{aligned} (K + \frac{\Delta t_i}{2}\nu B)\mathbf{c}(t_{i+1}) \quad & + \quad \frac{\Delta t_i}{2}\mathbf{c}(t_{i+1})^T\mathbf{B}\mathbf{c}(t_{i+1}) \\ & = \quad (K - \frac{\Delta t_i}{2}\nu B)\mathbf{c}(t_i) + \frac{\Delta t_i}{2}(F(t_i) + F(t_{i+1})) \\ & \quad + \frac{\Delta t_i}{2}\mathbf{c}(t_i)^T\mathbf{B}\mathbf{c}(t_i) + KO((\Delta t_i)^3). \end{aligned} \tag{3.15}$$

This leads to a nonlinear Crank-Nicholson method for Navier-Stokes' equation:

$$\begin{aligned} (K + \frac{\Delta t_i}{2}\nu B)\mathbf{c}_{i+1} \quad & + \quad \frac{\Delta t_i}{2}\mathbf{c}_{i+1}^T\mathbf{B}\mathbf{c}_{i+1} \\ = (K - \frac{\Delta t_i}{2}\nu B)\mathbf{c}_i \quad & + \quad \frac{\Delta t_i}{2}(F(t_i) + F(t_{i+1})) + \frac{\Delta t_i}{2}\mathbf{c}_i^T\mathbf{B}\mathbf{c}_i \end{aligned} \tag{3.16}$$

for $i = 0,1,2,\ldots,n-1$ with $\mathbf{c}_0 = \mathbf{c}(\mathbf{g})$ which is the coefficient vector of $S_{\mathbf{g}}$. For each i, this is a nonlinear system which may be solved by using Newton's method. Writing

$$G(\mathbf{c}) = (K + \frac{\Delta t_i}{2}\nu B)\mathbf{c} + \Delta t_i\mathbf{c}^T\mathbf{B}\mathbf{c} - \mathbf{h}_i$$

with $\mathbf{h}_i$ being the right-hand side of (3.13), we need to solve $G(\mathbf{c}) = 0$.

Newton's method: Starting with an initial guess $\mathbf{c}^{(0)}$, e.g., the solution of $(K + \frac{\Delta t_i}{2}\nu B)\mathbf{c} = \mathbf{h}_i$, we have

$$\mathbf{c}^{(k+1)} = \mathbf{c}^{(k)} - G'(\mathbf{c}^{(k)})^{-1}G(\mathbf{c}^{(k)})$$

for $k = 1, 2, \cdots$ with $G'(\mathbf{c}) = K + \frac{\Delta t_i}{2}\nu B + \Delta t_i\left(\mathbf{c}^T\mathbf{B} + \mathbf{B}^T\mathbf{c}\right)$. If the initial guess $\mathbf{c}^{(0)}$ is sufficiently closed to the solution then Newton's method will converge.

In the following, let us assume that the Newton method is convergent for each $i = 1, \cdots, n-1$. Letting $S_{\varphi,i}(x,y) = \sum_{j=1}^{N_1} c_{ij}\psi_j(x,y)$ with $\mathbf{c}_i = (c_{i,1}, \cdots, c_{i,N_1})^T$ being the solution of (3.14) for $i = 0, \cdots, n-1$, we would like to estimate the error

$$\|S_\varphi(\cdot, t_i) - S_{\varphi,i}(x,y)\|_{0,\Omega}.$$

To this end, we first note that $S_\varphi \in L_\infty(0, T, L_2(\Omega))$ by the proof of Theorem 3.1. Thus we have that $\|\mathbf{c}(t)\|_{\ell^2}$ is bounded independent of t. Let us make the assumption that

$$\|\mathbf{c}_i\|_{\ell^2} \le M, \quad \forall i = 0, \cdots, n-1, \tag{3.17}$$

for any fixed partition of $(0, T]$. Then we are able to prove the following

Lemma 3.11. *Suppose that* $\max\{\Delta t_i, 0 \le i \le n-1\} \le C\min\{\Delta t_i, 0 \le i \le n-1\}$. *Let* $\Delta t = \max\limits_{0\le i\le n-1} \Delta t_i$. *Suppose that* Δt *is small enough. Then we have*

$$\|S_\varphi(\cdot, t_i) - S_{\varphi,i}(x,y)\|_{0,\Omega} \le C|\Delta t|^2.$$

Proof. We first note that

$$\begin{aligned}\|\mathbf{c}^T\mathbf{B}\|_2 &\le C|S_\mathbf{c}|_{2,\Omega}\|\left[|\psi_i|_{2,\Omega}|\psi_j|_{2,\Omega}\right]_{N\times N}\|_2\\ &\le C|S_\mathbf{c}|_{2,\Omega} \le C\|\mathbf{c}\|_{\ell^2},\end{aligned}$$

where $S_\mathbf{c} = \sum_{j=1}^N c_j\psi_j$ with $\mathbf{c}_j = (c_1, \cdots, c_N)^T$.

Let $A_i = K + \frac{\Delta t_i}{2}B + \frac{\Delta t_i}{2}\mathbf{c}_{i+1}^T\mathbf{B}$ be the matrix on the left-hand side of (3.16). By the assumption (3.17), we know that A_i is invertible if Δt_i is small enough. Also, let $\tilde{A}_i = K - \frac{\Delta t_i}{2}B + \Delta t_i\mathbf{c}_i^T\mathbf{B}$ be the matrix on the right-hand side of (3.16). We observe that when Δt is small enough, $\|A_i^{-1}\tilde{A}_i\|_2 \le 1$. Next let

$$E_i = A_i^{-1}KO(\Delta t_i)^3 = \left(I + K^{-1}\frac{\Delta t_i}{2}B + \Delta t_iK^{-1}\mathbf{c}_{i+1}^T\mathbf{B}\right)^{-1}O(|\Delta t_i|^3).$$

Thus when Δt is small enough, $\|E_i\|_2 \le (1+\epsilon)O(|\Delta t_i|^3)$. Letting $\mathbf{e}_i = \mathbf{c}(t_i) - \mathbf{c}_i$, we have, from (3.15) and (3.16)

$$\mathbf{e}_{i+1} = A_i^{-1}\tilde{A}_i\mathbf{e}_i + E_i = \prod_{j=0}^{i} A_j^{-1}\tilde{A}_j\mathbf{e}_0 + \sum_{j=1}^{i} E_j = \sum_{j=1}^{i} E_j.$$

