Constr. Approx. (1990) 6: 399–419

CONSTRUCTIVE APPROXIMATION

On Bivariate Super Vertex Splines

Charles K. Chui and Mingjun Lai

Abstract. A vertex spline basis of the super-spline subspace

$$\hat{S}^r_d := S_d^{r, r+\lfloor (d-2r-1)/2 \rfloor}(\triangle)$$

of $S^r_d(\triangle)$, where $d \ge 3r+2$ and $\triangle$ is an arbitrary triangulation in $\mathbf{R}^2$, is constructed, so that the full approximation order of $d+1$ can be achieved via an approximation formula using this basis.

1. Introduction

Although the order of approximation of univariate (polynomial) spline functions of degree d is always $d+1$, it is well known that the approximation order of piecewise polynomial functions of total degree d in $C^r(\mathbf{R}^s)$ where $s > 1$ and $r \ge 1$ may depend on both the degree d and the order of smoothness r. We denote the space of such functions by $S^r_d := S^r_d(\triangle)$ where $\triangle$ is the grid partition that separates the polynomial pieces. For instance, while S^0_d always has approximation order $d+1$, the approximation order of S^1_3, even for a three-direction mesh $\triangle$ in $\mathbf{R}^2$, is only 3 instead of 4 (see [dBH1]). In general, it is at least intuitively clear that if the degree d is sufficiently larger than the smoothness order r, then the approximation order should be $d+1$. Indeed, on a simplicial partition, a parallelepiped partition, or some mixed partitions $\triangle$ in $\mathbf{R}^s$, this conclusion is true provided $d \ge 2^s r + 1$ (see [Z2], [M], and [CL2]). In particular, if $\triangle$ is an arbitrary (regular) triangulation in $\mathbf{R}^2$, the full order of approximation of $d+1$ is achieved by S^r_d for $d \ge 4r+1$ (see [Z1]). Here, a triangulation is said to be regular if there exists a unique interpolant from S^0_1 to any given data on the vertices of $\triangle$, or, equivalently, if none of the edges of $\triangle$ contains a vertex of $\triangle$ in its interior. Throughout this paper, $\triangle$ will always denote a regular, but otherwise arbitrary, triangulation in $\mathbf{R}^2$.

Recently, de Boor and Höllig [dBH2] proved that S^r_d already has approximation order $d+1$ provided that $d \ge 3r+2$. This important discovery not only improves the old result of $d \ge 4r+1$ but is also sharp in the sense that on the three-direction mesh $\triangle$ the approximation order of S^r_{3r+1} is no longer $3r+2$ as shown in [dBH2]

Date received: April 25, 1988. Date revised: March 15, 1989. Communicated by Wolfgang Dahmen.
AMS classification: 41A15, 41A25, 41A63.
Key words and phrases: Multivariate, Splines, Super splines, Approximation order, Bézier net, Interpolation.

for $r = 1, 2, 3$ and in [J] in general. Our paper may be considered as a follow-up of the important work of [dBH2] in the sense that the technique of "disentangling the rings" in [dBH2] will be one of the basic tools in constructing the "vertex spline" basis of the "super-spline" subspace $\hat{S}_d^r := S_d^{r,r+\lfloor (d-2r-1)/2\rfloor}$ of S_d^r, so that the full order of approximation $d+1$ can be achieved via an approximation formula using this vertex spline basis for $d \geq 3r+2$.

The notion of super splines in S_d^r was introduced in [CL2] and generalized by Schumaker in [S2] as follows. Let $r \leq l$ and set

$$S_d^{r,l} = \{f \in S_d^r \colon D^\alpha f(\mathbf{v}) \text{ exists for } |\alpha| \leq l \text{ at every vertex } \mathbf{v} \text{ of } \triangle\}.$$

If $r > l$, then each $f \in S_d^{r,l}$ is called a super spline. For $d \geq 3r+2$ and $l = \rho(r, d)$, where

$$\rho(r, d) := r + \lfloor (d-2r-1)/2 \rfloor,$$

we construct a basis of $\hat{S}_d^r$ consisting of functions with the smallest possible supports. More precisely, the support of each basis function will contain at most one vertex of $\triangle$ in its interior. Such a piecewise polynomial function is called a vertex spline, a notion introduced in [CL1].

This paper is organized as follows: a collection a preliminary lemmas is provided in Section 2, vertex splines in the super-spline subspaces $\hat{S}_d^r$ of S_d^r, $d \geq 3r+2$, are constructed in Section 3, and the main results of this paper are proved in Section 4. We refer the reader to [CL4] for explicit formulations of vertex splines in $\hat{S}_8^2 = S_8^{2,3}$ in terms of their Bézier nets and their representative pictures.

2. Preliminary Lemmas

In this section we list all the lemmas which are necessary for constructing our super vertex splines and deriving the main results in the next two sections. Some of these lemmas are known and will not be proved here. Throughout, Bézier representation of the polynomial pieces of the vertex splines are used.

Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be the vertices of a triangle δ. For any $\mathbf{x} \in \mathbf{R}^2$, we write

$$\mathbf{x} = \lambda_1(\mathbf{x})\mathbf{v}_1 + \lambda_2(\mathbf{x})\mathbf{v}_2 + \lambda_3(\mathbf{x})\mathbf{v}_3$$

with $\lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x}) + \lambda_3(\mathbf{x}) = 1$ for all $\mathbf{x}$, where λ_1, λ_2, λ_3 are linear polynomials in $\mathbf{x}$. The triple $(\lambda_1, \lambda_2, \lambda_3)$ is called the barycentric coordinate of $\mathbf{x}$ with respect to the triangle δ. Hence, for any multi-integer $\beta \in \mathbf{Z}_+^3$ with $|\beta| := \beta_1 + \beta_2 + \beta_3$,

$$\varphi_\beta(\lambda_1, \lambda_2, \lambda_3) = \varphi_\beta(\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \lambda_3(\mathbf{x})) := \frac{|\beta|!}{\beta!} \lambda_1^{\beta_1}(\mathbf{x})\lambda_2^{\beta_2}(\mathbf{x})\lambda_3^{\beta_3}(\mathbf{x})$$

