Construction of Bivariate Compactly Supported Biorthogonal Box Spline Wavelets with Arbitrarily High Regularities

by

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Abstract. We give a simple formula for the duals of the filters associated with bivariate box spline functions. We show how to construct bivariate non-separable compactly supported biorthogonal wavelets associated with box spline functions which have arbitrarily high regularities.

1. Introduction

Let $B_{l,m,n}$ be the bivariate box spline function whose Fourier transform is

$$\widehat{B}_{l,m,n}(\omega_1,\omega_2) = \left(\frac{1-e^{i\omega_1}}{i\omega_1}\right)^l \left(\frac{1-e^{i\omega_2}}{i\omega_2}\right)^m \left(\frac{1-e^{i(\omega_1+\omega_2)}}{i(\omega_1+\omega_2)}\right)^n .$$

(For properties of box spline functions, see [3] and [2]. For computation of these bivariate box spline functions, see [4] and [13].) It is known that $B_{l,m,n}$ generates a multi-resolution approximation of $L_2(\mathbf{R}^2)$ (cf. [14]). We are interested in constructing a compactly supported function $\tilde{B}_{l,m,n}$ generating a multi-resolution approximation of $L_2(\mathbf{R}^2)$ which is a biorthogonal dual to $B_{l,m,n}$ in the following sense:

$$\int\int_{\mathbf{R}^2} B_{l,m,n}(x-j,y-k)\tilde{B}_{l,m,n}(x-j',y-k')dxdy = \delta_{j,j'}\delta_{k,k'} \qquad (1.1)$$

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for all integers $j, k \in \mathbf{Z}$, where $\delta_{j,k}$ is the standard Kronecker notation defined by $\delta_{j,k} = 0$ if $j \neq k$ and $\delta_{j,k} = 1$ if $j = k$ and $\mathbf{Z}$ is the collection of all integers.

In the univariate setting, for B-spline function B_n, biorthogonal dual function $\tilde{B}_n$ were constructed in [7]. Also, compactly supported biorthogonal wavelets associated with B_n were constructed there. Since bivariate box splines are a natural generalization of B-spline functions, several attempts have been made to construct these types of biorthogonal wavelets associated with box spline function $B_{l,m,n}$. See, e.g., [6], [15], [8] and [16]. So far compactly supported biorthogonal wavelets associated with box spline $B_{1,1,1}$ have been constructed ([8]). The construction of these wavelets associated with general $B_{l,m,n}$ has remained a challenge since then.

We are furthermore interested in constructing compactly supported biorthogonal wavelets $\psi_j, j = 1,2,3$ and $\tilde{\psi}_j, j = 1,2,3$ and two families of FIR filters $\{M_j, j = 0,1,2,3\}$ and $\{J_j, j = 0,1,2,3\}$ with

$$\widehat{\psi}_j(\omega_1, \omega_2) = M_j(e^{i\frac{\omega_1}{2}}, e^{i\frac{\omega_2}{2}})\widehat{B}_{l,m,n}\left(\frac{\omega_1}{2}, \frac{\omega_2}{2}\right), \quad j = 1,2,3, \tag{1.2}$$

and

$$\widehat{\tilde{\psi}}_j(\omega_1, \omega_2) = J_j\left(e^{i\frac{\omega_1}{2}}, e^{i\frac{\omega_2}{2}}\right)\widehat{\tilde{B}}_{l,m,n}\left(\frac{\omega_1}{2}, \frac{\omega_2}{2}\right), \quad j = 1,2,3, \tag{1.3}$$

such that the dilations and translates of the ψ_j's and $\tilde{\psi}_j$'s form two dual Riesz bases for $L_2(\mathbf{R}^2)$ (cf. [7] for the univariate setting) and the two families form an exact reconstruction of synthesis/analysis filter bank for image/data processing (see [21] and [8].)

In this paper, we shall give an explicit formula for $\widehat{\tilde{B}}_{l,m,n}$ for any given positive integers l, m, and n in §2 and a matrix extension scheme to construct M_j's and J_j's which lead to compactly supported biorthogonal wavelets with arbitrarily high regularities in §3. Finally, we shall give examples of these wavelets in §4. The regularities of these biorthogonal wavelets are studied in §2. The estimate of the regularities is based on an excellent theory developed in [10]. The proof of the fact that these ϕ_j's and ψ_j's generate two dual Riesz bases may be based on a straightforward generalization of the arguments for the univariate setting in [7] or based on the multivariate theory in [5]. We sincerely thank the pioneer researchers for their theories which lay a solid foundation for biorthogonal wavelets. We thus concentrate ourselves on the construction of concrete examples while omitting the details of the generalization here. Our contribution in this paper is just the explicit formula together with the matrix extension scheme.

2. Construction of Compactly Supported Biorthogonal Dual Functions

2.1. Construction of a Biorthogonal Mask J_0 and Dual $\tilde{B}_{l,m,n}$

Denote $z_1 = e^{i\omega_1}$ and $z_2 = e^{i\omega_2}$. Let

$$M_0(z_1, z_2) = \left(\frac{1+z_1}{2}\right)^l \left(\frac{1+z_2}{2}\right)^m \left(\frac{1+z_1 z_2}{2}\right)^n$$

be a mask associated with the box spline function $B_{l,m,n}$. We look for a mask $J_0(z_1, z_2)$ in the form

$$\overline{J_0(z_1, z_2)} = \left(\frac{1+z_1}{2}\right)^{\tilde{n}-l} \left(\frac{1+z_2}{2}\right)^{\tilde{n}-m} \left(\frac{1+z_1 z_2}{2}\right)^{\tilde{m}-n} H(z_1, z_2) D(z_1 z_2) \quad (2.1)$$

with $\tilde{n} > l, \tilde{n} > m$ and odd integer $\tilde{m} > n$ such that

$$\begin{aligned} &M_0(z_1, z_2)\overline{J_0(z_1, z_2)} + M_0(-z_1, z_2)\overline{J_0(-z_1, z_2)} \\ &\quad + M_0(z_1, -z_2)\overline{J_0(z_1, -z_2)} + M_0(-z_1, -z_2)\overline{J_0(-z_1, -z_2)} = 1. \end{aligned} \quad (2.2)$$

Let us first recall a well-known fact that there exists a polynomial $P_N(y)$ of degree $< N$ such that

$$(1-y)^N P_N(y) + y^N P_N(1-y) = 1. \quad (2.3)$$

An explicit formula for $P_N(y)$ was given in [9] which leads to the construction of the well-known compactly supported orthonormal wavelets. That is,

$$P_N(y) = \sum_{k=0}^{N-1} \binom{N-1+k}{k} y^k. \quad (2.4)$$

We shall give another derivation of this polynomial P_N which ultimately leads to the formulation for H and D above. We have

Theorem 2.1. *Let $\tilde{n} > n$ and $\tilde{m} = 2\hat{m} + 1$. Let $J_0(z_1, z_2)$ be defined in (2.1) with H and D defined by*

$$H(z_1, z_2) = \sum_{k=0}^{\tilde{n}-1} \binom{2\tilde{n}-1}{k} \left(\frac{1+z_1}{2}\frac{1+z_2}{2}\right)^{\tilde{n}-1-k} \left(\frac{1-z_1}{2}\frac{1-z_2}{2}\right)^k, \quad (2.5)$$

and

$$D(e^{i(\omega_1+\omega_2)}) = e^{-i(\omega_1+\omega_2)N} P_{\tilde{n}+\hat{m}} \left(\sin^2\left(\frac{\omega_1+\omega_2}{2}\right)\right). \quad (2.6)$$

Then J_0 is a dual of M_0 satisfying (2.2).

