Bivariate Spline Method for Numerical Solution of Steady State Navier-Stokes Equations over Polygons in Stream Function Formulation

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Abstract

We use the bivariate spline finite elements to numerically solve the steady state Navier-Stokes equations. The bivariate spline finite element space we use in this paper is the space of splines of smoothness r and degree $3r$ over triangulated quadrangulations. The stream function formulation for the steady state Navier-Stokes equations is employed. Galerkin's method is applied to the resulting nonlinear fourth order equation, and Newton's iterative method is then used to solve the resulting nonlinear system. We show the existence and uniqueness of the weak solution in $H^2(\Omega)$ of the nonlinear fourth order problem and give an estimate of how fast the numerical solution converges to the weak solution. The Galerkin method with C^1 cubic splines is implemented in MATLAB. Our numerical experiments show that the method is effective and efficient.

1 Introduction

We are interested in using bivariate spline finite elements to numerically solving the steady state Navier-Stokes' equations over a planar polygon Ω. The aim of this research is to provide an efficient numerical tool for fluid dynamics simulation. We intend to make our paper as self-contained as possible so that it may be accessible to engineers, computer programmers, and applied mathematics graduate students as well.

Let $\Omega \subseteq \mathbf{R}^2$ be a simply connected polygonal domain and $\mathbf{u} = (u_1, u_2)^T$ be the planar velocity of a fluid flow over Ω. Also, let p be the pressure function, $\mathbf{f} = (f_1, f_2)^T$ be the external body force of the fluid and $\mathbf{g} = (g_1, g_2)^T$ be the velocity of the fluid flow on the boundary $\partial\Omega$. Then the steady state Navier-Stokes equations are

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, & (x,y)\in\Omega \\ \mathrm{div}\mathbf{u} = 0, & (x,y)\in\Omega \\ \mathbf{u} = \mathbf{g}, & (x,y)\in\partial\Omega, \end{cases} \tag{1.1}$$

where Δ denotes the usual Laplacian operator and ∇ the gradient operator. To motivate our study, we shall first consider the numerical solution of Stokes' equations which are a linearized version of the Navier-Stokes equations. That is, after omitting the nonlinear terms, we have the steady state Stokes' equations:

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, & (x,y)\in\Omega \\ \mathrm{div}\mathbf{u} = 0, & (x,y)\in\Omega \\ \mathbf{u} = \mathbf{g} & (x,y)\in\partial\Omega\ . \end{cases} \tag{1.2}$$

Recall the fact that there exists a stream function φ such that $\mathbf{u} = \mathbf{curl}\ \varphi$, i.e., $u_1 = \dfrac{\partial\varphi}{\partial y}$, $u_2 = -\dfrac{\partial\varphi}{\partial x}$. Such φ is unique up to a constant (cf. [Girault and Raviart'86, pp. 37–39].) Thus we may simplify the above Stokes and Navier-Stokes equations by cancelling the pressure term. Consider the Stokes equations (1.2) first. Replacing $\mathbf{u}$ by $\mathbf{curl}\varphi$ and then differentiating the first equation with respect to y and the second with respect to x, we subtract the second equation from the first one to obtain the following fourth order equation

$$\nu\Delta^2\varphi = h$$

with $h = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$. Thus, the Stokes equations become a fourth order biharmonic equation:

$$\begin{cases} \nu\Delta^2\varphi = h, & \text{in } \Omega \\ \dfrac{\partial\varphi}{\partial x} = -g_2, & \text{on } \partial\Omega \\ \dfrac{\partial\varphi}{\partial y} = g_1, & \text{on } \partial\Omega \\ \varphi = h_2, & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where h_2 is an anti-derivative of the tangential derivative of φ along $\partial\Omega$ and will be examined in detail later. By a similar calculation, we easily see that the Navier-Stokes equations become the following fourth order nonlinear equation

$$\begin{cases} \nu\Delta^2\varphi - \dfrac{\partial}{\partial y}\left(\dfrac{\partial\varphi}{\partial y}\dfrac{\partial^2\varphi}{\partial x\partial y} - \dfrac{\partial\varphi}{\partial x}\dfrac{\partial^2\varphi}{\partial y^2}\right) & \\ \qquad -\dfrac{\partial}{\partial x}\left(\dfrac{\partial\varphi}{\partial y}\dfrac{\partial^2\varphi}{\partial x^2} - \dfrac{\partial\varphi}{\partial x}\dfrac{\partial^2\varphi}{\partial x\partial y}\right) = h, & \text{in } \Omega \\ \dfrac{\partial\varphi}{\partial x} = -g_2, & \text{on } \partial\Omega \\ \dfrac{\partial\varphi}{\partial y} = g_1, & \text{on } \partial\Omega \\ \varphi = h_2, & \text{on } \partial\Omega\ . \end{cases} \tag{1.4}$$

Let $H^2(\Omega)$ be the usual Sobolev space and $H_0^2(\Omega)$ be the subspace of $H^2(\Omega)$ of functions whose derivatives of order less than or equal to one all vanish on the boundary $\partial\Omega$. Define the bilinear form $a_2(\varphi,\psi)$ and trilinear form $b(\theta,\varphi,\psi)$ by

$$\begin{aligned} a_2(\varphi,\psi) &= \int_\Omega \Delta\varphi(x,y)\Delta\psi(x,y)dxdy \\ b(\theta,\varphi,\psi) &= \int_\Omega \Delta\theta(x,y)\left(\frac{\partial\varphi(x,y)}{\partial x}\frac{\partial\psi(x,y)}{\partial y} - \frac{\partial\varphi(x,y)}{\partial y}\frac{\partial\psi(x,y)}{\partial x}\right)dxdy \end{aligned}$$

and denote the $L_2(\Omega)$ inner product by

$$\langle h,\psi\rangle = \int_\Omega h(x,y)\psi(x,y)dxdy.$$

We say $\varphi \in H^2(\Omega)$ is a weak solution of the Stokes equations (1.3) if φ satisfies the following

$$\begin{cases} \nu a_2(\varphi,\psi) = \langle h,\psi\rangle, & \forall\psi \in H_0^2(\Omega) \\ \dfrac{\partial\varphi}{\partial x} = -g_2, & \text{on } \partial\Omega \\ \dfrac{\partial\varphi}{\partial y} = g_1, & \text{on } \partial\Omega \\ \varphi = h_2, & \text{on } \partial\Omega\ . \end{cases} \tag{1.5}$$

Similarly, a function $\varphi \in H^2(\Omega)$ is a weak solution of the Navier-Stokes equations (1.4) if φ satisfies

$$\begin{cases} \nu a_2(\varphi,\psi) + b(\varphi,\varphi,\psi) = \langle h,\psi\rangle, & \forall\psi \in H_0^2(\Omega) \\ \dfrac{\partial\varphi}{\partial x} = -g_2, & \text{on } \partial\Omega \\ \dfrac{\partial\varphi}{\partial y} = g_1, & \text{on } \partial\Omega \\ \varphi = h_2, & \text{on } \partial\Omega\ . \end{cases} \tag{1.6}$$

Such weak formulations are referred as the stream function formulation of the Stokes and Navier-Stokes equations, respectively. It is known that the weak solution for Stokes' equations exists and is unique for any $\nu > 0$. (See, e.g., [Girault and Raviart'86].) For the Navier-Stokes equations, it can be shown that such a weak solution exists for any $\nu > 0$, and is unique when ν is sufficiently large.

By taking divergence, div, of the equations in (1.1) and (1.2), we can easily see that the pressure functions p of the Stokes and Navier-Stokes equations satisfy the following Poisson equations with nonhomogeneous Neumann boundary conditions involving the stream functions.

$$\begin{cases} -\Delta p = -\mathrm{div}(\mathbf{f}), & \text{in } \Omega \\ \dfrac{\partial p}{\partial n} = \nu\Delta(n\cdot\mathbf{curl})\varphi + n\cdot\mathbf{f} & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

for the Stokes equation and

$$\begin{cases} -\Delta p = -\mathrm{div}(\mathbf{f}) + \mathrm{div}[(\mathbf{curl}\varphi\cdot\nabla)\mathbf{curl}(\varphi)], & \text{in } \Omega \\ \dfrac{\partial p}{\partial n} = \nu\Delta(n\cdot\mathbf{curl})\varphi + n\cdot\mathbf{f} + \\ \qquad n\cdot[(\mathbf{curl}\varphi\cdot\nabla)(\mathbf{curl}\varphi)], & \text{on } \partial\Omega \end{cases} \tag{1.8}$$

for the Navier-Stokes equations.

An advantage of using the stream function formulation instead of the traditional velocity-pressure formulation and stream function-vorticity formulation is that the system of nonlinear equations has much smaller size. Indeed, let N, M, and L be the dimension of the C^0 quadratic finite element space, the bivariate spline space to be introduced below, and the linear finite element space, respectively. In the velocity-pressure formulation (cf. [Temam'77]), one needs $2N$ unknowns to approximate the velocity vector and L unknowns for the pressure function and thus, the system of nonlinear equations is of size $(2N + L) \times (2N + L)$. In the stream function-vorticity formulation, one needs N unknowns to approximate the stream function and N unknowns to approximate the vorticity function and hence, the size of the system is $2N \times 2N$. In the stream function formulation, the size of the system is $M \times M$. Since the bivariate splines have a much better approximation power, one may choose $M << N$ to get the same accuracy. Since the system is nonlinear, we have to use Newton's iterative method and hence, the linearized system has to be solved many times. Thus, the smaller the size of the system, the more robust the fluid flow simulation becomes.

Since we are going to solve the Stokes and Navier-Stokes equations in the stream function formulation, higher order finite elements are definitely needed. It is known that setting up a linear system using higher order finite elements takes more time than that using linear or

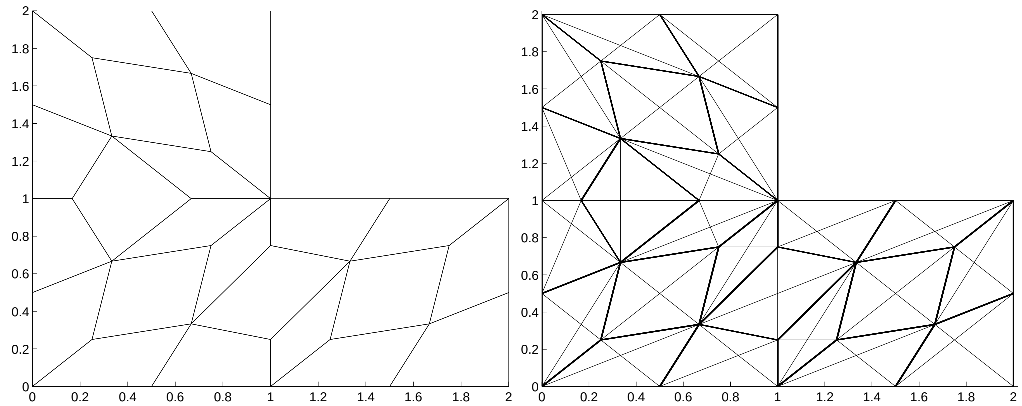

Figure 1. Quadrangulation and Derived triangulation of L-shaped Domain

quadratic finite elements. However, we should note that the time for setting up a linear system using the bivariate spline functions is linearly increasing as the number of the triangles in the underlying triangulation while the time for solving a linear system is cubically increasing (direct methods) or quadratically increasing (iterative methods). Since the spline functions usually give a much better approximation than linear and quadratic finite elements, for the same tolerance the spline method requires a smaller number of triangles in a triangulation of the underlying polygonal domain Ω than the linear and quadratic finite element method. Hence, for sufficiently small tolerances the total time for computing numerical solutions using the spline method will be less than that of the linear or quadratic finite element method.

Refer to Tables 1a, 1b, 2a and 2b, for the accuracy which the spline method can achieve. Once we compute the stream function, we then solve for the pressure function. It should be noted that as the weak form of the boundary conditions for this problem involve second order derivatives of the stream function, a high order approximation of the stream function is necessary. Refer to Table 4 for the accuracy of the pressure function approximations by the standard linear finite element method.

Let us now introduce the bivariate spline space to be used to approximate the stream functions for the Stokes and Navier-Stokes equations. Let $\diamondsuit$ be a quadrangulation of Ω which consists of convex quadrilaterals and $\diamondplus$ be the special triangulation obtained from $\diamondsuit$ by adding the two diagonals of each quadrilateral in $\diamondsuit$. See, e.g., Figure 1 for an example of a quadrangulation of an L-shape domain and a derived triangulation. Let

$$S_{3r}^{r}(\diamondplus) = \{s \in C^{r}(\Omega) : s\,|_{t} \in \mathbf{P}_{3r},\ \forall\ t \in \diamondplus\}$$

be the C^r spline space over a triangulation $\diamondplus$, where $\mathbf{P}_{3r}$ is the space of all polynomials of total degree $\leq 3r$ and t denotes any triangle in $\diamondplus$. When $r = 1$, this spline space is not new. It is the space of Fraeijs en Veubeke and Sander's finite elements which was introduced in the mid 60's. (See [Fraeijs en Veubeke'65] and [Sander'64].) In a recent work [Lai and Schumaker'99], the construction of locally supported basis functions for a subspace of $S^r_{3r}(\diamondplus)$ and the approximation properties of the subspace were given. This enables us to use it for solving Navier-Stokes equations.

Although quadrangulations may not be as flexible as triangulations, a quadrangulation can always be derived from a triangulation. We may triangulate any given polygonal domain Ω first and then convert it into a quadrangulation by connecting the center of each triangle t to the midpoint of the three edges of t. An advantage of quadrangulations over triangulations is that one may easily throw the slim quadrilaterals away by making the two vertices of the shorter diagonal of a slim quadrilateral into one. Bezier nets provide an easy vehicle to manipulate the spline functions in $S^r_{3r}(\diamondplus)$. (See [Farin '86] for reference on the definition and the properties of Bezier nets.) In particular, Bezier nets have been used to construct locally supported basis functions for $S^1_3(\diamondplus)$ (cf. [Lai'96]). Any quadrangulation may be refined as follows: For each quadrilateral q, connect the midpoint of each of the four sides to the intersection of the two diagonals of q. We obtain four subquadrilaterals for each quadrilateral. Hence, letting $\diamondplus_1$ be the special triangulation obtained from the refinement of $\diamondplus$, it is easy to see that $S^r_{3r}(\diamondplus) \subset S^r_{3r}(\diamondplus_1)$. Also, any quadrangulation admits local refinements (c.f. [Lai and Wenston'96]) which is necessary for computing solutions with singularities.

In the following sections, we shall show that there exists a weak solution S_φ in $S^r_{3r}(\diamondplus)$ for the Stokes and Navier-Stokes equations and that the solution is unique for any ν for the Stokes equations and for sufficiently large ν for the Navier-Stokes equations. We shall also show that $\mathbf{curl}(S_\varphi)$ converges to $\mathbf{u}$ when $|\diamondplus| \to 0$, where $|\diamondplus|$ denotes the largest diameter of the triangles in $\diamondplus$. In particular, we shall show that when $\varphi \in H^k(\Omega)$ for $4 \le k \le 3r+1$,

$$\|\mathbf{u} - \mathbf{curl}(S_\varphi)\|_{L_2(\Omega)} \le C|\diamondplus|^{k-2}$$

for the Stokes equations and for the Navier-Stokes equations when ν is sufficiently large. Here, C is a positive constant independent of $\mathbf{u}$.

Besides spline space $S^r_{3r}(\diamondplus)$, we shall also use the linear finite element space. Let $S^0_1(\diamondplus)$ be the standard linear finite element space over the triangulation $\diamondplus$. For the pressure function p, let $S_p \in S^0_1(\diamondplus)$ be the weak solution of the Poisson equations (1.7) and (1.8) associated with the Stokes and Navier-Stokes equations. We shall show that

$$\|p - S_p\|_{L_2(\Omega)} \le C|\diamondplus|,$$

where C is a positive constant independent of p. In general, when p is sufficiently smooth, a much better convergence rate can be obtained by using $S^{r-1}_{3r-3}(\diamondplus)$.

We note that the numerical experiments presented in this paper indicate a much better convergence rate than the above estimates for many functions over an L-shape domain. It is known that for Navier-Stokes equations, the Reynolds number ($= 1/\nu$) plays an important role. We shall note that when a triangulation is refined, we are able to compute a numerical solution of the Navier-Stokes equations with larger Reynolds number. For example, we have tested the well-known cavity fluid flow problem. We are able to compute a reasonable numerical solution with Reynolds number over 30,000(see §6.). For another example, over the L-shape domain inside $[0, 2] \times [0, 2]$, we are able to compute a good approximation to many stream functions for Navier-Stokes equations with Reynolds number 10,000. See Table 3.

The organization of this paper is as follows. After reviewing the weak solution of biharmonic equations and introducing the approximation properties of $S^r_{3r}(\diamondplus)$ in Section 2, the study of numerical solutions in $S^r_{3r}(\diamondplus)$ for the Stokes equations will be presented in Section 3. The analysis of the homogeneous Navier-Stokes equations is contained in Section 4 while the analysis of the general nonhomogeneous Navier-Stokes equations is in Section 5. It is straightforward to compute numerical approximations in $S^r_{3r}(\diamondplus)$ for the Stokes equations and we shall present several numerical experiments in §3. To compute the weak solution in $S^r_{3r}(\diamondplus)$ for the Navier-Stokes equations, we have to use Newton's iteration method. We shall show that the iterates from Newton's method converge to a weak solution. This part of the analysis will be contained in Section 6. Finally, in section 7, we will study the numerical approximation of the pressure functions. All numerical examples demonstrate that the spline method studied in this paper provides an effective and efficient approximation of the solution of these linear and nonlinear partial differential equations.