is a polynomial of total degree $|\beta|$ where $\beta! := \beta_1!\,\beta_2!\,\beta_3!$. It is well known that $\{\varphi_\beta : |\beta| = n\}$ forms a basis of π_n, the space of polynomials of total degree at most n in $\mathbf{R}^2$. That is, any polynomial P_n in π_n can be written uniquely as

$$P_n(\mathbf{x}) = \sum_{|\beta|=n} a_\beta \varphi_\beta(\lambda_1, \lambda_2, \lambda_3)$$

which is called its Bézier representation with Bézier coefficients a_β's. The three-dimensional set $\{((\beta_1/n)\mathbf{x}_1+(\beta_2/n)\mathbf{x}_2+(\beta_3/n)\mathbf{x}_3, a_\beta): |\beta|=n\}$, or brevity $\{a_\beta\}$, is called the Bézier net of P_n. (See [F], [dB], and [C] for the properties of Bézier representations and Bézier nets.)

Now let P_n and Q_n be two polynomials in π_n defined on two adjacent triangles $\delta_1=\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\rangle$ and $\delta_2=\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\rangle$, and consider their Bézier representations

$$P_n(\mathbf{x})=\sum_{|\beta|=n} a_\beta \varphi_\beta(\lambda_1, \lambda_2, \lambda_3)$$

and

$$Q_n(\mathbf{x})=\sum_{|\beta|=n} b_\beta \varphi_\beta(\nu_1, \nu_2, \nu_3),$$

where $(\lambda_1, \lambda_2, \lambda_3)$ and (ν_1, ν_2, ν_3) are the barycentric coordinates of $\mathbf{x}$ with respect to δ_1 and δ_2, respectively (see Fig. 1 for $n=5$). Instead of showing the Bézier nets in $\mathbf{R}^3$, it is sometimes more convenient to display their Bézier coefficients on triangular arrays in $\mathbf{R}^2$ as in Fig. 1.

Let $\lambda^0:=(\lambda_1^0, \lambda_2^0, \lambda_3^0)$ be the barycentric coordinates of $\mathbf{x}_4$ with respect to δ_1; that is, $\mathbf{x}_4=\lambda_1^0\mathbf{x}_1+\lambda_2^0\mathbf{x}_2+\lambda_3^0\mathbf{x}_3$ with $\lambda_1^0+\lambda_2^0+\lambda_3^0=1$, and define f on $\delta_1\cup\delta_2$ by $f|_{\delta_1}=P_n$ and $f|_{\delta_2}=Q_n$. The following result on smoothness matching conditions is well known. (See [F], [dB], or [C].)

Lemma 1. *The polynomials P_n and Q_n are joined smoothly across the edge $[\mathbf{x}_1, \mathbf{x}_2]$ up to order r in the sense that $f\in C^r(\delta_1\cup\delta_2)$ if and only if the Bézier nets $\{a_\beta\}$ and $\{b_\beta\}$ satisfy*

$$b_{(\beta_1,\beta_2,\beta_3)}=\sum_{|\alpha|=\beta_3} a_{\alpha+(\beta_1,\beta_2,0)}\varphi_\alpha(\lambda^0) \tag{2.1}$$

for all $\beta=(\beta_1, \beta_2, \beta_3)$ with $0\le\beta_3\le r$ and $|\beta|=n$.

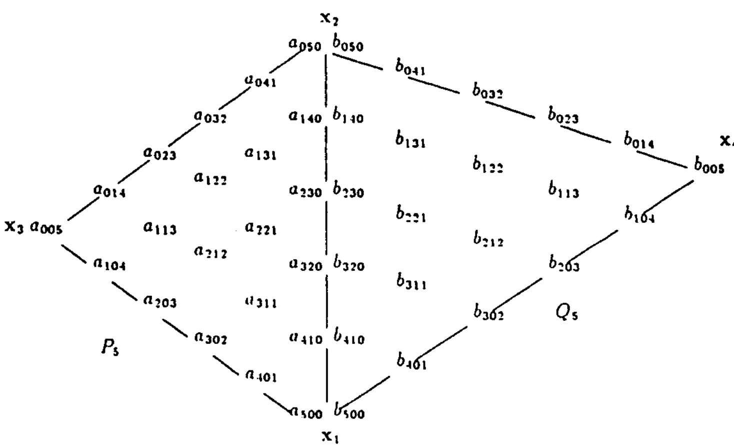

Fig. 1

A set $M \subset \mathbf{Z}_+^2$ is said to be a lower set if $\alpha \in M$ whenever $\beta \in M$ and $0 \le \alpha \le \beta$. Define the mappings A_i^n, $i = 1, 2, 3$, of $\mathbf{Z}_+^2$ into $\mathbf{Z}_+^3$ by

$$A_1^n(\beta_1, \beta_2) = (n - \beta_1 - \beta_2, \beta_1, \beta_2),$$
$$A_2^n(\beta_1, \beta_2) = (\beta_1, n - \beta_1 - \beta_2, \beta_2),$$

and

$$A_3^n(\beta_1, \beta_2) = (\beta_1, \beta_2, n - \beta_1 - \beta_2).$$

Lemma 2. *Let $M_{n,r} = \{\alpha \in \mathbf{Z}_+^3 : \alpha_3 \le r, |\alpha| = n\}$ and let M_1, $M_2 \subset \mathbf{Z}_+^2$ be lower sets satisfying $A_1^n M_1 \cap A_2^n M_2 = \varnothing$ and $A_1^n M_1 \cup A_2^n M_2 = M_{n,r}$. Then P_n and Q_n are joined smoothly across the edge $[\mathbf{x}_1, \mathbf{x}_2]$ up to order r in the sense that $f \in C^r(\delta_1 \cup \delta_2)$ if and only if they satisfy the following interpolatory matching conditions:*

$$(2.2)_1 \qquad (D_{\mathbf{x}_4 - \mathbf{x}_1})^i (D_{\mathbf{x}_2 - \mathbf{x}_1})^j Q_n(\mathbf{x}_1) = (\lambda_2^0 D_{\mathbf{x}_2 - \mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_3 - \mathbf{x}_1})^i (D_{\mathbf{x}_2 - \mathbf{x}_1})^j P_n(\mathbf{x}_1)$$

for $(i, j) \in M_1$ and

$$(2.2)_2 \qquad (D_{\mathbf{x}_4 - \mathbf{x}_2})^i (D_{\mathbf{x}_1 - \mathbf{x}_2})^j Q_n(\mathbf{x}_2) = (\lambda_2^0 D_{\mathbf{x}_1 - \mathbf{x}_2} + \lambda_3^0 D_{\mathbf{x}_3 - \mathbf{x}_2})^i (D_{\mathbf{x}_1 - \mathbf{x}_2})^j P_n(\mathbf{x}_2)$$

for $(i, j) \in M_2$.