Proof. We first note that

$$1 = (1-y+y)^{2N-1} = \sum_{k=0}^{2N-1} \binom{2N-1}{k} (1-y)^{2N-1-k} y^k$$
$$= \sum_{k=0}^{N-1} \binom{2N-1}{k} (1-y)^{2N-1-k} y^k + \sum_{\ell=0}^{N-1} \binom{2N-1}{2N-1-\ell} (1-y)^{\ell} y^{2N-1-\ell}$$
$$= (1-y)^N \sum_{k=0}^{N-1} \binom{2N-1}{k} (1-y)^{N-1-k} y^k$$
$$+ y^N \sum_{k=0}^{N-1} \binom{2N-1}{k} (1-y)^{k} y^{N-1-k}.$$

We have thus obtained another formulation for P_N in (2.4):

$$P_N(y) = \sum_{k=0}^{N-1} \binom{2N-1}{k} (1-y)^{N-1-k} y^k. \tag{2.7}$$

By the uniqueness of the solution for equation (2.3) with $\deg(P_N) \le N-1$, we can conclude that the two formulas (2.4) and (2.7) are equivalent. We need to use this fact later. We then note that

$$\frac{1+z_1z_2}{2} = \frac{1+z_1}{2}\frac{1+z_2}{2} + \frac{1-z_1}{2}\frac{1-z_2}{2}.$$

By letting $H(z_1,z_2)$ be defined in (2.5), we have, similar to the new derivation of $P_N(y)$ above,

$$\left(\frac{1+z_1z_2}{2}\right)^{2\tilde{n}-1} = \left(\frac{1+z_1}{2}\frac{1+z_2}{2} + \frac{1-z_1}{2}\frac{1-z_2}{2}\right)^{2\tilde{n}-1}$$
$$= \sum_{k=0}^{\tilde{n}-1} \binom{2\tilde{n}-1}{k} \left(\frac{1+z_1}{2}\frac{1+z_2}{2}\right)^{2\tilde{n}-1-k} \left(\frac{1-z_1}{2}\frac{1-z_2}{2}\right)^{k}$$
$$+ \sum_{\ell=0}^{\tilde{n}-1} \binom{2\tilde{n}-1}{2\tilde{n}-1-\ell} \left(\frac{1+z_1}{2}\frac{1+z_2}{2}\right)^{\ell} \left(\frac{1-z_1}{2}\frac{1-z_2}{2}\right)^{2\tilde{n}-1-\ell}$$
$$= \left(\frac{1+z_1}{2}\frac{1+z_2}{2}\right)^{\tilde{n}} H(z_1,z_2) + \left(\frac{1-z_1}{2}\frac{1-z_2}{2}\right)^{\tilde{n}} H(-z_1,-z_2)$$

and similarly,

$$\left(\frac{1-z_1z_2}{2}\right)^{2\tilde{n}-1} = \left(\frac{1-z_1}{2}\frac{1+z_2}{2} + \frac{1+z_1}{2}\frac{1-z_2}{2}\right)^{2\tilde{n}-1}$$
$$= \left(\frac{1-z_1}{2}\frac{1+z_2}{2}\right)^{\tilde{n}} H(-z_1,z_2) + \left(\frac{1+z_1}{2}\frac{1-z_2}{2}\right)^{\tilde{n}} H(z_1,-z_2)$$

With the definition of J_0 in (2.1), (2.2) may be simplified as follows:

$$\begin{aligned}
&M_0(z_1,z_2)\overline{J_0(z_1,z_2)} + M_0(-z_1,-z_2)\overline{J_0(-z_1,-z_2)}\\
&+ M_0(-z_1,z_2)\overline{J_0(-z_1,z_2)} + M_0(z_1,-z_2)\overline{J_0(z_1,-z_2)}\\
=& \left[\left(\frac{1+z_1}{2}\frac{1+z_2}{2}\right)^{\tilde n} H(z_1,z_2) + \left(\frac{1-z_1}{2}\frac{1-z_2}{2}\right)^{\tilde n} H(-z_1,-z_2)\right]\times\\
&\left(\frac{1+z_1z_2}{2}\right)^{\tilde m} D(z_1z_2)\\
+& \left[\left(\frac{1-z_1}{2}\frac{1+z_2}{2}\right)^{\tilde n} H(-z_1,z_2) + \left(\frac{1+z_1}{2}\frac{1-z_2}{2}\right)^{\tilde n} H(z_1,-z_2)\right]\times\\
&\left(\frac{1-z_1z_2}{2}\right)^{\tilde m} D(-z_1z_2)\\
=& \left(\frac{1+z_1z_2}{2}\right)^{2\tilde n+\tilde m-1} D(z_1z_2) + \left(\frac{1-z_1z_2}{2}\right)^{2\tilde n+\tilde m-1} D(-z_1z_2).
\end{aligned}$$

Let $\tilde m = 2\widehat m + 1$ and $N = \tilde n + \widehat m$. Recall $z_1 = e^{i\omega_1}$ and $z_2 = e^{i\omega_2}$. Then the last equation may be simplified further:

$$\begin{aligned}
&\left(\cos^2\frac{\omega_1+\omega_2}{2}\right)^N e^{i(\omega_1+\omega_2)N} D(e^{i(\omega_1+\omega_2)})\\
&+ \left(\sin^2\frac{\omega_1+\omega_2}{2}\right)^N (-1)^N e^{i(\omega_1+\omega_2)N} D(-e^{i(\omega_1+\omega_2)}).
\end{aligned}$$

Let $y = \sin^2\left(\frac{\omega_1+\omega_2}{2}\right)$ and recognize that $e^{i(\omega_1+\omega_2)N}D(e^{i(\omega_1+\omega_2)}) = P_N(y)$. We can see that the above equation is just the left-hand side of (2.3). Therefore, we have established the results of Theorem 2.1. ■

We remark here that the filter $J_0(z_1,z_2)$ is a linear phase filter. It is known that

$$\widehat{B}_{l,m,n}(\omega_1,\omega_2) = \prod_{k=1}^{\infty} M_0\left(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}\right) \in L_2(\mathbf{R}^2). \tag{2.8}$$

We now construct the dual functions $\tilde B_{l,m,n}$ associated with box spline $B_{l,m,n}$ by letting

$$\widehat{\tilde B}_{l,m,n}(\omega_1,\omega_2) = \prod_{k=1}^{\infty} J_0\left(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}\right). \tag{2.9}$$

We shall study the regularity of the dual functions $\tilde B_{l,m,n}$ in the next subsection.

2.2. Fourier Based Techniques for the Smoothness of the Dual $\tilde B_{l,m,n}$

To make $\tilde{B}_{l,m,n} \in L_2(\mathbf{R}^2)$, we split J_0 into $\tilde{J}_0$ and $\widehat{J}_0$ as follows and estimate their infinite products:

$$\overline{\tilde{J}_0(z_1,z_2)} = \left(\frac{1+z_1z_2}{2}\right)^{\tilde{m}-n} D(z_1z_2)$$

and

$$\overline{\widehat{J}_0(z_1,z_2)} = \left(\frac{1+z_1}{2}\right)^{\tilde{n}-l} \left(\frac{1+z_2}{2}\right)^{\tilde{n}-m} H(z_1,z_2).$$

We first consider $\overline{\widehat{J}_0(z_1,z_2)}$. For $H(z_1,z_2)$, it is clear that

$$\begin{aligned}
|H(z_1,z_2)| &\le \sum_{k=0}^{\tilde{n}-1} \binom{2\tilde{n}-1}{k} \left|\frac{1+z_1}{2}\frac{1+z_2}{2}\right|^{\tilde{n}-1-k} \left|\frac{1-z_1}{2}\frac{1-z_2}{2}\right|^{k} \\
&= \sum_{k=0}^{\tilde{n}-1} \binom{2\tilde{n}-1}{k} \left|\cos\frac{\omega_1}{2}\cos\frac{\omega_2}{2}\right|^{\tilde{n}-1-k} \left|\sin\frac{\omega_1}{2}\sin\frac{\omega_2}{2}\right|^{k} \\
&\le \left(\sum_{k=0}^{\tilde{n}-1} \binom{2\tilde{n}-1}{k} \left(\cos^2\frac{\omega_1}{2}\right)^{\tilde{n}-1-k} \left(\sin^2\frac{\omega_1}{2}\right)^{k}\right)^{\frac{1}{2}} \times \\
&\quad \left(\sum_{k=0}^{\tilde{n}-1} \binom{2\tilde{n}-1}{k} \left(\cos^2\frac{\omega_2}{2}\right)^{\tilde{n}-1-k} \left(\sin^2\frac{\omega_2}{2}\right)^{k}\right)^{\frac{1}{2}} \\
&= \left(P_{\tilde{n}}\left(\sin^2\frac{\omega_1}{2}\right) P_{\tilde{n}}\left(\sin^2\frac{\omega_2}{2}\right)\right)^{\frac{1}{2}},
\end{aligned}$$

where $P_{\tilde{n}}$ is the polynomial defined in (2.7) which is equivalent to (2.4). Similarly, we have

$$\begin{aligned}
&\left|H\left(e^{i2\omega_1}, e^{i2\omega_2}\right)\right| \le \left(P_{\tilde{n}}(\sin^2\omega_1)P_{\tilde{n}}(\sin^2\omega_2)\right)^{\frac{1}{2}} \\
&= \left(P_{\tilde{n}}\left(4\sin^2\frac{\omega_1}{2}\left(1-\sin^2\frac{\omega_1}{2}\right)\right) P_{\tilde{n}}\left(4\sin^2\frac{\omega_2}{2}\left(1-\sin^2\frac{\omega_2}{2}\right)\right)\right)^{\frac{1}{2}}.
\end{aligned}$$

By applying the results developed in [10], i.e., in Lemmas 7.1.1 – 7.1.8, we have

$$\begin{aligned}
\left|\prod_{k=1}^{\infty} \widehat{J}_0\left(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}\right)\right| &\le C\left((1+|\omega_1|)(1+|\omega_2|)\right)^{-\tilde{n}+\max(l,m)+\frac{1}{2}\log_2 P_{\tilde{n}}(\frac{3}{4})} \\
&\le C\left((1+|\omega_1|)(1+|\omega_2|)\right)^{-\tilde{n}+\max(l,m)+\frac{\log 3}{2\log 2}\tilde{n}} \\
&= C\left((1+|\omega_1|)(1+|\omega_2|)\right)^{-(1-\frac{\nu}{2})\tilde{n}+\max(l,m)}
\end{aligned}$$

where $\nu = \frac{\log 3}{\log 2} < 2$ and $1-\frac{\nu}{2} > 0$. Here, we have used the fact $P_{\tilde{n}}(3/4) \le 3^{\tilde{n}}$.

