The Convergence of a Central-Difference Discretization of Rudin-Osher-Fatemi Model for Image Denoising

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Abstract. We study the connection between minimizers of the discrete and the continuous Rudin-Osher-Fatemi models. We use a central-difference total variation term in the discrete ROF model and treat the discrete input data as a projection of the continuous input data into the discrete space. We employ a method developed in [13] with slight adaption to the setting of the central-difference total variation ROF model. We obtain an error bound between the discrete and the continuous minimizer in L^2 norm under the assumption that the continuous input data are in $W^{1,2}$.

1 Introduction

One of the most influential variational models for image denoising is the total variation–based model proposed by Rudin, Osher and Fatemi(ROF) [10]. This model studies the following constrained minimization problem:

$$\begin{gathered}\arg\min_{\mathbf{u}} |u|_{BV} \\ \text{with } \int_\Omega u = \int_\Omega g \quad \text{and} \quad \int_\Omega |u-g|^2 = \sigma^2\end{gathered} \tag{1}$$

where g is the input data, σ is the standard deviation of the noise, Ω is the unit square $[0,1]^2$, and $|u|_{BV}$ is the total variation (TV) of u defined as follows. We consider functions ϕ in the space of C^1 functions from Ω to $\mathbb{R}^2$ with compact support, i.e., $[C_0^1(\Omega)]^2$. The variation of a function $u \in L^1(\Omega)$ is then defined to be

$$|u|_{BV} := \int_\Omega |Du| := \sup_{\phi\in[C_0^1(\Omega)]^2,\ |\phi|\le 1 \text{ point-wise}} \int_\Omega u\nabla\cdot\phi.$$

For more details on functions of bounded variation, we refer the reader to [9].

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The existence and uniqueness of the minimizer of (1) have been studied by Lions, Osher and Rudin [11] and more completely by Acar and Vogel [1]. Chambolle and Lions [4] proved that the constrained problem (1) is equivalent to the following unconstrained problem:

$$\arg \quad \min_{\mathbf{u}} |u|_{BV} + \frac{1}{2\lambda}\int_{\Omega} |u-g|^2 . \tag{2}$$

They also proved more general results of existence and uniqueness of (1). We later call

$$E(u) = |u|_{BV} + \frac{1}{2\lambda}\int |u-g|^2 \tag{3}$$

the ROF energy functional.

On the computing side, the most commonly used discrete variational model is based on the discrete energy

$$E_k(u) = \sum_{i,j=0}^{k-1} \mu_{i,j} |(\nabla u)_{i,j}| + \frac{1}{2\lambda} \sum_{i,j=0}^{k-1} \mu_{i,j} (u_{i,j} - g_{i,j})^2 \, , \tag{4}$$

where u is defined by a 2-dimensional matrix of size $k \times k$, $\mu_{i,j}$ is related to the scale k. A simple choice of $\mu_{i,j}$ is $\mu_{i,j} = 1/k^2$. There are several possible choices for the discrete gradient operator ∇u [3], [5], and [13]. A common choice is

$$(\nabla u)_{i,j} = ((\nabla_x u)_{i,j}, (\nabla_y u)_{i,j}) \, ,$$

with

$$(\nabla_x u)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h}, \quad (\nabla_y u)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{h} \, ,$$

where $h = 1/k$. On the boundary, u is assumed to satisfy the discrete Neumann boundary conditions:

$$u_{-1,j} = u_{0,j}, \quad u_{k,j} = u_{k-1,j}, \tag{5}$$

$$u_{i,-1} = u_{i,0}, \quad u_{i,k} = u_{i,k-1}. \tag{6}$$

The discrete function $g_{i,j}$ is the input image. Many efficient algorithms have been developed to find the numerical minimizer of (4) [6], [2], [3].

It is not hard to show that E_k Γ-converges to E (for the definition of Γ-convergence, we refer the reader to [7]), therefore, the sequence $\{u^k\}$, minimizers of E_k, converges to u, the minimizer of E, in $L^1(\Omega)$ and $E_k(u^k)$ converges to $E(u)$ as k tends to ∞ (cf. [7]).