2 Preliminaries

Let $L_2(\Omega)$ ($L_2(\partial\Omega)$) be the space of all square integrable function on Ω ($\partial\Omega$). Let $\alpha = (\alpha_1, \alpha_2)$ be a nonnegative bi-integer, $|\alpha| = \alpha_1 + \alpha_2$ and

$$\partial^\alpha = \left(\frac{\partial}{\partial x}\right)^{\alpha_1} \left(\frac{\partial}{\partial y}\right)^{\alpha_2}$$

denote the usual partial derivative operator. For each integer $m \geq 0$, we define the Sobolev space

$$H^m(\Omega) = \{f \in L_2(\Omega);\ \partial^\alpha f \in L_2(\Omega), \forall |\alpha| \leq m\}$$

which is a Hilbert space equipped with the norm

$$||f||_{m,\Omega} = \left(\sum_{|\alpha|\le m} \int_\Omega |\partial^\alpha f(x,y)|^2 dx dy \right)^{1/2}$$

and the scalar inner product

$$(f,g)_{m,\Omega} = \sum_{|\alpha|\le m} \int_\Omega \partial^\alpha f(x,y) \partial^\alpha g(x,y) dx dy.$$

We also use the following inner product and seminorm:

$$\langle f, g\rangle = \int_\Omega f(x,y) g(x,y) dx dy$$

$$|f|_{m,\Omega} = \left(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha f|^2 dx dy \right)^{1/2}.$$

For later applications, we shall use the following subspace of $H^m(\Omega)$.

$$H_0^m(\Omega) := \{f \in H^m(\Omega) : \partial^\alpha f|_{\partial\Omega} = 0, \forall |\alpha| \le m-1\}.$$

It is a consequence of the well-known Poincaré inequality (cf. Lemma 2.2 below), that the semi-norm $|f|_{m,\Omega}$ is equivalent to the norm $||f||_{m,\Omega}$ on $H_0^m(\Omega)$. The following five lemmas are well-known and will be used over and over. We quote them here for convenience. We shall give a proof or justification for Lemmas 2.7 and 2.8.

LEMMA 2.1. (The Lax-Milgram Theorem) Let V be a closed subspace of $H^m(\Omega)$. Let $a(f,g)$ be a bilinear form on V for which there exist constants α and M such that

$$a(f,g) \le M\|f\|_{m,\Omega}\|g\|_{m,\Omega}, \quad \forall f,g \in V$$

and

$$a(f,f) \ge \alpha \|f\|_{m,\Omega}^2, \quad \forall f \in V.$$

Let $F(f)$ be a bounded linear functional on V. Then there exists a unique $u \in V$ such that

$$a(u,g) = F(g), \quad \forall g \in V.$$

LEMMA 2.2. (Poincaré's inequality) For any $f \in H_0^1(\Omega)$,

$$\|f\|_{0,\Omega} \le K_0 |f|_{1,\Omega}$$

where K_0 denotes a positive constant dependent only on Ω.

LEMMA 2.3. (The second Poincaré's inequality) For any $p \in H^1(\Omega)$ such that $\int_\Omega p\, dxdy = 0$,

$$\|p\|_{0,\Omega} \le K_1 |p|_{1,\Omega}$$

for a constant K_1 independent of p.

LEMMA 2.4. (The trace Theorem) For any $p \in H^1(\Omega)$,

$$\|p\|_{0,\partial\Omega} \le K_2 \|p\|_{1,\Omega}$$

for a constant K_2 independent of p.

We also denote by $H^{-m}(\Omega)$ the dual space of $H_0^m(\Omega)$ normed by

$$\|f\|_{-m,\Omega} := \sup_{\substack{g \in H_0^m(\Omega) \\ g \neq 0}} \frac{\langle f, g\rangle}{|g|_{m,\Omega}}.$$

It is easy to prove the following by using the previously mentioned Lax-Milgram Theorem.

LEMMA 2.5. For any given $h \in H^{-2}(\Omega)$, there exists a unique $\varphi \in H_0^2(\Omega)$ satisfying the homogeneous biharmonic equation in the following weak formulation:

$$a_2(\varphi, \psi) = \langle h, \psi\rangle, \quad \forall \psi \in H_0^2(\Omega).$$

Since Ω is a polygon, $\partial\Omega$ is a collection of line segments, $\Gamma_i = [w_i, w_{i+1}], 1 \le i \le J$ with $w_{J+1} := w_1$, arranged in the counter clockwise direction. Let τ_i and n_i be the unit tangent and unit outward normal vectors on Γ_i and let $\frac{\partial}{\partial\tau} f$ denote the tangential derivative of f on $\partial\Omega$, i.e., $\frac{\partial}{\partial\tau} f|_{\Gamma_i} = \frac{\partial}{\partial\tau_i} f|_{\Gamma_i} 1 \le i \le J$. Similarly, let $\frac{\partial}{\partial n} f$ denote the normal derivative of f on $\partial\Omega$.

Thus, the boundary conditions of both the Stokes and Navier-Stokes equations may be rewritten as

$$\frac{\partial}{\partial n_i} \varphi|_{\Gamma_i} = -(g_1, g_2) \cdot \tau_i, \quad \varphi|_{\Gamma_i} = h_{2,i}, \quad i = 1, \ldots, J. \tag{2.1}$$

where $\varphi(w_1) = 0$ and

$$h_{2,i}(x, y) = \int_{\cup_{j=1}^{i-1} \Gamma_j \cup [w_i, (x,y)]} (g_1, g_2) \cdot n_i ds, \quad \forall (x, y) \in \Gamma_i.$$

Since Ω is a simply connected, $h_{2,J}(w_{J+1}) = 0$ by the Divergence theorem and the divergence free condition div$\mathbf{u} = 0$. In general, we consider the following nonhomogeneous boundary value problem for the biharmonic equation:

$$\begin{cases} \Delta^2 \varphi = h & \text{in } \Omega \\ \dfrac{\partial}{\partial n_i}\varphi|_{\Gamma_i} = h_{1,i}, & 1 \le i \le J \\ \varphi|_{\Gamma_i} = h_{2,i}, & 1 \le i \le J \end{cases} \tag{2.2}$$

with $h_{2,i}(w_{i+1}^-) = h_{2,i+1}(w_{i+1}^+), i = 1, \ldots, J$, and $w_{J+1} = w_1$. For convenience, let $h_1(x,y) = h_{1,j}(x,y), (x,y) \in \Gamma_j$ and $h_2(x,y) = h_{2,j}(x,y), (x,y) \in \Gamma_j$, $j = 1, \cdots, J$.

Let us introduce the following concept of the compatibility of the boundary conditions. In this way, we are able to concentrate on the discussion of numerical solutions of the Stokes and Navier-stokes equations. (See [Grisvard'85] for a detailed explanation of necessary and sufficient conditions on boundary functions to ensure a solution of the Stokes equations over polygonal domains.)

Definition 2.6. Let Ω be a polygon and let h_1 and h_2 be two functions defined on $\partial\Omega$. We say that h_1 and h_2 are compatible with the polygonal domain Ω if there exists a $\varphi_{h_1,h_2} \in H^2(\Omega)$ such that

$$\frac{\partial \varphi_{h_1,h_2}}{\partial n} = h_1, \text{ and } \varphi_{h_1,h_2} = h_2, \text{ on } \partial\Omega$$

Note that the linear independence of the tangential directions τ_i allow us to write the normal directions n_i as the linear combinations

$$n_i = \alpha_i \tau_i + \beta_i \tau_{i+1}, \quad n_{i+1} = \eta_i \tau_i + \delta_i \tau_{i+1}.$$

Compatibility of the functions h_1 and h_2 implies the equations

$$h_1(w_{i+1}^-) = \alpha_i \frac{\partial h_2}{\partial \tau_i}(w_{i+1}^-) + \beta_i \frac{\partial h_2}{\partial \tau_{i+1}}(w_{i+1}^+)$$

and

$$h_1(w_{i+1}^+) = \eta_i \frac{\partial h_2}{\partial \tau_i}(w_{i+1}^-) + \delta_i \frac{\partial h_2}{\partial \tau_{i+1}}(w_{i+1}^+)$$

for $i = 1, \cdots, J$. With Definition 2.6, we define two norms of any pair of compatible boundary conditions (h_1, h_2) by

$$\|(h_1, h_2)\|_2 = \inf\left\{|\varphi_{h_1,h_2} - s|_{2,\Omega} : s \in H_0^2(\Omega)\right\}$$

and

$$\|(h_1,h_2)\|_1 = \inf\left\{|\varphi_{h_1,h_2} - s|_{2,\Omega} + |\varphi_{h_1,h_2} - s|_{1,\Omega} : s \in H_0^2(\Omega)\right\}, \tag{2.3}$$

where φ_{h_1,h_2} is as defined in Definition 2.6.

LEMMA 2.7. Suppose that h_1 and h_2 are compatible. Then there exists an unique weak solution $\varphi \in H^2(\Omega)$ such that

$$a_2(\varphi,\psi) = 0, \quad \forall \psi \in H_0^2(\Omega) \tag{2.4}$$

with $\frac{\partial}{\partial n}\varphi| = h_1$ and $\varphi = h_2$ on $\partial\Omega$. Furthermore, φ satisfies the following inequalities:

$$|\varphi|_{2,\Omega} \le 2\|(h_1,h_2)\|_2$$

and

$$|\varphi|_{1,\Omega} \le K_3\|(h_1,h_2)\|_1.$$

for a constant $K_3 = \max\{K_0, 1\}$ with K_0 as in Lemma 2.2.

Proof: Since h_1 and h_2 are compatible, let φ_{h_1,h_2} be a function in $H^2(\Omega)$ satisfying the boundary conditions. By the well-known Lax-Milgram theorem, there exists unique $\theta \in H_0^2(\Omega)$ satisfying

$$a_2(\theta,\psi) = a_2(\varphi_{h_1,h_2},\psi), \quad \forall \psi \in H_0^2(\Omega).$$

It follows that

$$|\theta|_{2,\Omega}^2 = a_2(\theta,\theta) = a_2(\varphi_{h_1,h_2},\theta) \le |\varphi_{h_1,h_2}|_{2,\Omega}|\theta|_{2,\Omega}.$$

Thus, we have

$$|\theta|_{2,\Omega} \le |\varphi_{h_1,h_2}|_{2,\Omega}.$$

Similarly, for any $s \in H_0^2(\Omega)$, $\varphi_{h_1,h_2} - s$ satisfies the boundary conditions and $\theta - s$ is the unique solution in $H_0^2(\Omega)$ satisfying

$$a_2(\theta - s,\psi) = a_2(\varphi_{h_1,h_2} - s,\psi), \quad \forall \psi \in H_0^2(\Omega)$$

and

$$|\theta - s|_{2,\Omega} \le |\varphi_{h_1,h_2} - s|_{2,\Omega}.$$

Let $\varphi = \varphi_{h_1,h_2} - s - \theta + s$. Then φ satisfies (2.4) with the nonhomogeneous boundary conditions. Furthermore,

$$|\varphi|_{2,\Omega} \le 2|\varphi_{h_1,h_2} - s|_{2,\Omega}$$

for any $s \in H_0^2(\Omega)$. Thus,

$$\begin{aligned} |\varphi|_{2,\Omega} &\leq 2\inf\left\{|\varphi_{h_1,h_2} - s|_{2,\Omega}, s \in H_0^2(\Omega)\right\} \\ &= 2\|(h_1,h_2)\|_2. \end{aligned}$$

Furthermore, by Lemma 2.2, we have

$$|\theta - s|_{1,\Omega} \leq K_0|\theta - s|_{2,\Omega} \leq K_0|\varphi_{h_1,h_2} - s|_{2,\Omega}.$$

Thus,

$$\begin{aligned} |\varphi|_{1,\Omega} &\leq |\theta - s|_{1,\Omega} + |\varphi_{h_1,h_2} - s|_{1,\Omega} \\ &\leq K_3(|\varphi_{h_1,h_2} - s|_{1,\Omega} + |\varphi_{h_1,h_2} - s|_{2,\Omega}) \end{aligned}$$

for any $s \in H_0^2(\Omega)$ with $K_3 = \max\{K_0, 1\}$. Hence, we obtain

$$|\varphi|_{1,\Omega} \leq K_3\|(h_1,h_2)\|_1.$$

The uniqueness of φ is an immediate consequence of Lemma 2.5. ■

Next let us introduce the approximation properties of the bivariate spline space $S_{3r}^r(\triangle)$, In [Lai and Schumaker'99], a locally supported basis $\{\phi_\xi, \xi \in \Gamma\}$ for a super spline subspace of $S_{3r}^r(\triangle)$ was constructed. The basis functions ϕ_ξ's satisfy the following properties: for any triangulation $\triangle$ of Ω,

- H1) the ϕ_ξ are locally supported, that is, the support $\sigma(\phi)$ is contained in $\text{star}^\ell(v)$ for a vertex $v \in \triangle$;
- H2) the functions $\phi_\xi, \xi \in \Gamma$ are uniformly bounded on Ω by a constant K_4;
- H3) for each triangle $T \in \triangle$, the number of ϕ_ξ's whose support contains T is bounded by a constant K_5;

where ℓ is a constant independent of Ω and constants K_4 and K_5 may only be dependent on the smallest angle of triangles in $\triangle$. Here, we recall that given a vertex v in a triangulation, $\text{star}^0(v)$ is the union of all triangles which share the vertex v. The *star of order* ℓ, $\text{star}^\ell(v)$, is defined recursively as $\text{star}^\ell(v) := \{\cup T : T \text{ shares a vertex with } \tilde{T} \in \text{star}^{\ell-1}(v)\}$. (See [Lai and Schumaker'99].)

With these basis functions, we are able to construct a quasi-interpolant operator of the form

$$Q_m f = \sum_{\xi \in \Gamma} (\lambda_{\xi,m} f) \phi_\xi$$

with $1 \le m \le 3r$ satisfying the following properties:

- H4) there is a constant K_6 such that for each $\xi \in \Gamma$, there is a triangle T_ξ contained in the support of ϕ_ξ such that

$$|\lambda_{\xi,m} f| \le \frac{K_6}{\sqrt{A_{T_\xi}}} \|f\|_{0,T_\xi} \quad \text{for all } f \in L_2(\Omega),$$

  where A_{T_ξ} denotes the area of triangle T_ξ.

- H5) the operator Q_m reproduces $\mathbf{P}_m$ in the sense that $Q_m P = P$ for all $P \in \mathbf{P}_m$.

In fact, we may choose

$$\lambda_{\xi,m} f = \gamma_\xi I_{\xi,m} f$$

where γ_ξ is a linear functional which picks off the Bézier coefficient of spline functions in $S^0_{3r}(\Diamond)$ associated with index ξ and $I_{\xi,m}$ denotes the polynomial interpolation operator associated with the index ξ which reproduces all polynomials of degree $\le m$. By Theorem 5.3 in [Lai and Schumaker'98], we are able to conclude the following

Lemma 2.8. Fix $r \ge 1$ and $1 \le m \le 3r$. There exists a linear quasi-interpolation operator Q_m mapping $L_1(\Omega)$ into $S^r_{3r}(\Diamond)$ such that if f is in the Hilbert space $H^{m+1}(\Omega)$, then

$$\|f - Q_m f\|_{k,\Omega} \le K_7 |\Diamond|^{m+1-k} \, |f|_{m+1,\Omega},$$

for all $0 \le k \le m$. Here $|\Diamond|$ is the maximum of the diameters of the triangles in $\Diamond$. If Ω is convex, then the constant K_7 depends only on r, p, m, and on the smallest angle in $\Diamond$. If Ω is nonconvex, C also depends on the Lipschitz constant $L_{\partial\Omega}$ associated with the boundary of Ω.

Note that $\|f - Q_m f\|_{k,\Omega}$ is a mesh dependent norm when $k > r$ in the sense that

$$\|f - Q_m f\|^2_{k,\Omega} = \sum_{t \in \Diamond} \|f - Q_m f\|^2_{k,t}.$$

We note that for $r = 1$, this result was given in [Ciavaldini and Nedelec'74]. We are now able to generalize it to arbitrary integer $r \ge 1$.

3 Numerical Solution of the Stokes Equations

In this section, we consider the following biharmonic equation with nonhomogeneous boundary conditions h_1 and h_2, where h_1 and h_2 are assumed to be compatible in the sense of Definition 2.6:

$$\begin{cases} \nu\Delta^2\varphi = h & \text{in } \Omega \\ \dfrac{\partial}{\partial n}\varphi = h_1, & \text{on } \partial\Omega \\ \varphi = h_2, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We are going to solve the above equation by using spline functions in $S_{3r}^r(\triangle)$. Although in general we may not be able to find a spline function $s \in S_{3r}^r(\triangle)$ satisfying the boundary conditions exactly, we can find a spline $S_b \in S_{3r}^r(\triangle)$ approximating the boundary conditions in the following sense: letting $v_1, \cdots, v_B$ be the boundary vertices of the triangulation $\triangle$ arranged in the counter-clockwise direction,

$$\begin{aligned} S_b(v_i + j(v_{i+1} - v_i)/(3r)) &= h_2(v_i + j(v_{i+1} - v_i)/(3r)), j = 0, \cdots, 3r \\ \frac{\partial S_b}{\partial n_i}\Big|_{\Gamma_i}\left(v_i + \frac{j}{3r-1}(v_{i+1} - v_i)\right) &= h_1\left(v_i + \frac{j}{3r-1}(v_{i+1} - v_i)\right), j = 0, \cdots, 3r-1. \end{aligned}$$

These equations determine the coefficients of the polynomial pieces of S_b on $\partial\Omega$. For convenience let $\tilde{h}_1 := \frac{\partial}{\partial n} S_b|_{\partial\Omega}$ and $\tilde{h}_2 := S_b|_{\partial\Omega}$. Let

$$V_0 := S_{3r}^r(\triangle) \cap H_0^2(\Omega)$$

be the subspace of $S_{3r}^r(\triangle)$ consisting of those splines which vanish and have vanishing normal derivative on $\partial\Omega$. We now are in a position to prove the following:

THEOREM 3.1. There exists an unique spline function $S_\varphi \in S_{3r}^r(\triangle)$ satisfying the following weak formulation

$$\nu a_2(S_\varphi, \psi) = \langle h, \psi \rangle, \forall \psi \in V_0$$

and the approximate boundary conditions

$$S_\varphi|_{\partial\Omega} = \tilde{h}_2, \frac{\partial}{\partial n} S_\varphi|_{\partial\Omega} = \tilde{h}_1.$$