Next we consider $\overline{\tilde{J}_0(z_1,z_2)}$. We note that $|D(z_1z_2)| = P_N\left(\sin^2\frac{\omega_1+\omega_2}{2}\right)$ and apply the results in [10] again, i.e., Lemmas 7.1.1–7.1.8 to get

$$\begin{aligned}\left|\prod_{k=1}^{\infty}\tilde{J}_0\left(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}\right)\right| &\le C(1+|\omega_1+\omega_2|)^{-\tilde{m}+n+\log_2 P_N\left(\frac{3}{4}\right)}\\ &\le C(1+|\omega_1+\omega_2|)^{-2\widehat{m}+n+\frac{\log 3}{\log 2}(\widehat{m}+\tilde{n})-1}\\ &= C(1+|\omega_1+\omega_2|)^{(\nu-2)\widehat{m}+\nu\tilde{n}+n-1}.\end{aligned}$$

Since $\nu = \frac{\log 3}{\log 2} < 2$, for any fixed $\tilde{n}$, we can choose $\widehat{m}$, i.e., $\tilde{m}$ large enough such that

$$(\nu-2)\widehat{m}+\nu\tilde{n}+n-1 \le 0,$$

and hence,

$$\left|\prod_{k\ge 1}^{\infty}\tilde{J}_0\left(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}\right)\right| \le C$$

for a positive constant C. Therefore, we can choose $\tilde{n}$ large enough that $\max(l,m)-(1-\frac{\nu}{2})\tilde{n} < -1/2$ and then $\tilde{m}$ large enough that $(\nu-2)\widehat{m}+\nu\tilde{n}+n-1 \le 0$ such that

$$\left|\prod_{k=1}^{\infty}J_0\left(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}\right)\right| \le C\left((1+|\omega_1|)(1+|\omega_2|)\right)^{-(1-\frac{\nu}{2})\tilde{n}+\max(l,m)} \in L_2(\mathbf{R}^2).$$

By choosing $\tilde{n}$ even larger, especially, for any $\alpha > 0$, $\tilde{n} > (\max(l,m)+1+\alpha)/(1-\nu/2)$, we can make

$$\left((1+|\omega_1|)(1+|\omega_2|)\right)^{\alpha}\left|\prod_{k=1}^{\infty}J_0\left(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}\right)\right| \in L^1(\mathbf{R}^2).$$

Finally, by a straightforward generalization of Lemma 6.2.2 in [10] in the bivariate setting, we can show that $\tilde{B}_{l,m,n}$ is a compactly supported function. Summarizing the discussions above, we have obtained the following

Theorem 2.2. *Let $\tilde{n}$ and $\tilde{m}$ be large enough. Then $\tilde{B}_{l,m,n}$ is a well-defined compactly supported L_2 function. Furthermore, for any $\alpha > 0$, $\tilde{B}_{l,m,n} \in C^{\alpha}(\mathbf{R}^2)$ if $\tilde{n}$ and $\tilde{m}$ sufficiently large, e.g.,*

$$\tilde{n} > \frac{2(\max(l,m)+2+\alpha)}{2-\log(3)/\log(2)}, \qquad \tilde{m} > 2\frac{\tilde{n}\log(3)/\log(2)+n-1}{2-\log(3)/\log(2)}+1.$$

2.3. Biorthogonality and Riesz Basis Property

We next show that $\tilde{B}_{l,m,n}$ defined in (2.9) is a biorthogonal dual to $B_{l,m,n}$ in the sense of (1.1). We have

Theorem 2.3. *Let $\tilde{n}$ and $\tilde{m}$ be sufficiently large. Then $\tilde{B}_{l,m,n}$ generates a multi-resolution approximation of $L_2(\mathbf{R}^2)$. Also, $\tilde{B}_{l,m,n}$ is a biorthogonal dual of $B_{l,m,n}$.*

We shall use the results, more precisely, Theorem 3.3 developed in [5] to prove Theorem 2.3 although there is a more general result available in the more current literature (cf. [20]). By Lemma 3.2 in [5] which is a generalization of the univariate result in [7], we first see that $\widehat{\tilde{B}}_{l,m,n}$ is continuous and $\widehat{\tilde{B}}_{l,m,n}(0,0)\widehat{B}_{l,m,n}(0,0) = 1$ and then we need to show the following

Lemma 2.4. *For any sufficiently large integers $\tilde{n}$ and $\tilde{m}$,*

$$\sum_{\ell\in \mathbf{Z}^2} |\widehat{B}_{l,m,n}((\omega_1,\omega_2)+2\pi\ell)\widehat{\tilde{B}}_{l,m,n}((\omega_1,\omega_2)+2\pi\ell)|^2 \geq C_2 > 0. \tag{2.10}$$

Proof. Recall from (2.1) that

$$\begin{aligned}
&\left|M_0\left(e^{i\omega_1},e^{i\omega_2}\right)J_0\left(e^{i\omega_1},e^{i\omega_2}\right)\right| \\
&=\left|\frac{1+e^{i\omega_1}}{2}\right|^{\tilde{n}}\left|\frac{1+e^{i\omega_2}}{2}\right|^{\tilde{n}}\left|\frac{1+e^{i(\omega_1+\omega_2)}}{2}\right|^{\tilde{m}}\left|H\left(e^{i\omega_1},e^{i\omega_2}\right)\right|\left|D\left(e^{i\omega_1+\omega_2)}\right)\right| \\
&\geq \left|\cos\frac{\omega_1}{2}\right|^{\tilde{n}}\left|\cos\frac{\omega_2}{2}\right|^{\tilde{n}}\left|\cos\frac{\omega_1+\omega_2}{2}\right|^{\tilde{m}}\left|H\left(e^{i\omega_1},e^{i\omega_2}\right)\right|
\end{aligned}$$

since $\left|D\left(e^{i\omega_1+\omega_2)}\right)\right| \geq 1$ for any $(\omega_1,\omega_2)\in\mathbf{R}^2$.

Note that the sum on the left of the inequality (2.10) is a periodic function. We only need to show (2.10) for $(\omega_1,\omega_2)\in[-\pi,\pi]^2$. It is easy to see that $|\widehat{\tilde{B}}_{l,m,n}(\omega_1,\omega_2)| = |\widehat{\tilde{B}}_{l,m,n}(-\omega_1,-\omega_2)|$. Thus, we only need to consider the inequality (2.10) for $(\omega_1,\omega_2)\in[-\pi,\pi]\times[0,\pi]$.