It is interesting to know the rate of convergence and the convergence in other norm, e.g., in L^2 norm. It is also interesting see the difference between the continuous minimizer and the discrete minimizer. The authors in [13] proved that if the discrete energy E_k is equipped with a symmetrical discrete total variation as defined in (7) and the discrete input data g^k is the projection of the continuous input data g by taking average of g on each pixel, one can bound the error between the discrete minimizer u^k and the continuous u in L^2 norm by the Lipschitz norm of g provided that g is in some Lipschitz space.

$$
\begin{aligned}
|u^k|_{\mathrm{TV}} = \sum_{i,j=0}^{k-1} \frac{h^2}{4} \Bigg\{ & \left(\left(\frac{u^k_{i+1,j} - u^k_{i,j}}{h} \right)^2 + \left(\frac{u^k_{i,j+1} - u^k_{i,j}}{h} \right)^2 \right)^{1/2} + \\
& \left(\left(\frac{u^k_{i+1,j} - u^k_{i,j}}{h} \right)^2 + \left(\frac{u^k_{i,j} - u^k_{i,j-1}}{h} \right)^2 \right)^{1/2} + \\
& \left(\left(\frac{u^k_{i,j} - u^k_{i-1,j}}{h} \right)^2 + \left(\frac{u^k_{i,j+1} - u^k_{i,j}}{h} \right)^2 \right)^{1/2} + \\
& \left(\left(\frac{u^k_{i,j} - u^k_{i-1,j}}{h} \right)^2 + \left(\frac{u^k_{i,j} - u^k_{i,j-1}}{h} \right)^2 \right)^{1/2} \Bigg\} \qquad (7)
\end{aligned}
$$

In this paper, we extend the study in [13], [12] to the discrete ROF model equipped with a central-difference TV term which is much simpler than the symmetrical discrete TV term. The ideas for the study in this paper is exactly the same to the ones in [13]. However, a problem of the central-difference model is that it does not deal well with some non-smooth data, for example, a chessboard image. Thus we have to adapt the study in [13] slightly to this situation and put a stronger assumption on the input data g in order to establish the convergence. We can still get a similar error bound if the input data $g \in W^{1,2}$. More precisely, our main results are

Theorem 1. If $g \in W^{1,2}$, u is the minimizer of E in (3) and u^k is the minimizer of E_k in (4) equipped with the central-difference TV operator, we will give the definition in (10), then

$$|E(u) - E_k(u^k)| \leq C(1 + \frac{1}{\lambda})(\|g\|_{W^{1,2}} + \|g\|^2_{W^{1,2}})h^{1/2}.$$

and

Theorem 2. If $g \in W^{1,2}$, u is the minimizer of the functional E in (3) and u^k is the minimizer of the functional E_k in (10), then

$$\|I_h u^k - u\|^2 \leq C(\lambda + 1)(\|g\|_{W^{1,2}} + \|g\|^2_{W^{1,2}})h^{1/2}.$$

where $I_h u^k$ is the piecewise constant injection of u^k into L^2 space. The definition of $I_h u^k$ will be given in (14) in the next secion.

2 Preliminaries

A continuous image u is defined as a L^2 function on $\Omega \subset \mathbb{R}^2$. In practice, we always assume Ω to be the unit square $[0, 1] \times [0, 1]$.

We assume the output of denoised image to be in the space of bounded variation. In the discrete settings, we consider the discrete set Ω^k to be the set of all pairs $i = (i_1, i_2) \in Z^2$ with $0 \le i_1, i_2 \le k$. A discrete image u^k is defined as a function on Ω^k. We always use superscripts to indicate a function is a discrete function through this paper. For discrete functions, we define the discrete $\ell^p(\Omega^k)$ norms

$$\|u^k\|_{\ell^p(\Omega^k)} := \left(\sum_{i \in \Omega^k} |u_i^k|^p \, \mu_i \right)^{\frac{1}{p}} \quad \text{for } 1 \le p \le \infty$$