Proof: Let $\{\psi_i\}_{i=1}^N \subset V_0$ be a locally supported spline basis and $\{\psi_i\}_{i=N+1}^{N+M} \subset S_{3r}^r(\triangle)$ be a set of locally supported functions such that

$$\{\psi_i\}_{i=1}^N \cup \{\psi_{i+N}\}_{i=1}^M$$

is a locally supported basis of $S_{3r}^r(\diamondsuit)$. For each $S_\varphi \in S_{3r}^r(\diamondsuit)$ satisfying the boundary conditions, we may write $S_\varphi = \sum_{i=1}^N e_i\psi_i + \sum_{i=1}^M b_{i+N}\psi_{i+N}$ with known coefficients b_{i+N}'s which are determined by the approximate boundary conditions, i.e., $S_\varphi|_{\partial\Omega} = \sum_{i=1}^M b_{i+N}\psi_{i+N}|_{\partial\Omega} = \tilde{h}_2$ and $\frac{\partial}{\partial n}S_\varphi|_{\partial\Omega} = \sum_{i=1}^M b_{i+N}\frac{\partial}{\partial n}\psi_{i+N} = \tilde{h}_1$. To find S_φ satisfying the weak formulation, we need to solve

$$\nu \sum_{i=1}^N c_i a_2(\psi_i, \psi_j) = \langle h, \psi_j\rangle - \sum_{i=1}^M b_{i+N} a_2(\psi_{i+N}, \psi),$$

for $j = 1, \ldots, N$. This linear system is nonsingular. Indeed, if there is a vector $\mathbf{c}_0 = (c_1^0, c_2^0, \ldots, c_N^0)^T$ such that

$$[a_2(\psi_i, \psi_j)]\,\mathbf{c} = 0,$$

then, $0 = \mathbf{c}_0^T\,[a_2(\psi_i, \psi_j)]\,\mathbf{c}_0$ and hence,

$$0 = \int_\Omega (\Delta \sum_{i=1}^N c_i^0\psi_i)^2 dxdy$$

which implies that $\Delta(\sum_{i=1}^N c_i^0\psi_i) \equiv 0$ or $\sum_{i=1}^N c_i^0\psi_i$ is a linear polynomial which has to be zero since $\sum_{i=1}^N c_i^0\psi_i \in V_0$. That is, $\sum_{i=1}^N c_i^0\psi_i \equiv 0$. Since $\{\psi_i\}_{i=1}^N$ is a basis of V_0, $\mathbf{c}_0$ must be a zero vector. This completes the proof. ■

Next we need to show that S_φ is a good approximation of φ. To this end we need an auxiliary function $\tilde{\varphi} \in H^2(\Omega)$ satisfying

$$\nu a_2(\tilde{\varphi}, \psi) = \langle h, \psi\rangle, \quad \forall \psi \in H_0^2(\Omega)$$

with $\dfrac{\partial\tilde{\varphi}}{\partial n}|_{\partial\Omega} = \tilde{h}_1$ and $\tilde{\varphi}|_{\partial\Omega} = \tilde{h}_2$. The existence of such solution $\tilde{\varphi}$ follows from the fact that there exists a unique solution $\theta \in H_0^2(\Omega)$ satisfying

$$\nu a_2(\theta, \psi) = \langle h, \psi\rangle - a_2(S_\varphi, \psi), \quad \forall \psi \in H_0^2(\Omega)$$

and then $\tilde{\varphi} := \theta + S_\varphi$.

LEMMA 3.2. Let φ be the weak solution of boundary value problem (3.1). Then

$$|\varphi - \tilde{\varphi}|_{2,\Omega} \le 2\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_2$$

and

$$|\varphi - \tilde{\varphi}|_{1,\Omega} \le K_3\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_1$$

Proof: Recall that

$$\nu a_2(\varphi, \psi) = \langle h, \psi \rangle, \forall \psi \in H_0^2(\Omega)$$

Let $\theta = \varphi - \tilde{\varphi} \in H^2(\Omega)$. Then θ is the weak solution of the following biharmonic equation

$$a_2(\theta, \psi) = 0, \quad \forall \psi \in H_0^2(\Omega)$$

with $\frac{\partial}{\partial n}\theta|_{\partial\Omega} = h_1 - \tilde{h}_1$ and $\theta|_{\partial\Omega} = h_2 - \tilde{h}_2$. Thus, by Lemma 2.7, the results of this lemma follow. This completes the proof. ■

We are now ready to prove the following:

THEOREM 3.3. Let φ and S_φ be the weak solution of (3.1) in $H^2(\Omega)$ and $S_{3r}^r(\diamondsuit)$, respectively. Suppose that $\varphi \in H^k(\Omega)$ with $3 \le k \le 3r+1$. Then

$$|\varphi - S_\varphi|_{2,\Omega} \le C|\diamondsuit|^{k-2}$$

for a constant C dependent on $|\varphi|_{k,\Omega}$.

Proof: We first note that for any $v \in V_0$, $a_2(\varphi - S_\varphi, v) = 0$ and $\tilde{\varphi} - S_\varphi \in H_0^2(\Omega)$. Note that $a_2(\varphi - \tilde{\varphi}, \psi) = 0, \forall \psi \in H_0^2(\Omega)$. We also note that by Lemma 2.8, we have

$$\begin{aligned}
&\inf_{v \in V_0} |\varphi - S_\varphi - v|_{2,\Omega} \\
=\ & \inf\left\{ |\varphi - v|_{2,\Omega} : v \in S_{3r}^r(\diamondsuit), v|_{\partial\Omega} = \tilde{h}_2, \frac{\partial}{\partial n} v|_{\partial\Omega} = \tilde{h}_1 \right\} \\
\le\ & K_7 |\diamondsuit|^{k-2} |\varphi|_{k,\Omega}.
\end{aligned}$$

Hence, we have for any $v \in V_0$,

$$\begin{aligned}
&|\varphi - S_\varphi|_{2,\Omega}^2 \\
&\le 2|\varphi - \tilde{\varphi}|_{2,\Omega}^2 + 2|\tilde{\varphi} - S_\varphi|_{2,\Omega}^2 \\
&= 2|\varphi - \tilde{\varphi}|_{2,\Omega}^2 + 2a_2(\tilde{\varphi} - S_\varphi, \tilde{\varphi} - S_\varphi) \\
&= 2|\varphi - \tilde{\varphi}|_{2,\Omega}^2 + 2a_2(\tilde{\varphi} - \varphi, \tilde{\varphi} - S_\varphi) + 2a_2(\varphi - S_\varphi, \tilde{\varphi} - S_\varphi) \\
&= 2|\varphi - \tilde{\varphi}|_{2,\Omega}^2 + 2a_2(\varphi - S_\varphi, \tilde{\varphi} - \varphi) + 2a_2(\varphi - S_\varphi, \varphi - S_\varphi) \\
&\le 2|\varphi - \tilde{\varphi}|_{2,\Omega}^2 + 2|\varphi - S_\varphi|_{2,\Omega}|\tilde{\varphi} - \varphi|_{2,\Omega} + 2a_2(\varphi - S_\varphi, \varphi - S_\varphi - v) \\
&\le 2|\varphi - \tilde{\varphi}|_{2,\Omega}^2 + 2|\varphi - S_\varphi|_{2,\Omega}|\tilde{\varphi} - \varphi|_{2,\Omega} + 2|\varphi - S_\varphi|_{2,\Omega}|\varphi - S_\varphi - v|_{2,\Omega}.
\end{aligned}$$

That is, we have

$$
\begin{aligned}
& |\varphi - S_\varphi|^2_{2,\Omega} \\
\leq \;& 2|\varphi - \tilde{\varphi}|^2_{2,\Omega} + 2|\varphi - S_\varphi|_{2,\Omega}|\tilde{\varphi} - \varphi|_{2,\Omega} + 2|\varphi - S_\varphi|_{2,\Omega} K_7 |\diamondsuit|^{k-2}|\varphi|_{k,\Omega}.
\end{aligned}
$$

For convenience, let $\alpha = |\varphi - S_\varphi|_{2,\Omega}$, $\beta = |\varphi - \tilde{\varphi}|_{2,\Omega}$ and $\gamma = K_7|\diamondsuit|^{k-2}|\varphi|_{k,\Omega}$. Then the above equality may be written in short as

$$\alpha^2 \leq 2\beta^2 + 2\alpha\beta + 2\alpha\gamma$$

or

$$(\alpha - (\beta + \gamma))^2 \leq (\beta + \gamma)^2 + 2\beta^2.$$

Thus, we get

$$\alpha \leq \beta + \gamma + \sqrt{2\beta^2 + (\beta + \gamma)^2} \leq \beta + \gamma + \sqrt{2}\beta + \beta + \gamma \leq 4(\beta + \gamma).$$

Hence, in terms of the original notations, we have

$$
\begin{aligned}
|\varphi - S_\varphi|_{2,\Omega} &\leq 4(|\varphi - \tilde{\varphi}|_{2,\Omega} + K_7|\diamondsuit|^{k-2}|\varphi|_{k,\Omega}) \\
&\leq 8\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2\|_2 + 4K_7|\diamondsuit|^{k-2}|\varphi|_{k,\Omega} \\
&= 8\inf\left\{\|\varphi - S_\varphi - s\|_{2,\Omega} : s \in H^2_0(\Omega)\right\} + 4K_7|\diamondsuit|^{k-2}|\varphi|_{k,\Omega} \\
&\leq 8\inf\{\|\varphi - S_\varphi - s\|_{2,\Omega} : s \in V_0\} + 4K_7|\diamondsuit|^{k-2}|\varphi|_{k,\Omega} \\
&\leq 12C|\diamondsuit|^{k-2}|\varphi|_{k,\Omega}
\end{aligned}
$$

by Lemma 2.8. This completes the proof. ■

THEOREM 3.4. Under the same assumptions as Theorem 3.3, we have

$$\|\mathbf{u} - \mathbf{curl}(S_\varphi)\|_{0,\Omega} = |\varphi - S_\varphi|_{1,\Omega} \leq K_8|\diamondsuit|^{k-2}$$

for a constant K_8 depending on $|\varphi|_{k,\Omega}$.

Proof: Note that $\mathbf{u} = \mathbf{curl}(\varphi)$ and that $\mathbf{curl}(S_\varphi)$ is a numerical velocity approximating $\mathbf{u}$. We use the well known Poincaré's inequality (cf. Lemma 2.2) and Lemma 2.7 to get

$$
\begin{aligned}
\|\mathbf{u} - \mathbf{curl}(S_\varphi)\|_{0,\Omega} &= |\varphi - S_\varphi|_{1,\Omega} \\
&\leq |\varphi - \tilde{\varphi}|_{1,\Omega} + |\tilde{\varphi} - S_\varphi|_{1,\Omega}
\end{aligned}
$$

$$
\begin{aligned}
&\le |\varphi - \tilde{\varphi}|_{1,\Omega} + K_0|\tilde{\varphi} - S_\varphi|_{2,\Omega} \\
&\le |\varphi - \tilde{\varphi}|_{1,\Omega} + K_0|\varphi - \tilde{\varphi}|_{2,\Omega} + K_0|\varphi - S_\varphi|_{2,\Omega} \\
&\le K_0 C|\diamondsuit|^{k-2} + K_3\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_{1,\Omega)} \\
&\quad +2K_0\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_{2,\Omega)}
\end{aligned}
$$

By Lemma 2.8, we have

$$
\begin{aligned}
\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_2 &= \inf\left\{\|\varphi - S_\varphi - s\|_{2,\Omega} : s \in H_0^2(\Omega)\right\} \\
&\le \inf\{\|\varphi - S_\varphi - s\|_{2,\Omega} : s \in V_0\} \\
&\le K_7|\diamondsuit|^{k-2}|\varphi|_{k,\Omega}.
\end{aligned}
$$

Similarly, we have

$$
\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_1 \le 2K_7|\diamondsuit|^{k-2}|\varphi|_{k,\Omega},
$$

and the conclusion of this theorem follows. ■

We have implemented the C^1 cubic spline method in MATLAB and have used it to solve the Stokes equations (1.3) in the stream function formulation. In fact, our programs are able to solve any nonhomogeneous biharmonic equation. We have tested our programs on several known stream functions over polygonal domains, in particular L shaped ones. By calculating the right hand side and boundary conditions from the known stream functions and feeding these into our nonhomogeneous equation solver, we obtained numerical approximations of the known stream functions. In Table 1, we list the maximum error of the numerical approximations against the exact solution, where the maximum errors are computed based on 30401 points equally-spaced over the L-shape domain for the quadrangulation as shown in Figure 1 and its three refinements. Here, $\nu = 0.01$.

Table 1a. Maximum Errors of the C^1 Cubic Spline Method for Boundary Value Problem (1.3)

Matrix Sizes	150×150	527×527	1971×1971	7619×7619
$x^4 + y^4$	5.669×10^{-3}	4.894×10^{-4}	3.039×10^{-5}	1.901×10^{-6}
$\sin(x + 3y)$	1.296×10^{-2}	9.789×10^{-4}	8.053×10^{-5}	6.841×10^{-6}
$\exp(2x + y)$	2.387×10^{-1}	1.886×10^{-2}	1.330×10^{-3}	8.842×10^{-5}
$1/(1 + x + y)$	1.640×10^{-3}	1.612×10^{-4}	1.303×10^{-5}	8.991×10^{-7}
$r^{5/2}\sin(5/2\theta)$	1.931×10^{-3}	3.404×10^{-4}	5.921×10^{-5}	1.046×10^{-5}
$r^{3/2}\sin(3/2\theta)$	1.193×10^{-2}	4.221×10^{-3}	1.492×10^{-3}	4.980×10^{-4}

For the last two functions, $x = r\cos(\theta)$ and $y = r\sin(\theta)$. It is clear to see from Table 1a that the convergence of the numerical solution is almost fourth order convergence when the solution is sufficiently smooth. When the exact solution has less smoothness, the convergence rate of the numerical solution is slower. To compensate, we need to apply adaptive refinement techniques. In Table 1b, we list the maximum error for the two less smooth functions in Table 1a with successive refinements consisting of a local refinement of all quadrilaterals having $(0,0)$ as a vertex followed by a global refinement of all quadrilaterals.

Table 1b. Maximum Errors of the C^1 Cubic Spline Method with Adaptive Mesh for Boundary Value Problem (1.3)

Matrix Sizes	150×150	615×615	2395×2395	9355×9355
$r^{5/2}\sin(5\theta/2)$	1.931×10^{-3}	5.896×10^{-5}	3.804×10^{-6}	3.710×10^{-7}
Matrix Sizes	150×150	246×246	751×751	3067×3067
$r^{3/2}\sin(3\theta/2)$	1.193×10^{-2}	2.387×10^{-4}	4.053×10^{-5}	2.501×10^{-6}

4 Numerical Solutions of the Navier-Stokes equations

In this section, we shall show the existence and uniqueness of the weak solution in $H^2(\Omega)$ of the Navier-Stokes equations. We shall also discuss the approximation properties of the weak solution in $S^r_{3r}(\diamondsuit)$. Since we have to solve a system of nonlinear equations, we shall discuss how to solve the nonlinear system using Newton's method in Section 6.

Let us begin with the steady state Navier-Stokes equations with zero boundary conditions, i.e., $\mathbf{g} = (0,0)^T$ in (1.1). The nonhomogeneous Navier-Stokes equations will be studied in the next section. Let $V \subset H^2_0(\Omega)$ be any closed subspace in $H^2_0(\Omega)$ and $\varphi \in V$ be a solution of the Navier-Stokes' equations in the following weak formulation:

$$\begin{cases} \nu a_2(\varphi,\psi) + b(\varphi,\varphi,\psi) =< h,\psi >, & \forall \psi \in V \\ \dfrac{\partial \varphi}{\partial x} = 0, & \text{on } \partial\Omega \\ \dfrac{\partial \varphi}{\partial y} = 0, & \text{on } \partial\Omega \\ \varphi = 0, & \text{on } \partial\Omega\ . \end{cases} \tag{4.1}$$

We first recall the following well-known result.

Lemma 4.1. For any $f \in H^1_0(\Omega)$,

$$\int_\Omega |f(x,y)|^4 dx dx \le 2 \int_\Omega |f(x,y)|^2 dx dy \int_\Omega |\nabla f(x,y)|^2 dx dy.$$

For a proof, cf. [Ladyzhenskaya'69]. A simple observation gives the following two Lemmas.

Lemma 4.2. $b(\theta,\phi,\phi) = 0$, for any $\theta \in H^2(\Omega)$ and $\phi \in H^1(\Omega)$.

Lemma 4.3. $b(\theta,\phi,\psi) = -b(\theta,\psi,\phi)$.

Lemma 4.4. Let $\varphi \in V$ be a weak solution of the Navier-Stokes equations in (4.1). Then

$$|\varphi|_2 \le \frac{K_9}{\nu}\|h\|_{0,\Omega}.$$

for some constant K_9 dependent on Ω.

Proof: Since $\nu a_2(\varphi,\varphi) =< h,\varphi >$ by Lemma 4.2, we have, by Lemma 2.2,

$$\nu|\varphi|^2_{2,\Omega} =< h,\varphi > \le \|h\|_{0,\Omega}\|\varphi\|_{0,\Omega} \le \|h\|_{0,\Omega} K_0^2 |\varphi|_{2,\Omega}.$$

Thus $|\varphi|_{2,\Omega} \le \frac{K_9}{\nu}\|h\|$ with $K_9 = K_0^2$. This completes the proof. ■

LEMMA 4.5. For $\theta, \phi, \psi \in H_0^2(\Omega)$,

$$|b(\theta,\phi,\psi)| \leq K_{10}|\theta|_{2,\Omega}|\phi|_{2,\Omega}|\psi|_{2,\Omega}.$$

Proof: We use the Cauchy-Schwarz inequality, Lemma 4.1, and Poincaré's inequality (cf. Lemma 2.2) to get

$$\begin{aligned}
|b(\theta,\phi,\psi)| &\leq |\theta|_{2,\Omega}\left(\int_\Omega (\nabla\phi)^2(\nabla\psi)^2\,dxdy\right)^{1/2} \\
&\leq |\theta|_{2,\Omega}\left(\int_\Omega\left(\left(\frac{\partial\varphi}{\partial x}\right)^4+\left(\frac{\partial\varphi}{\partial y}\right)^4\right)dxdy\right)^{1/4}\times \\
&\qquad \left(\int_\Omega\left(\left(\frac{\partial\psi}{\partial x}\right)^4+\left(\frac{\partial\psi}{\partial y}\right)^4\right)dxdy\right)^{1/4} \\
&\leq \sqrt{2}|\theta|_{2,\Omega}\sqrt{K_0}|\phi|_{2,\Omega}|\psi|_{2,\Omega}.
\end{aligned}$$

This completes the proof with $K_{10} = \sqrt{2}\sqrt{K_0}$. ■

We are now ready to prove one of the main results in this section.