Note $\|E_i\|_2 = O(|\Delta t_i|^3)$. Hence, $\|\mathbf{e}_{i+1}\|_{\ell^2} \le nO((\Delta t)^3) \le O(|\Delta t|^2)$. Furthermore,

$$\|S_\varphi(\cdot, t_i) - S_{\varphi,i}(x,y)\|_{0,\Omega} \le \|\mathbf{e}_i\|_{\ell^2}\sqrt{\sum_{j=1}^{N_1}\int_\Omega |\psi_j|^2 dxdy} \le O(|\Delta t|^2)$$

since all ψ_j are locally supported and uniformly bounded by a constant which is dependent only on the smallest angle of the underlying triangulation (cf. [11]). This completes the proof of Lemma 3.11. ∎

We have implemented in MATLAB the above numerical method for solving the Navier-Stokes' equation using the bivariate C^1 cubic spline space $S_3^1(\diamondsuit)$. In the following example, we consider the K-shape domain again. We start with a quadrangulated triangulation as shown in Fig. 1 and then refine it twice. As before, the time interval $[0,1]$ is first divided into 10 subintervals and then 20 and 40 subintervals. We tested many artificial exact solutions by first computing the right-hand side functions and boundary functions using MATHEMATICA and then feeding the resulting functions into our programs to find numerical solutions. We evaluate the numerical solutions at those 201×201 equally spaced points over $[0,2] \times [0,2]$ which are inside the K-shaped domain. Table IV is a list of the maximum errors of the numerical solutions against the exact solutions.

TABLE IV. Maximum Errors of Some Stream Functions

Matrix Sizes	150×150	527×527	1971×1971
$\sin(1+t(x+y))$	2.1307×10^{-3}	1.3624×10^{-4}	8.4965×10^{-6}
$(x^2+y^2)^{5/2}\sin(t\pi/2)$	2.2305×10^{-1}	1.5598×10^{-2}	2.0127×10^{-3}
$\exp(1+t(x+y))$	2.1053×10^{-1}	1.6672×10^{-2}	1.8070×10^{-3}
$\sin(t\pi/2)\sin(x+y)$	1.6503×10^{-3}	1.2829×10^{-4}	8.6280×10^{-6}
$(x^2+y^2)^{5/2}(1+t^2)$	2.2305×10^{-1}	1.5598×10^{-2}	1.9737×10^{-3}
$(x^4+y^4)^{5/2}(1+\sin(t\pi/2))$	1.1081×10^{-2}	8.6303×10^{-4}	1.0522×10^{-4}

We can see that the convergence is very fast and appears to be fourth order. Let us also show the convergence of the derivatives of the numerical solutions to the corresponding derivatives of the exact solutions. Table V lists the maximum errors of the derivative with respect to x.

TABLE V. The Maximum Errors of the x-Derivative of the Stream Functions

Matrix Sizes	150×150	527×527	1971×1971
$\sin(1+t(x+y))$	9.6419×10^{-3}	1.1937×10^{-3}	1.5063×10^{-4}
$(x^2+y^2)^{5/2}\sin(t\pi/2)$	1.8497	2.6421×10^{-1}	4.4226×10^{-2}
$\exp(1+t(x+y))$	1.2316	1.2959×10^{-1}	2.5115×10^{-2}
$\sin(t\pi/2)\sin(x+y)$	6.0855×10^{-3}	1.0731×10^{-3}	1.7180×10^{-4}
$(x^2+y^2)^{5/2}(1+t^2)$	1.8497	2.6421×10^{-1}	4.4226×10^{-2}

Next we consider the following Navier-Stokes' equation:

$$\begin{cases} \frac{\partial}{\partial t}\Delta\varphi - \nu\Delta^2\varphi - \frac{\partial}{\partial x}\left(\Delta\varphi\frac{\partial}{\partial y}\varphi\right) + \frac{\partial}{\partial y}\left(\Delta\varphi\frac{\partial}{\partial x}\varphi\right) = 1, & \text{in } \Omega \times (0,T) \\ \frac{\partial}{\partial n}\varphi = 0, \varphi = 0, & \text{on } \partial\Omega \times (0,T) \\ \varphi(x,y,0) = 0. & \text{in } \Omega \end{cases}$$

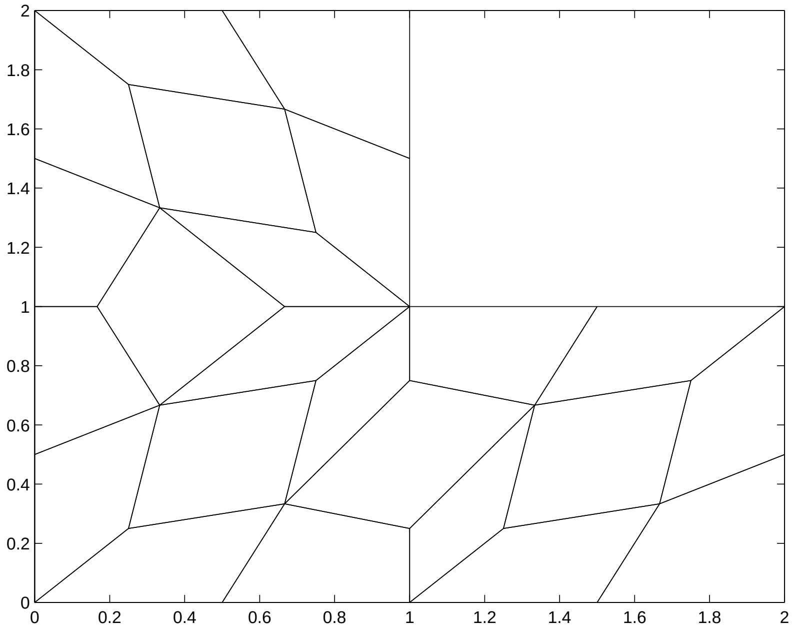

FIG. 5. An Initial Quadrangulation of an L-shape Domain

Here, Ω is a L-shape domain. The solution will be singular at the re-entrance $(1, 1)$. We start with a quadrangulation as in Fig. 5. Then we refine it two times uniformly. We use the numerical solution from second refinement as an approximate exact solution and compute the maximum error against the numerical solutions from the initial quadrangulation and the first refinement. We list them below as well as the associated matrix size. On the other hand, we use a local refinement technique (cf. [13]) to refine the initial quadrangulation at and nearby the singular point $(1, 1)$ as shown on the graph in Fig. 6. We list the maximum errors of the numerical solutions based on the irregular refinement against the approximate exact solution.