where μ_i is the measure of the discrete space at each index i. The simplest choice of μ_i is

$$\mu_i = 1 \quad \text{for } i \in \Omega^k.$$

In analogue of Sobolev norm, we define the discrete Sobolev norm as follows. The first order forward finite differences of u^k at point $i = (i_1, i_2)$ are

$$\Delta_x^+ u_i^k = \frac{u^k_{i_1+1,i_2} - u^k_{i_1,i_2}}{h}; \quad \Delta_y^+ u_i^k = \frac{u^k_{i_1,i_2+1} - u^k_{i_1,i_2}}{h},$$

where $h = 1/k$ is the step size. We can also define backward finite difference as

$$\Delta_x^- u_i^k = \frac{u^k_{i_1,i_2} - u^k_{i_1-1,i_2}}{h}; \quad \Delta_y^- u_i^k = \frac{u^k_{i_1,i_2} - u^k_{i_1,i_2-1}}{h}.$$

One can define the second order finite difference as

$$\Delta_{xx} u_i^k = \frac{\Delta_x^+ u_i^k - \Delta_x^- u_i^k}{h}.$$

Also $\Delta_{yy} u_i^k$ can be similarly defined.

We define $\|\nabla u^k\|_{\ell^1}$, $\|\Delta_{xx} u^k\|_{\ell^1}$, $\|\Delta_{yy} u^k\|_{\ell^1}$ as

$$\|\nabla u^k\|_{\ell^1} := \sum_i (|\Delta_x^+ u_i^k| + |\Delta_y^+ u_i^k|)\mu_i; \tag{8}$$

$$\|\Delta_{xx} u^k\|_{\ell^1} := \sum_i |\Delta_{xx} u_i^k| \mu_i, \quad \|\Delta_{yy} u^k\|_{\ell^1} := \sum_i |\Delta_{yy} u_i^k| \mu_i. \tag{9}$$

In this paper, we shall study the error bound for the central-difference discrete ROF model of which the energy functional is defined as follows

$$E_c(u^k) = J_c(u^k) + \frac{1}{2\lambda} \|u^k - g^k\|_c^2. \tag{10}$$

where the BV term J_c is defined by

$$J_c(u^k) := \sum_{i \in \Omega^k} \sqrt{|\Delta_x^c u_i^k|^2 + |\Delta_y^c u_i^k|^2} \; \mu_i, \tag{11}$$

and $\Delta_x^c u_i^k$ and $\Delta_y^c u_i^k$ at $i := (i_1, i_2)$ are defined by

$$\Delta_x^c u_i^k = \frac{u_{i_1+1,i_2}^k - u_{i_1-1,i_2}^k}{2h}, \quad \Delta_y^c u_i^k = \frac{u_{i_1,i_2+1}^k - u_{i_1,i_2-1}^k}{2h}.$$

Here u^k satisfies the discrete Neumann boundary condition:

$$\begin{aligned} u_{-1,j}^k = u_{1,j}^k, &\quad u_{k+1,j}^k = u_{k-1,j}^k, \\ u_{i,-1}^k = u_{i,1}^k, &\quad u_{i,k+1}^k = u_{i,k-1}^k. \end{aligned}$$

The discrete space measure $\mu_i = |\Omega_i|$ where Ω_i is the intersection of Ω and the square with center ih and size h.

$$\Omega_i := \Omega \cap [i_1 h - \frac{h}{2}, i_1 h + \frac{h}{2}] \times [i_2 h - \frac{h}{2}, i_2 h + \frac{h}{2}]. \tag{12}$$

It is straightforward to calculate

$$\mu_i = \begin{cases} h^2/4 & (i_1, i_2) \in \{(0,0), (0,k), (k,0), (k,k)\} \\ h^2/2 & i_1 = 0, k; 0 < i_2 < k \text{ or } i_2 = 0, k; 0 < i_1 < k \\ h^2 & 0 < i_1, i_2 < k \end{cases} \tag{13}$$