The proof of this lemma follows immediately from the fact that two univariate polynomials of total degree $(n - i)$ (i.e., the ith transversal derivatives of P_n and Q_n along $[\mathbf{x}_1, \mathbf{x}_2]$) agree if they agree j_1-fold at $\mathbf{x}_1$ and j_2-fold at $\mathbf{x}_2$, where $j_1 + j_2 = n - i + 1$. (See [CL3].)

The following two lemmas are essentially the same as Lemmas 4.1 and 4.2 in [dBH2]. They are presented a little differently here for our later applications.

Lemma 3. *Assume $\mathbf{x}_2 \notin [\mathbf{x}_3, \mathbf{x}_4]$. Let $\{a_\beta\}$ and $\{b_\beta\}$, $\beta = (\beta_1, \beta_2, \beta_3)$, be the Bézier nets of P_n and Q_n, respectively. Assume that the values $\{\alpha_\beta, b_\beta : \beta_2 \ge 1\}$ and $\{a_\beta, b_\beta : \beta_2 = 0$ and $0 \le \beta_3 \le n - 2l - 2\}$ are given, for some $l \le (n-2)/2$, and that the Bézier nets $\{a_\beta, b_\beta : |\beta| = n\}$ satisfy the smoothness conditions (2.1) of order $n - 2l - 2$. If $\{a_\beta : \beta_2 \ge 1\}$ and $\{b_\beta : \beta_2 \ge 1\}$ also satisfy the smoothness conditions (2.1) of order $n - 1$, then, for any given $\{a_\beta, b_\beta : \beta_2 = 0$ and $0 \le \beta_1 \le l\}$, there exists a unique set of coefficients $\{a_\beta, b_\beta : \beta_2 = 0$ and $l + 1 \le \beta_1 \le 2l + 1\}$ such that $\{a_\beta\}$ and $\{b_\beta\}$ satisfy the smoothness conditions (2.1) of order n.*

Proof. By the assumption, we only need to prove that there exists a unique solution $\{a_\beta, b_\beta : \beta_2 = 0$ and $l = 1 \le \beta_1 \le 2l + 1\}$ such that the smoothness conditions that only involve any of the values $\{a_\beta, b_\beta : \beta_2 = 0, 0 \le \beta_1 \le l + 1\}$ hold. By Lemma 1, the smoothness conditions are

$$(2.3) \qquad b_{(i,0,n-i)} = \sum_{|\alpha| = n-i} a_{(i,0,0)+\alpha} \varphi_\alpha(\lambda^0), \qquad i = 0, \ldots, 2l+1.$$

Thus, we have $2l+2$ conditions and $2l+2$ unknowns $\{a_\beta, b_\beta : \beta_2 = 0, l+1 \le \beta_1 \le 2l+1\}$. The linear system (2.3) may be decomposed into two smaller linear subsystems:

$$(2.4) \qquad b_{(i,0,n-i)} = \sum_{|\alpha| = n-i} a_{\alpha+(i,0,0)} \varphi_\alpha(\lambda^0), \qquad i = l+1, \ldots, 2l+1,$$

and

$$(2.5)\qquad b_{(k,0,n-k)} = \sum_{|\alpha|=n-k} a_{\alpha+(k,0,0)}\varphi_\alpha(\lambda^0), \qquad k=0,\ldots,l,$$

which may be rewritten as

$$(2.6)\qquad \sum_{i=l+1}^{2l+1} a_{(i,0,n-i)}\binom{n-k}{i-k}(\lambda_1^0)^{i-k}(\lambda_3^0)^{n-i} = c_k, \qquad k=0,\ldots,l,$$

where $c_0,\ldots,c_l$ are certain constants involving the given a_α's and b_β's. The latter linear system (2.6) has a unique solution $\{a_\beta: \beta_2=0, l+1\le\beta_1\le 2l+1\}$ because the determinant of its coefficient matrix can be simplified to be

$$(2.7)\qquad \det\begin{bmatrix} \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{(l+1)!} \\ \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(l+2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(l+1)!} & \frac{1}{(l+2)!} & \cdots & \frac{1}{(2l+1)!} \end{bmatrix} = \frac{\prod_{i=1}^{l+1}(l+1-i)!}{\prod_{i=1}^{l+1}(2l+2-i)!} \neq 0.$$

Then substituting the values $\{a_\beta: \beta_2=0, l+1\le\beta_1\le 2l+1\}$ into the subsystem (2.4), we have also uniquely determined $\{b_\beta: \beta_2=0, l+1\le\beta_1\le 2l+1\}$. This completes the proof of the lemma. ■

Lemma 4. *Assume* $\mathbf{x}_2\in[\mathbf{x}_3,\mathbf{x}_4]$ *and that the Bézier coefficients* $\{a_\beta: \beta_2\ge 1\}$ *and* $\{b_\beta: \beta_2\ge 1\}$ *are given and satisfy the smoothness conditions* (2.1) *up to order* $n-1$. *Furthermore, assume that* $\{a_\beta: \beta_2=0$ *and* $0\le\beta_3\le l\}$ *and* $\{b_\beta: \beta_2=0$ *and* $0\le\beta_3\le l\}$ *are given and satisfy the smoothness conditions* (2.1) *of order* l, *where* $l<n$. *Then, for any* $\{a_\beta: \beta_2=0$ *and* $0\le\beta_1\le n-l-1\}$, *there exists a unique set of coefficients* $\{b_\beta: \beta_2=0$ *and* $0\le\beta_1\le n-l-1\}$ *such that* $\{a_\beta: |\beta|=n\}$ *and* $\{b_\beta: |\beta|=n\}$ *satisfy the smoothness conditions* (2.1) *of order* n.