Note also that

$$\left|H\left(e^{i\omega_1},e^{i\omega_2}\right)\right| = \left|\sum_{k=0}^{\tilde{n}}\binom{2\tilde{n}-1}{k}\left(-\sin\frac{\omega_1}{2}\sin\frac{\omega_2}{2}\right)^k\left(\cos\frac{\omega_1}{2}\cos\frac{\omega_2}{2}\right)^{\tilde{n}-1-k}\right|.$$

For $(\omega_1,\omega_2)\in[-\pi,0]\times[0,\pi]$, we have $|H(e^{i\omega_1},e^{i\omega_2})| \geq |\cos\frac{\omega_1}{2}\cos\frac{\omega_2}{2}|^{\tilde{n}-1}$, and hence,

$$\begin{aligned}
\prod_{k=1}^{\infty}\left|H\left(e^{i\frac{\omega_1}{2^k}},e^{i\frac{\omega_2}{2^k}}\right)\right| &\geq \prod_{k=1}^{\infty}\left|\cos\frac{\omega_1}{2^{k+1}}\cos\frac{\omega_2}{2^{k+1}}\right|^{\tilde{n}-1} \\
&= \left|\frac{\sin\frac{\omega_1}{2}}{\frac{\omega_1}{2}}\frac{\sin\frac{\omega_1}{2}}{\frac{\omega_2}{2}}\right|^{\tilde{n}-1} \\
&\geq \left(\frac{2}{\pi}\right)^{2\tilde{n}-2}
\end{aligned}$$

by an elementary inequality $\sin x \geq \frac{2}{\pi}x$ for $x \in [0, \frac{\pi}{2}]$. Therefore, for $(\omega_1, \omega_2) \in [-\pi, 0] \times [0, \pi]$, we have

$$\begin{aligned}
&\left|\widehat{B}_{l,m,n}(\omega_1, \omega_2)\widehat{\tilde{B}}_{l,m,n}(\omega_1, \omega_2)\right| \\
&\geq \prod_{k=1}^{\infty} \left|\cos\frac{\omega_1}{2^{k+1}}\right|^{\tilde{n}} \left|\cos\frac{\omega_2}{2^{k+1}}\right|^{\tilde{n}} \left|\cos\frac{\omega_1+\omega_2}{2^{k+1}}\right|^{\tilde{m}} \prod_{k=1}^{\infty} \left|H\left(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}\right)\right| \\
&\geq \left(\frac{\sin\frac{\omega_1}{2}}{\frac{\omega_1}{2}}\right)^{\tilde{n}} \left(\frac{\sin\frac{\omega_2}{2}}{\frac{\omega_2}{2}}\right)^{\tilde{n}} \left(\frac{\sin\frac{(\omega_1+\omega_2)}{2}}{\frac{\omega_1+\omega_2}{2}}\right)^{\tilde{m}} \cdot \left(\frac{2}{\pi}\right)^{2\tilde{n}-2} \\
&\geq \left(\frac{2}{\pi}\right)^{4\tilde{n}+\tilde{m}-2} > 0.
\end{aligned}$$

Therefore, the inequality (2.10) holds for $(\omega_1, \omega_2) \in [-\pi, 0] \times [0, \pi]$.

For $(\omega_1, \omega_2) \in [0, \pi] \times [0, \pi]$, we first note that $|H(1, 1)| = 1$ and $H(e^{i\omega_1}, e^{i\omega_2})$ is continuous. There exists a $\delta > 0$ such that for $(\omega_1, \omega_2) \in [0, \delta]^2$, $|H(e^{i\omega_1}, e^{i\omega_2})| \geq \frac{1}{2}$. On the other hand, we have

$$|H(e^{i\omega_1}, e^{i\omega_2}) - 1| \geq C(|\omega_1| + |\omega_2|)$$

or

$$|H(e^{i\omega_1}, e^{i\omega_2})| \geq 1 - C(|\omega_1| + |\omega_2|).$$

There exist an integer k_0 such that for $C\left(\left|\frac{\omega_1}{2^{k_0}}\right| + \left|\frac{\omega_2}{2^{k_0}}\right|\right) \leq \frac{1}{2}$. By another elementary inequality: $1 - x \geq e^{-2x}$ for $0 \leq x \leq \frac{1}{2}$, we have for $(\omega_1, \omega_2) \in [0, \delta] \times [0, \delta]$,

$$\begin{aligned}
\prod_{k=1}^{\infty} \left|H\left(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}\right)\right| &= \prod_{k=1}^{k_0} \left|H\left(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}\right)\right| \prod_{k=k_0+1}^{\infty} \left|H\left(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}\right)\right| \\
&\geq \left(\frac{1}{2}\right)^{k_0} \prod_{k=1}^{\infty} \left(1 - C\frac{|\omega_1| + |\omega_2|}{2^{k_0+k}}\right) \\
&\geq \left(\frac{1}{2}\right)^{k_0} e^{-2C\sum_{k=1}^{\infty} \frac{1}{2^k}\left(\frac{|\omega_1|+|\omega_2|}{2^{k_0}}\right)} \\
&= \left(\frac{1}{2}\right)^{k_0} e^{-2\cdot C\frac{|\omega_1|+|\omega_2|}{2^{k_0}}} \\
&\geq \left(\frac{1}{2}\right)^{k_0} e^{-1} > 0.
\end{aligned}$$

Here, we have assumed that $\delta < \frac{\pi}{4}$. Therefore,

$$\left|\widehat{B}_{l,mn}(\omega_1, \omega_2)\widehat{\tilde{B}}_{l,m,n}(\omega_1, \omega_2)\right| \geq \left(\frac{2}{\pi}\right)^{2\tilde{n}+\tilde{m}} \left(\frac{1}{2}\right)^{k_0} e^{-1} > 0,$$

for $(\omega_1,\omega_2)\in[0,\delta]\times[0,\delta]$.

For $(\omega_1,\omega_2)\in[0,\pi]\times[\delta,\pi]$, we consider a term

$$\widehat{B}_{l,m,n}((\omega_1,\omega_2)+(0,-2\pi))\widehat{\tilde{B}}_{l,m,n}((\omega_1,\omega_2)+(0,-2\pi)).$$

Note that $0\le\frac{\omega_1}{2}\le\frac{\pi}{2}$, $\frac{\delta-2\pi}{2}\le\frac{\omega_2-2\pi}{2}\le-\frac{\pi}{2}$, and $\frac{\delta}{2}-\pi\le\frac{\omega_1+\omega_2-2\pi}{2}\le 0$. We have

$$\left|\frac{\sin\frac{\omega_1}{2}}{\frac{\omega_1}{2}}\right|^{\tilde{n}}\ge\left(\frac{2}{\pi}\right)^{\tilde{n}},\quad\left|\frac{\sin\frac{\omega_2-2\pi}{2}}{\frac{\omega_2}{2}-\pi}\right|^{\tilde{n}}\ge\left(\frac{\sin\frac{\delta}{2}}{\pi-\frac{\delta}{2}}\right)^{\tilde{n}}$$

and

$$\left(\frac{\sin\frac{\omega_1+\omega_2-2\pi}{2}}{\frac{\omega_1+\omega_2-2\pi}{2}}\right)^{\tilde{m}}\ge\left(\frac{\sin(\delta/2)}{\pi-\delta/2}\right)^{\tilde{m}}.$$

Also, we have

$$\begin{aligned}&|H(e^{i\frac{\omega_1}{2^j}},e^{i\frac{\omega_2-2\pi}{2^j}})|\\&=\left|\sum_{k=0}^{\tilde{n}-1}\binom{2\tilde{n}-1}{k}\left(-\sin\frac{\omega_1}{2^{j+1}}\sin\frac{\omega_2-2\pi}{2^{j+1}}\right)^k\left(\cos\frac{\omega_1}{2^{j+1}}\cos\frac{\omega_2-2\pi}{2^{j+1}}\right)^{\tilde{n}-1-k}\right|\\&\ge\left|\cos\frac{\omega_1}{2^{j+1}}\cos\frac{\omega_2-2\pi}{2^{j+1}}\right|^{\tilde{n}-1}\end{aligned}$$

for $j\ge1$. Thus, for $(\omega_1,\omega_2)\in[0,\pi]\times[\delta,\pi]$, we have

$$\begin{aligned}&\left|\widehat{B}_{l,m,n}(\omega_1,\omega_2-2\pi)\widehat{\tilde{B}}_{l.m,n}(\omega_1,\omega_2-2\pi)\right|\\&\ge\left(\frac{2}{\pi}\right)^{\tilde{n}}\left(\frac{\sin\frac{\delta}{2}}{\pi-\frac{\delta}{2}}\right)^{\tilde{n}+\tilde{m}}\left|\frac{\sin\frac{\omega_1}{2}}{\frac{\omega_1}{2}}\frac{\sin\frac{\omega_2-2\pi}{2}}{\frac{\omega_2-2\pi}{2}}\right|^{\tilde{n}-1}\\&\ge\left(\frac{2}{\pi}\right)^{2\tilde{n}-1}\left(\frac{\sin\frac{\delta}{2}}{\pi-\frac{\delta}{2}}\right)^{2\tilde{n}+\tilde{m}-1}>0.\end{aligned}$$

Similarly for the case $(\omega_1,\omega_2)\in[\delta,\pi]\times[0,\pi]$. Therefore, we conclude the proof of Lemma 2.4. ■

The same arguments in the proof above can also show that

$$\sum_{\ell\in\mathbf{Z}^2}|\widehat{\tilde{B}}_{l,m,n}((\omega_1,\omega_2)+2\pi\ell)|^2\ge C_1. \tag{2.11}$$