The ℓ^2 term is defined by

$$\|u^k - g^k\|_c^2 = \sum_{i,j=0}^{k} |u_{i,j}^k - g_{i,j}^k|^2 \mu_{i,j}.$$

We often need to extend $u \in L^p(\Omega)$ and $u^k \in \ell^p(\Omega^k)$ to all of $\mathbb{R}^2$ and $\mathbb{Z}^2$, respectively; we denote the extensions by $\operatorname{Ext} u$ and $\operatorname{Ext}_k u^k$. For $u \in L^p(\Omega)$, we use the following procedure. First,

$$\operatorname{Ext} u(x) = u(x), \quad x \in \Omega.$$

We then reflect horizontally across the line $x_1 = 1$,

$$\operatorname{Ext} u(x_1, x_2) = \operatorname{Ext} u(2 - x_1, x_2), \quad 1 \le x_1 \le 2,\ 0 \le x_2 \le 1,$$

and reflect again vertically across the line $x_2 = 1$,

$$\operatorname{Ext} u(x_1, x_2) = \operatorname{Ext} u(x_1, 2 - x_2), \quad 0 \le x_1 \le 2,\ 1 \le x_2 \le 2.$$

Having defined $\operatorname{Ext} u$ on 2Ω, we then extend $\operatorname{Ext} u$ periodically with period $(2,2)$ on all of $\mathbb{R}^2$.

We use a similar construction for discrete functions u^k. First we extend u^k to

$$2\Omega^k := \{i = (i_1, i_2) \in \mathbb{Z}^2 \mid 0 \le i_1, i_2 \le 2k\}$$

as follows:

$$\operatorname{Ext}_k u_i^k = u_i^k, \quad i \in \Omega^k;$$

then we reflect horizontally

$$\mathrm{Ext}_k\, u^k_{(i_1,i_2)} = \mathrm{Ext}_k\, u^k_{(2k-i_1,i_2)}, \quad k+1 \le i_1 \le 2k,\ 0 \le i_2 \le k,$$

and then vertically

$$\mathrm{Ext}_k\, u^k_{(i_1,i_2)} = \mathrm{Ext}_k\, u^k_{(i_1,2k-i_2)}, \quad 0 \le i_1 \le 2k,\ k+1 \le i_2 \le 2k.$$

Now that $\mathrm{Ext}_k\, u^k$ is defined on $2\Omega^k$, we extend it periodically with period $(2k, 2k)$ to all of $\mathbb{Z}^2$. Note that with this definition, the value of $\mathrm{Ext}_k\, u^k$ at any point immediately "outside" Ω^k is the same as the value of u^k at the closest point "inside" Ω^k.

We sometimes need to inject or project functions into $L^2(\Omega)$ or discrete space $\ell^2(\Omega^k)$ respectively. We use the piecewise constant injector to inject discrete function u^k into $L^p(\Omega)$:

$$(I_h u^k)(x) = u^k_i \quad \text{for } x \in \Omega_i. \tag{14}$$

We also define an injector L_h into a space of continuous, piecewise linear functions. In fact, L_h is the linear interpolation of discrete points $\{u^k_i\}$ on a triangulation of vertices $h\mathbb{Z}^2$.

$$L_h u^k = \sum_{i\in\Omega^k} u^k_i \phi^k_i. \tag{15}$$

Here ϕ^k_i is a dilated and translated tent function,

$$\phi^k_i(x) := \phi^k_{i_1,i_2}(x_1, x_2) := \phi(x_1/h - i_1, x_2/h - i_2), \tag{16}$$

where ϕ is the tent function which is continuous on $\mathbb{R}^2$, supported in the hexagon shown in Fig. 1, linear on each triangle as shown in Fig. 1, and satisfies the following

$$\phi(i_1, i_2) = \begin{cases} 0 & (i_1, i_2) \in \mathbb{Z}^2 \backslash (0, 0) \\ 1 & (i_1, i_2) = (0, 0) \end{cases}$$

We also consider the piecewise constant projector of $u \in L^1(\Omega)$ onto the space of discrete functions, defined by

$$(P_k u)_i = \frac{1}{|\Omega_i|} \int_{\Omega_i} u, \quad i \in \Omega^k,$$

where $|\Omega_i| = \mu_i$ is the measure of Ω_i defined in (12).