THEOREM 4.6. Let $V_0 = H_0^2(\Omega)$ or $V_0 = S_{3r}^r(\diamondsuit) \cap H_0^2(\Omega)$. For any $\nu > 0$, there exists a weak solution φ in V_0 of the steady state Navier-Stokes equation.

Proof: Let us first consider $V_0 = S_{3r}^r(\diamondsuit) \cap H_0^2(\Omega)$ which is a finite dimensional space. Fix $\phi \in V_0$. We define a continuous linear functional

$$[P(\phi),\psi] := \nu a_2(\phi,\psi) + b(\phi,\phi,\psi) - < h,\psi >, \forall\psi \in V.$$

Since V_0 is a Hilbert space, by Riesz's representation theorem there exists a $u_\phi \in V_0$ such that $[P(\phi),\psi] = (u_\phi,\psi)$ where $(\cdot,\cdot)$ denotes the inner product in $V_0 \subset H_0^2(\Omega)$. Thus P defines a nonlinear operator mapping V_0 to V_0. Clearly, P is a continuous map since

$$[P(\phi_1) - P(\phi_2),\psi] = \nu a_2(\phi_1-\phi_2,\psi) + b(\phi_1,\phi_1-\phi_2,\psi) + b(\phi_1-\phi_2,\phi_2,\psi)$$

and

$$\begin{aligned}
\|P(\phi_1) - P(\phi_2)\| &= \max_{|\psi|_{2,\Omega}\leq 1}[P(\phi_1)-P(\phi_2),\psi] \\
&\leq \nu|\phi_1-\phi_2|_{2,\Omega} + C_2|\phi_1|_{2,\Omega}|\phi_1-\phi_2|_{2,\Omega} + C_2|\phi_1-\phi_2|_{2,\Omega}|\phi_2|_{2,\Omega} \\
&= (\nu + C_2|\phi_1|_{2,\Omega} + C_2|\phi_2|_{2,\Omega})|\phi_1-\phi_2|_{2,\Omega}
\end{aligned}$$

by Lemma 4.5. Note that by Lemma 2.2

$$
\begin{aligned}
[P(\phi), \phi] &= \nu|\phi|_{2,\Omega}^2 - < h, \phi > \\
&\geq \nu|\phi|_{2,\Omega}^2 - K_0^2\|h\|_{0,\Omega}|\phi|_{2,\Omega} \\
&= \nu|\phi|_{2,\Omega}\left(|\phi|_{2,\Omega} - \frac{K_0^2}{\nu}\|h\|_{0,\Omega}\right) \\
&> 0
\end{aligned}
$$

for $|\phi|_{2,\Omega} > \frac{K_0^2}{\nu}\|h\|_{0,\Omega}$. Let

$$L = \frac{K_0^2}{\nu}\|h\|_{0,\Omega} + 1$$

and $D = \{\phi \in V, |\phi|_{2,\Omega} \leq L\}$. Suppose that $P(\phi) \neq 0$ for any $\phi \in D$. We consider the map

$$S(\phi) := -L\frac{P(\phi)}{\|P(\phi)\|}.$$

Clearly this map is well defined on D and maps D into D. Since S is a continuous map by Brower's fixed point theorem there exists a $\phi_0 \in D$ such that

$$\phi_0 = S(\phi_0) = -L\frac{P(\phi_0)}{\|P(\phi_0)\|}.$$

But this implies that

$$
\begin{aligned}
|\phi_0|_{2,\Omega}^2 &= (\phi_0, \phi_0) = \left(\phi_0, -L\frac{P(\phi_0)}{\|P(\phi_0)\|}\right) \\
&= -\frac{L}{\|P(\phi_0)\|}[P(\phi_0), \phi_0] < 0
\end{aligned}
$$

since $\|\phi_0\| = L$ which implies that $[P(\phi_0), \phi_0] > 0$. This is a contradiction which implies that $P(\phi)$ must vanish at some point in D. Thus there exists a $\varphi \in D$ such that

$$[P(\varphi), \psi] = \nu a_2(\varphi, \psi) + b(\varphi, \varphi, \psi) - < h, \psi >= 0, \text{ for all } \psi \in V_0.$$

This finishes the proof for $V_0 = S_{3r}^r(\Omega) \cap H_0^2(\Omega)$.

Next we consider $V_0 = H_0^2(\Omega)$. Let $\diamondplus_n$ be the n^{th} refinement of $\diamondplus$ and $S_{3r}^r(\diamondplus_n)$ be the bivariate spline space over the quadrangulation $\diamondplus_n$. Let $\varphi_n \in S_{3r}^r(\diamondplus_n) \cap H_0^2(\Omega)$ be the weak solution of the Navier Stokes equations as discussed above. Since $\{\varphi_n\}$ is a bounded sequence in $H_0^2(\Omega)$ by Lemma 4.4, there exists a weakly convergent subsequence. For simplicity let us say that $\{\varphi_n\}$ is convergent weakly. That is, there exists a $\hat{\varphi} \in H_0^2(\Omega)$,

$$a_2(\varphi_n, \psi) \rightarrow a_2(\hat{\varphi}, \psi), \quad n \rightarrow \infty, \forall \psi \in H_0^2(\Omega).$$

We now claim that $b(\varphi_n, \varphi_n, \psi)$ is convergent to $b(\hat{\varphi}, \hat{\varphi}, \psi)$ for any $\psi \in C^1(\overline{\Omega}) \cap H_0^2(\Omega)$ which is dense in $H_0^2(\Omega)$. Indeed, since $\{\varphi_n\}$ converges weakly in $H_0^2(\Omega)$ and $\{\varphi_n\} \subset H_0^1(\Omega)$, $\{\varphi_n\}$ converges strongly in $H_0^1(\Omega)$ by the Rellich-Konderachov theorem ([Adams'75, p.144]). Since for any $\psi \in C^1(\overline{\Omega}) \cap H_0^2(\Omega)$,

$$\begin{aligned}
& |b(\varphi_n, \varphi_n, \psi) - b(\hat{\varphi}, \hat{\varphi}, \psi)| \\
\leq \; & |b(\varphi_n, \varphi_n - \hat{\varphi}, \psi)| + |b(\varphi_n - \hat{\varphi}, \hat{\varphi}, \psi)| \\
\leq \; & |\varphi_n|_{2,\Omega} \left(\| \frac{\partial \psi}{\partial x} \|_{L_\infty(\Omega)} + \| \frac{\partial \psi}{\partial y} \|_{L_\infty(\Omega)} \right) |\varphi_n - \hat{\varphi}|_{1,\Omega} + |b(\varphi_n - \hat{\varphi}, \hat{\varphi}, \psi)|,
\end{aligned}$$

the claim will follow by showing that $|b(\varphi_n - \hat{\varphi}, \hat{\varphi}, \psi)| \to 0$. Note that by Lemma 2.5 there exists an unique solution $g \in H_0^2(\Omega)$ of the homogeneous boundary value problem

$$\begin{cases}
\Delta^2 g = \Delta \left(\dfrac{\partial \hat{\varphi}}{\partial x} \dfrac{\partial \psi}{\partial y} - \dfrac{\partial \hat{\varphi}}{\partial y} \dfrac{\partial \psi}{\partial x} \right), & \text{in } \Omega \\
\dfrac{\partial g}{\partial n} = 0, & \text{on } \partial\Omega \\
g = 0, & \text{on } \partial\Omega \; .
\end{cases}$$

That is

$$a_2(g, f) = \langle \Delta \left(\frac{\partial \hat{\varphi}}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \hat{\varphi}}{\partial y} \frac{\partial \psi}{\partial x} \right), f \rangle, \forall f \in H_0^2(\Omega).$$

Since $f \in H_0^2(\Omega)$, by Green's formula we have

$$a_2(g, f) = \langle \left(\frac{\partial \hat{\varphi}}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \hat{\varphi}}{\partial y} \frac{\partial \psi}{\partial x} \right), \Delta f \rangle, \quad \forall f \in H_0^2(\Omega)$$

Thus

$$\begin{aligned}
|b(\varphi_n - \hat{\varphi}, \hat{\varphi}, \psi)| &= \langle \Delta(\varphi_n - \varphi), \frac{\partial \hat{\varphi}}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \hat{\varphi}}{\partial y} \frac{\partial \psi}{\partial x} \rangle \\
&= \langle \Delta(\varphi_n - \hat{\varphi}), \Delta g \rangle \\
&= a_2(\varphi_n - \varphi, g) \\
&\to 0
\end{aligned}$$

as $n \to +\infty$. Thus $b(\varphi_n, \varphi_n, \psi) \to b(\hat{\varphi}, \varphi, \psi)$ for any $\psi \in C^1(\overline{\Omega}) \cap H_0^2(\Omega)$. Hence

$$\nu a_2(\hat{\varphi}, \psi) + b(\hat{\varphi}, \hat{\varphi}, \psi) = \langle h, \psi \rangle, \quad \forall \psi \in C^1(\overline{\Omega}) \cap H_0^2(\Omega).$$

It immediately follows that the above equation holds for all $\psi \in H_0^2(\Omega)$ because $C^1(\overline{\Omega}) \cap H_0^2(\Omega)$ is dense in $H_0^2(\Omega)$. Therefore $\hat{\varphi} \in H_0^2(\Omega)$ is a solution of the Navier-Stokes equations. This completes the proof. ∎

Regarding the uniqueness of the weak solutions in $V_0 \subset H_0^2(\Omega)$, we have

THEOREM 4.7. Let K_9 and K_{10} be the constants in Lemmas 4.4 and 4.5. If $\nu^2 > K_9 K_{10} \|h\|_{0,\Omega}$, then the weak solution in V_0 is unique.

Proof: Suppose that there exists two solution ϕ_1 and ϕ_2 such that

$$\nu a_2(\phi_1, \psi) + b(\phi_1, \phi_1, \psi) = < h, \psi >, \quad \forall \psi \in V_0$$

and

$$\nu a_2(\phi_2, \psi) + b(\phi_2, \phi_2, \psi) = < h, \psi >, \forall \psi \in V_0.$$

The difference of the above two equations is

$$\nu a_2(\phi_1 - \phi_2, \psi) = b(\phi_1, \phi_1 - \phi_2, \psi) + b(\phi_2 - \phi_1, \phi_2, \psi).$$

In particular, letting $\psi = \phi_1 - \phi_2$ and using Lemma 4.2 and Lemma 4.5 we have

$$\begin{aligned} \nu|\phi_1 - \phi_2|_{2,\Omega}^2 &= \nu a_2(\phi_1 - \phi_2, \phi_1 - \phi_2) \\ &= b(\phi_2 - \phi_1, \phi_2, \phi_1 - \phi_2) \\ &\leq K_{10}|\phi_2 - \phi_1|_{2,\Omega}|\phi_2|_{2,\Omega}|\phi_1 - \phi_2|_{2,\Omega} \\ &\leq K_{10}\frac{K_9}{\nu}\|h\|_{0,\Omega}|\phi_1 - \phi_2|_{2,\Omega}^2. \end{aligned}$$

If $\nu^2 > K_9 K_{10} \|h\|_{0,\Omega}$, the above inequality will only hold for $|\phi_1 - \phi_2|_{2,\Omega} = 0$. This completes the proof. ■

Let $\varphi \in H_0^2(\Omega)$ and $S_\varphi \in S_{3r}^r(\oplus) \cap H_0^2(\Omega)$ be the weak solutions of the steady state Navier-Stokes equation (4.1), respectively. We would like to see how close S_φ is to φ. We have

LEMMA 4.8. Let $\alpha = \nu^2 - K_9 K_{10} \|h\|_{0,\Omega} > 0$ and $M = \nu^2 + 2K_9 K_{10} \|h\|_{0,\Omega}$. Then

$$|\varphi - S_\varphi|_{2,\Omega} \leq \frac{M}{\alpha} \inf_{s \in S_{3r}^r(\oplus) \cap H_0^2(\Omega)} |\varphi - s|_{2,\Omega}.$$

Proof: It is easy to see that

$$\nu a_2(\varphi - S_\varphi, \psi) = b(\varphi, \varphi - S_\varphi, \psi) + b(\varphi - S_\varphi, S_\varphi, \psi),$$

for $\psi \in S_{3r}^r(\oplus) \cap H_0^2(\Omega)$. Then we have, for any $s \in S_{3r}^r(\oplus) \cap H_0^2(\Omega)$,

$$\nu|\varphi - S_\varphi|_{2,\Omega}^2 = \nu a_2(\varphi - S_\varphi, \varphi - S_\varphi)$$

$$
\begin{aligned}
&= \nu a_2(\varphi - S_\varphi, \varphi - s) + \nu a_2(\varphi - S_\varphi, s - S_\varphi) \\
&= \nu a_2(\varphi - S_\varphi, \varphi - s) + b(\varphi, \varphi - S_\varphi, s - S_\varphi) + b(\varphi - S_\varphi, S_\varphi, s - S_\varphi) \\
&= \nu a_2(\varphi - S_\varphi, \varphi - s) + b(\varphi, \varphi - S_\varphi, s - \varphi) + b(\varphi - S_\varphi, S_\varphi, s - \varphi) \\
&\quad + b(\varphi - S_\varphi, S_\varphi, \varphi - S_\varphi)
\end{aligned}
$$

by Lemma 4.2. Note that by Lemmas 4.4 and 4.5,

$$
\begin{aligned}
|b(\varphi, \varphi - S_\varphi, s - \varphi)| &\leq \frac{K_9 K_{10}}{\nu} \|h\|_{0,\Omega} |\varphi - S_\varphi|_{2,\Omega} |s - \varphi|_{2,\Omega} \\
|b(\varphi - S_\varphi, S_\varphi, s - \varphi)| &\leq \frac{K_9 K_{10}}{\nu} \|h\|_{0,\Omega} |\varphi - S_\varphi|_{2,\Omega} |s - \varphi|_{2,\Omega} \\
|b(\varphi - S_\varphi, S_\varphi, \varphi - S_\varphi)| &\leq \frac{K_9 K_{10}}{\nu} \|h\|_{0,\Omega} |\varphi - S_\varphi|^2_{2,\Omega}.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
\nu|\varphi - S_\varphi|^2_{2,\Omega} \leq\ & \nu|\varphi - S_\varphi|_{2,\Omega} |\varphi - s|_{2,\Omega} + \frac{2K_9 K_{10}}{\nu} \|h\|_{0,\Omega} |\varphi - S_\varphi|_{2,\Omega} |\varphi - s|_{2,\Omega} \\
& + \frac{K_9 K_{10}}{\nu} \|h\|_{0,\Omega} |\varphi - S_\varphi|^2_{2,\Omega}.
\end{aligned}
$$

Hence we conclude that

$$
|\varphi - S_\varphi|_{2,\Omega} \leq \frac{\nu^2 + 2K_9 K_{10} \|h\|_{0,\Omega}}{\nu^2 - K_9 K_{10} \|h\|_{0,\Omega}} \inf_{s \in S^r_{3r}(\diamondsuit) \cap H^2_0(\Omega)} |\varphi - s|_{2,\Omega}.
$$

This completes the proof. ■

THEOREM 4.9. Suppose that $\alpha = \nu^2 - K_9 K_{10}\|h\| > 0$ and that $\varphi \in H^k(\Omega)$ with $3 \leq k \leq 3r + 1$. Then $\mathbf{curl}(S_\varphi)$ is an approximation of the velocity $\mathbf{u}$ satisfying

$$
\|\mathbf{u} - \mathbf{curl}(S_\varphi)\|_{0,\Omega} \leq K_{11} |\diamondsuit|^{k-2} |\varphi|_{k,\Omega}.
$$

for a constant $K_{11} > 0$.

Proof: Since $\varphi \in H^k(\Omega) \cap \varphi \in H^2_0(\Omega)$ there exists $s \in S^r_{3r}(\diamondsuit) \cap H^2_0(\Omega)$ by Lemma 2.8 such that

$$
|\varphi - s|_{2,\Omega} \leq K_7 |\diamondsuit|^{k-2} |\varphi|_{k,\Omega}.
$$

Then by Lemma 4.8 we have

$$
|\varphi - S_\varphi|_{2,\Omega} \leq \frac{M}{\alpha} K_7 |\diamondsuit|^{k-2} |\varphi|_{k,\Omega}.
$$

Therefore, by Lemma 2.2 we have

$$\begin{aligned}
& \|\mathbf{u} - \mathbf{curl}(S_\varphi)\|_{0,\Omega} \\
= & \|\mathbf{curl}(\varphi) - \mathbf{curl}(S_\varphi)\|_{0,\Omega} \\
= & |\varphi - S_\varphi|_{1,\Omega} \\
\le & K_0 K_7 \frac{M}{\alpha} |\diamondplus|^{k-2} |\varphi|_{k,\Omega}
\end{aligned}$$

which completes the proof with $K_{11} = K_0 K_7 M/\alpha$. ■

In the following, we shall present the results of several numerical experiments on the steady state Navier Stokes equations. (The discussion above only provides a weak formulation of the steady state Navier Stokes equations and a proof of the existence and an analysis of the approximation properties of the weak solution. The numerical implementation of the weak formulation will be addressed in Section 6.) We have implemented in MATLAB the C^1 cubic spline functions and have used them to numerically compute weak solutions of the nonlinear systems. We tested many known stream functions by calculating the right-hand sides and boundary conditions and feeding them into our nonlinear biharmonic equation solver. The numerical solutions we thus obtained from our computer programs are compared against the exact solution. In Table 2, we list the maximum errors, computed based on 30401 points equally-spaced over the L-shape domain, of the numerical approximations against the exact solutions, where the L-shape domain is quadrangulated as shown in Figure 1 and is refined two times and $\nu = 0.01$.