Proof. This result is a simple consequence of Lemma 1. ■

Remark. The solution set $\{a_\beta, b_\beta\}$ in Lemma 3 actually depends on the geometry of the triangles $\langle\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\rangle$ and $\langle\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_4\rangle$. More precisely, each a_β or b_β depends on certain powers of $(\lambda_1^0)^{-1}$ and $(\lambda_3^0)^{-1}$ (cf. (2.6)). Thus, if the area $|\langle\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\rangle|$ of the triangle $\langle\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\rangle$ is very small so that λ_1^0 is very close to zero, then the magnitude of a_β or b_β would be very large. For this reason we need the notion of "near-singularity." An edge $[\mathbf{x}_1,\mathbf{x}_2]$ is called a *near-singular edge at* $\mathbf{x}_2$ if $|\langle\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\rangle|>0$ is near zero; e.g., $0<\lambda_1^0\ll a$, where $a=\max\{\lambda_3^0,(\lambda_3^0)^{-1}\}$. If $|\langle\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\rangle|=0$, then the edge $[\mathbf{x}_1,\mathbf{x}_2]$ is called a *singular edge at* $\mathbf{x}_2$. An interior vertex $\mathbf{v}$ is called a *near-singular vertex* if it is the point of intersection of four near-singular or singular edges with at least three distinct slopes (see Fig. 3). Also, an interior vertex $\mathbf{v}$ is said to be a *singular vertex* if it is the point of intersection of four edges with only two distinct slopes (see Fig. 2).

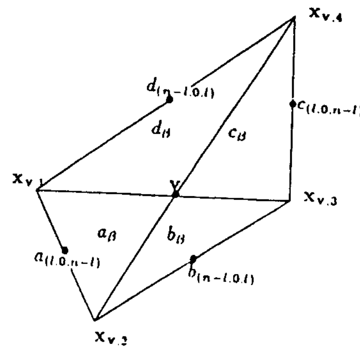

Fig. 2

Lemma 5. *Let* $\mathbf{v}$ *be a singular vertex. Assume that the Bézier coefficients* $\{a_\beta: \beta_2 \geq 1\}$, $\{b_\beta: \beta_2 \geq 1\}$, $\{c_\beta: \beta_2 \geq 1\}$, *and* $\{d_\beta: \beta_2 \geq 1\}$ *on the four triangles that share the common vertex* $\mathbf{v}$ *are given and satisfy the smoothness conditions of order* $n-1$ *(see Fig. 2); i.e., they are considered as the Bézier coefficients of a polynomial of degree* $<n$. *Then, for any* $0 \leq l \leq n$ *and any given* $a_{(l,0,n+l)}$, *there exists a unique set of coefficients* $b_{(n-l,0,l)}$, $c_{(l,0,n-l)}$, *and* $d_{(n-l,0,l)}$ *so that the* nth *order smoothness conditions* (2.1) *involving these coefficients are satisfied.*

Proof. By Lemma 1, for any given $a_{(l,0,n-l)}$, the values $b_{(n-l,0,l)}$, $c_{(l,0,n-l)}$, and $d_{(n-l,0,l)}$ are consecutively determined by (2.1). To show that $d_{(n-l,0,l)}$ and $a_{(l,0,n-l)}$ satisfy the smoothness condition, we may assume without loss of generality that a_β, b_β, c_β, and d_β, with $\beta_2 \geq 1$, are zero and obtain

$$b_{(n-l,0,l)} = \left(\frac{|\mathbf{x}_3 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_1|}\right)^{n-l} a_{(l,0,n-l)},$$

$$c_{(l,0,n-l)} = \left(\frac{|\mathbf{x}_4 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_2|}\right)^{l} b_{(n-l,0,l)},$$

$$d_{(n-l,0,l)} = \left(\frac{|\mathbf{x}_1 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_3|}\right)^{n-l} c_{(l,0,n-l)}.$$

Hence,

$$\begin{aligned} d_{(n-l,0,l)} &= \left(\frac{|\mathbf{x}_1 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_3|}\right)^{n-l} \left(\frac{|\mathbf{x}_4 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_2|}\right)^{l} \left(\frac{|\mathbf{x}_3 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_1|}\right)^{n-l} a_{(l,0,n-l)} \\ &= \left(\frac{|\mathbf{x}_1 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_2|}\right)^{l} a_{(l,0,n-l)} \end{aligned}$$

completing the proof of the lemma. ■

Lemmas 1, 3, 4, and 5 are used for constructing our vertex splines. In the following we establish three lemmas for studying "stability" in the presence of near-singular vertices.

Lemma 6. *Assume $\mathbf{x}_2 \notin [\mathbf{x}_3, \mathbf{x}_4]$ and that the values in*

$$\{a_\beta, b_\beta : \beta_2 = 0 \text{ and } 0 \le \beta_3 \le n - 2l - 2\}$$

are given, where $l \le (n-2)/2$, and $a_\beta = b_\beta = 0$ for $\beta_2 \ge 1$. Furthermore, assume that these Bézier coefficients satisfy the smoothness conditions (2.1) of order $n-2l-2$. Then if $a_\beta = b_\beta = 0$ for $\beta_2 = 0$ and $0 \le \beta_1 \le l$, the Bézier coefficients a_β and b_β with $\beta_2 = 0$, $l+1 \le \beta_1 \le 2l+1$, and $|\beta| = n$, which are uniquely determined by the smoothness conditions (2.1) up to order n, are bounded by a constant which depends only on certain linear combinations of the powers of the ratios $|\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle| / |\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4 \rangle|$ and $|\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4 \rangle| / |\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle|$.