We need both continuous and discrete smoothing operators, which we define as follows. Assume that $\eta(x)$ is a a fixed non-negative, rotationally symmetric, mollifier with support in the unit disk that is C^∞ and has integral 1. For $\epsilon > 0$ we define the scaled function

$$\eta_\epsilon(x) := \frac{1}{\epsilon^2}\eta\Big(\frac{x}{\epsilon}\Big), \quad x \in \mathbb{R}^2;$$

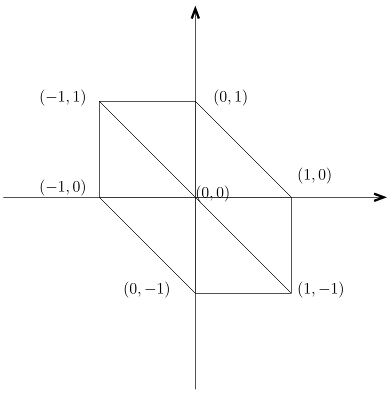

Fig. 1. The Support of ϕ

we smooth a function $u \in L^p(\Omega)$, $1 \le p \le \infty$, by computing

$$(\eta_\epsilon * \operatorname{Ext} u)(x) = \int_{\mathbb{R}^2} \eta_\epsilon(x-y)\ \operatorname{Ext} u(y)\, dy, \quad x \in 2\Omega.$$

The discrete smoothing operator S_L is defined by

$$(S_L u^k)_i = \frac{1}{(2L+1)^2} \sum_{j_1, j_2 = -L}^{L} u^k_{i+(j_1,j_2)} \quad \text{for } i \in \Omega^k$$

For $u \in L^p(\Omega)$ we define the (first-order) $L^p(\Omega)$ modulus of smoothness by

$$\omega(u,t)_{L^p(\Omega)} = \sup_{\tau \in \mathbb{R}^2,\ |\tau|<t} \left(\int_{x, x+\tau \in \Omega} |u(x+\tau) - u(x)|^p\, dx \right)^{\frac{1}{p}}.$$

We also define

$$\omega(\operatorname{Ext} u, t)_{L^p(2\Omega)} := \sup_{\tau \in \mathbb{R}^2,\ |\tau|<t} \| \operatorname{Ext} u(\cdot + \tau) - \operatorname{Ext} u \|_{L^p(2\Omega)}.$$

We also have need of a discrete modulus of smoothness. To begin, we define the translation operator

$$(T_\ell(u^k))_i := u^k_{i+\ell} \quad \text{for any } \ell = (\ell_1, \ell_2) \in \mathbb{Z}^2. \tag{17}$$

We define the norm $|\ell| = |\ell_1| + |\ell_2|$ on $\mathbb{Z}^2$, and then the discrete ℓ^p modulus of smoothness is

$$\omega(u^k, m)_{\ell^p} := \sup_{\ell \in \mathbb{Z}^2,\ |\ell| \le m} \left(\sum_{i, i+\ell \in \Omega^k} |u^k_{i+\ell} - u^k_i|^p\, \mu_i \right)^{\frac{1}{p}}.$$

For $\operatorname{Ext}_k u^k$ we define similarly

$$\omega(u^k, m)_{\ell^p(2\Omega^k)} = \sup_{\ell \in \mathbb{Z}^2,\ |\ell| \le m} \|T_\ell u^k - u^k\|_{\ell^p(2\Omega^k)}.$$

3 Basic Properties

We begin with the following properties.