Table 2a. Maximum Errors of the C^1 Cubic Spline Method
for nonlinear Biharmonic Equations (1.6)

Matrix Sizes	150×150	527×527	1971×1971
x^3y^3	3.805×10^{-2}	2.709×10^{-3}	3.254×10^{-4}
$x^4 + y^4$	1.999×10^{-2}	1.199×10^{-3}	1.809×10^{-4}
$\sin(x + 3y)$	5.587×10^{-2}	3.343×10^{-3}	1.706×10^{-4}
$\exp(2x + y)$	4.795×10^{-1}	6.569×10^{-2}	3.758×10^{-3}
$1/(1 + x + y)$	1.572×10^{-3}	1.577×10^{-4}	1.291×10^{-5}
$r^{5/2}\sin(5\theta/2)(1 + x)$	2.051×10^{-2}	5.429×10^{-4}	5.826×10^{-5}
$r^{3/2}\sin(3\theta/2)(1 + x)$	1.208×10^{-2}	4.319×10^{-3}	1.503×10^{-3}

It can be also seen from Table 2a that the convergence of the numerical solutions is close to fourth order convergence when the solution is sufficiently smooth. For the last two functions, the convergence rate is very slow. We apply a local refinement prior to the global refinements to speedup the convergences. Refer to Table 2b. In general, one should refine the quadrangulation locally at all corners.

Table 2b. Maximum Errors of the C^1 Cubic Spline Method with Adaptive Mesh for Nonlinear Biharmonic Equations (1.6)

Matrix Sizes	150×150	615×615	2395×2395
$r^{5/2}\sin(5\theta/2)(1+x)$	2.051×10^{-2}	5.417×10^{-4}	3.048×10^{-5}
Matrix Sizes	150×150	222×222	897×897
$r^{3/2}\sin(3\theta/2)(1+x)$	1.208×10^{-2}	5.038×10^{-4}	3.959×10^{-5}

5 Numerical Solution of Nonhomogeneous Navier-Stokes Equations

We consider the nonhomogeneous Navier Stokes equation: find $\varphi \in H^2(\Omega)$ satisfying

$$\begin{cases} \nu a_2(\varphi,\psi) + b(\varphi,\varphi,\psi) = \langle h,\psi\rangle, & \forall\ \psi \in H_0^2(\Omega) \\ \dfrac{\partial \varphi}{\partial n} = h_1 & \text{on } \partial\Omega \\ \varphi = h_2 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

We first need the following crucial lemma. The proof of this lemma is lengthy and hence postponed to the end of this section.

Lemma 5.1. Suppose that h_1 and h_2 are compatible. For any $\gamma > 0$, there exists $\varphi_0 \in H^2(\Omega)$ satisfying $\frac{\partial \varphi_0}{\partial n}|_{\partial\Omega} = h_1$ and $\varphi_0|_{\partial\Omega} = h_2$ such that

$$|b(\theta,\varphi_0,\theta)| \le \gamma |\theta|_{2,\Omega}^2, \quad \text{for } \theta \in \mathrm{H}_0^2(\Omega).$$

We next consider a new homogeneous nonlinear biharmonic equation: find $\theta \in H_0^2(\Omega)$ satisfying the following

$$\begin{cases} \nu a_2(\theta,\psi) + b(\theta,\theta,\psi) + b(\theta,\varphi_0,\psi) + b(\varphi_0,\theta,\psi) & \\ \qquad = \langle h,\psi\rangle - \nu a_2(\varphi_0,\psi) - b(\varphi_0,\varphi_0,\psi), & \forall\ \psi \in H_0^2(\Omega) \\ \dfrac{\partial \theta}{\partial n} = 0 & \text{on } \partial\Omega \\ \theta = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2}$$

We will show that this boundary value problem has a solution θ. Then it is easy to see that $\varphi = \theta + \varphi_0$ satisfies the nonhomogeneous Navier-Stokes equation (5.1).

To this end, we define

$$\begin{aligned}[P(\theta),\psi] &:= \nu a_2(\theta,\psi)+b(\theta,\theta,\psi)+b(\theta,\varphi_0,\psi)+b(\varphi_0,\theta,\psi)-\langle h,\psi\rangle \\ &+ \nu a_2(\varphi_0,\psi)+b(\varphi_0,\varphi_0,\psi).\end{aligned}$$

By Lemma 4.5 and Lemma 2.2, we have

$$\begin{aligned}& [P(\theta),\theta] \\ =\ & \nu a_2(\theta,\theta)+b(\theta,\theta,\theta)+b(\theta,\varphi_0,\theta)+b(\varphi_0,\theta,\theta)-\langle h,\theta\rangle+\nu a_2(\varphi_0,\theta)+b(\varphi_0,\varphi_0,\theta) \\ =\ & \nu a_2(\theta,\theta)+b(\theta,\varphi_0,\theta)-\langle h,\theta\rangle+\nu a_2(\varphi_0,\theta)+b(\varphi_0,\varphi_0,\theta) \\ \geq\ & \nu|\theta|_{2,\Omega}^2-\gamma|\theta|_{2,\Omega}^2-K_0^2|h|_{0,\Omega}|\theta|_{2,\Omega}-\nu|\varphi_0|_{2,\Omega}|\theta|_{2,\Omega}-C_2|\varphi_0|_{2,\Omega}^2|\theta|_{2,\Omega} \\ =\ & |\theta|_{2,\Omega}\left((\nu-\gamma)|\theta|_{2,\Omega}-(K_0^2|h|_{0,\Omega}+\nu|\varphi_0|_{2,\Omega}+C_2|\varphi_0|_{2,\Omega}^2)\right) \\ >\ & 0\end{aligned}$$

if $\gamma<\nu$ and $|\theta|_{2,\Omega}\geq\dfrac{1}{\nu-\gamma}(K_0^2|h|_{0,\Omega}+\nu|\varphi_0|_{2,\Omega}+C_2|\varphi_0|_{2,\Omega}^2)$. We may use the same argument used in the proof of Theorem 4.6 to show that for any $\nu>0$, (5.2) has a solution θ in $H_0^2(\Omega)$. Hence we may conclude

THEOREM 5.2. For any $\nu>0$, there exists a weak solution $\varphi\in H^2(\Omega)$ satisfying (5.1).

Let $\tilde{h}_1$ and $\tilde{h}_2$ be the two spline functions defined on $\partial\Omega$ in §3, and let $\diamondsuit_n$ denote the n^{th} refinement of $\diamondsuit$. We first consider the following nonhomogeneous Navier-Stokes equations: letting $V_n=S_{3r}^r(\diamondsuit_n)$ for simplicity, find $\tilde{\varphi}\in V_n$ such that

$$\begin{cases}\nu a_2(\tilde{\varphi},\psi)+b(\tilde{\varphi},\tilde{\varphi},\psi)=\langle h,\psi\rangle, & \forall\ \psi\in V_n\cap H_0^2(\Omega)\\ \dfrac{\partial\tilde{\varphi}}{\partial n}=\tilde{h}_1 & \text{on }\ \partial\Omega\\ \tilde{\varphi}=\tilde{h}_2 & \text{on }\ \partial\Omega\end{cases}\tag{5.3}$$

We need the following lemma which is an analog to Lemma 5.1. Again we leave the proof to the end of this section.

LEMMA 5.3. Let $\gamma>0$. Then for n large enough, there exists a $\varphi_0\in V_n$ satisfying the boundary conditions: $\frac{\partial\varphi_0}{\partial n}|_{\partial\Omega}=\tilde{h}_1$ and $\varphi_0|_{\partial\Omega}=\tilde{h}_2$ such that

$$|b(\psi,\varphi_0,\psi)|\leq\gamma|\psi|_{2,\Omega}^2,\ \text{for}\ \psi\in \mathrm{V_n}\cap \mathrm{H_0^2}(\Omega).$$

With Lemma 5.3, the argument used in the proof of Theorem 5.2 yields the following.

THEOREM 5.4. For sufficiently large n, problem (5.3) has a solution $\tilde{\varphi} \in V_n$.

Next we show that the weak solutions in $H^2(\Omega)$ or in V_n are bounded.

THEOREM 5.5. Let $\varphi \in H^2(\Omega)$ be the solution of (5.1). Then if ν is large enough,

$$|\varphi|_{2,\Omega} \le \frac{K_0^2\|h\|_{0,\Omega} + 2\nu\|(h_1,h_2)\|_2}{\nu - K_{10}\|(h_1,h_2)\|_2}.$$

Similarly, let $S_\varphi \in S_{3r}^r(\oplus_n)$ be the solution of (5.3). Then

$$|S_\varphi|_{2,\Omega} \le \frac{K_0^2\|h\|_{0,\Omega} + 2\nu\|(\tilde{h}_1,\tilde{h}_2)\|_2}{\nu - K_{10}\|(\tilde{h}_1,\tilde{h}_2)\|_2}.$$

Proof: Let $\varphi_\ell \in H^2(\Omega)$ be the unique solution of following linear biharmonic equation

$$\begin{cases} a_2(\varphi_\ell,\psi) = 0, & \forall \psi \in H_0^2(\Omega) \\ \dfrac{\partial \varphi_\ell}{\partial n} = h_1, & \text{on } \partial\Omega \\ \varphi_\ell = h_2, & \text{on } \partial\Omega \end{cases}$$

By Lemma 2.5 we know that

$$|\varphi_\ell|_{2,\Omega} \le 2\|(h_1,h_2)\|_2.$$

The fact that $\varphi - \varphi_\ell \in H_0^2(\Omega)$ and Lemmas 2.2 and 4.5 now imply that

$$\begin{aligned} \nu|\varphi-\varphi_\ell|_{2,\Omega}^2 &= \nu a_2(\varphi-\varphi_\ell,\varphi-\varphi_\ell) \\ &= \nu a_2(\varphi,\varphi-\varphi_\ell) - \nu a_2(\varphi_\ell,\varphi-\varphi_\ell) \\ &= \langle h,\varphi-\varphi_\ell\rangle - b(\varphi,\varphi,\varphi-\varphi_\ell) \\ &= \langle h,\varphi-\varphi_\ell\rangle - b(\varphi,\varphi_\ell,\varphi-\varphi_\ell) \\ &\le K_0^2|h|_{0,\Omega}|\varphi-\varphi_\ell|_{2,\Omega} + K_{10}|\varphi|_{2,\Omega}|\varphi_\ell|_{2,\Omega}|\varphi-\varphi_\ell|_{2,\Omega}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\nu|\varphi|_{2,\Omega} \le \nu|\varphi-\varphi_\ell|_{2,\Omega} + \nu|\varphi_\ell|_{2,\Omega} \\ \le\ & K_0^2|h|_{0,\Omega} + K_{10}|\varphi|_{2,\Omega}|\varphi_\ell|_{2,\Omega} + \nu 2\|(h_1,h_2)\|_2 \end{aligned}$$

or

$$\begin{aligned} &(\nu - 2K_{10}\|(h_1,h_2)\|_2)\,|\varphi|_{2,\Omega} \\ \le\ & K_0^2|h|_{0,\Omega} + 2\nu\|(h_1,h_2)\|_2. \end{aligned}$$

If ν is sufficiently large, we have established the estimate for φ. Similarly, we can show the same result for S_φ. This completes the proof. ■

To see how close S_φ is to φ, we consider first how close $\tilde{\varphi}$ is to φ, where $\tilde{\varphi} \in H^2(\Omega)$ is the solution of (5.1) with h_i replaced by $\tilde{h}_i$.

It is easy to see that $\hat{\theta} = \varphi - \tilde{\varphi}$ satisfies the following linear boundary value problem

$$\begin{cases} \nu a_2(\hat{\theta}, \psi) + b(\hat{\theta}, \varphi, \psi) + b(\tilde{\varphi}, \hat{\theta}, \psi) = 0, & \forall\ \psi \in H_0^2(\Omega) \\ \dfrac{\partial \hat{\theta}}{\partial n} = h_1 - \tilde{h}_1 & \text{on } \ \partial\Omega \\ \hat{\theta} = h_2 - \tilde{h}_2 & \text{on } \ \partial\Omega. \end{cases} \tag{5.4}$$

Clearly, by Lemma 2.5, there exists a $\theta_0 \in H^2(\Omega)$ satisfying

$$\begin{cases} \nu a_2(\theta_0, \psi) = 0 & \forall\ \psi \in H_0^2(\Omega) \\ \dfrac{\partial \theta_0}{\partial n} = h_1 - \tilde{h}_1 & \text{on } \ \partial\Omega \\ \theta_0 = h_2 - \tilde{h}_2 & \text{on } \ \partial\Omega \end{cases}$$

and

$$|\theta_0|_{2,\Omega} \ \leq \ 2\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_2 \tag{5.5}$$

$$|\theta_0|_{1,\Omega} \ \leq \ K_3\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_1. \tag{5.6}$$

Let $\theta = \hat{\theta} - \theta_0 \in H_0^2(\Omega)$. Then θ is a solution of the fourth order problem

$$\begin{cases} \nu a_2(\theta, \psi) + b(\theta, \varphi, \psi) + b(\tilde{\varphi}, \theta, \psi) \\ \qquad = -b(\theta_0, \varphi, \psi) - b(\tilde{\varphi}, \theta_0, \psi), & \forall\ \psi \in H_0^2(\Omega) \\ \dfrac{\partial \theta}{\partial n} = 0, & \text{on } \partial\Omega \\ \theta = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.7}$$

The corresponding bilinear form in θ and ψ is coercive when ν is large enough since, by Lemma 4.5 and Theorem 5.5,

$$\begin{aligned} \nu a_2(\theta, \theta) + b(\theta, \varphi, \theta) + b(\tilde{\varphi}, \theta, \theta) \ &\geq \ \nu|\theta|_{2,\Omega}^2 - K_{10}|\varphi|_{2,\Omega}|\theta|_{2,\Omega}^2 \\ &= \ (\nu - K_{10}|\varphi|_{2,\Omega})|\theta|_{2,\Omega}^2 \\ &\geq \ \beta|\theta|_{2,\Omega}^2. \end{aligned}$$

It follows that the solution θ of (5.7) satisfies

$$\begin{aligned} \nu|\theta|_{2,\Omega}^2 \ &= \ \nu a_2(\theta, \theta) \\ &= \ \nu a_2(\theta, \theta) + b(\theta, \varphi, \theta) + b(\tilde{\varphi}, \theta, \theta) - b(\theta, \varphi, \theta) \\ &= \ -b(\theta_0, \varphi, \theta) - b(\tilde{\varphi}, \theta_0, \theta) - b(\theta, \varphi, \theta) \\ &\leq \ K_{10}|\theta_0|_{2,\Omega}(|\varphi|_{2,\Omega} + |\tilde{\varphi}|_{2,\Omega})|\theta|_{2,\Omega} + K_{10}|\theta|_{2,\Omega}^2|\varphi|_{2,\Omega}. \end{aligned}$$

That is, we have

$$(\nu - K_{10}|\varphi|_{2,\Omega})|\theta|_{2,\Omega} \le K_{10}(|\varphi|_{2,\Omega} + |\tilde{\varphi}|_{2,\Omega})2\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_2).$$

By Theorem 5.5, it follows that there exists a constant C, dependent upon h_1 and h_2 such that

$$|\theta|_{2,\Omega} \le C\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_2.$$

Hence, because of $|\hat{\theta}|_{2,\Omega} \le |\theta|_{2,\Omega} + |\theta_0|_{2,\Omega}$, we have obtained the following estimate:

$$|\hat{\theta}|_{2,\Omega} \le (C+2)\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_2.$$

Similarly, we have

$$\begin{aligned} |\hat{\theta}|_{1,\Omega} &\le |\theta|_{1,\Omega} + |\theta_0|_{1,\Omega} \\ &\le K_0|\theta|_{2,\Omega} + K_0\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_1 \\ &\le K_0 C\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_2 + K_0\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_1 \\ &\le K_0(C+1)\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_1. \end{aligned}$$

Therefore, we have obtained the following by recalling that $\hat{\theta} = \varphi - \tilde{\varphi}$.

LEMMA 5.6. Let $\varphi \in H^2(\Omega)$ and $\tilde{\varphi} \in H^2(\Omega)$ be the weak solution of (5.1) and (5.3). If ν is large enough, then

$$|\varphi - \tilde{\varphi}|_{2,\Omega} \le C\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_2$$

and

$$|\varphi - \tilde{\varphi}|_{1,\Omega} \le C\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_1$$

for a positive constant C.

Next we consider the difference $\tilde{\varphi} - S_\varphi$. We have

LEMMA 5.7. Suppose that ν is large enough. Then

$$|\tilde{\varphi} - S_\varphi|_{2,\Omega} \le C_6 \inf_{s \in S^r_{3r}(\diamondsuit) \cap H^2_0(\Omega)} |\varphi - S_\varphi - s|_{2,\Omega} + C_6\|(h_1 - \tilde{h}_1, h_2 - \tilde{h}_2)\|_2.$$

Proof: Since $\tilde{\varphi} - S_\varphi \in H^2_0(\Omega)$, we have for any $s \in H^2_0(\Omega) \cap S^r_{3r}(\diamondsuit)$

$$\begin{aligned} \nu|\tilde{\varphi} - S_\varphi|^2_{2,\Omega} &= \nu a_2(\tilde{\varphi} - S_\varphi, \tilde{\varphi} - S_\varphi) \\ &= \nu a_2(\tilde{\varphi} - S_\varphi, \tilde{\varphi} - S_\varphi - s) + \nu a_2(\tilde{\varphi} - S_\varphi, s) \end{aligned}$$

$$
\begin{aligned}
&= \nu a_2(\tilde\varphi - S_\varphi, \tilde\varphi - \varphi) + \nu a_2(\tilde\varphi - S_\varphi, \varphi - S_\varphi - s) \\
&\qquad -b(\tilde\varphi, \tilde\varphi - S_\varphi, s) - b(\tilde\varphi - S_\varphi, S_\varphi, s) \\
&\le \nu|\tilde\varphi - S_\varphi|_{2,\Omega}|\tilde\varphi - \varphi|_{2,\Omega} + \nu|\tilde\varphi - S_\varphi|_{2,\Omega}|\varphi - S_\varphi - s|_{2,\Omega} \\
&\qquad -b(\tilde\varphi, \tilde\varphi - S_\varphi, s - (\tilde\varphi - S_\varphi)) - b(\tilde\varphi - S_\varphi, S_\varphi, s - (\tilde\varphi - S_\varphi)) \\
&\qquad -b(\tilde\varphi - S_\varphi, S_\varphi, \tilde\varphi - S_\varphi) \\
&\le \nu|\tilde\varphi - S_\varphi|_{2,\Omega}|\tilde\varphi - \varphi|_{2,\Omega} + \nu|\tilde\varphi - S_\varphi|_{2,\Omega}|\varphi - S_\varphi - s|_{2,\Omega} \\
&\qquad +K_{10}|\tilde\varphi|_{2,\Omega}|\tilde\varphi - S_\varphi|_{2,\Omega}(|\varphi - \tilde\varphi|_{2,\Omega} + |\varphi - S_\varphi - s|_{2,\Omega}) \\
&\qquad +K_{10}|S_\varphi|_{2,\Omega}|\tilde\varphi - S_\varphi|_{2,\Omega}|\tilde\varphi - S_\varphi - s|_{2,\Omega} \\
&\qquad +K_{10}|S_\varphi|_{2,\Omega}|\tilde\varphi - S_\varphi|^2_{2,\Omega}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
(\nu \; - \; & K_{10}|S_\varphi|_{2,\Omega})|\tilde\varphi - S_\varphi|_{2,\Omega} \\
&\le (\nu + K_{10}|\tilde\varphi|_{2,\Omega} + K_{10}|S_\varphi|_{2,\Omega})|\tilde\varphi - \varphi|_{2,\Omega} \\
&+ (\nu + K_{10}|S_\varphi|_{2,\Omega} + K_{10}|\tilde\varphi|_{2,\Omega})|\varphi - S_\varphi - s|_{2,\Omega}
\end{aligned}
$$

for any $s \in S^r_{3r}(\diamondsuit) \cap H^2_0(\Omega)$. The proof now follows from Theorem 5.5 and Lemma 5.6. ■

We can now prove the main result in this section.