TABLE VI. Comparison of the Maximum Errors Base on Regular and Irregular Refinements

Matrix Sizes	150×150	527×527	378×378
		uniform refinement	irregular refinement
Maximum Errors	4.5883×10^{-4}	9.9497×10^{-5}	8.9765×10^{-5}

IV. PRESSURE FUNCTION APPROXIMATION

In this section we consider the numerical solution of the pressure function in the time evolution Stokes and Navier-Stokes equations. We first recall the following basic lemmas.

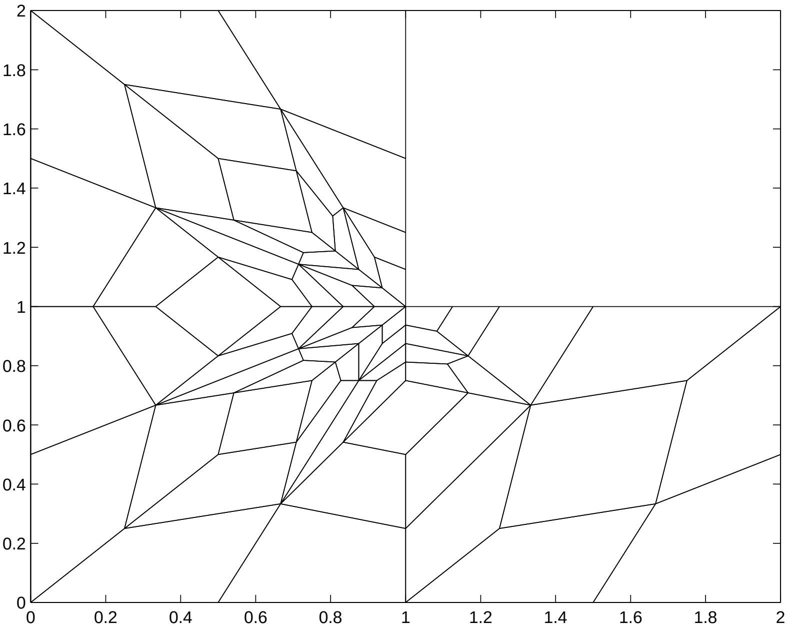

FIG. 6. An Irregular Refinement of the L-shape Domain

Lemma 4.1. *Let $\mathcal{V} = \{\mathbf{u} : \text{div } \mathbf{u} = 0,\ u_1, u_2 \in C_0^\infty(\Omega)\}$. Given a vector function $\mathbf{f} = (f_1, f_2) \in H^{-1}(\Omega)^2$, if $\langle \mathbf{f}, \mathbf{u}\rangle = \langle f_1, u_1\rangle + \langle f_2, u_2\rangle = 0$ for all test functions $\mathbf{u}$ in $\mathcal{V}$, then there exists a distribution $p \in L_2(\Omega)$ such that $\mathbf{f} = \nabla p$.*

See [[6], p. 25] for a proof. See [[17], p.15] for a proof of the following lemma.

Lemma 4.2. *If a distribution p has its first derivatives $\frac{\partial}{\partial x}p$ and $\frac{\partial}{\partial y}p$ in $H^{-1}(\Omega)$, then $p \in L_2(\Omega)$ and*

$$|p|_{0,\Omega} \le C|p|_{-1,\Omega}$$

for a constant C independent of p.

Theorem 4.3. *Suppose that $\mathbf{f} \in L_2(0, T; H^{-1}(\Omega))^2$ and that the solution $\mathbf{u} = \mathbf{curl}(\varphi) \in L_2(0, T; H^1(\Omega))^2$, where $\mathbf{u}$ is the planar velocity vector of the Stokes equations. Then there exists a scalar function $p \in L_2(\Omega)$ such that*

$$\nabla p = \mathbf{f} - \frac{\partial}{\partial t}(\mathbf{curl}\varphi) + \nu\Delta(\mathbf{curl}\varphi). \tag{4.1}$$

Furthermore, if $\mathbf{u} \in C^1[0, T; H^2(\Omega)]^2$ and $\mathbf{f} \in L_2(0, T; L_2(\Omega))^2$ then $p \in L_2(0, T; H^1(\Omega))$.

Proof. We apply Lemma 4.1. For any $\mathbf{v} \in \mathcal{V}$, since $\text{div}(\mathbf{v}) = 0$ there exists a function ψ such that $\mathbf{v} = \mathbf{curl}(\psi)$. Then we have

$$\begin{aligned}&\int_\Omega (\nu\Delta(\mathbf{curl}\varphi) + \mathbf{f} - \mathbf{curl}\dot{\varphi}) \cdot \mathbf{v}\, dxdy \\ = &\ \nu \int_\Omega \mathbf{curl}(\Delta\varphi) \cdot \mathbf{curl}\psi dxdy + \int_\Omega \mathbf{f} \cdot \mathbf{curl}\psi\, dxdy\end{aligned}$$

$$
\begin{aligned}
& - \int_\Omega \mathbf{curl}\dot{\varphi} \cdot \mathbf{curl}\psi dxdy \\
= & -\nu \int_\Omega \Delta\varphi\Delta\psi \; dxdy + \int_\Omega \mathbf{f}\mathbf{curl}\psi dxdy - \int_\Omega \nabla\dot{\varphi}\nabla\psi dxdy \\
= & -a_1(\dot{\varphi}, \psi) - \nu a_2(\varphi, \psi) + \langle \mathbf{f}, \mathbf{curl}(\psi)\rangle \\
= & \; 0
\end{aligned}
$$

by (2.4). Thus Lemma 4.1 implies that there exists a scalar function $p \in L_2(\Omega)$ for each $t \in (0, T]$ which satisfies (4.1).

When $\mathbf{u} \in C^1[0, T; H^2(\Omega)]^2$, and $\mathbf{f} \in L_2(\Omega))^2$ for each $t \in (0, T]$, we immediately know that $\nabla p \in L_2(\Omega)^2$ and hence $p \in H^1(\Omega)$ for each $t \in (0, T]$. This completes the proof. ∎

Similarly we can show the following

Theorem 4.4. *Let $\mathbf{u} = \mathbf{curl}(\varphi)$ be the weak solution of the Navier-Stokes equations (3.1). Suppose that $\mathbf{f} \in H^{-1}(\Omega)^2$. Then there exists a scalar function $p \in L_2(\Omega)$ for each time $t \in (0, T)$ satisfying*

$$
\nabla p = \mathbf{f} - \frac{\partial}{\partial t}(\mathbf{curl}\varphi) + \nu\Delta(\mathbf{curl}\varphi) + (\mathbf{curl}\varphi \cdot \nabla)\mathbf{curl}\varphi. \tag{4.2}
$$

Furthermore, if $\mathbf{f} \in L_2(\Omega)^2$ and if $\varphi \in C^2(\Omega) \cap H^3(\Omega)$ for each $t \in (0, T]$, then $p \in H^1(\Omega)$ for each $t \in (0, T]$.