Lemma 1. (Contraction) *Let u, v be the minimizers for input data f and g in problem (2) respectively,*

$$\|u-v\|_{L^2} \leq \|f-g\|_{L^2}\,.$$

See a proof in [13] or [12]. With the above property, one can have the following

Lemma 2. (Continuity of translation) *Assume u is the minimizer of E in problem (2) for input data g. Extend u to Ext u over $\mathbb{R}^2$ by symmetric extension as defined before. Then*

$$\|\mathit{Ext}\ u(x+h) - \mathit{Ext}\ u(x)\|_{L^2(\Omega)} \leq \omega(g,|h|)_{L^2(\Omega)}.$$

Remark 1. One can conclude from Lemma 2 that

$$\omega(u,|h|)_{L^2(\Omega)} \leq \omega(g,|h|)_{L^2(\Omega)}. \tag{18}$$

Remark 2. Similar techniques allow one to show that this result also holds for the discrete case of u^k and g^k where u^k is the minimizer of the discrete energy E_k with the symmetric discrete TV operator J_c, and u^k is extended on $\mathbb{Z}^2$ as before. In fact, the corresponding discrete version is.

$$\|T_\ell(u^k) - u^k\|_{\ell^2(A)} \leq C\omega(g^k,|\ell|)_{\ell^2(A)}, \tag{19}$$

where A is the index set $\{i := (i_1,i_2) : 0 \leq i_1, i_2 \leq k\}$. For any discrete image v^k, the discrete modulus of continuity is

$$\omega_1(v^k,m)_{\ell^2(A)} := \sup_{0<|\ell|\leq m} \|T_\ell(v^k) - v^k\|_{\ell^2(A_{n_1,n_2})} \tag{20}$$

with T_ℓ being the translation operator defined in (17) and

$$A_{n_1,n_2} := \{(i,j) : (i,j) \in A, (i+n_1, j+n_2) \in A\}.$$

Lemma 3. (Maximum principle)
Suppose u^k is the minimizer of E_k. If $g^k \in L^\infty$. Then

$$\|u^k\|_\infty \leq \|g^k\|_\infty.$$

The following lemmas bound the errors introduced by injectors and projectors defined before respectively.

Lemma 4. *Let $u \in L^2(\Omega)$ and $u^k \in \ell^2(\Omega^k)$. Then there exists a constant C such that the following properties hold:*

a)

$$\|P_k u\|_{\ell^2} \leq \|u\|_{L^2};$$

b)

$$\omega(P_k u, m)_{\ell^2} \leq C\omega(u, mh)_{L^2}.$$

c)

$$\|u^k\|_{\ell^2} = \|I_h u^k\|_{L^2};$$

d)

$$\omega(I_h u^k, mh)_{\ell^2} \leq C\omega(u^k, m)_{L^2}.$$

e)

$$\|u - I_h P_k u\|_{L^2} \leq C\omega(u, h)_{L^2}.$$

The following lemma bounds the difference between the two injectors we defined in (14) and (15).

Lemma 5

$$\|L_h u^k - I_h u^k\|_{L^2} \leq C\omega(u^k, 1)_{\ell^2}$$

The following lemmas show the properties of the smoothing operators

Lemma 6

$$\|S_L u^k - u^k\|_{\ell^2} \leq \omega(u^k, L)_{\ell^2}, \tag{21}$$

$$J_c(S_L u^k) \leq J_c(u^k), \tag{22}$$

and

$$\|\Delta_{xx} S_L u^k\|_{\ell^1} + \|\Delta_{yy} S_L u^k\|_{\ell^1} \leq \frac{C}{Lh}\|\nabla u^k\|_{\ell^1}. \tag{23}$$

The first inequality in Lemma 6 shows the error between u^k and smoothed u^k can be bounded by its discrete modulus of continuity. The second inequality shows smoothing does not increase the discrete total variation. The last inequality shows the the second order difference of the smoothed function can be bounded by its first order finite difference.

Lemma 7 is the continuous case of Lemma 6.

Lemma 7

$$\|\eta_\epsilon * u - u\|_{L^2} \leq \omega(u, \epsilon)_{L^2}, \tag{24}$$

$$|\eta_\epsilon * u|_{BV} \leq |u|_{BV}, \tag{25}$$

and

$$\|D_{xx} u^\epsilon\|_{L^1} + \|D_{yy} u^\epsilon\|_{L^1} \leq \frac{C}{\epsilon}|u|_{BV}. \tag{26}$$

4 Proof of the Main Result

4.1 Main Idea

Recall the ROF continuous and discrete energy functionals are defined by

$$E(v) = |v|_{BV} + \frac{1}{2\lambda}\|v - g\|^2; \tag{27}$$

$$E_k(v^k) = J_c(v^k) + \frac{1}{2\lambda}\|v^k - g^k\|_c^2 \tag{28}$$

with input image $g^k = P_k g$.