Proof. From the proof of Lemma 3, we know that

$$\{a_\beta, b_\beta : \beta_2 = 0 \text{ and } l+1 \le \beta_1 \le 2l+1\}$$

satisfy the linear systems:

$$b_{(i,0,n-i)} = \sum_{|\beta| = n-i} a_{\beta+(i,0,0)} \varphi_\beta^0, \qquad i = l+1, \ldots, 2l+1, \tag{2.8}$$

and

$$b_{(k,0,n-k)} = \sum_{|\beta| = n-k} a_{\beta+(k,0,0)} \varphi_\beta^0, \qquad k = 0, \ldots, l,$$

the latter of which, in view of the assumption on b_β, may be rewritten as

$$0 = b_{(k,0,n-k)} = \sum_{\substack{|\beta| = n-k \\ \beta_2 = 0}} a_{\beta+(k,0,0)} \frac{(n-k)!}{\beta!} (\lambda_1^0)^{\beta_1} (\lambda_3^0)^{\beta_3}, \qquad k = 0, \ldots, l,$$

or, equivalently,

$$\begin{aligned} \sum_{\beta_3=0}^{l} a_{(l+1+\beta_3,0,n-l-1-\beta_3)} \binom{n-k}{n-l-1-\beta_3} (\lambda_1^0)^{\beta_3} (\lambda_3^0)^{n-l-1-\beta_3} \\ = -\sum_{\beta_3=0}^{n-2l-2} a_{(n-\beta_3,0,\beta_3)} \binom{n-k}{\beta_3} (\lambda_1^0)^{n-k-\beta_3} (\lambda_3^0)^{\beta_3}, \qquad k = 0, \ldots, l. \end{aligned} \tag{2.9}$$

By using (2.7), we may now solve these linear equations for

$$a_{(l+1+\beta_3,0,n-l-1-\beta_3)} (\lambda_1^0)^{\beta_3} (\lambda_3^0)^{n-l-1-\beta_3}, \qquad \beta_3 = 0, \ldots, l,$$

each of which is a linear combination of $\sum_{i=0}^{n-2l-2} a_{(n-i,0,i)} \binom{n-k}{i} (\lambda_1^0)^{n-l-1-i} (\lambda_3^0)^i$ where $k = 0, \ldots, l$. Note that the quantities $(\lambda_1^0)^{n-l-1-i}$, $i = 0, \ldots, n-2l-2$, have a common factor $(\lambda_1^0)^{l+1}$. Therefore, by canceling these factors, we conclude that $a_{(l+1+\beta_3,0,n-l-1-\beta_3)}$, $\beta_3 = 0, \ldots, l$, are bounded by a constant which only depends on some powers of λ_3^0. Thus, from (2.8), we may also conclude that $b_{(i,0,n-i)}$, $i = l+1, \ldots, 2l+1$, are bounded by a constant depending only on certain powers of λ_3^0 and $(\lambda_3^0)^{-1}$. This establishes the lemma. ■

We first consider the space $\hat{S}_8^2 = S_8^{2,3}$. Hence, in the presence of near-singular vertices, we need to study $n = 4$.

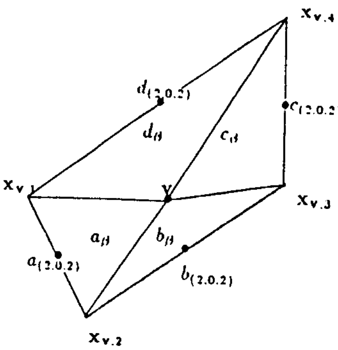

Fig. 3

Lemma 7. *Let* $\mathbf{v}$ *be a near-singular vertex and assume that* $|\langle \mathbf{x}_{v,1}, \mathbf{v}, \mathbf{x}_{v,3}\rangle| \le |\langle \mathbf{x}_{v,2}, \mathbf{v}, \mathbf{x}_{v,4}\rangle|$. *Let* $\{a_\beta\}, \{b_\beta\}, \{c_\beta\}$, *and* $\{d_\beta\}$ *be the Bézier nets of the four polynomial pieces as shown in Fig.* 3. *Assume that* $\{a_\beta, b_\beta, c_\beta, d_\beta \colon \beta_2 \ge 1\}$ *are given and satisfy the smoothness conditions of order* 3. *Furthermore, assume that* $b_{(4,0,0)} = c_{(0,0,4)}$ *is also given. Then, for any given choice of* $d_{(0,0,4)}, d_{(1,0,3)}$, *and* $a_{(k,0,4-k)}$, $0 \le k \le 4$, *the Bézier coefficients* $b_{(k,0,4-k)}, c_{(k,0,4-k)}$, *and* $d_{(j,0,4-j)}$, *where* $0 \le k \le 4$ *and* $2 \le j \le 4$, *are uniquely determined by the smoothness conditions up to order* 2, *and are bounded by a constant which depends only on certain powers of*

$$A := \max_{1 \le i \le 4} \left\{ \frac{|\langle \mathbf{x}_{v,i}, \mathbf{v}, \mathbf{x}_{v,i+1}\rangle|}{|\langle \mathbf{x}_{v,i+1}, \mathbf{v}, \mathbf{x}_{v,i+2}\rangle|}, \frac{|\langle \mathbf{x}_{v,i+1}, \mathbf{v}, \mathbf{x}_{v,i+2}\rangle|}{|\langle \mathbf{x}_{v,i}, \mathbf{v}, \mathbf{x}_{v,i+1}\rangle|} \right\}. \tag{2.10}$$

Proof. Without loss of generality we may assume that $a_\beta = 0$ for all $|\beta| = 4$. Then we also have $b_\beta = c_\beta = d_\beta = 0$, $\beta_2 \ge 1$. By using Lemma 1, $c_{(4,0,0)}$, $c_{(3,0,1)}$, and $c_{(2,0,2)}$ may be computed in terms of $d_{(0,0,4)}$ and $d_{(1,0,3)}$, and

$$c_{(2,0,2)} = O(A d_{(1,0,3)} \eta_1 / |\langle \mathbf{x}_{v,1}, \mathbf{v}, \mathbf{x}_{v,4}\rangle|) + O(d_{(0,0,4)} (\eta_1 / |\langle \mathbf{x}_{v,1}, \mathbf{v}, \mathbf{x}_{v,4}\rangle|)^2),$$

where

$$\eta_1 = |\langle \mathbf{x}_{v,1}, \mathbf{v}, \mathbf{x}_{v,3}\rangle|.$$

Set

$$\eta_2 = |\langle \mathbf{x}_2, \mathbf{v}, \mathbf{x}_4\rangle|.$$

Then we apply Lemma 3 to solve for $b_{(3,0,1)}$ and $c_{(1,0,3)}$. By using the same method as in the proof of Lemma 6, we conclude that $b_{(3,0,1)} = c_{(1,0,3)} = O(\eta_1/\eta_2 A^2) = O(A^2)$. Thus, the lemma is established. ■

We next consider the space $\hat{S}^3_{11} = S^{3,5}_{11}$. In this situation we need to study the case $n = 6$.