Under the conditions in Theorems 4.3 and 4.4, and if the solution and the boundary are smooth enough, the pressure functions p of the Stokes and Navier-Stokes equations may be obtained by solving the Poisson equations (4.3) and (4.4) with nonhomogeneous Neumann boundary condition.

$$
\begin{cases}
\Delta p = \operatorname{div}(\mathbf{f}), & \text{in } \Omega \\
\frac{\partial}{\partial n} p = n \cdot (\mathbf{f} - \mathbf{curl}(\dot{\varphi}) + \nu\Delta\mathbf{curl}(\varphi)), & \text{on } \partial\Omega
\end{cases} \tag{4.3}
$$

and

$$
\begin{cases}
\Delta p = \operatorname{div}(\mathbf{f} - (\mathbf{curl}\varphi \cdot \nabla)\mathbf{curl}\varphi), & \text{in } \Omega \\
\frac{\partial}{\partial n} p = n \cdot (\mathbf{f} - \mathbf{curl}(\dot{\varphi}) + \nu\Delta\mathbf{curl}(\varphi) & \\
\qquad + (\mathbf{curl}\varphi \cdot \nabla)\mathbf{curl}\varphi), & \text{on } \partial\Omega.
\end{cases} \tag{4.4}
$$

Let us consider the weak formulation for problems (4.3) and (4.4):

$$
a_1(p, q) = \langle \mathbf{f}, \nabla q\rangle + \nu\langle \Delta\mathbf{curl}\varphi, \nabla q\rangle - \langle \mathbf{curl}(\dot{\varphi}), \nabla q\rangle, \tag{4.5}
$$

for the Stokes equation, and

$$
\begin{aligned}
a_1(p, q) \;=\; & \langle \mathbf{f}, \nabla q\rangle - \langle (\mathbf{curl}\varphi \cdot \nabla)\mathbf{curl}\varphi, \nabla q\rangle \\
& + \nu\langle \Delta\mathbf{curl}\varphi, \nabla q\rangle - \langle \mathbf{curl}(\dot{\varphi}), \nabla q\rangle, \quad q \in \tilde{H}^1(\Omega)
\end{aligned} \tag{4.6}
$$

for the Navier-Stokes equations, where $\tilde{H}^1(\Omega) = \{g \in H^1(\Omega) : \int_\Omega g dxdy = 0\}$.

We next recall the numerical solution of the standard Poisson equation with nonhomogeneous Neumann boundary condition:

$$
\begin{cases}
-\Delta p = h & \text{in } \Omega \\
\frac{\partial p}{\partial n} = g & \text{on } \partial\Omega \\
\int_\Omega p \; dxdy = 0 &
\end{cases}
$$

whose weak formulation is

$$a_1(p,q) = \langle h, q\rangle + \langle g, q\rangle_{\partial\Omega}, \quad q \in \tilde{H}^1(\Omega) \tag{4.7}$$

where $h \in H^{-1}(\Omega)$ and $g \in L_2(\partial\Omega)$. It is well known that the Lax-Milgram Lemma implies the existence of an unique $p \in \tilde{H}^1(\Omega)$ satisfying (4.7).

Let

$$\tilde{S}^r(\Diamond) = \begin{cases} \{s \in S_1^0(\Diamond) : \int_\Omega s \; dxdy = 0\}, & \text{if } r = 0 \\ \{s \in S_{3r}^r(\Diamond) : \int_\Omega s \; dxdy = 0\}, & \text{if } r \geq 1. \end{cases}$$

Note that $\partial\Omega \cap \Diamond$ consists of a finite number of line segments $[v_i, v_{i+1}], i = 1, \cdots, N$. If a function g is in $L_2([v_i, v_{i+1}])$ for all $i = 1, \cdots, N$, we will denote this by $g \in L_2(\partial\Omega \cap \Diamond)$. Similarly, if $f|_\triangle \in L_2(\triangle)$ for each triangle $\triangle \in \Diamond$, we will say that $f \in L_2(\Omega \cap \Diamond)$ and define a mesh dependent norm by

$$\|f\|_{0,\Omega}^2 = \sum_{\triangle \in \Diamond} \|f\|_{0,\triangle}^2.$$

We have the following

Lemma 4.5. *For any given $h \in L_2(\Omega \cap \Diamond)$ and $g \in L_2(\partial\Omega \cap \Diamond)$, there exists a unique $S_p \in \tilde{S}^r(\Diamond)$ satisfying*

$$a_1(S_p, q) = \langle h, q\rangle + \langle g, q\rangle, \quad \forall\; q \in \tilde{S}^r(\Diamond). \tag{4.8}$$

Proof. By the second Poincaré's inequality, $a_1(q,q) \geq C\|q\|_{1,\Omega}$ and hence a_1 is coercive on $\tilde{S}^r(\Diamond)$. By the second Poincaré's inequality again and the trace theorem,

$$\langle h, q\rangle + \langle g, q\rangle_{\partial\Omega} \leq \|h\|_{0,\Omega} C \|q\|_{1,\Omega} + \|g\|_{L_2(\partial\Omega)} C \|q\|_{1,\Omega}$$

for $h \in L_2(\Omega \cap \Diamond)$ and $g \in L_2(\partial\Omega \cap \Diamond)$. By the well-known Lax-Milgram Lemma there exists an unique weak solution of (4.8). Thus we have completed the proof. ∎

Our numerical method for the pressure functions p of the Stokes and Navier-Stokes equations is to solve the Poisson equations (4.5) and (4.6) with φ replaced by $S_\varphi \in S_{3r}^r(\Diamond)$ which is the numerical solution of the linear and nonlinear time evolution biharmonic equations in §2 and §3. Since ΔS_φ is in $L_2(\Omega \cap \Diamond)$ for each $t \in [0,T]$, by Lemma 4.5, we know that the numerical solutions $S_p \in \tilde{S}^{r-1}(\Diamond)$ of (4.5) and (4.6) exist and are unique with S_φ in the place of φ.