To study the difference between $E_k(u^k)$ and $E(u)$, it should first be noticed that E_k and E are two different functionals defined on different spaces. E is defined on the continuous $BV(\Omega)$ space while E_k is a discrete operator defined on discrete function space. Therefore, some connection between these two operators should be built. We use two energy bounds to bridge them.

First, given a discrete minimizer u^k of functional E_k, we inject u^k into L^2 space by function $L_h S_L u^k$ with $E(L_h S_L u^k)$ less than $E_k(u^k)$ plus some error. The construction of $L_h S_L u^k$ is done by first "smoothing" u^k as $S_L u^k$, then linear-interpolating $S_L u^k$. Assuming u is the minimizer of E, we have

$$E(u) \le E(L_h S_L u^k) \le E_k(u^k) + e_{g,h}, \tag{29}$$

where $e_{g,h}$ is the error to be bounded in the next section, which depends on initial g and mesh size h, and tends to zero as h tends to zero.

The second energy bound is similar but taken in the opposite direction. Based on u, we construct a "smoothed" discrete function $P_k \eta_\epsilon * u$ by first "smoothing" it, then projecting it into discrete function space, with $E_k(P_k \eta_\epsilon * u)$ less than $E(u)$ plus an error term $e'_{g,h}$ similar to $e_{g,h}$. By the definition of u^k, we have

$$E_k(u^k) \le E_k(P_k \eta_\epsilon * u) \le E(u) + e'_{g,h}. \tag{30}$$

From (29) we see

$$E(u) - E_k(u^k) \le e_{g,h}\ ;$$

from (30)

$$E_k(u^k) - E(u) \le e'_{g,h}\ ;$$

then we conclude that

$$|E_k(u^k) - E(u)| \le \max\{e_{g,h}, e'_{g,h}\}\ .$$

This will complete our error bound.

4.2 Sketch of the Proof

Proposition 1. *If $g \in W^{1,2}$, and u^k, u are the minimizers of E_k, E in (28), (27) respectively, then*

$$E(u) \le E_k(u^k) + C(1 + \frac{1}{\lambda})(\|g\|_{W^{1,2}} + \|g\|^2_{W^{1,2}})h^{1/2}.$$

Proof. We shall bound the energy $E(L_h S_L u^k)$. It is straightforward to calculate its TV term (albeit, the computation is tedious) that

$$\left|L_h S_L u^k\right|_{BV} \le J_c(S_L u^k) + Ch\left(\|\Delta_{xx} S_L u^k\|_{\ell^1} + \|\Delta_{yy} S_L u^k\|_{\ell^1}\right).$$

By the property of discrete smoothing operator (22) and (23) in Lemma 6,

$$\left|L_h S_L u^k\right|_{BV} \le J_c(u^k) + \frac{C}{L}\|\nabla u^k\|_{\ell^1}.$$

By Holder's inequality and Lemma 2, $\|\nabla u^k\|_{\ell^1}$ is bounded by

$$\begin{aligned}
\|\nabla u^k\|_{\ell^1} &= \sum_i \left(|\Delta_x^+ u_i^k| + |\Delta_y^+ u_i^k|\right)\mu_i \\
&\le C\left(\left\{\sum_i |\Delta_x^+ u_i^k|^2 \mu_i\right\}^{1/2} + \left\{\sum_i |\Delta_y^+ u_i^k|^2 \mu_i\right\}^{1/2}\right) \\
&\le \frac{C}{h}\left(\|T_{(1,0)} u^k - u^k\| + \|T_{(0,1)} u^k - u^k\|\right) \\
&\le \frac{C}{h}\omega(g^k, 1)_{\ell^2} \qquad \text{by (19)} \\
&\le C\|g\|_{W^{1,2}}
\end{aligned}$$