It follows from Theorem 2.2 that

$$\sum_{\ell\in\mathbf{Z}^2}|\widehat{\tilde{B}}_{l,m,n}((\omega_1,\omega_2)+2\pi\ell)|^2\le C_2. \tag{2.12}$$

Thus, letting

$$V_0 = \mathrm{span}\{\tilde{B}_{l,m,n}(x-j, y-k), (j,k) \in \mathbf{Z}^2\},$$

the inequalities (2.11) and (2.12) imply that $\{\tilde{B}_{l,m,n}(x-j,y-k),(j,k) \in \mathbf{Z}^2\}$ is a Riesz basis for V_0. Letting $V_k := \{f(x/2^k, y/2^k) : \forall f(x,y) \in V_0\}$ for $k \in \mathbf{Z}$, we can show that $\bigcup_k V_k$ is dense in $L_2(\mathbf{R}^2)$ and $\bigcap_k V_k = \{0\}$. We leave the detail to the interested reader. Thus, we conclude that $\tilde{B}_{l,m,n}$ generates a multi-resolution approximation of $L_2(\mathbf{R}^2)$. These complete the proof of Theorem 2.3. ■

3. Construction of Compactly Supported Biorthogonal Wavelets

Let us start with the image/data analysis and synthesis filter bank in Fig. 1.

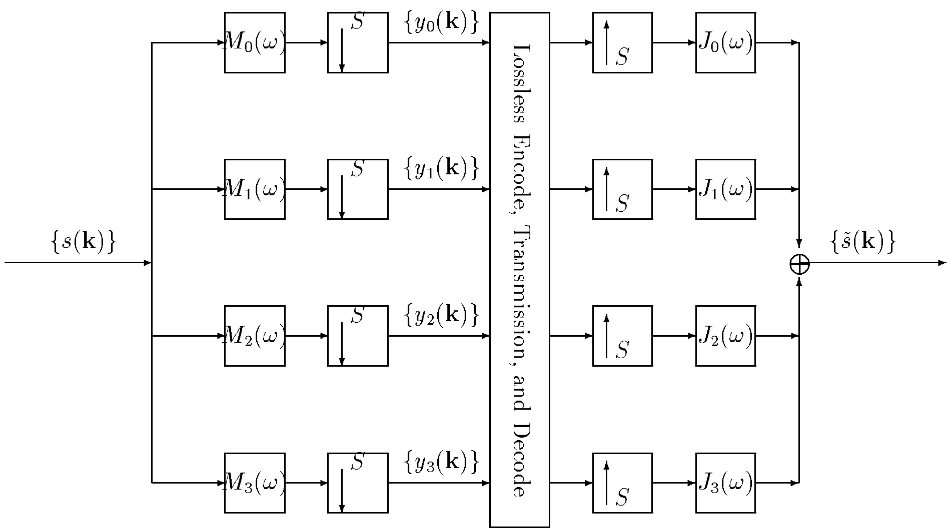

Fig. 1. A four band analysis/synthesis filter bank.

In Fig. 1, $S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ denotes a sampling matrix and $\{M_0, M_1, M_2, M_3\}$ and $\{J_0, J_1, J_2, J_3\}$ are two families of filters. In order to have $\tilde{s}$ in Fig. 1 reconstructed exactly, these two families of filters must satisfy the following

$$\begin{bmatrix} M_0(z_1,z_2) & M_1(z_1,z_2) & M_2(z_1,z_2) & M_3(z_1,z_2) \\ M_0(-z_1,z_2) & M_1(-z_1,z_2) & M_2(-z_1,z_2) & M_3(-z_1,z_2) \\ M_0(z_1,-z_2) & M_1(z_1,-z_2) & M_2(z_1,-z_2) & M_3(z_1,-z_2) \\ M_0(-z_1,-z_2) & M_1(-z_1,-z_2) & M_2(-z_1,-z_2) & M_3(-z_1,-z_2) \end{bmatrix} \times \begin{bmatrix} J_0(z_1,z_2) \\ J_1(z_1,z_2) \\ J_2(z_1,z_2) \\ J_3(z_1,z_2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.1}$$

(cf., e.g., [22], [21] and [8]). For convenience, let us denote by $A(M_0, M_1, M_2, M_3)$ the coefficient matrix in (3.1).

If we have these two families of filters, we may define biorthogonal wavelets as in (1.2) and (1.3). In order to have compactly supported wavelets we need $M_j, J_j, j = 1, 2, 3$ to be polynomials in (z_1, z_2). Thus, the invertible matrix $A(M_0, M_1, M_2, M_3)$ must have a monomial determinant, i.e., $Cz_1^j z_2^k$.

To this end, we rewrite $M_j, j = 0, 1, 2, 3$ in its polyphase form (cf. [23])

$$M_j(z_1, z_2) = f_{j0}(z_1^2, z_2^2) + z_1 f_{j1}(z_1^2, z_2^2) + z_2 f_{j2}(z_1^2, z_2^2) + z_1 z_2 f_{j3}(z_1^2, z_2^2).$$

Similarly we have

$$\begin{aligned}
M_j(-z_1, z_2) &= f_{j0}(z_1^2, z_2^2) - z_1 f_{j1}(z_1^2, z_2^2) + z_2 f_{j2}(z_1^2, z_2^2) - z_1 z_2 f_{j3}(z_1^2, z_2^2) \\
M_j(z_1, -z_2) &= f_{j0}(z_1^2, z_2^2) + z_1 f_{j1}(z_1^2, z_2^2) - z_2 f_{j2}(z_1^2, z_2^2) - z_1 z_2 f_{j3}(z_1^2, z_2^2) \\
M_j(-z_1, -z_2) &= f_{j0}(z_1^2, z_2^2) - z_1 f_{j1}(z_1^2, z_2^2) - z_2 f_{j2}(z_1^2, z_2^2) + z_1 z_2 f_{j3}(z_1^2, z_2^2).
\end{aligned}$$

We can easily check

$$A(M_0, M_1, M_2, M_3)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_1 z_2 \end{bmatrix} \begin{bmatrix} f_{00} & f_{10} & f_{20} & f_{30} \\ f_{01} & f_{11} & f_{21} & f_{31} \\ f_{02} & f_{12} & f_{22} & f_{32} \\ f_{03} & f_{13} & f_{23} & f_{33} \end{bmatrix},$$

where $f_{jk} := f_{jk}(z_1^2, z_2^2)$'s. Thus we have the following well-known fact (cf. [10, p. 318]).

Lemma 3.1. *Given M_0, the existence of the matrix $A(M_0, M_1, M_2, M_3)$ such that its determinant is a monomial $Cz_1^{2\mu} z_2^{2\nu}$ is equivalent to the existence of $[f_{jk}]_{0 \le j,k \le 3}$ whose determinant is a monomial.*

It is clear from the expression of $M_0(z_1, z_2)$ associated with box spline $B_{l,m,n}$ that $M_0(z_1, z_2)$, $M_0(-z_1, z_2), M_0(z_1, -z_2), M_0(-z_1, -z_2)$ have no common zeros in $\mathbf{C}^2$, where $\mathbf{C}$ denotes the usual complex space. It follows that $f_{00}, f_{01}, f_{02}, f_{03}$ have no common zeros.

We further claim that for $M_0(z_1, z_2)$, the first three polyphase terms $f_{00}(z_1^2, z_2^2)$, $f_{01}(z_1^2, z_2^2)$, $f_{02}(z_1^2, z_2^2)$ have no common zero in $(\mathbf{C})^2$. Indeed, let us suppose $(\hat{z}_1^2, \hat{z}_2^2) \in (\mathbf{C})^2$ is a common zero of f_{00}, f_{01}, f_{02}. Then we have

$$\begin{aligned}
M_0(\hat{z}_1, \hat{z}_2) &= \hat{z}_1 \hat{z}_2 f_{03}(\hat{z}_1^2, \hat{z}_2^2) \\
M_0(-\hat{z}_1, \hat{z}_2) &= -\hat{z}_1 \hat{z}_2 f_{03}(\hat{z}_1^2, \hat{z}_2^2) \\
M_0(\hat{z}_1, -\hat{z}_2) &= -\hat{z}_1 \hat{z}_2 f_{03}(\hat{z}_1^2, \hat{z}_2^2) \\
M_0(-\hat{z}_1, -\hat{z}_2) &= \hat{z}_1 \hat{z}_2 f_{03}(\hat{z}_1^2, \hat{z}_2^2)
\end{aligned} \tag{3.2}$$