We finally consider how well S_p approximates p. Let us first consider the pressure function of the Stokes equations. Let $S_\varphi \in S_{3r}^r(\Diamond)$ be the numerical approximation of the stream function φ as in §2 and $S_p \in \tilde{S}^{r-1}(\Diamond)$ satisfy the following

$$a_1(S_p, q) = \langle \mathbf{f}, \nabla q\rangle + \nu\langle \Delta \mathbf{curl} S_\varphi, \nabla q\rangle - \langle \mathbf{curl}(\dot{S_\varphi}), \nabla q\rangle,$$

for $q \in \tilde{S}^{r-1}(\Diamond)$. Let $\tilde{p} \in \tilde{H}^1(\Omega)$ be the weak solution of the following Poisson equation:

$$a_1(\tilde{p}, q) = \langle \mathbf{f}, \nabla q\rangle + \nu\langle \Delta \mathbf{curl} S_\varphi, \nabla q\rangle - \langle \mathbf{curl}(\dot{S_\varphi}), \nabla q\rangle,$$

for all $q \in \tilde{H}^1(\Omega)$. As before, such $\tilde{p}$ exists and is unique. In general the solution φ is in $L_\infty(0,T; H^1(\Omega)) \cap L_2(0,T; H^2(\Omega))$. When the domain Ω is convex, the solution φ will be in $L_\infty(0,T;H^2(\Omega))$(cf. Lemma 3.6). In the case where φ is sufficiently smooth, we have the following result:

Lemma 4.6. *Suppose that $\varphi \in L_\infty(0,T;H^m(\Omega))$ with $m \geq 3$. Then*

$$|p-\tilde{p}|_{1,\Omega} \leq C|\diamondsuit|^{m-3}$$

for a constant C independent of p.

Proof. We have for any $q \in \tilde{H}^1(\Omega)$,

$$a_1(p-\tilde{p},q) = \nu \int_\Omega \mathbf{curl}\Delta(\varphi - S_\varphi)\cdot\nabla q\, dxdy - \langle \mathbf{curl}(\dot{\varphi} - \dot{S_\varphi}), \nabla q\rangle.$$

Thus, it follows that

$$\begin{aligned} &|p-\tilde{p}|^2_{1,\Omega} \\ =\ & \nu\int_\Omega \mathbf{curl}\Delta(\varphi - S_\varphi)\cdot\nabla(p-\tilde{p})dxdy + \int_\Omega \mathbf{curl}(\dot{\varphi}-\dot{S_\varphi})\cdot\nabla(p-\tilde{p})dxdy \\ \leq\ & \nu\left(\left\|\frac{\partial}{\partial x}\Delta(\varphi - S_\varphi)\right\|_{L_2(\Omega)} + \left\|\frac{\partial}{\partial y}\Delta(\varphi - S_\varphi)\right\|_{L_2(\Omega)}\right)|p-\tilde{p}|_{1,\Omega} \\ & +|\dot{\varphi}-\dot{S_\varphi}|_{1,\Omega}|p-\tilde{p}|_{1,\Omega}. \end{aligned}$$

By Theorem 2.10, we have

$$|\dot{\varphi}-\dot{S_\varphi}|_{1,\Omega} \leq C|\diamondsuit|^{m-3}|\varphi|_{m,\Omega}.$$

Recall from Theorem 2.9 that $|\varphi - S_\varphi|_{2,\Omega} \leq C|\diamondsuit|^{m-2}$. Since $\varphi \in H^m(\Omega)$ with $m \geq 3$, there exists a best approximation $S_{\varphi,a} \in S^r_{3r}(\diamondsuit)$ such that $|\varphi - S_{\varphi,a}|_{\ell,\Omega} \leq C|\diamondsuit|^{m-\ell}$. Thus, assuming that $\diamondsuit$ is a quasi-uniform triangulation, we have

$$\begin{aligned} \left\|\frac{\partial}{\partial x}\Delta(\varphi - S_\varphi)\right\|_{L_2(\Omega)} &\leq \left\|\frac{\partial}{\partial x}\Delta(\varphi - S_{\varphi,a})\right\|_{L_2(\Omega)} + \left\|\frac{\partial}{\partial x}\Delta(S_{\varphi,a} - S_\varphi)\right\|_{L_2(\Omega)} \\ &\leq C_1|\diamondsuit|^{m-3} + \frac{C_2}{|\diamondsuit|}\|\Delta(S_{\varphi,a}-S_\varphi)\|_{L_2(\Omega)} \\ &\leq C1_7|\diamondsuit|^{m-3} + \frac{C_2}{|\diamondsuit|}\|\Delta(S_{\varphi,a}-\varphi)\|_{L_2(\Omega)} + \frac{C_2}{|\diamondsuit|}\|\Delta(\varphi - S_\varphi)\|_{L_2(\Omega)} \\ &\leq C_1|\diamondsuit|^{m-3} + \frac{C_2}{|\diamondsuit|}C|\diamondsuit|^{m-2} + \frac{C_2}{|\diamondsuit|}C|\diamondsuit|^{m-2} \\ &= C_3|\diamondsuit|^{m-3}. \end{aligned}$$

Similarly, we also have

$$\left\|\frac{\partial}{\partial y}\Delta(\varphi - S_\varphi)\right\| \leq C_3|\diamondsuit|^{m-3}.$$

Therefore, $|p-\tilde{p}|^2_{1,\Omega} \leq 2\nu C|\diamondsuit|^{m-3}|p-\tilde{p}|_{1,\Omega}$ which completes the proof. ∎

Next we note that

$$a_1(\tilde{p} - S_p, q) = 0, \qquad \forall q \in \tilde{S}^{r-1}(\diamondsuit).$$

It follows that

$$\begin{aligned} |\tilde{p}-S_p|^2_{1,\Omega} &= \int_\Omega \nabla(\tilde{p}-S_p)\nabla(\tilde{p}-p)dxdy + \int_\Omega \nabla(\tilde{p}-S_p)\nabla(p-S_p-q)dxdy \\ &\le |\tilde{p}-S_p|_{1,\Omega}|p-\tilde{p}|_{1,\Omega} + |\tilde{p}-S_p|_{1,\Omega}|p-S_p-q|_{1,\Omega} \end{aligned}$$

for any $q \in \tilde{S}^{r-1}(\diamondsuit)$. Hence, if $p \in H^{m-2}(\Omega)$, we have

$$\begin{aligned} |\tilde{p}-S_p|_{1,\Omega} &\le |p-\tilde{p}|_{1,\Omega} + \inf_{q\in\tilde{S}^{r-1}(\diamondsuit)} |p-q|_{1,\Omega} \\ &\le C|\diamondsuit|^{m-3} + C|\diamondsuit|^{m-3}. \end{aligned}$$

By the second Poincaré's inequality, we also have $|p-S_p|_{0,\Omega} \le C|\diamondsuit|^{m-3}$. Therefore, we may conclude the following.