THEOREM 5.8. Suppose that ν is large enough and that $\varphi \in H^k(\Omega)$ with $3 \le k \le 3r+1$. Then

$$|\varphi - S_\varphi|_{2,\Omega} \le K_{12}|\diamondsuit|^{k-2} \tag{5.8}$$

and

$$\|\mathbf{u} - \mathbf{curl}(S_\varphi)\|_{0,\Omega} \le K_{13}|\diamondsuit|^{k-2} \tag{5.9}$$

for two positive constants K_{12} and K_{13} dependent on $|\varphi|_{k,\Omega}$.

Proof: By Lemma 2.8 there exists a quasi-interpolant $Q_{k-1}\varphi \in S^r_{3r}(\diamondsuit)$ such that

$$|\varphi - Q_{k-1}\varphi|_{2,\Omega} \le K_7|\diamondsuit|^{k-2}|\varphi|_{k,\Omega}.$$

Since $Q_{k-1}\varphi = S_\varphi + s$ for some $s \in S^r_{3r}(\diamondsuit) \cap H^2_0(\Omega)$, we combine the above fact with Lemmas 5.6 and 5.7 to get (5.8). Also, by Lemma 2.2, we have

$$
\begin{aligned}
|\varphi - S_\varphi|_{1,\Omega} &\le |\varphi - \tilde\varphi|_{1,\Omega} + |\tilde\varphi - S_\varphi|_{1,\Omega} \\
&\le |\varphi - \tilde\varphi|_{1,\Omega} + K_0|\tilde\varphi - S_\varphi|_{2,\Omega}.
\end{aligned}
$$

Thus, by Lemmas 5.6 and 5.7, we conclude (5.9). This completes the proof. ■

We now prove Lemmas 5.1 and 5.3 in detail.

Proof of Lemma 5.1: Since h_1 and h_2 are compatible there exists $\varphi_b \in H^2(\Omega)$ satisfying the boundary conditions given in definition 2.1. Let $\rho(x,y) := \mathrm{dist}((\mathrm{x},\mathrm{y}),\partial\Omega)$ be the distance from (x,y) to $\partial\Omega$. For any $\varepsilon > 0$, let $\delta(\varepsilon) = \exp\left(-\frac{1}{\varepsilon}\right)$ and $\Omega_\varepsilon = \{(x,y) \in \Omega;\, \rho(x,y) \le 2\delta(\varepsilon)\}$. Then it is easy to see that Ω_ε is a union of a finite number of quadrilaterals each of which has two parallel sides. Let θ_ε be a continuously twice differentiable function on $\Omega \cup \partial\Omega$ such that

$$\begin{cases} \theta_\varepsilon(x,y) = 1 & \text{in } \{(x,y) \in \Omega, \rho(x,y) \le \delta(\varepsilon)\} \\ \theta_\varepsilon(x,y) = 0 & \text{in } \Omega \setminus \Omega_\varepsilon \\ \left|\frac{\partial}{\partial x}\theta_\varepsilon(x,y)\right| \le \frac{\varepsilon}{\rho(x,y)} & \text{in } \Omega_\varepsilon \\ \left|\frac{\partial}{\partial y}\theta_\varepsilon(x,y)\right| \le \frac{\varepsilon}{\rho(x,y)} & \text{in } \Omega_\varepsilon \end{cases}$$

Existence of such function θ_ε follows from [Temam'77]. Clearly, $\left.\frac{\partial\theta_\varepsilon}{\partial n}\right|_{\partial\Omega} = 0$ and hence $\phi_b\theta_\varepsilon \in H^2(\Omega)$ also satisfies the boundary conditions. By Sobolev's imbedding theorem (cf. [Adams'75]), we have

$$\left(\int_\Omega |\nabla\varphi_b|^8 dxdy\right)^{1/8} \le C_1\|\varphi_b\|_{2,\Omega}$$

and $\|\varphi_b\|_{L_\infty(\Omega)} \le C_2\|\varphi_b\|_{2,\Omega}$. With these inequalities, we have

$$\begin{aligned} \int_\Omega \left|\frac{\partial}{\partial x}(\varphi_b\theta_\varepsilon)\frac{\partial}{\partial y}\theta\right|^2 dxdy &\le 2\int_\Omega \left|\frac{\partial\varphi_b}{\partial x}\theta_\varepsilon\frac{\partial\theta}{\partial y}\right|^2 dxdy + 2\int_\Omega \left|\varphi_b\frac{\partial\theta_\varepsilon}{\partial x}\frac{\partial\theta}{\partial y}\right|^2 dxdy \\ &\le 2\left(\int_{\Omega_\varepsilon}\left|\frac{\partial\varphi_b}{\partial x}\theta_\varepsilon\right|^4 dxdy\right)^{1/2}\left(\int_{\Omega_\varepsilon}\left|\frac{\partial\theta}{\partial y}\right|^4 dxdy\right)^{1/2} \\ &+ 2\max_{(x,y)\in\Omega_\varepsilon}\left|\rho(x,y)\varphi_b(x,y)\frac{\partial\theta_\varepsilon}{\partial x}(x,y)\right|^2\int_{\Omega_\varepsilon}\left|\frac{1}{\rho}\frac{\partial\theta}{\partial y}\right|^2 dxdy \\ &\le 2C\delta(\varepsilon)^{1/4}\left(\int_{\Omega_\varepsilon}\left|\frac{\partial\varphi_b}{\partial x}\right|^8 dxdy\right)^{1/4}\sqrt{2}K_0|\theta|^2_{2,\Omega} \\ &+ 2\varepsilon^2\|\varphi_b\|^2_{L_\infty(\Omega)}\int_{\Omega_\varepsilon}\left|\frac{1}{\rho}\frac{\partial\theta}{\partial y}\right|^2 dxdy \\ &\le \left(2C\delta(\varepsilon)^{1/4}\sqrt{2}K_0C_1^2\|\varphi_b\|^2_{2,\Omega} + 2\varepsilon^2C_2^2\|\varphi_b\|^2_{2,\Omega)}\right)|\theta|^2_{2,\Omega} \end{aligned}$$

by using Lemmas 4.1 and 2.2 and the well-known Hardy inequality. Similarly, we have

$$\int_\Omega \left|\frac{\partial}{\partial y}(\varphi_b\theta_\varepsilon)\frac{\partial}{\partial x}\theta\right|^2 dxdy \le C\left(\delta(\varepsilon)^{1/4} + \varepsilon^2\right)\|\varphi_b\|^2_{2,\Omega}|\theta|^2_{2,\Omega}.$$

Therefore, we conclude that

$$\begin{aligned}|b(\theta,\varphi_b\theta_\varepsilon,\theta)|^2 &\le |\Delta\theta|^2_{0,\Omega}\Big(\int_\Omega \left|\frac{\partial}{\partial x}(\varphi_b\theta_\varepsilon)\frac{\partial\theta}{\partial y}\right|^2 dxdy + \int_\Omega \left|\frac{\partial}{\partial y}(\varphi_b\theta_\varepsilon)\frac{\partial\theta}{\partial x}\right|^2 dxdy\\ &\le \mu^2(\varepsilon)|\theta|^2_{2,\Omega}|\theta|^2_{2,\Omega}\end{aligned}$$

with $\mu^2(\varepsilon) = 2C(\delta(\varepsilon)^{1/4}+\varepsilon^2)\|\varphi_b\|^2_{2,\Omega}$. For any $\gamma > 0$, let ε be small enough to guarantee $\mu(\varepsilon) \le \gamma$. Then

$$|b(\theta,\varphi_b\theta_\varepsilon,\theta)| \le \mu(\varepsilon)|\theta|^2_{2,\Omega} \le \gamma|\theta|^2_{2,\Omega}.$$

This completes the proof. ■

Proof of Lemma 5.3: Let $\tilde\varphi_b \in H^2(\Omega)$ satisfy the boundary conditions with $\tilde h_1$ and $\tilde h_2$. We may assume that $\tilde\varphi_b \in C^1(\overline{\Omega})$, for example $\tilde\varphi_b \in S^r_{3r}(\diamondsuit)$. Let θ_ε be the same function as in the proof of Lemma 5.1. Let $\diamondsuit_n$ be the nth refinement of $\diamondsuit$ and $S_{\tilde\varphi_b\theta_\varepsilon} \in S^r_{3r}(\diamondsuit_n)$ be the spline quasi-interpolant of $\tilde\varphi_b\theta_\varepsilon$. Since $\tilde\varphi_b\theta_\varepsilon \in C^1(\overline{\Omega})$ and $\frac{\partial}{\partial x}S_{\tilde\varphi_b\theta_\varepsilon}$ converges to $\frac{\partial}{\partial x}(\tilde\varphi_b\theta_\varepsilon)$ in L_∞ norm when $n \to +\infty$, we have

$$\begin{aligned}&\int_\Omega \left|\frac{\partial}{\partial x}(S_{\tilde\varphi_b\theta_\varepsilon})\frac{\partial\theta}{\partial y}\right|^2 dxdy\\ \le\ & 2\int_{\Omega_\varepsilon}\left|\frac{\partial}{\partial x}(S_{\tilde\varphi_b\theta_\varepsilon}) - \frac{\partial}{\partial x}\tilde\varphi_b\theta_\varepsilon\right|^2\left(\frac{\partial\theta}{\partial y}\right)^2 dxdy\\ &\quad +2\int_{\Omega_\varepsilon}\left(\frac{\partial}{\partial x}(\tilde\varphi_b\theta_\varepsilon)\right)^2\left|\frac{\partial\theta}{\partial y}\right|^2 dxdy\\ \le\ & 2\max_{(x,y)\in\Omega_\varepsilon}\left|\rho(x,y)\left(\frac{\partial}{\partial x}(S_{\tilde\varphi_b\theta_\varepsilon}) - \frac{\partial}{\partial x}(\tilde\varphi_b\theta_\varepsilon)\right)\right|^2\int_{\Omega_\varepsilon}\left|\frac{1}{\rho}\frac{\partial\theta}{\partial y}\right|^2 dxdy\\ +\ & C\left(\delta(\varepsilon)^{1/4}+\varepsilon^2\right)\|\tilde\varphi_b\|^2_{2,\Omega}|\theta|^2_{2,\Omega}\end{aligned}$$

by using the same argument in the proof of Lemma 5.1. Similarly, we have the same estimate for $\int_\Omega \left|\frac{\partial}{\partial y}(S_{\tilde\varphi_b\theta_\varepsilon})\frac{\partial\theta}{\partial x}\right|^2 dxdy$. Hence, for any $\theta \in S^r_{3r}(\diamondsuit)\cap H^2_0(\Omega)$ we have,

$$\begin{aligned}|b(\theta,S_{\tilde\varphi_b\theta_\varepsilon},\theta)|^2 &\le |\Delta\theta|^2_{0,\Omega}\Big(\int_\Omega\left|\frac{\partial}{\partial y}(S_{\tilde\varphi_b\theta_\varepsilon})\frac{\partial\theta}{\partial x}\right|^2 dxdy + \int_\Omega\left|\frac{\partial}{\partial x}(S_{\tilde\varphi_b\theta_\varepsilon})\frac{\partial\theta}{\partial y}\right|^2 dxdy\Big)\\ &\le 2\Big[\max_{(x,y)\in\Omega_\varepsilon}\left|\rho(x,y)\left(\frac{\partial}{\partial x}(S_{\tilde\varphi_b\theta_\varepsilon}) - \frac{\partial}{\partial x}(\tilde\varphi_b\theta_\varepsilon)\right)\right|^2\\ &\qquad + \max_{(x,y)\in\Omega_\varepsilon}|\rho(x,y)\left(\frac{\partial}{\partial y}(S_{\tilde\varphi_b\theta_\varepsilon}) - \frac{\partial}{\partial y}(\tilde\varphi\theta_\varepsilon)\right)|^2\Big]|\theta|^2_{2,\Omega}\\ &\qquad +2C\Big[\delta(\varepsilon)^{1/4}+\varepsilon^2\Big]\|\tilde\varphi_b\|^2_{2,\Omega)}|\theta|^2_{2,\Omega}\\ &=: \mu_1^2(\diamondsuit_n)|\theta|^2_{2,\Omega} + \mu_2^2(\varepsilon)|\theta|^2_{2,\Omega}.\end{aligned}$$

For any $\gamma > 0$, let ε be small enough to guarantee that $\mu_2(\varepsilon) < \gamma/2$ and, fixing θ_ε, let n be large enough to guarantee that $\mu_1(\triangle_n) < \gamma/2$. This completes the proof. ■

6 Numerical Implementation of the Navier-Stokes Equations

In this section we shall show that Newton's iterations converge to the spline $S_\varphi \in S^r_{3r}(\triangle)$ satisfying

$$\nu a_2(S_\varphi, \psi) + b(S_\varphi, S_\varphi, \psi) = \langle h, \psi \rangle, \forall \psi \in S^r_{3r}(\triangle) \cap H^2_0(\Omega)$$

with $S_\varphi|_{\partial\Omega} = 0$ and $\dfrac{\partial}{\partial n} S_\varphi|_{\partial\Omega} = 0$. Note that the Navier-Stokes equations with nonhomogeneous boundary conditions can be solved in a similar manner.

Recall that $\{\psi_i\}_{i=1}^N \subset S^r_{3r}(\triangle) \cap H^2_0(\Omega)$ is a locally supported basis and that we may write

$$S_\varphi(x, y) = \sum_{i=1}^N c_i \psi_i(x, y).$$

Then the weak formulation of the Navier-Stokes equations can be written as a system of nonlinear equations

$$\begin{aligned} \nu \sum_{i=1}^N c_i \int_\Omega \Delta\psi_i \Delta\psi_j dxdy \quad &+ \quad \int_\Omega \sum_{i=1}^N c_i \Delta\psi_i \left(\sum_{k=1}^N c_k \left(\frac{\partial\psi_k}{\partial y}\frac{\partial\psi_j}{\partial x} - \frac{\partial\psi_k}{\partial x}\frac{\partial\psi_j}{\partial y} \right) \right) \\ &= \quad \int_\Omega h\psi_j dxdy, \quad j = 1, \ldots, N. \end{aligned}$$

Let $a_{ij} = \int_\Omega \Delta\psi_i \Delta\psi_j dxdy$ and $b_{ijk} = \int_\Omega \Delta\psi_i \left(\frac{\partial\psi_k}{\partial y}\frac{\partial\psi_j}{\partial x} - \frac{\partial\psi_k}{\partial x}\frac{\partial\psi_j}{\partial y} \right) dxdy$, and $h_j = \int_\Omega h\psi_j dxdy$. Then the above nonlinear system is

$$\nu \sum_{i=1}^N c_i a_{ij} + \sum_{i,k=1}^N c_i c_k b_{i,j,k} = h_j, \quad j = 1, 2, \ldots, N$$

In matrix form we have

$$\nu A\mathbf{c} + \mathbf{c}^T \mathbf{B} \mathbf{c} = \mathbf{h}$$

with $\mathbf{c} = (c_1, \ldots, c_N)^T, A = (a_{ij})_{N\times N}, \mathbf{B} = (b_{i,k,j})_{N\times N\times N}$ and $\mathbf{h} = (h_1, \ldots, h_N)^T$. Thus, we need to solve the nonlinear equations

$$F(\mathbf{c}) = \nu A\mathbf{c} + \mathbf{c}^T \mathbf{B}\mathbf{c} - \mathbf{h} = 0.$$

Newton's method: Starting with an initial guess $\mathbf{c}^{(0)}$, define

$$\mathbf{c}^{(n+1)} = \mathbf{c}^{(n)} - F'(\mathbf{c}^{(n)})^{-1} F(\mathbf{c}^{(n)})$$

with $F'(\mathbf{c}) = \nu A + \mathbf{c}^T\mathbf{B} + \mathbf{B}^T\mathbf{c}$.