That is,

$$\left(\frac{1+\hat{z}_1}{2}\right)^l \left(\frac{1+\hat{z}_2}{2}\right)^m \left(\frac{1+\hat{z}_1\hat{z}_2}{2}\right)^n = -\left(\frac{1-\hat{z}_1}{2}\right)^l \left(\frac{1+\hat{z}_2}{2}\right)^m \left(\frac{1-\hat{z}_1\hat{z}_2}{2}\right)^n$$

$$= -\left(\frac{1+\hat{z}_1}{2}\right)^l \left(\frac{1-\hat{z}_2}{2}\right)^m \left(\frac{1-\hat{z}_1\hat{z}_2}{2}\right)^n = \left(\frac{1-\hat{z}_1}{2}\right)^l \left(\frac{1-\hat{z}_2}{2}\right)^m \left(\frac{1+\hat{z}_1\hat{z}_2}{2}\right)^n. \tag{3.3}$$

It is easy to see that none of the above four terms is zero. Thus it follows that

$$(1+\hat{z}_1)^l(1+\hat{z}_2)^m = (1-\hat{z}_1)^l(1-\hat{z}_2)^m$$

and

$$(1-\hat{z}_1)^l(1+\hat{z}_2)^m = (1+\hat{z}_1)^l(1-\hat{z}_2)^m.$$

It follows that

$$(1-(\hat{z}_1)^2)^l(1+\hat{z}_2)^{2m} = (1-(\hat{z}_1)^2)^l(1-\hat{z}_2)^{2m}$$

and

$$(1+\hat{z}_1)^{2l}(1-(\hat{z}_2)^2)^m = (1-\hat{z}_1)^{2l}(1-(\hat{z}_2)^2)^m$$

That is, $|1+\hat{z}_2| = |1-\hat{z}_2|$ and $|1+\hat{z}_1| = |1-\hat{z}_1|$. Therefore, it follows $\hat{z}_1 = ai$ and $\hat{z}_2 = bi$ with $i = \sqrt{-1}$ and a, b being real numbers. Putting $\hat{z}_1 = ai$ and $\hat{z}_2 = bi$ back into the four terms (3.3), we have

$$\left(\frac{1+ai}{2}\right)^l \left(\frac{1+bi}{2}\right)^m \left(\frac{1-ab}{2}\right)^n = -\left(\frac{1-ai}{2}\right)^l \left(\frac{1+bi}{2}\right)^m \left(\frac{1+ab}{2}\right)^n$$

or

$$\left(\frac{1+ai}{2}\right)^l \left(\frac{1-ab}{2}\right)^n = -\left(\frac{1-ai}{2}\right)^l \left(\frac{1+ab}{2}\right)^n$$

By taking the absolute value of both sides, we get

$$|1-ab| = |1+ab|.$$

Thus, it follows $ab = 0$. Then $\hat{z}_1 = 0$ or $\hat{z}_2 = 0$ or both. If $\hat{z}_1 = 0$, we will get a contradiction $\hat{z}_2 = -1$ after putting $\hat{z}_1 = 0$ into (3.3). Similar for $\hat{z}_2 = 0$ or both $\hat{z}_1 = 0$ and $\hat{z}_2 = 0$. Therefore, we have verified the claim. Let us formulate the claim as follows.

Lemma 3.2. *Write*

$$
\begin{aligned}
M_0(z_1,z_2) &= \left(\frac{1+z_1}{2}\right)^n \left(\frac{1+z_2}{2}\right)^n \left(\frac{1+z_1z_2}{2}\right)^m \\
&= f_{00}(z_1^2,z_2^2) + z_1 f_{01}(z_1^2,z_2^2) + z_2 f_{02}(z_1^2,z_2^2) + z_1z_2 f_{03}(z_1^2,z_2^2)
\end{aligned}
$$

in the polyphase form. Then f_{00}, f_{01}, f_{02} have no common zeros in $(\mathbf{C})^2$.

Lemma 3.3. *Suppose that $f_{0j}, j=0,\ldots,3$ are polynomials in (z_1,z_2). Suppose that f_{00}, f_{01}, f_{02} have no common zeros in $(\mathbf{C})^2$. Then there exist $f_{k,j}, j=0,1,2,3$ and $k=1,2,3$ such that the matrix $[f_{k,j}]_{0\le k,j\le 3}$ is of determinant ± 1.*

Proof. By the well-known Hilbert Nullstellensatz (cf. [11]), there exist polynomials p_0, p_1, p_2 such that $p_0 f_{00} + p_1 f_{01} + p_2 f_{02} = 1$. Then it is easy to check that

$$
\begin{aligned}
&\begin{bmatrix} f_{00} & 1 & 0 & 0 \\ f_{01} & 0 & 1 & 0 \\ f_{02} & 0 & 0 & 1 \\ f_{03} & -p_0(1-f_{03}) & -p_1(1-f_{03}) & -p_2(1-f_{03}) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -p_0(1-f_{03}) & -p_1(1-f_{03}) & -p_2(1-f_{03}) & 1 \end{bmatrix} \times \begin{bmatrix} f_{00} & 1 & 0 & 0 \\ f_{01} & 0 & 1 & 0 \\ f_{02} & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

which is obviously of determinant -1. This completes the proof. ■

By Lemma 3.2, for M_0, we can find $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3$ such that $A(M_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3)$ has a determinant which is monomial $Cz_1^j z_2^k$ for some j and k and a constant $C \neq 0$. By the definition of determinant of matrices,

$$
\begin{aligned}
Cz_1^j z_2^k &= \det(A(M_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3)) \\
&= M_0(z_1,z_2)\tilde{A}_0(z_1,z_2) + M_0(-z_1,z_2)\tilde{A}_0(-z_1,z_2) \\
&+ M_0(z_1,-z_2)\tilde{A}_0(z_1,-z_2) + M_0(-z_1,-z_2)\tilde{A}_0(-z_1,-z_2)
\end{aligned} \tag{3.4}
$$

which has only the terms whose exponents of z_1 and z_2 are even, where $\tilde{A}_0$ denotes the cofactor of $M_0(z_1,z_2)$. That is, $j = 2j'$, and $k = 2k'$ and hence, $\det(A(M_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3)) = Cz_1^{2j'}, z_2^{2k'}$. Without loss of generality, we may simply assume

$$\det(A(M_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3)) = 1$$

by absorbing $Cz_1^{2j'} z_2^{2k'}$ in one of $\tilde{M}_1, \tilde{M}_2$, and $\tilde{M}_3$. Let us invert the matrix $(A(M_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3))^T$. From the definition of the inverse matrix, we know there exists $\tilde{J}_0, \overline{J}_1, \overline{J}_2$, and $\overline{J}_3$ such that

$$(A(M_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3)^T)^{-1} = A(\tilde{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3)$$

or equivalently,

$$\left(A(\tilde{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3)\right)^{-1} = (A(M_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3))^T.$$

Since the determinant is 1, we know that, by Cramer's rule, M_0 is equal to the cofactor of $\tilde{J}_0$ in matrix $A(\tilde{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3)$. In particular, we have

$$M_0(z_1, z_2) = \det \begin{bmatrix} \overline{J}_1(-z_1, z_2) & \overline{J}_2(-z_1, z_2) & \overline{J}_3(-z_1, z_2) \\ \overline{J}_1(z_1, -z_2) & \overline{J}_2(z_1, -z_2) & \overline{J}_3(z_1, -z_2) \\ \overline{J}_1(-z_1, -z_2) & \overline{J}_2(-z_1, -z_2) & \overline{J}_3(-z_1, -z_2) \end{bmatrix}. \tag{3.5}$$

Note that expanding according to the first column of $A(\tilde{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3)$ and by using the definition of the inverse matrix, we have

$$\begin{aligned} 1 &= \det(A(\tilde{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3) \\ &= \tilde{J}_0(z_1, z_2)M_0(z_1, z_2) + \tilde{J}_0(-z_1, z_2)M_0(-z_1, z_2) + \tilde{J}_0(z_1, -z_2)M_0(z_1, -z_2) \\ &\quad + \tilde{J}_0(-z_1, -z_2)M_0(-z_1, -z_2). \end{aligned}$$

Replacing the first column of matrix $A(\tilde{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3)$ by a column $[\overline{J_0(z_1, z_2)}, \overline{J_0(-z_1, z_2)}, \overline{J_0(z_1, -z_2)}, \overline{J_0(-z_1, -z_2)}]^T$, we get a new matrix $A(\overline{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3)$ whose determinant is

$$\begin{aligned} &\det(A(\overline{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3) \\ =&\overline{J}_0(z_1, z_2)M_0(z_1, z_2) + \overline{J}_0(-z_1, z_2)M_0(-z_1, z_2) + \overline{J}_0(z_1, -z_2)M_0(z_1, -z_2) \\ &\quad + \overline{J}_0(-z_1, -z_2)M_0(-z_1, -z_2) \\ =&1 \end{aligned}$$

by (2.2). We compute the inverse of $A(\overline{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3)$ and write

$$A(\overline{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3)^{-1} = A(q_0, M_1, M_2, M_3)^T.$$