Theorem 4.7. *Suppose that $\diamondsuit$ is a quasi-uniform triangulation. Suppose $\varphi \in H^m(\Omega)$ with $m \ge 3$ and $p \in H^{m-2}(\Omega)$. Then we have*

$$|p-S_p|_{1,\Omega} \le C|\diamondsuit|^{m-3} \text{ and } |p-S_p|_{0,\Omega} \le C|\diamondsuit|^{m-3}.$$

Next we consider the numerical approximation of the pressure term in the time evolution Navier-Stokes equations. Let S_p be the weak solution in $\tilde{S}^{r-1}(\diamondsuit)$ of the following Poisson equation

$$\begin{aligned} a_1(S_p,q) &= \langle \mathbf{f},\nabla q\rangle + \langle(\mathbf{curl}(S_\varphi)\cdot\nabla)\mathbf{curl}(S_\varphi),\nabla q\rangle \\ &\quad +\nu\langle\Delta\mathbf{curl}(S_\varphi),\nabla q\rangle - \langle\mathbf{curl}\dot{S_\varphi},\nabla q\rangle \end{aligned}$$

for all $q \in \tilde{S}^{r-1}(\diamondsuit)$, where S_φ is the weak solution of Navier-Stokes equations in §3. Let $\tilde{p} \in \tilde{H}^1(\Omega)$ be the weak solution of the following Poisson problem with nonhomogeneous Neumann boundary condition:

$$\begin{aligned} a_1(\tilde{p},q) &= \langle \mathbf{f},\nabla q\rangle + \langle(\mathbf{curl}(S_\varphi)\cdot\nabla)\mathbf{curl}(S_\varphi),\nabla q\rangle \\ &\quad +\nu\langle\Delta\mathbf{curl}(S_\varphi),\nabla q\rangle - \langle\mathbf{curl}\dot{S_\varphi},\nabla q\rangle \end{aligned}$$

for all $q \in \tilde{S}^{r-1}(\diamondsuit)$. We first prove the following

Lemma 4.8. *Suppose that $\varphi \in L_2(0,T;H^m(\Omega))$ with $m \ge 3$ satisfies the assumptions in Theorem 3.3. Suppose that $p \in L_1(0,T;H^{m-2}(\Omega))$. Then*

$$\int_0^T |p-\tilde{p}|_{1,\Omega}dt \le C|\diamondsuit|^{m-3}$$

for some constant C independent of $\diamondsuit$.

Proof. Based on the weak formulations above, we have

$$\begin{aligned} a_1(p-\tilde{p},q) &= \int_\Omega \nabla(p-\tilde{p})\nabla q\, dxdy \\ &= \int_\Omega \nu\Delta\mathbf{curl}(\varphi-S_\varphi)\nabla q dxdy \end{aligned}$$

$$+\int_\Omega (\mathbf{curl}(\varphi - S_\varphi)\cdot\nabla)\mathbf{curl}\varphi\cdot\nabla q\ dxdy$$
$$+\int_\Omega (\mathbf{curl}\ S_\varphi\cdot\nabla)\mathbf{curl}(\varphi - S_\varphi)\cdot\nabla q\ dxdy$$
$$-\int_\Omega \mathbf{curl}(\dot\varphi - \dot S_\varphi)\cdot\nabla q dxdy$$

for any $q \in \tilde H^1(\Omega)$. Letting $q = p - \tilde p$ we have

$$\begin{aligned}|p-\tilde p|_{1,\Omega}^2 &\le \nu\|\varphi - S_\varphi\|_{3,\Omega}|p-\tilde p|_{1,\Omega}\\ &+|\varphi - S_\varphi|_{1,\Omega}\max_{(x,y)\in\Omega}\left(\left|\frac{\partial^2\varphi(x,y)}{\partial x^2}\right| + \left|\frac{\partial^2\varphi(x,y)}{\partial x\partial y}\right| + \left|\frac{\partial^2\varphi(x,y)}{\partial y^2}\right|\right)|p-\tilde p|_{1,\Omega}\\ &+\max_{(x,y)\in\Omega}\left(\left|\frac{\partial S_\varphi(x,y)}{\partial x}\right| + \left|\frac{\partial S_\varphi(x,y)}{\partial y}\right|\right)|\varphi - S_\varphi|_{2,\Omega}|p-\tilde p|_{1,\Omega}\\ &+|\dot\varphi - \dot S_\varphi|_{1,\Omega}|p-\tilde p|_{1,\Omega}.\end{aligned} \tag{4.9}$$

By Sobolev's embedding theorem,

$$\max_{(x,y)\in\Omega}\left(\left|\frac{\partial^2\varphi(x,y)}{\partial x^2}\right| + \left|\frac{\partial^2\varphi(x,y)}{\partial x\partial y}\right| + \left|\frac{\partial^2\varphi(x,y)\varphi}{\partial y^2}\right|\right) \le C|\varphi|_{4,\Omega}$$

and

$$\max_{(x,y)\in\Omega}\left(\left|\frac{\partial S_\varphi(x,y)}{\partial x}\right| + \left|\frac{\partial S_\varphi(x,y)}{\partial y}\right|\right) \ \le\ |S_\varphi|_{3,\Omega} \le |S_\varphi - \varphi|_{3,\Omega} + |\varphi|_{3,\Omega}.$$

By cancelling $|p-\tilde p|_{1,\Omega}$ on both sides of (4.9) and integrating with respect to t, we get

$$\begin{aligned}&\int_0^T |p-\tilde p|_{1,\Omega}dt\\ \le\ & \nu T^{1/2}\left(\int_0^T|\varphi - S_\varphi|_{3,\Omega}^2 dt\right)^{1/2} + C\left(\int_0^T|\varphi - S_\varphi|_{1,\Omega}^2 dt\right)^{1/2}\left(\int_0^T\|\varphi\|_{4,\Omega}^2 dt\right)^{1/2}\\ &+C\left(\left(\int_0^T\|S_\varphi - \varphi\|_{3,\Omega}^2 dt\right)^{1/2} + \left(\int_0^T|\varphi|_{3,\Omega}^2 dt\right)^{1/2}\right)\left(\int_0^T|\varphi - S_\varphi|_{2,\Omega}^2 dt\right)^{1/2} +\\ &+\int_0^T|\dot\varphi - \dot S_\varphi||_{1,\Omega}dt.\end{aligned}$$