We have

$$\left|L_h S_L u^k\right|_{BV} \le J_c(u^k) + \frac{C}{L}\|g\|_{W^{1,2}}.$$

The L^2 term of $E\left(L_h S_L u^k\right)$ can be written as

$$\begin{aligned}
\|L_h S_L u^k - g\|_{L^2} = \|&(L_h S_L u^k - I_h S_L u^k) + (I_h S_L u^k - I_h u^k) \\
&+ (I_h u^k - I_h g^k) + (I_h g^k - g)\|_{L^2} \\
\le \|&u^k - g^k\|_c + C(Lh)\|g\|_{W^{1,2}}
\end{aligned}$$

Applying properties of injectors and projectors, Lemma 4 and Lemma 5 and noting the assumption $Lh \le 1$ and the fact that

$$\|u^k - g^k\|_c \le \|g^k\|_c \le \|g\|,$$

we obtain

$$\|L_h S_L u^k - g\|_{L^2}^2 \le \|u^k - g^k\|_c^2 + C(Lh)\|g\|_{W^{1,2}}^2.$$

Thus

$$\begin{aligned}
E(L_h S_L u^k) &= \left|L_h S_L u^k\right|_{BV} + \frac{1}{2\lambda}\|L_h S_L u^k - g\|_{L^2}^2 \\
&\le J_c(u^k) + \frac{C}{L}\|g\|_{W^{1,2}} + \frac{1}{2\lambda}\|u^k - g^k\|_c^2 + \frac{C}{\lambda}(Lh)\|g\|_{W^{1,2}}^2 \\
&= E_k(u^k) + \frac{C}{L}\|g\|_{W^{1,2}} + \frac{C}{\lambda}(Lh)\|g\|_{W^{1,2}}^2.
\end{aligned}$$

Setting

$$L = h^{-1/2},$$

we obtain the result of this proposition.

Using similar method we prove the following

Proposition 2. *If $g \in W^{1,2}$, and u, u^k are the minimizers of E, E_k in (27), (28) respectively, then*

$$E_k(u^k) \leq E(u) + C(1 + \frac{1}{\lambda})(\|g\|_{W^{1,2}} + \|g\|^2_{W^{1,2}})h^{1/2}.$$

Combining Propositions 1 and 2 immediately yields the following

Theorem 1. *If $g \in W^{1,2}$, and u, u^k are the minimizers of E, E_k in (27), (28) respectively, then*

$$|E(u) - E_k(u^k)| \leq C(1 + \frac{1}{\lambda})(\|g\|_{W^{1,2}} + \|g\|^2_{W^{1,2}})h^{1/2}.$$

Next we need the following lemma

Lemma 8. *If u is the minimizer of E in (27), then for any $v \in BV$,*

$$\|v - u\|^2 \leq 2\lambda(E(v) - E(u)). \tag{31}$$

A proof of this Lemma can be found in [13] or [12]. It then follows

Theorem 2. *If $g \in W^{1,2}$, and u, u^k are the minimizers of E, E_k in (27), (28) respectively, then*

$$\|I_h u^k - u\|^2 \leq C(\lambda + 1)(\|g\|_{W^{1,2}} + \|g\|^2_{W^{1,2}})h^{1/2}.$$

Remark 3. In this paper, we have proved the error bound for the discrete ROF model equipped with a central-difference TV term using the method suggested in [13]. This model is simpler in form than the model studied in [13], where a symmetrical TV term is used. This model is also slightly easier to be computed by Chambolle's method (cf. [3]). However we notice that the central-difference model fails to deal with a class of data, for example a chessboard image. Thus we have to put some stronger assumption on the initial data(in $W^{1,2}$)) to obtain the error bound which may not be satisfied by all real images. However this result still shows the method in [13] can be extended to other symmetric discrete TV operators. It is also interesting to study further if a similar error bound for this model can be obtained without this assumption imposed.

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