Lemma 8. *Let* $\mathbf{v}$ *be a near-singular vertex and assume that* $|\langle \mathbf{x}_{v,1}, \mathbf{v}, \mathbf{x}_{v,3}\rangle| \le |\langle \mathbf{x}_{v,2}, \mathbf{v}, \mathbf{x}_{v,4}\rangle|$. *Let* $\{a_\beta\}, \{b_\beta\}, \{c_\beta\}$, *and* $\{d_\beta\}$ *be the Bézier nets of the four polynomial pieces on the four triangles sharing* $\mathbf{v}$ *(see Fig.* 3*). Assume that* $\{a_\beta, b_\beta, c_\beta, d_\beta \colon \beta_2 \ge 1\}$

are given and satisfy the smoothness conditions of order 5. *Furthermore, assume that* $b_{(6,0,0)} = c_{(0,0,6)}$, $b_{(5,0,1)}$, *and* $c_{(1,0,5)}$ *are given and satisfy the smoothness conditions of order* 1. *Then for any given* $d_{(0,0,6)}$, $d_{(1,0,5)}$, $d_{(2,0,4)}$, *and* $a_{(k,0,6-k)}$, $0 \le k \le 6$, *the Bézier coefficients* $d_{(j,0,6-j)}$, $b_{(k,0,6-k)}$, *and* $c_{(k,0,6-k)}$, *where* $3 \le j \le 6$ *and* $0 \le k \le 6$, *which are uniquely determined by the smoothness conditions of order* 3, *are bounded by a constant which depends only on certain powers of A as defined in* (2.10).

The proof of this lemma follows the same lines of argument as that of Lemma 7 and is omitted here.

The above eight lemmas provide the necessary machinery for constructing our super vertex splines and studying "stability" in the presence of near-singular vertices. To verify that the approximation order of the super-spline subspace spanned by these vertex splines is indeed full for $d \ge 3r+2$, we need two additional lemmas.

For $N_i \subset \mathbf{Z}_+^2$, $i = 1, 2, 3$, we say that $\{N_i: i = 1, 2, 3\}$ induces a partition of

$$\Lambda_n = \{\beta \in \mathbf{Z}_+^3: |\beta| = n\}$$

if the following conditions are satisfied:

(1) $\bigcup_{i=1}^3 A_i^n N_i = \Lambda_n$, and

(2) $A_i^n N_i \cap A_j^n N_j = \varnothing$, $i \ne j$.

Lemma 9. *Assume that* $\{N_i \subset \mathbf{Z}_+^2: i = 1, 2, 3\}$ *is a collection of lower sets that induces a partition of* A_n. *Then to any given data* $\{f_{i,\beta}: \beta \in N_i\}$, $i = 1, 2, 3$, *there exists a unique polynomial* $P_n \in \pi_n$ *that satisfies the following interpolation conditions*:

$$(D_{\mathbf{x}_2-\mathbf{x}_1})^{\beta_1}(D_{\mathbf{x}_3-\mathbf{x}_1})^{\beta_2}P_n(\mathbf{x}_1) = f_{1,(\beta_1,\beta_2)}, \qquad (\beta_1, \beta_2) \in N_1,$$
$$(D_{\mathbf{x}_1-\mathbf{x}_2})^{\beta_1}(D_{\mathbf{x}_3-\mathbf{x}_2})^{\beta_2}P_n(\mathbf{x}_2) = f_{2,(\beta_1,\beta_2)}, \qquad (\beta_1, \beta_2) \in N_2,$$

and

$$(D_{\mathbf{x}_1-\mathbf{x}_3})^{\beta_1}(D_{\mathbf{x}_2-\mathbf{x}_3})^{\beta_2}P_n(\mathbf{x}_3) = f_{3,(\beta_1,\beta_2)}, \qquad (\beta_1, \beta_2) \in N_3.$$

Since N_i, $i = 1, 2, 3$, are lower sets, the interpolation data $\{f_{i,\beta}: \beta \in N_i\}$ determine the Bézier net of a polynomial $P_n \in \pi_n$ with indices in $A_i^n N_i$ uniquely. Hence, since $\{N_i: i = 1, 2, 3\}$ induces a partition of Λ_n, it can be seen that P_n satisfying the above interpolation conditions exists and is unique. We refer the reader to [CL3] for the more general results along this line. The following result is usually attributed to Bramble and Hilbert [BH].

Lemma 10. *Let F be a linear functional on* $C^{k+1}(G)$ *that satisfies the following two properties*:

(i) $|F(f)| \le C \sum_{l=0}^k h^l |f|_l$ *where C is a constant independent of f and h and*

$$|f|_l := \sup_{\mathbf{x} \in G} \sum_{|\alpha|=l} |D^\alpha f(\mathbf{x})|,$$

and

(ii) $F(p) = 0$ *for all* $p \in \pi_k$.

Then there exists a positive constant K independent of f and h such that

$$|F(f)| \le Kh^{k+1}|f|_{k+1}.$$

3. Construction of Vertex Splines

In this section we outline a procedure for constructing a basis of $\hat{S}_d^r$ consisting of vertex splines. Each vertex spline will be specified by interpolatory parameters at the corresponding vertex. In the following we introduce the notion of derivatives relative to an edge and a triangle. In the construction of each polynomial piece of a vertex spline, we subdivide the indices of the Bézier coefficients of this polynomial into four parts as indicated by I, II, III, and IV in Fig. 4. The Bézier coefficients with indices in I are either zero or will be determined by the interpolation parameters, those with indices in II will be determined by derivatives relative to the triangle, those with indices in III by derivatives relative to the edges, and those with indices in IV by using Lemmas 3 or 4. We first introduce the necessary definitions and notations and then specify the interpolation parameters of these vertex splines. We only discuss the special case where $d = 3r+2$, since it will be clear that our construction procedure is also valid for $3r+2 \le d < 4r+1$. Of course, the construction of a vertex spline basis of $S_d^{r,2r}$ for $d \ge 4r+1$ is much easier and can even be carried out in the multivariate setting as already discussed in [CL3].