By the definition of the inverse matrices, it is now easy to recognize that $q_0 = M_0$ since (3.5). That is, we have

$$A(\overline{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3)A(M_0, M_1, M_2, M_3)^T = I$$

where I denotes the identity matrix of 4×4 or

$$A(M_0, M_1, M_2, M_3)A(\overline{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3)^T = I$$

which implies (3.1). Therefore, we have obtained the following

Theorem 3.4. *Let* $M_0(z_1,z_2) = \left(\frac{1+z_1}{2}\right)^l \left(\frac{1+z_2}{2}\right)^m \left(\frac{1+z_1z_2}{2}\right)^n$ *and* J_0 *given in Theorem 2.1. Then there exist* M_1, M_2, M_3 *and* J_1, J_2, J_3 *such that the exact reconstruction condition (3.1) holds.*

By extending the arguments in [7] to the two dimensional setting or using a result in [5] or [19], we can conclude the following. The details of the proof or verifications are omitted here.

Theorem 3.5. *Let* ψ_j, $j = 1,2,3$ *and* $\tilde{\psi}_j$, $j = 1,2,3$ *be defined in (1.2) and (1.3) using the* M_j*'s and* J_j*'s constructed above. Let*

$$\psi_{j,k,(\ell_1,\ell_2)}(x,y) = 2^{-k}\psi_j(2^{-k}x - \ell_1, 2^{-k}y - \ell_2)$$
$$\tilde{\psi}_{j,k,(\ell_1,\ell_2)}(x,y) = 2^{-k}\tilde{\psi}_j(2^{-k}x - \ell_1, 2^{-k}y - \ell_2)$$

for $(\ell_1,\ell_2) \in \mathbf{Z}^2$, $k \in Z$, *and* $j = 1,2,3$. *Then the* $\psi_{j,k,(\ell_1,\ell_2)}$*'s and* $\tilde{\psi}_{j,k,(\ell_1,\ell_2)}$*'s constitute two dual Riesz bases of* $L_2(\mathbf{R}^2)$.

4. Examples and Remarks

Let us explain the detail for constructing compactly supported biorthogonal box spline wavelets. For a box spline function $B_{l,m,n}$, we have

$$M_0(z_1,z_2) = \left(\frac{1+z_1}{2}\right)^l \left(\frac{1+z_2}{2}\right)^m \left(\frac{1+z_1z_2}{2}\right)^n.$$

We first compute $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3$ such that $A(M_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3)$ has a determinant $Cz_1^{2j}z_2^{2k}$ for a constant C and some j and k. To this end, we express M_0 in its polyphase form:

$$M_0(z_1,z_2) = f_0(z_1^2,z_2^2) + z_1f_1(z_1^2,z_2^2) + z_2f_2(z_1^2,z_2^2) + z_1z_2f_3(z_1^2,z_2^2)$$

and we find polynomials p_0, p_1, p_2 such that

$$p_0f_0 + p_1f_1 + p_2f_2 = 1$$

by Gröbner's basis method (cf. [1, pp. 53–56.]). Such p_0, p_1, p_2 are not unique. They are dependent on the ordering of monomial basis of bivariate polynomials. For example, we have, for the case associated with the box spline function $B_{1,1,1}$,

$$p_0 = 4, p_1 = -4z_1^2, p_2 = 4.$$

For the case associated with the box spline $B_{2,2,1}$, we have

$$p_0 = 24 + 16z_2^2,$$
$$p_1 = 10 - 4z_2^2,$$
$$p_2 = -6 - 36z_2^2.$$

For the case associated with $B_{2,2,2}$, we have

$$p_0 = 46 + 60z_1^2 + 22z_1^2 z_2^2,$$
$$p_1 = \frac{5}{2} - 65z_1^2 - \frac{15z_1^4}{2} + 22z_1^2 z_2^2,$$
$$p_2 = \frac{13}{2} - \frac{15z_1^2}{2} - \frac{61z_2^2}{2} - \frac{67z_1^2 z_2^2}{2} - 11z_1^2 z_2^4.$$

(In fact, we have implemented the Gröbner basis method in MATHEMATICA and we are able to produce p_0, p_1, p_2 for any box spline function $B_{l,m,n}$. We have tested our programs for all $l, m, n \leq 4$.) With these p_0, p_1, p_2, we first form

$$A(M_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_1 z_2 \end{bmatrix} \times \begin{bmatrix} f_0 & 1 & 0 & 0 \\ f_1 & 0 & 1 & 0 \\ f_2 & 0 & 0 & 1 \\ f_3 & -p_0(1-f_3) & -p_1(1-f_3) & -p_2(1-f_3) \end{bmatrix}.$$

Next we find the inverse of $A(M_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3)^T$:

$$A(\tilde{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3) = (A(M_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3)^T)^{-1}.$$

For any desirable smoothness $\alpha \geq 0$, we choose $\tilde{n}$ and $\tilde{m}$ as in §2 such that $\tilde{B}_{l,m,n} \in C^\alpha(\mathbf{R}^2)$. For the $\tilde{n}$ and $\tilde{m}$, we have J_0 as defined in (2.1) together with (2.5) and (2.6). Replacing the first column of $A(\tilde{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3)^T$ by a vector $[\overline{J_0(z_1, z_2)}, \overline{J_0(-z_1, z_2)}, \overline{J_0(z_1, -z_2)}, \overline{J_0(-z_1, -z_2)}]^T$, we finally compute

$$A(M_0, M_1, M_2, M_3) = (A(\overline{J}_0, \overline{J}_1, \overline{J}_2, \overline{J}_3)^T)^{-1}.$$

Once we have M_j's and J_j's, we use (1.2) and (1.3) to obtain wavelets ψ_j's and ϕ_j's. All the computations above for small l, m, n have been performed by MATHEMATICA. We include our MATHEMATICA program in the appendix.

In the following, we list $\tilde{n}$ and $\tilde{m}$ such that $\tilde{B}_{l,m,n} \in C^{\alpha}(\mathbf{R}^2)$ for small values (l, m, n). For these small values (l, m, n), we may improve the estimate of $\tilde{n}$ and $\tilde{m}$ in Theorem 2.2 by using the results in [18]. Indeed, recall from Table 1 in [18], we have, for Daubechies' scaling function ϕ_n with "minimum phase", the largest exponent $\alpha(n)$ such that

$$\int_{-\infty}^{\infty} (1+|\omega|)^{\alpha(n)} |\hat{\phi}_n(\omega)| d\omega < \infty$$

for $n = 3, \cdots, 9$. That is,

n	3	4	5	6	7	8	9
$\alpha(n)$	1.0831	1.6066	1.9424	2.1637	2.4348	2.7358	3.0432

For example, for $B_{1,1,1}$, we choose $\tilde{n} = 3$ and write $J_0(z_1, z_2) = \hat{J}_0(z_1, z_2)\tilde{J}_0(z_1, z_2)$ with

$$\overline{\hat{J}_0(z_1, z_2)} = \left(\frac{1+z_1}{2}\right)^2 \left(\frac{1+z_2}{2}\right)^2 H(z_1, z_2), \overline{\tilde{J}_0(z_1, z_2)} = \left(\frac{1+z_1 z_2}{2}\right)^{\tilde{m}-1} D(z_1 z_2).$$

Then we have

$$\begin{aligned}
&\int_{\mathbf{R}^2} \prod_{k=1}^{\infty} |\hat{J}_0(e^{i\omega_1/2^k}, e^{i\omega_2/2^k})| d\omega_1 d\omega_2 \\
&\le \int_{\mathbf{R}^2} \prod_{k=1}^{\infty} \left| \left(\frac{1+e^{\frac{i\omega_1}{2^k}}}{2}\right)^2 \left(\frac{1+e^{\frac{i\omega_2}{2^k}}}{2}\right)^2 \right| \left[P_3(\sin^2(\frac{\omega_1}{2^{k+1}}))P_3(\sin^2(\frac{\omega_2}{2^{k+1}}))\right]^{1/2} d\omega_1 d\omega_2 \\
&\le C \int_{\mathbf{R}^2} (1+|\omega_1|)(1+|\omega_2|) |\hat{\phi}_3(\omega_1)| |\hat{\phi}_3(\omega_2)| d\omega_1 d\omega_2 \\
&\le C \int_{-\infty}^{\infty} (1+|\omega_1|)^{\alpha(3)} |\hat{\phi}_3(\omega_1)| d\omega_1 \int_{-\infty}^{\infty} (1+|\omega_2|)^{\alpha(3)} |\hat{\phi}_3(\omega_2)| d\omega_2 \\
&< \infty
\end{aligned}$$