Using the same argument as in the proof of Lemma 4.6, we can show the following inequality:

$$\left(\int_0^T|\varphi - S_\varphi|_{3,\Omega}^2 dt\right)^{1/2} \le C|\triangle|^{m-3}.$$

Therefore we conclude that

$$\int_0^T|p-\tilde p|_{1,\Omega}dt \le \nu C|\triangle|^{m-3}$$

for a positive constant C dependent on φ. ∎

Finally, we summarize the discussion above with the following:

Theorem 4.9. *Let φ be the weak solution of the nonlinear time evolution biharmonic equations in §3. Suppose that $\mathbf{f} \in L_\infty(0,T;L_2(0,T))$ and that $\varphi \in C^1(0,T;H^m(\Omega))$ with $m \geq 4$. Let p and S_p be the pressure function and the approximation in $\tilde{S}^{r-1}(\diamondsuit)$, where $\diamondsuit$ is a quasi-uniform triangulation. Suppose that $p \in L_1(0,T;H^{m-2}(\Omega))$. Then*

$$\int_0^T |p - S_p|_{1,\Omega} dt \leq K|\diamondsuit|^{m-3} \text{ and } \int_0^T |p - S_p|_{0,\Omega} dt \leq K|\diamondsuit|^{m-3}.$$

Proof. We first note that

$$\int_\Omega \nabla(\tilde{p} - S_p)\nabla q \; dxdy = 0, \quad \forall \; q \in \tilde{S}^{r-1}(\diamondsuit)$$

and hence

$$\begin{aligned} |\tilde{p} - S_p|^2_{1,\Omega} &= \int_\Omega \nabla(\tilde{p} - S_p)\nabla(\tilde{p} - p)dxdy + \int_\Omega \nabla(\tilde{p} - S_p)\nabla(p - S_p - q)dxdy \\ &\leq |\tilde{p} - S_p|_{1,\Omega}|p - \tilde{p}|_{1,\Omega} + |\tilde{p} - S_p|_{1,\Omega}|p - S_p - q|_{1,\Omega} \end{aligned}$$

for any $q \in \tilde{S}^{r-1}(\diamondsuit)$. Since $p \in H^{m-2}(\Omega)$ we have that $\inf_{q \in \tilde{S}^{r-1}(\diamondsuit)} |p - S_p - q|_{1,\Omega} \leq C|\diamondsuit|^{m-3}$. Thus, it follows that

$$\int_0^T |\tilde{p} - S_p|_{1,\Omega} dt \leq \int_0^T |p - \tilde{p}|_{1,\Omega} dt + C|\diamondsuit|^{m-3} \leq C_1|\diamondsuit|^{m-3}$$

by Lemma 4.8. Hence, we have

$$\begin{aligned} \int_0^T |p - S_p|_{1,\Omega} dt &\leq \int_0^T |p - \tilde{p}|_{1,\Omega} dt + \int_0^T |\tilde{p} - S_p|_{1,\Omega} dt \\ &\leq C_1|\diamondsuit|^{m-3} + C_2|\diamondsuit|^{m-3}. \end{aligned}$$

By Lemma 4.2, we get $\int_0^T |p - S_p|_{0,\Omega} dt \leq C \int_0^T |p - S_p|_{1,\Omega} dt \leq C_1|\diamondsuit|^{m-3}$. We have thus established the proof. ■

We have implemented in MATLAB the standard linear finite element method for computing numerically the pressure function after obtaining a numerical approximation of the velocity vector from the programs mentioned in §2 and §3. By putting the artificial velocity vector and pressure into the Navier-Stokes equations (1.1), we calculate the right-hand side function $\mathbf{f}$. Using the $\mathbf{f}$, the boundary values of the artificial velocity, and the initial values over the K-shape domain, we compute the numerical approximation of the stream function with our MATLAB programs and then compute the numerical approximation of the pressure function. Then we compute the error against the artificial solution. For the velocity vectors, we compute the error in $L_2(0,T;L_2(\Omega))$ and for the pressure functions, we measure the error in $L_1(0,T;L_2(\Omega))$. We used the Trapezoidal rule for $L_2(0,T)$ and $L_1(0,T)$ and the Gauss-Legendre rule over each triangle in $\diamondsuit$ for $L_2(\Omega)$. In Tables VII, VIII and IX, we list the errors over the K-shape domain for three different artificial velocity vectors and pressure functions. The domain is triangulated as shown in Fig. 7 and is refined three times as shown in Fig. 8. The Reynolds number $\nu = 1$ and the time $T = 1$.