Let us first divide the underlying index set $\{\beta \in \mathbf{Z}_+^3 : |\beta| = 3r+2\}$ of the Bézier net on a triangle $\delta = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ into four parts (see Fig. 4 for $r = 5$). In the following we use the notation

$$l(r) := \rho(r, 3r+2) = r + \lfloor r + \tfrac{1}{2} \rfloor.$$

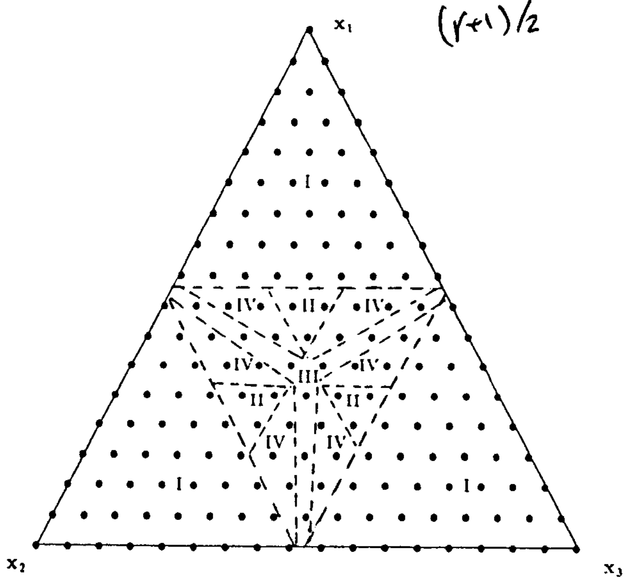

Fig. 4

Part I is the union of the collections $A_i^{3r+2}C_1$, $i=1,2,3$, where $C_1=\{(l,m)\in \mathbf{Z}_+^2\colon l+m\le l(r)\}$.

Part II is the union of the collections $A_i^{3r+2}C_2$, $i=1,2,3$, where $C_2=\{(l,m)\in \mathbf{Z}_+^2\colon l+m\ge l(r)+1 \text{ and } l, m\le r\}$.

Part III is the union of the collections $A_i^{3r+2}C_3$, $i=1,2,3$, where $C_3=\{(r-2m, r+1+m)\in \mathbf{Z}_+^2\colon m=0,\dots,\lfloor r/2\rfloor\}$.

Part IV consists of the remaining Bézier coefficients on δ; i.e., the union of the collection $A_i^{3r+2}C_4\cup A_i^{3r+2}\bar{C}_4$, $i=1,2,3$, where

$$C_4=\bigcup_{i=1}^{\lfloor r/2\rfloor}\{(r+1,r-i),\dots,(r+1+i-1,r-i-(i-1))\}$$

and $\bar{C}_4=\{(l,m)\colon (m,l)\in C_4\}$.

We then introduce the notion of derivatives relative to an edge or relative to a triangle. For an edge $e=[\mathbf{x}_{e,1},\mathbf{x}_{e,2}]$ and a triangle $\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,3}\rangle$ with e as one of its edges, *the derivatives relative to the edge e* (corresponding to the triangle $\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,3}\rangle$) are defined by

$$D_e^\alpha=(D_{\mathbf{x}_{e,3}-\mathbf{x}_{e,1}})^{\alpha_1}(D_{\mathbf{x}_{e,2}-\mathbf{x}_{e,1}})^{\alpha_2},\qquad \alpha=(\alpha_1,\alpha_2)\in \mathbf{Z}_+^2,$$

where the derivatives are taken within the triangle. For a triangle $t=\langle \mathbf{x}_i,\mathbf{x}_j,\mathbf{x}_k\rangle$, *the derivatives relative to the triangle t at* $\mathbf{x}_i$ are defined by

$$D_{t(\mathbf{x}_i)}^\alpha=(D_{\mathbf{x}_j-\mathbf{x}_i})^{\alpha_1}(D_{\mathbf{x}_k-\mathbf{x}_i})^{\alpha_2},\qquad \alpha=(\alpha_1,\alpha_2)\in \mathbf{Z}_+^2,$$

where $\mathbf{x}_j$ and $\mathbf{x}_k$ are labeled according to the counterclockwise orientation of $\{\mathbf{x}_i,\mathbf{x}_j,\mathbf{x}_k\}$ and the derivatives are taken within the triangle.

For a given arbitrary triangulation Δ, denote the collections of all vertices and edges of Δ by $\mathcal{V}$ and $\mathcal{E}$, respectively. The vertices of Δ are denoted by $\mathcal{V}=\{\mathbf{v}_1,\dots,\mathbf{v}_N\}$. For each $e\in\mathcal{E}$ with vertices $\mathbf{v}_i$ and $\mathbf{v}_j$, we assign the direction of e according to the order of increasing indices of $\mathbf{v}_i$ and $\mathbf{v}_j$; i.e., $e=[\mathbf{x}_{e,1},\mathbf{x}_{e,2}]$, where

$$\mathbf{x}_{e,1}:=\mathbf{v}_{\min\{i,j\}}\quad\text{and}\quad \mathbf{x}_{e,2}:=\mathbf{v}_{\max\{i,j\}}.$$

If $e=[\mathbf{x}_{e,1},\mathbf{x}_{e,2}]$ is an interior edge of Δ, let $\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,3}\rangle$ and $\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,4}\rangle$ be the two triangles of Δ with e as the common edge and label the vertices so that $\mathbf{x}_{e,1}$, $\mathbf{x}_{e,2}$, $\mathbf{x}_{e,3}$ and $\mathbf{x}_{e,1}$, $\mathbf{x}_{e,4}$, $\mathbf{x}_{e,2}$ are both in the counterclockwise direction. If e is a boundary edge, we denote by $\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,3}\rangle$ the triangle of Δ with e as one of its edges. The interpolation parameters will be specified at the vertex $\mathbf{x}_{e,1}$ (for the corresponding vertex spline).