To make $\prod_{k=1}^{\infty} \left|\tilde{J}_0(e^{i\omega_1/2^k}, e^{i\omega_2/2^k})\right| \le C$, we use Theorem 2.2 to choose $\tilde{m} = 25 > 2\left(\frac{3\nu}{2-\nu})\right) + 1$ with $\nu = \log 3/\log 2$. Therefore, $\tilde{B}_{1,1,1} \in C^0$. Similarly, we can find other $\tilde{n}$ and $\tilde{m}$ to make $\tilde{B}_{1,1,1} \in C^1$ or C^2 and so on. We summarize our computation as follows:

$B_{l,m,n}$	$\tilde{B}_{l,m,n}$	$\tilde{B}_{l,m,n}$	$\tilde{B}_{l,m,n}$
	C^0	C^1	C^2
$(1,1,1)$	$\tilde{n} = 3, \tilde{m} = 25$	$\tilde{n} = 6, \tilde{m} = 47$	$\tilde{n} = 9, \tilde{m} = 71$
$(2,2,1)$	$\tilde{n} = 6, \tilde{m} = 47$	$\tilde{n} = 9, \tilde{m} = 71$	
$(2,2,2)$	$\tilde{n} = 6, \tilde{m} = 53$	$\tilde{n} = 9, \tilde{m} = 75$	

Remark 1. It is easy to see that the size of the two low-pass filters M_0 and J_0 are quite different. To balance these two filters, we may factor D into two square roots and factor H into H_1 and H_2 with $|H_1| = |H_2|$. The details will be discussed in our forthcoming paper on the construction of two filters M_0 and J_0 which have the same size.

Remark 2. It is also easy to see that the construction of biorthogonal wavelets in the bivariate setting may be generalized to higher dimensional setting. For details, the reader is referred to another forthcoming paper of ours.

Remark 3. In the above construction, we require that J_0 have the same exponent $\tilde{n}$ for the terms $(1+z_1)$ and $(1+z_2)$ in (2.1). It is interesting to find the explicit formula for J_0 in the form of

$$\overline{J_0(z_1,z_2)} = \left(\frac{1+z_1}{2}\right)^{\tilde{l}-l} \left(\frac{1+z_2}{2}\right)^{\tilde{m}-m} \left(\frac{1+z_1z_2}{2}\right)^{\tilde{n}-n} L(z_1,z_2)$$

for any integers $\tilde{l}$ and $\tilde{m}$ with $\tilde{l}-l>0$ and $\tilde{m}-m>0$. In fact, we are able to construct this polynomial L by using MATHEMATICA. However, a general explicit formulation for L is still under investigation.

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Appendix

The following is a program in MATHEMATICA which produces the masks M_1, M_2, M_3 and J_0, J_1, J_2, J_3 for the given mask associated with box spline function $B_{1,1,1}$ or $B_{2,2,1}$ or $B_{2,2,2}$ and for a given smoothness.

```
(* This MATHEMATICA program computes dual filters associated with box
spline functions with three direction sets (1,1,1), (2,2,1), (2,2,2). *)
n=1; m=1; (* For (1,1,1), choose n=1 and m=1. For (2,2,1), choose n=2,m=1.
For (2,2,2), choose n=2 and m=2. Choose nt and mt large enough for a
smooth biorthogonal wavelets. If m=1, mt=m+ even integer.
If m=2, mt=m+odd integer since mt must be an odd integer. *)
nt=n+2;mt=m+10; mh=mt-1;
(* The mask associated with box spline is *)
Box[x_,y_]:=(1+x)^ n (1+y)^ n (1+x y)^ m/(2^ (2 n+m));
(* Compute the polyphases of the mask. *)
f0[x_,y_]:=(Box[x,y]+Box[-x,y]+Box[x,-y]+Box[-x,-y])/4;
f1[x_,y_]:=(Box[x,y]-Box[-x,y]+Box[x,-y]-Box[-x,-y])/(4 x);
f2[x_,y_]:=(Box[x,y]+Box[-x,y]-Box[x,-y]-Box[-x,-y])/(4 y);
f3[x_,y_]:=(Box[x,y]-Box[-x,y]-Box[x,-y]+Box[-x,-y])/(4 x y);
(* Check if polyphases are right.*)
test1=Expand[Box[x,y]-(f0[x,y]+x f1[x,y]+y f2[x,y] +x y f3[x,y])];
(* Input the known polynomials p0, p1, p2. *)
If[ n==1 && m==1,
{ p0[x_,y_]:=4;
p1[x_,y_]:=-4 x^ 2;
p2[x_,y_]:=4; },
If[ n==2 && m==1,
{ p0[x_,y_]:=24+16*y^ 2;
p1[x_,y_]:=10-4*y^ 2;
p2[x_,y_]:=-6-36*y^ 2; },
If[ n==2 && m==2,
p0[x_,y_]:=46 + 60*x^ 2 + 22*x^ 2*y^ 2;
p1[x_,y_]:=5/2 - 65*x^ 2 - 15*x^ 4/2+ 22*x^ 2*y^ 2;
p2[x_,y_]:=13/2 -15*x^ 2/2-61*y^ 2/2-67*x^ 2*y^ 2/2-11*x^ 2*y^ 4; ]]];
If[m > 2||n > 2, Print["Error: m and n must be less than 3."]; Abort[ ];]
(* Check if Hilbert Nullstellensatz applies correctly. *)
test2=Expand[p0[x,y] f0[x,y]+p1[x,y] f1[x,y]+p2[x,y] f2[x,y]];
A1={{ f0[x,y],f1[x,y],f2[x,y],f3[x,y] },
{ 1,0,0,-p0[x,y] (1-f3[x,y])},
{0,1,0,-p1[x,y] (1-f3[x,y])},
{ 0,0,1,-p2[x,y] (1-f3[x,y])}};
```

```
A2={{1,0,0,0},{0,x,0,0},{0,0,y,0},{0,0,0,xy}};
A3={{1,1,1,1},{1,-1,1,-1},{1,1,-1,-1},{1,-1,-1,1}};
AMT=A3 . A2 . Transpose[A1];
(* AMT is a desirable matrix extension from the mask. *)
AJT=Inverse[AMT];
(* Compute the dual mask. *)
Boxt[x_,y_]:=(1+x)^ nt (1+y)^ nt (1+x y)^ mt/(2^ (nt+nt+mt));
H[x_,y_]:= Sum[ Binomial[2 nt-1,k] ((1+x) (1+y))^ (nt-1-k)
((1-x) (1-y))^ k, { k,0,nt-1}]/4^ (nt-1);
N1=nt+mh/2;
Dau[z_]:=Sum[ Binomial[N1-1+k,k] z^ k, {k,0,N1-1}];
DD[z_]=z^ (-N1) Dau[1/2-z/4-1/(4 z)];
(* Check if box spline mask is dual to the mask we need. *)
See[x_,y_]=Boxt[x,y] H[x,y] DD[x y];
test3=Simplify[Expand[See[x,y]+See[-x,y]+See[x,-y]+See[-x,-y]]];
(* Now the dual mask J0 is *)
nw=nt-n; mw=mt-m;
J0[x_,y_]:= (1+x)^ nw (1+y)^ nw (1+x y)^ mw/(2^ (nw+nw+mw))
H[x,y] DD[x y];
(* Absorb the determinant of AJT into J3. *)
DET=Det[AJT]; jt3=AJT[[4]]; jt4=DET^ (-1) jt3;
JT=Append[Drop[AJT,{4,4}],jt4];
(* Replace jt1 by J0 and its variants. *)
aa={J0[x,y],J0[-x,y],J0[x,-y],J0[-x,-y]};
JN=Prepend[Drop[JT,{1,1}],aa];
MN=Inverse[Transpose[JN]];
(* Check if the first row consists of box spline mask and its variants. *)
test4=Expand[MN[[1]]];
(* The four masks M0, M1, M2, M3 are given below. *)
MS=Factor[Expand[Transpose[MN][[1]]]];
(* The four masks J0, J1, J2, J3 are given below. *)
JS=Factor[Expand[Transpose[JN][[1]]]];
(* Check if MN and JN are inverse each other. *)
test5=Expand[Transpose[MN] . JN];
```