Let $\mathbf{c}^*$ be the solution of $F(\mathbf{c}^*) = 0$. As we knew from Section 4 such $\mathbf{c}^*$ exists. Then by Taylors' expansion we have

$$\begin{aligned} 0 = F(\mathbf{c}^*) &= F(\mathbf{c}^{(n)}) + F'(\mathbf{c}^{(n)})(\mathbf{c}^{(*)} - \mathbf{c}^{(n)}) \\ &+ \frac{1}{2}(\mathbf{c}^* - \mathbf{c}^{(n)})^T F''(\xi)(\mathbf{c}^{(*)} - \mathbf{c}^{(n)}). \end{aligned}$$

Here the second order Frechet's derivative $F''(\xi)$ is actually a constant matrix of size $N \times N \times N$, i.e. $F''(\xi) = \mathbf{B} + \mathbf{B}^T$. From Newton's method we have $F'(\mathbf{c}^{(n)})(\mathbf{c}^{(n+1)} - \mathbf{c}^{(n)}) = -F(\mathbf{c}^{(n)})$. Thus,

$$0 = F'(\mathbf{c}^{(n)})(\mathbf{c}^* - \mathbf{c}^{(n+1)}) + \frac{1}{2}(\mathbf{c}^{(*)} - \mathbf{c}^{(n)})^T(\mathbf{B} + \mathbf{B}^T)(\mathbf{c}^{(*)} - \mathbf{c}^{(n)})$$

or

$$\mathbf{c}^{(n+1)} - \mathbf{c}^* = \frac{1}{2}F'(\mathbf{c}^{(n)})^{-1}(\mathbf{c}^* - \mathbf{c}^{(n)})^T(\mathbf{B} + \mathbf{B}^T)(\mathbf{c}^{(*)} - \mathbf{c}^{(n)}).$$

Let $\|\cdot\|_2$ denote the usual ℓ^2 norm of vectors and the corresponding induced norm for matrices. Suppose that $\|F'(\mathbf{c})^{-1}\|_2 \le C_4$ in a neighborhood of $\mathbf{c}^*$ and note that

$$(\mathbf{c}^* - \mathbf{c}^{(n)})^T\mathbf{B} = \left[\int_\Omega \Delta\left(S_\phi^{(*)} - S_\phi^{(n)}\right)\left(\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial x} - \frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial y}\right)dxdy\right]_{N\times N}.$$

By Lemma 4.5 we have

$$\begin{aligned} &\left|\int_\Omega \Delta\left(S_\phi^{(*)} - S_\phi^{(n)}\right)\left(\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial x} - \frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial y}\right)dxdy\right| \\ \le\ & K_{10}|S_\phi^* - S_\phi^{(n)}|_{2,\Omega}|\psi_i|_{2,\Omega}|\psi_j|_{2,\Omega}. \end{aligned}$$

it follows that

$$\begin{aligned} \|(\mathbf{c}^* - \mathbf{c}^{(n)})^T\mathbf{B}\|_2 &\le C_2|S_\phi^* - S_\phi^{(n)}|_{2,\Omega}\|\left[|\psi_i|_{2,\Omega}|\psi_j|_{2,\Omega}\right]_{N\times N}\|_2 \\ &\le C_5\|\mathbf{c}^{(*)} - \mathbf{c}^{(n)}\|_2 \end{aligned}$$

Therefore we get

$$\|\mathbf{c}^{(n+1)} - \mathbf{c}^{(*)}\|_2 \le C_5C_4\|\mathbf{c}^{(n)} - \mathbf{c}^{(*)}\|_2^2.$$

Let $d_n = C_4C_5\|\mathbf{c}^{(n)} - \mathbf{c}^{(*)}\|^2$. The above inequality becomes

$$d_{n+1} \le d_n^2$$

and hence

$$d_{n+1} \le d_m^{2^{n-m+1}}.$$

If $d_m < 1$, then Newton's method converges. Hence we have obtained the following.

THEOREM 6.1. Suppose that an initial guess $\mathbf{c}^{(0)}$ is sufficiently close to the exact solution $\mathbf{c}^{(*)}$ in the sense that $d_0 < 1$. Then Newton's iteration method converges and for all $n = 0, 1, 2 \ldots$.

$$\|\mathbf{c}^{(n+1)} - \mathbf{c}^{(*)}\|_2 \leq \|\mathbf{c}^{(0)} - \mathbf{c}^{(*)}\|_2^{2^{n+1}}.$$

To make Newton's method successful we use a homotopy idea to find a good initial point as follows. It is apparent that Newton's iteration method will converge if ν is sufficiently large. Suppose that for a value ν_0, the Newton's method converges to a solution of the equation

$$F_{\nu_0}(\mathbf{c}_{\nu_0}) = 0.$$

where $F_\nu(\mathbf{c}) = \nu A\mathbf{c} + \mathbf{c}^T\mathbf{B}\mathbf{c} - \mathbf{h}$. Then for ν sufficiently close to ν_0 we use $\mathbf{c}_{\nu_0}$ as an initial vector for Newton's method applied to nonlinear system $F_\nu(\mathbf{c}) = 0$. Since $F_\nu(\mathbf{c}_{\nu_0}) = F_{\nu_0}(\mathbf{c}_{\nu_0}) + (\nu - \nu_0)A\mathbf{c}_{\nu_0} = (\nu - \nu_0)A\mathbf{c}_{\nu_0}$ and $\|A\mathbf{c}_{\nu_0}\|_2$ is bounded, $F_\nu(\mathbf{c}_{\nu_0})$ will be sufficiently close to zero and $\mathbf{c}_{\nu_0}$ will be sufficiently close to the solution of $F_\nu(\mathbf{c}) = 0$ if ν is sufficiently close to ν_0. Starting from ν_0, we find a $\nu_1 < \nu_0$ such that Newton's method converges. Then we find a $\nu_2 < \nu_1$ and so on.

We have implemented Newton's iteration method to solve the Navier-Stokes equations. We found that the homotopy idea described above provides an excellent way of obtaining a good initial vector. For example, for the well-known test case of the cavity flow problem over the unit square (cf. [Greenspan'69], our method converges when the Reynolds number increases to over $50,000$.

We also find that as the triangulation is refined, we are able to solve the Navier-Stokes' equations with a larger Reynolds number. For example, over the L-shape domain, we set our target Reynolds number to be $1/\nu_t := 10,000$ and started from a Reynolds number $1/\nu_0 = 100$. Our programs gradually reduce the ν, i.e., increase the Reynolds number in the following way: Initially, let $\nu_1 = \nu_t$, the target viscosity. Find a $\nu_0 > \nu_1$ for which our numerical method converges. We use the solution corresponding to this ν_0 as an initial value for the Newton's iterations used to solve the Navier-Stokes equations with ν_1. If Newton's iterations are convergent and $\nu_1 = \nu_t$, we are done. If Newton's iterations are convergent and $\nu_1 \neq \nu_t$, we set $\delta = \nu_1 - \nu_0$, $\nu_0 = \nu_1$, and $\nu_1 = \max\{\nu_t, \nu_1 - \delta\}$. Otherwise, i.e., if Newton's iterations failed to converge for initial value ν_0, we set $\nu_1 = (\nu_0 + \nu_1)/2$. Then we solve Navier-Stokes equations with the new ν_1 and repeat the above steps.

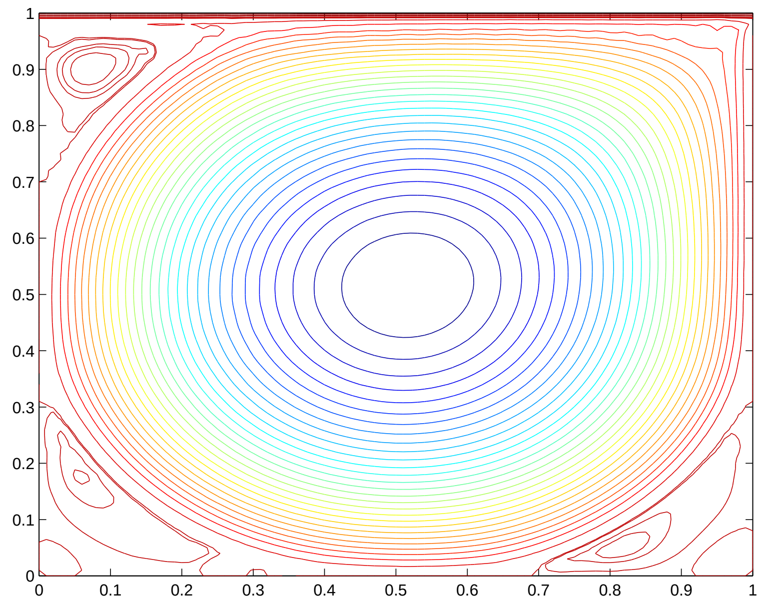

Figure 2. Streamline of the cavity flow at Reynolds number 30,000

We tabulate the numerical experiments in Table 3 of the maximum errors based on 30401 points over L-shape domains and Reynolds numbers $1/\nu$ for two test functions. Note how the Reynolds number increases as the domain is refined.

Table 3. Reynolds Numbers and Maximum Errors of the C^1 Cubic Spline Method for Nonlinear Biharmonic Equations (1.6)

Matrix Sizes	x^3y^3	x^4+y^4
150x150	$1/\nu = 1663$	$1/\nu = 1035$
	2.8903×10^{-1}	3.342×10^{-1}
527x527	$1/\nu = 2716$	$1/\nu = 5512$
	3.244×10^{-2}	7.637×10^{-2}
1971x1971	$1/\nu = 10,000$	$1/\nu = 10,000$
	1.837×10^{-2}	4.033×10^{-3}

7 Numerical Solution of Pressure Functions

In this section, we consider the numerical solution of the pressure function in the steady state Stokes and Navier-Stokes equations. We first recall the following basic lemmas.

LEMMA 7.1. Let $\mathcal{V} = \{\mathbf{u} : \text{div } \mathbf{u} = 0,\ u_1, u_2 \in C_0^\infty(\Omega)\}$. Given a vector function $\mathbf{f} = (f_1, f_2) \in H^{-1}(\Omega)^2$, if $\langle \mathbf{f}, \mathbf{u} \rangle = \langle f_1, u_1 \rangle + \langle f_2, u_2 \rangle = 0$ for all test functions in $\mathcal{V}$, then there exists a distribution $p \in L_2(\Omega)$ such that $\mathbf{f} = \nabla p$.

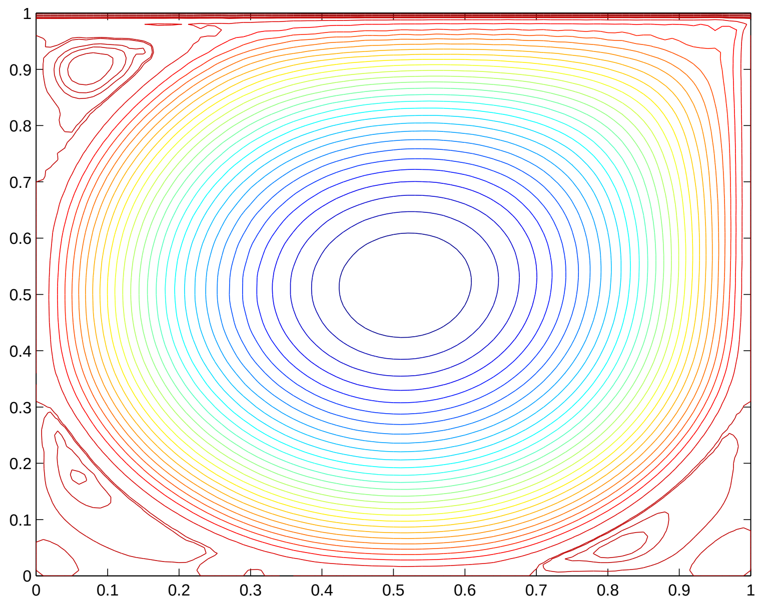

Figure 3. Streamline of the cavity flow at Reynolds number 60,000

See [Girault and Raviart'86, p.25] for a proof. See [Temam'77, p.15] for a proof of the following lemma.

LEMMA 7.2. If a distribution p has its first derivatives $\frac{\partial}{\partial x}p$ and $\frac{\partial}{\partial y}p$ in $H^{-1}(\Omega)$, then $p \in L_2(\Omega)$ and

$$|p|_{0,\Omega} \leq C|p|_{-1,\Omega}$$

for a constant C independent of p.

We next recall the numerical solution of the standard Poisson equation with Neumann boundary condition:

$$\begin{cases} -\Delta p = h & \text{in } \Omega \\ \frac{\partial p}{\partial n} = g & \text{on } \partial\Omega \\ \int_\Omega p \, dxdy = 0 \end{cases}$$

whose weak formulation is

$$a_1(p,q) = \langle h, q\rangle + \langle g, q\rangle_{\partial\Omega}, \quad q \in \tilde{H}^1(\Omega) \tag{7.1}$$

where $a_1(p,q) = \int_\Omega \nabla p \cdot \nabla q \; dxdy$, $\langle g, q\rangle_{\partial\Omega} = \int_{\partial\Omega} gq \; ds$, and $\tilde{H}^1(\Omega) = \{f \in H^1(\Omega) : \int_\Omega f \; dxdy = 0\}$. Here, we have to assume that h and g satisfy a compatibility condition:

$$\int_\Omega h dxdy + \int_{\partial\Omega} gds = 0.$$

Let us sketch a short proof for the existence and uniqueness of the weak solution of the Poisson equation.

LEMMA 7.3. For any given $h \in L_2(\Omega)$ and $g \in L_2(\partial\Omega)$, there exists a unique $p \in \tilde{H}^1(\Omega)$ satisfying (7.1).

Proof: By Lemma 2.3, $a_1(q,q) \geq C\|q\|_{1,\Omega}$ and hence, (7.1) is coercive on $\tilde{H}^1(\Omega)$. By Lemmas 2.3 and 2.4,

$$\langle h, q\rangle + \langle g, q\rangle_{\partial\Omega} \leq \|h\|_{0,\Omega} C\|q\|_{1,\Omega} + \|g\|_{L_2(\partial\Omega)} C\|q\|_{1,\Omega}$$

if $h \in L_2(\Omega)$ and $g \in L_2(\partial\Omega)$. By Lemma 2.1, the Lax-Milgram Theorem, there exists an unique weak solution of (7.1). Thus, we have completed the proof. ■

Let

$$\tilde{S}^r(\diamondplus) = \begin{cases} \{s \in S_1^0(\diamondplus) : \int_\Omega s\ dxdy = 0\}, & \text{if } r = 0 \\ \{s \in S_{3r}^r(\diamondplus) : \int_\Omega s\ dxdy = 0\}, & \text{if } r \geq 1. \end{cases}$$

Note that $\partial\Omega \cup \diamondplus$ consists of a finite number of line segments $[v_i, v_{i+1}], i = 1, \cdots, N$. If a function g is in $L_2([v_i, v_{i+1}])$ for all $i = 1, \cdots, N$, we will denote this by $g \in L_2(\partial\Omega \cap \diamondplus)$. Similarly we will say $f \in L_2(\Omega \cap \diamondplus)$ if $\forall t \in \diamondplus f|_t \in L_2(t)$ and we define

$$\|f\|_{0,\Omega}^2 = \sum_{t \in \diamondplus} \|f\|_{0,t}^2.$$

Similar to the proof of Lemma 7.3 we have

LEMMA 7.4. For any given $h \in L_2(\Omega \cap \diamondplus)$ and $g \in L_2(\partial\Omega \cap \diamondplus)$, there exists a unique $S_p \in \tilde{S}^r(\diamondplus)$ satisfying

$$a_1(S_p, q) = \langle h, q\rangle + \langle g, q\rangle_{\partial\Omega}, \quad \forall\ q \in \tilde{S}^r(\diamondplus).$$

We are now ready to discuss the numerical solution of the pressure terms in the Stokes and Navier-Stokes equation.

THEOREM 7.5. Suppose that $\mathbf{f} \in H^{-1}(\Omega)^2$ and that the solution $\mathbf{u} = \mathbf{curl}(\varphi) \in H^1(\Omega)^2$, where $\mathbf{u}$ is the planar velocity vector of the steady state Stokes equations. Then there exists a $p \in L_2(\Omega)$ such that

$$\nabla p = \nu\Delta(\mathbf{curl}\ \varphi) + \mathbf{f}. \tag{7.2}$$

Furthermore, if $\mathbf{u} \in H^2(\Omega)^2$ and $\mathbf{f} \in L_2(\Omega)^2$ then $p \in H^1(\Omega)$.

Proof. We apply Lemma 7.1. For any $\mathbf{v} \in \mathcal{V}$, since $\mathrm{div}(\mathbf{v}) = 0$, let ψ be a function such that $\mathbf{v} = \mathbf{curl}(\psi)$. Then recalling that $h = \dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y}$, we have

$$\int_\Omega (\nu\Delta(\mathbf{curl}\ \varphi) + \mathbf{f}) \cdot \mathbf{v}\ dxdy$$

$$= \nu \int_\Omega \mathbf{curl}(\Delta\varphi) \cdot \mathbf{curl}\ \psi\ dxdy + \int_\Omega \mathbf{f} \cdot \mathbf{curl}\ \psi\ dxdy$$
$$= -\nu \int_\Omega \Delta\varphi\Delta\psi\ dxdy + \int_\Omega h\psi\ dxdy$$
$$= -\nu a_2(\varphi, \psi) + \langle h, \psi \rangle$$
$$= 0.$$

Thus Lemma 7.1 implies that there exists a $p \in L_2(\Omega)$ such that p satisfies (7.2).

When $\mathbf{u} \in H^2(\Omega)^2$, and $\mathbf{f} \in L_2(\Omega)^2$, we immediately know that $\nabla p \in L_2(\Omega)$ and hence $p \in H^1(\Omega)$. This completes the proof. ■

Similarly, we can show the following

THEOREM 7.6. Let $\mathbf{u} = \mathbf{curl}\ \varphi$ be the weak solution of Navier-Stokes equations (4.1). Suppose that $\mathbf{f} \in H^{-1}(\Omega)^2$. Then there exists a $p \in L_2(\Omega)$ satisfying

$$\nabla p = \nu\Delta(\mathbf{curl}\varphi) + (\mathbf{curl}\ \varphi \cdot \nabla)\mathbf{curl}\varphi + \mathbf{f} \tag{7.3}$$

Furthermore, if $\mathbf{f} \in L_2(\Omega)^2$ and if $\varphi \in C^2(\Omega) \cap H^3(\Omega)$ then $p \in H^1(\Omega)$.

Under the conditions in Theorems 7.5 and 7.6, it is easy to see that the pressure functions p of the Stokes and Navier-Stokes equations satisfy the Poisson equations (1.7) and (1.8). Let us consider the weak formulations of the Poisson equations with Neumann boundary condition:

$$a_1(p, q) = \langle \mathbf{f}, \nabla q \rangle + \nu \langle \Delta\mathbf{curl}(\varphi), \nabla q \rangle, \quad q \in \tilde{H}^1(\diamondsuit) \tag{7.4}$$

for the Stokes equation and

$$a_1(p, q) \;\; = \;\; \langle \mathbf{f}, \nabla q \rangle - \langle (\mathbf{curl}\varphi \cdot \nabla)\mathbf{curl}\varphi, \nabla q \rangle \tag{7.5}$$
$$+\nu \langle \Delta\mathbf{curl})\varphi, \nabla q \rangle, \quad \forall\ q \in \tilde{H}^1(\diamondsuit) \tag{7.6}$$

for the Navier-Stokes equations, where $p \in \tilde{H}^1(\Omega)$.