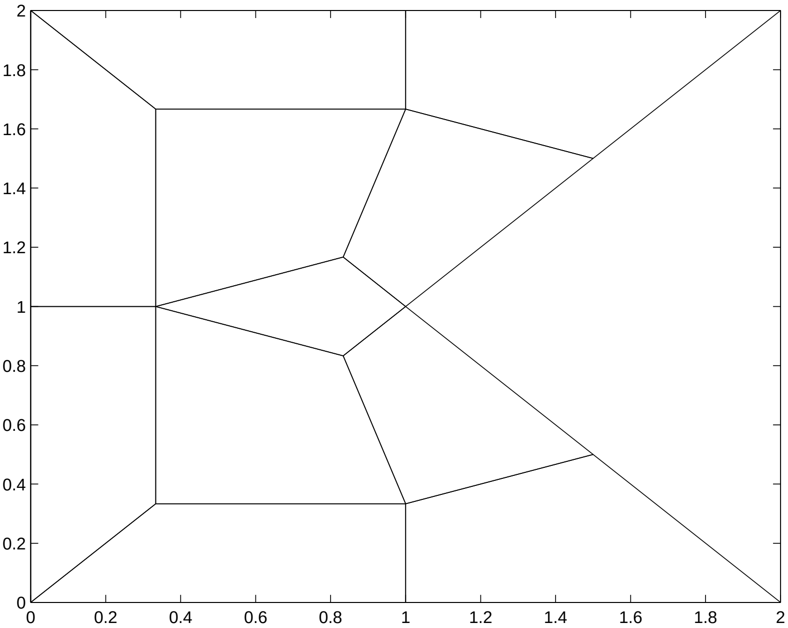

FIG. 7. A quadrangulation of K-shape domain

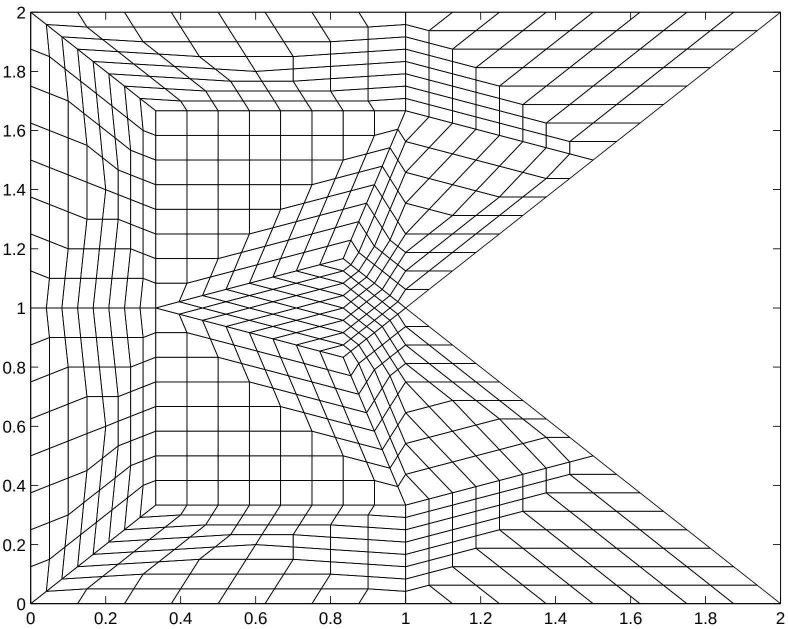

FIG. 8. The fourth regular refinement of the K-shape domain

TABLE VII. Maximum Errors of the Pressure Function and Velocity Vector of Navier-Stokes Equations

Matrix Sizes	$\mathbf{u} = [-0.1e^{0.1t}\pi(1+\cos(\pi x))\sin(\pi y),$	$p = (x^2+y^2)e^{-t}$
	$0.1e^{0.1t}\pi(1+\cos(\pi y))\sin(\pi x)]$	
78x78	2.351×10^{-2}	2.118×10^{-1}
263x263	6.662×10^{-3}	1.283×10^{-1}
963x963	1.114×10^{-3}	4.196×10^{-2}
3684x3684	1.565×10^{-4}	1.410×10^{-2}

TABLE VIII. Maximum Errors of the Pressure Function and Velocity Vector of Navier-Stokes Equations

	Error in Velocity	Error in Pressure
Matrix Sizes	$\mathbf{u} = [(1+e^{-0.1t})\cos(x+y)$	$p = (x^2+y^2)(1+\frac{1}{1+t^2})$
	$-(1+e^{-0.1t})\cos(x+y)]$	
78x78	9.329×10^{-3}	8.227×10^{-1}
263x263	1.400×10^{-3}	2.883×10^{-1}
963x963	2.002×10^{-4}	9.655×10^{-2}
3684x3684	2.702×10^{-5}	2.926×10^{-2}

TABLE IX. Maximum Errors of the Pressure Function and Velocity Vector of Navier-Stokes Equations

	Error in Velocity	Error in Pressure
Matrix Sizes	$\mathbf{u} = [\cos(x+y+t), -\cos(x+y+t)]$	$p = x^2+y^2+xyt^2$
78x78	5.537×10^{-3}	5.095×10^{-1}
263x263	7.505×10^{-4}	1.778×10^{-1}
963x963	1.055×10^{-4}	5.896×10^{-2}
3684x3684	1.638×10^{-5}	1.786×10^{-2}

V. REMARKS

Remark 5.1. Let $\diamondsuit$ be a non-degenerate convex quadrangulation with V_i and V_b being the numbers of interior and boundary vertices of $\diamondsuit$, respectively. Let $\triangle$ be any triangulation using the interior and boundary vertices of $\diamondsuit$. Let $\diamondsuit\!\!\!\!+$ be the triangulated quadrangulation obtained by adding the two diagonals of each quadrilateral. In this remark, we give dimension formula for several elementary bivariate spline spaces.

Let N_t, E_t, be the number of triangles and edges of $\triangle$, N_q, E_q be the number of quadrilaterals and edges of $\Diamond$, and $N_{\diamondplus}$, $E_{\diamondplus}$ be the number of triangles and edges of $\diamondplus$. Then it is clear from Euler formula that

$$\begin{aligned}
N_t &= 2N_q = 2V_i + V_b - 2; \\
E_t &= E_q + N_q = 3V_i + 2V_b - 3; \\
N_{\diamondplus} &= 4N_q = 4V_i + 2V_b - 4; \\
E_{\diamondplus} &= E_q + 4N_q = 6V_i + 7V_b/2 - 6; \\
V_{\diamondplus} &= 2V_i + 3V_b/2 - 1 \\
N_q &= V_i + V_b/2 - 1; \\
E_q &= 2V_i + 3V_b/2 - 2,
\end{aligned}$$

where $V_{\diamondplus}$ denotes the total number of vertices of $\diamondplus$. Therefore, we have the following dimension formulae for various spline spaces.

$$\begin{aligned}
\dim(S_3^1(\diamondplus)) &= 3V_i + 3V_b + E_q = 5V_i + 9V_b/2 - 2; \\
\dim(S_2^0(\diamondplus)) &= 5N_q + V_i + V_b + E_q = 8V_i + 5V_b - 7; \\
\dim(S_1^0(\diamondplus)) &= 2V_i + 3V_b/2 - 1.
\end{aligned}$$

The conclusion is the dimension of $S_3^1(\diamondplus)$ is smaller than that of $S_2^0(\diamondplus)$.

Remark 5.2. Again let $\Diamond$ be a non-degenerate convex quadrangulation with V_i and V_b being the numbers of interior and boundary vertices of $\Diamond$, respectively. Again let $\triangle$ be any triangulation using the interior and boundary vertices of $\Diamond$. Let $\triangle_{CT}$ be the well-known Clough-Tocher refinement of $\triangle$ which is obtained by connecting the center of each triangle $t \in \triangle$ to the three vertices of t. Then

$$\dim(S_3^1(\triangle_{CT}) = 6V_i + 5V_b - 3.$$

That is, the well-known C^1 cubic finite element space $S_3^1(\triangle_{CT})$ has a larger dimension than that of $S_3^1(\diamondplus)$. Also, the triangulation $\triangle_{CT}$ has more triangles than that of $\diamondplus$.

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