Our numerical method for the pressure functions p of the Stokes and Navier-Stokes equations is to solve the Poisson equations (7.4) and (7.5) with φ replaced by $S_\varphi \in S^r_{3r}(\diamondsuit)$ which is the numerical solution of the linear and nonlinear biharmonic equations in §3 and §4 or §5. Since $\Delta\mathbf{curl}(S_\varphi)$ is in $L_2(\Omega \cap \diamondsuit)$, by Lemma 7.6 we know that the numerical solutions $S_p \in \tilde{S}^{r-1}(\diamondsuit)$ of (7.4) and (7.5) exist and are unique with S_φ in the place of φ.

We finally consider how well S_p approximates p. Let us first consider the pressure function of the steady state Stokes equation. Let $S_p \in \tilde{S}^{r-1}(\diamondsuit)$ satisfy the following

$$a_1(S_p, q) = \langle \mathbf{f}, \nabla q \rangle + \nu \langle \nu\Delta\mathbf{curl}S_\varphi, \nabla q \rangle, \quad q \in \tilde{S}^{r-1}(\diamondsuit).$$

Let $\tilde{p} \in \tilde{H}^1(\Omega)$ be the weak solution of the following Poisson equation:

$$a_1(\tilde{p}, q) = \langle \mathbf{f}, \nabla q \rangle + \nu \langle \Delta \mathbf{curl}(S_\varphi) \cdot \nabla q \rangle, \quad \forall\, q \in \tilde{H}^1(\Omega).$$

As before, $\tilde{p}$ exists and is unique. Then we have

LEMMA 7.7. Suppose that $\varphi \in H^k(\Omega)$ with $4 \le k \le 3r+1$. Then

$$|p - \tilde{p}|_{1,\Omega} \le C|\triangle|^{k-3}$$

for a constant C independent of p.

Proof: We have for any $q \in \tilde{H}^1(\Omega)$,

$$a_1(p - \tilde{p}, q) = \nu \int_\Omega \mathbf{curl} \Delta(\varphi - S_\varphi) \cdot \nabla q \; dxdy.$$

Thus it follows that

$$\begin{aligned} |p - \tilde{p}|_{1,\Omega}^2 &= \nu \int_\Omega \mathbf{curl}\; \Delta(\varphi - S_\varphi) \cdot \nabla(p - \tilde{p}) dxdy \\ &\le \nu \left(\left\| \frac{\partial}{\partial x} \Delta(\varphi - S_\varphi) \right\|_{L_2(\Omega)} + \left\| \frac{\partial}{\partial y} \Delta(\varphi - S_\varphi) \right\|_{L_2(\Omega)} \right) |p - \tilde{p}|_{1,\Omega}. \end{aligned}$$

Recall from Theorem 3.3 that $|\varphi - S_\varphi|_{2,\Omega} \le C|\triangle|^{k-2}$. Since $\varphi \in H^k(\Omega)$ with $4 \le k \le 3r+1$, there exists a best approximation $S_{\varphi,a} \in S^r_{3r}(\triangle)$ such that $|\varphi - S_{\varphi,a}|_{\ell,\Omega} \le K_7|\triangle|^{k-\ell}$ by Lemma 2.8. Thus, assuming that $\triangle$ is a quasi-uniform triangulation, we have

$$\begin{aligned} \left\| \frac{\partial}{\partial x} \Delta(\varphi - S_\varphi) \right\|_{L_2(\Omega)} &\le \left\| \frac{\partial}{\partial x} \Delta(\varphi - S_{p,a}) \right\|_{L_2(\Omega)} + \left\| \frac{\partial}{\partial x} \Delta(S_{\varphi,a} - S_\varphi) \right\|_{L_2(\Omega)} \\ &\le K_7|\triangle|^{k-3} + \frac{C_1}{|\triangle|} \|\Delta(S_{\varphi,a} - S_\varphi)\|_{L_2(\Omega)} \\ &\le K_7|\triangle|^{k-3} + \frac{C_1}{|\triangle|} \|\Delta(S_{\varphi,a} - \varphi)\| + \frac{C_1}{|\triangle|} \|\Delta(\varphi - S_\varphi)\| \\ &= C_1|\triangle|^{k-3} + \frac{C_1}{|\triangle|} K_7 |\triangle|^{k-2} + \frac{C_1}{|\triangle|} K_7 |\triangle|^{k-2} \\ &= C_2|\triangle|^{k-3}. \end{aligned}$$

Similarly, we also have

$$\left\| \frac{\partial}{\partial y} \Delta(\varphi - S_\varphi) \right\| \le C_2|\triangle|^{k-3}.$$

Therefore, $|p - \tilde{p}|_{1,\Omega}^2 \le 2\nu C_2 |\triangle|^{k-3} |p - \tilde{p}|_{1,\Omega}$ which completes the proof. ■

Next we note that

$$a_1(\tilde{p} - S_p, q) = 0, \qquad \forall q \in \tilde{S}^{r-1}(\triangle).$$

It follows that

$$\begin{aligned} |\tilde{p} - S_p|_{1,\Omega}^2 &= \int_\Omega \nabla(\tilde{p} - S_p)\nabla(\tilde{p} - p)dxdy + \int_\Omega \nabla(\tilde{p} - S_p)\nabla(p - S_p - q)dxdy \\ &\le |\tilde{p} - S_p|_{1,\Omega}|p - \tilde{p}|_{1,\Omega} + |\tilde{p} - S_p|_{1,\Omega}|p - S_p - q|_{1,\Omega} \end{aligned}$$

for any $q \in \tilde{S}^{r-1}(\triangle)$. Hence, if $p \in H^{k-2}(\Omega)$, we have

$$\begin{aligned} |\tilde{p} - S_p|_{1,\Omega} &\le |p - \tilde{p}|_{1,\Omega} + \inf_{q \in \tilde{S}^{r-1}(\triangle)} |p - q|_{1,\Omega} \\ &\le |\triangle|^{k-3} + K_7|\triangle|^{k-3} \end{aligned}$$

by Lemma 7.7 and Lemma 2.8. By Lemma 2.3 we also have $|p - S_p|_{0,\Omega} \le C|\triangle|^{k-3}$. Therefore, we may conclude the following.

THEOREM 7.8. Suppose that $\triangle$ is a quasi-uniform triangulation. Suppose that $\varphi \in H^k(\Omega)$ with $4 \le k \le 3r+1$ and that $p \in H^{k-2}(\Omega)$. Then we have

$$|p - S_p|_{1,\Omega} \le C|\triangle|^{k-3} \text{ and } |p - S_p|_{0,\Omega} \le C|\triangle|^{k-3}.$$

Next we consider the numerical approximation of the pressure term in Navier-Stokes equations. Let S_p be the weak solution in $\tilde{S}^{r-1}(\triangle)$ of the following Poisson equation

$$\begin{aligned} a_1(S_p, q) &= \langle \mathbf{f}, \nabla q\rangle + \langle(\mathbf{curl} S_\varphi \cdot \nabla)\mathbf{curl}(S_\varphi), \nabla q\rangle \\ &\quad + \nu\langle \Delta\mathbf{curl}(S_\varphi), \nabla q\rangle, \quad \forall\, q \in \tilde{S}^{r-1}(\triangle) \end{aligned}$$

where S_φ is the weak solution of Navier-Stokes equations in §4 or §5. Let $\tilde{p} \in \tilde{H}^1(\Omega)$ be the weak solution of the following Poisson problem with Neumann boundary condition:

$$\begin{aligned} a_1(\tilde{p}, q) &= \langle \mathbf{f}, \nabla q\rangle + \langle(\mathbf{curl} S_\varphi \cdot \nabla)\mathbf{curl}(S_\varphi), \nabla q\rangle \\ &\quad + \nu\langle \Delta\mathbf{curl}(S_\varphi), \nabla q\rangle_{\partial\Omega}, \quad \forall\, q \in \tilde{H}^1(\Omega). \end{aligned}$$

We first prove the following

LEMMA 7.9. Suppose that $\varphi \in H^k(\Omega)$ with $4 \le k \le 3r+1$, $p \in H^{k-2}(\Omega)$ and ν is sufficiently large. Then

$$|p - \tilde{p}|_{1,\Omega} \le C|\triangle|^{k-3}$$

for some constant C independent of $\diamondplus$.

Proof: Based on the weak formulations above we have

$$\begin{aligned} a_1(p-\tilde{p},q) &= \int_\Omega \nabla(p-\tilde{p})\nabla q\ dxdy \\ &= \int_\Omega \nu\Delta\mathbf{curl}(\varphi - S_\varphi)\nabla q\ dxdy \\ &\quad + \int_\Omega (\mathbf{curl}(\varphi - S_\varphi)\cdot\nabla)\mathbf{curl}\ \varphi\cdot\nabla q\ dxdy \\ &\quad + \int_\Omega (\mathbf{curl}\ S_\varphi\cdot\nabla)\mathbf{curl}(\varphi - S_\varphi)\cdot\nabla q\ dxdy \end{aligned}$$

for any $q \in \tilde{H}^1(\Omega)$. Thus

$$\begin{aligned} |p- \quad \tilde{p}| \ {}^2_{1,\Omega} &\le \nu\|\nabla(\Delta(\varphi - S_\varphi))\||p-\tilde{p}|_{1,\Omega} \\ &+|\varphi - S_\varphi|_{1,\Omega}\max_{(x,y)\in\Omega}\left(\left|\frac{\partial^2\varphi(x,y)}{\partial x^2}\right| + \left|\frac{\partial^2\varphi(x,y)}{\partial x\partial y}\right| + \left|\frac{\partial^2\varphi(x,y)}{\partial y^2}\right|\right)|p-\tilde{p}|_{1,\Omega} \\ &+\max_{(x,y)\in\Omega}\left(\left|\frac{\partial S_\varphi(x,y)}{\partial x}\right| + \left|\frac{\partial S_\varphi(x,y)}{\partial y}\right|\right)|\varphi - S_\varphi|_{2,\Omega}|p-\tilde{p}|_{1,\Omega}. \end{aligned}$$

By Theorem 4.8 we can show the following inequality using the same argument as in the proof of Lemma 7.7,

$$\|\nabla(\Delta(\varphi - S_\varphi))\| \le C|\diamondplus|^{k-3}.$$

By Sobolev's embedding theorem,

$$\max_{(x,y)\in\Omega}\left(\left|\frac{\partial^2\varphi(x,y)}{\partial x^2}\right| + \left|\frac{\partial^2\varphi(x,y)}{\partial x\partial y}\right| + \left|\frac{\partial^2\varphi(x,y)\varphi}{\partial y^2}\right|\right) \le C|\varphi|_{4,\Omega}$$

and

$$\begin{aligned} &\max_{(x,y)\in\Omega}\left(\left|\frac{\partial S_\varphi(x,y)}{\partial x}\right| + \left|\frac{\partial S_\varphi(x,y)}{\partial y}\right|\right) \\ &\qquad \le \quad S_\varphi|_{3,\Omega} \le |S_\varphi - \varphi|_{3,\Omega} + |\varphi|_{3,\Omega} \\ &\qquad \le \quad |\diamondplus|^{k-3} + |\varphi|_{3,\Omega}. \end{aligned}$$

Therefore, we conclude that

$$|p-\tilde{p}|_{1,\Omega} \quad \le \quad \nu|\diamondplus|^{k-3} + |\varphi|_{4,\Omega}C|\diamondplus|^{k-2} + (C|\diamondplus|^{k-3} + |\varphi|_{3,\Omega})|\diamondplus|^{k-2}.$$

which completes the proof. ■

Finally we summarize the discussion above to have the following:

THEOREM 7.10. Let φ be the weak solution of the nonlinear biharmonic equations in §4 or §5. Suppose that $\varphi \in H^k(\Omega)$ with $4 \leq k \leq 3r+1$ and that ν is large enough. Let p and S_p be the pressure function and the approximation in $\tilde{S}^{r-1}(\diamondplus)$, where $\diamondplus$ is a quasi-uniform triangulation. Suppose that $p \in H^{k-2}(\Omega)$. Then

$$|p - S_p|_{1,\Omega} \leq K|\diamondplus|^{k-3} \text{ and } |p - S_p|_{0,\Omega} \leq K|\diamondplus|^{k-3}.$$

Proof: We first note that

$$\int_\Omega \nabla(\tilde{p} - S_p)\nabla q \, dxdy = 0, \quad \forall\, q \in \tilde{S}^{r-1}(\diamondplus)$$

and hence

$$\begin{aligned}
|\tilde{p} - S_p|^2_{1,\Omega} &= \int_\Omega \nabla(\tilde{p} - S_p)\nabla(\tilde{p} - p)dxdy + \int_\Omega \nabla(\tilde{p} - S_p)\nabla(p - S_p - q)dxdy \\
&\leq |\tilde{p} - S_p|_{1,\Omega}|p - \tilde{p}|_{1,\Omega} + |\tilde{p} - S_p|_{1,\Omega}|p - S_p - q|_{1,\Omega}
\end{aligned}$$

for any $q \in \tilde{S}^{r-1}(\diamondplus)$. Thus, if $p \in H^{k-2}(\Omega)$ we have $\inf_{q \in \tilde{S}^{r-1}(\diamondplus)} |p - S_p - q|_{1,\Omega} \leq C|\diamondplus|^{k-3}$. It follows from Lemma 7.9 that

$$|\tilde{p} - S_p|_{1,\Omega} \leq |p - \tilde{p}|_{1,\Omega} + C|\diamondplus|^{k-3} \leq C_1|\diamondplus|^{k-3}.$$

Hence we have

$$\begin{aligned}
|p - S_p|_{1,\Omega} &\leq |p - \tilde{p}|_{1,\Omega} + |\tilde{p} - S_p|_{1,\Omega} \\
&\leq C_1|\diamondplus|^{k-3} + C_2|\diamondplus|^{k-3}.
\end{aligned}$$

By Lemma 2.3 we get $|p - S_p|_{0,\Omega} \leq C|p - S_p|_{1,\Omega} \leq C_1|\diamondplus|^{k-3}$. This completes the proof. ■

We have implemented in MATLAB the standard linear finite element method for solving the Poisson problem with nonhomogeneous Neumann boundary condition. We numerically compute the pressure functions for some artificial pressure and velocity vectors. By putting the artificial velocity vector and pressure into the Navier-Stokes equations (1.1), we calculate the right-hand side function **f**. Using the **f** and the boundary values of the artificial velocity over the boundary of the L-shape domain, we compute the numerical solutions by our MATLAB programs and then compute the error against the artificial solution. In Table 4, we list the maximum error, based on 30401 points equally spaced over the L-shape domain which is triangulated as shown in Figure 1 and is refined two times. In Table 4, the Reynolds number is 100. We are able to produce similar tables for Reynolds numbers 1,000, 10,000, and 100,000 with slower convergence rates. We omit the details.

Table 4. Maximum Errors of the Pressure Function
and Velocity Vector of Navier-Stokes Equations

	Error in Velocity	Error in Pressure
Matrix Size	$\mathbf{u} = (y^4, x^4)$	$p = x^2 + y^2$
150x150	$1.021 \times 10^{-1}, 1.303 \times 10^{-1}$	1.2110×10^{0}
527x527	$2.419 \times 10^{-2}, 1.857 \times 10^{-2}$	2.552×10^{-1}
1971x1971	$3.677 \times 10^{-3}, 2.392 \times 10^{-3}$	6.328×10^{-2}
	$\mathbf{u} = (\exp(y), \exp(x))$	$p = x^2 + y^2$
150x150	$1.396 \times 10^{-2}, 1.582 \times 10^{-2}$	1.710×10^{-1}
527x527	$3.120 \times 10^{-3}, 2.389 \times 10^{-3}$	5.145×10^{-2}
1971x1971	$5.311 \times 10^{-4}, 2.843 \times 10^{-4}$	1.542×10^{-2}
	$\mathbf{u} = 1/(1+y^2), 1/(1+x^2)$	$p = x^2 + y^2$
150x150	$6.142 \times 10^{-3}, 5.483 \times 10^{-3}$	1.226×10^{-1}
527x527	$8.285 \times 10^{-4}, 8.284 \times 10^{-4}$	3.655×10^{-2}
1971x1971	$9.709 \times 10^{-5}, 9.474 \times 10^{-5}$	1.043×10^{-2}
	$\mathbf{u} = (\sin(x+y), -\sin(x+y))$	$p = x^2 + y^2$
150x150	$1.227 \times 10^{-3}, 1.153 \times 10^{-3}$	9.692×10^{-2}
527x527	$1.980 \times 10^{-4}, 1.888 \times 10^{-4}$	3.071×10^{-2}
1971x1971	$3.194 \times 10^{-5}, 2.893 \times 10^{-5}$	8.915×10^{-3}

References

1. R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

2. J. F. Ciavaldini and J. C. Nedelec, Sur l'element de Fraeijs de Veubeke et Sander, R.A.I.R.O. **R-2**(1974), pp. 29-45.

3. G. Farin, Triangular Bernstein-Bézier patches, Comput. Aided Geom. Design 3(1986), 83–127.

4. B. Fraeijs de Veubeke, *Bending and stretching of plates*, Proc. Conf. Matrix Methods in Structure Mechanics, Air Force Institute of Technology, Dayton, Ohio, (1965).

5. P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman Publishing Co., Boston, 1985.

6. D. Greenspan, Numerical studies of prototype cavity flow problems, Computer Journal, 12(1969), pp. 88–93.

7. V. Girault and P.A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, 1986.

8. O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

9. M. J. Lai, Scattered data interpolation and approximation using bivariate C^1 piecewise cubic polynomials, Comput. Aided Geom. Design, 13(1996), pp. 81–88.

10. M. J. Lai and L. L. Schumaker, On the approximation power of splines on triangulated quadrangulations, SIAM Numer. Anal., 36(1999), pp. 143–159.

11. M. J. Lai and L. L. Schumaker, On the approximation power of bivariate splines, Advances in Comp. Math. 9(1998), pp. 251–279.

12. M. J. Lai and P. Wenston, On multi-level bases for elliptic boundary value problems, J. Comp. Applied Math., 71(1996), pp. 95–113.

13. G. Sander, Bornes supérieures et inférieures dans l'analyse matricielle des plaques en flexion-torsion, Bull. Soc. r. Sci. Liege 33(1964), pp. 456-494.

14. R. Teman, *Navier-Stokes Equations*, North Holland, Oxford, 1977.