Let $\mathbf{v}$ be a vertex in $\mathcal{V}$. We denote the triangles of Δ that share $\mathbf{v}$ as their common vertex by $T_{\mathbf{v},i}$, $i=1,\dots,l(\mathbf{v})$, where $T_{\mathbf{v},i}=\langle \mathbf{v},\mathbf{x}_{\mathbf{v},i},\mathbf{x}_{\mathbf{v},i+1}\rangle$, $i=1,\dots,l(\mathbf{v})$, and the vertices $\mathbf{x}_{\mathbf{v},1},\dots,\mathbf{x}_{\mathbf{v},l(\mathbf{v})+1}$ are labeled in the counterclockwise direction around $\mathbf{v}$. Here, if $\mathbf{v}$ is an interior vertex, we set $\mathbf{x}_{\mathbf{v},l(\mathbf{v})+1}:=\mathbf{x}_{\mathbf{v},1}$. We call $T_{\mathbf{v},i}$ a one-sided singular triangle relative to the vertex $\mathbf{v}$ if either $[\mathbf{v},\mathbf{x}_{\mathbf{v},i}]$ or $[\mathbf{v},\mathbf{x}_{\mathbf{v},i+1}]$ (but not both of them) is a singular or near-singular edge at $\mathbf{v}$, and a two-sided singular triangle relative to $\mathbf{v}$ if both $[\mathbf{v},\mathbf{x}_{\mathbf{v},i}]$ and $[\mathbf{v},\mathbf{x}_{\mathbf{v},i+1}]$ are singular or near-singular edges at $\mathbf{v}$. If $\mathbf{v}$ is a near-singular vertex, we choose $T_{\mathbf{v},1}$ such that $|\langle \mathbf{x}_{\mathbf{v},1},\mathbf{v},\mathbf{x}_{\mathbf{v},3}\rangle|\le|\langle \mathbf{x}_{\mathbf{v},2},\mathbf{v},\mathbf{x}_{\mathbf{v},4}\rangle|$ is satisfied. We relabel $T_{\mathbf{v},i}$, $i=1,\dots,l(\mathbf{v})$ to be $t_i(\mathbf{v})$, $i=1,\dots,m(\mathbf{v})$, as follows:

(1) If $\mathbf{v}$ is a singular vertex, $t_1(\mathbf{v})=T_{\mathbf{v},1}$ and $t_2(\mathbf{v})=T_{\mathbf{v},3}$. So $m(\mathbf{v})=2$.

(2) If $\mathbf{v}$ is a near-singular vertex, $t_1(\mathbf{v})=T_{\mathbf{v},1}$ and $t_2(\mathbf{v})=T_{\mathbf{v},4}$. That is, $m(\mathbf{v})=2$.

(3) If $\mathbf{v}$ is not a singular nor a near-singular vertex and one of the $T_{\mathbf{v},i}$, $1 \le i \le l(\mathbf{v})$, say $T_{\mathbf{v},j}$, is a two-sided singular triangle relative to $\mathbf{v}$, we denote $\{T_{\mathbf{v},1}, \ldots, T_{\mathbf{v},j-2}, T_{\mathbf{v},j}, T_{\mathbf{v},j+2}, \ldots, T_{\mathbf{v},l(\mathbf{v})}\}$ by $\{t_i(\mathbf{v}): i = 1, \ldots, m(\mathbf{v})\}$ where $m(\mathbf{v}) = l(\mathbf{v}) - 2$. Note that if $\mathbf{v}$ is a boundary vertex, $T_{\mathbf{v},1}, \ldots, T_{\mathbf{v},j-1}$ or $T_{\mathbf{v},j+1}, \ldots, T_{\mathbf{v},l(\mathbf{v})}$ may not exist.

(4) Assume that none of the $T_{\mathbf{v},i}$, $i = 1, \ldots, l(\mathbf{v})$, is a two-sided singular triangle relative to $\mathbf{v}$ and $\mathbf{v}$ is not a near-singular vertex. If $T_{\mathbf{v},i}$ is a one-sided singular triangle relative to $\mathbf{v}$, so is $T_{\mathbf{v},i+1}$ or $T_{\mathbf{v},i-1}$. If $T_{\mathbf{v},i}$ and $T_{\mathbf{v},i+1}$ share a common singular edge, we denote $\{T_{\mathbf{v},1}, \ldots, T_{\mathbf{v},i}, T_{\mathbf{v},i+2}, \ldots, T_{\mathbf{v},l(\mathbf{v})}\}$ by $\{t_i(\mathbf{v}): i = 1, \ldots, m(\mathbf{v})\}$ where $m(\mathbf{v}) = l(\mathbf{v}) - 1$.

(5) If none of the $T_{\mathbf{v},i}$, $i = 1, \ldots, l(\mathbf{v})$, is a one-sided or two-sided singular triangle relative to $\mathbf{v}$, we let $t_i(\mathbf{v}) = T_{\mathbf{v},i}$, $i = 1, \ldots, m(\mathbf{v})$, where $m(\mathbf{v}) = l(\mathbf{v})$.

Let $\mathcal{T} = \{t_i(\mathbf{v}): \mathbf{v} \in V, i = 1, \ldots, m(\mathbf{v})\}$. Note that some of triangles in $\triangle$ are accounted more than once in $\mathcal{T}$. Furthermore, set

$$I_e = \begin{cases} C_3 & \text{if } e \text{ is an interior edge,} \\ C_3^* & \text{if } e \text{ is a boundary edge,} \end{cases}$$

where

$$C_3^* = \{(l, m+n) \in \mathbf{Z}_+^2 : m = l(r) + 1 - l,\ r+1 \le l \le r+1+\lfloor r/2 \rfloor,\ 0 \le n \le \lfloor r/2 \rfloor\}.$$

Also, set

$$I_{\mathbf{v},i} = \begin{cases} C_2 & \text{if } t_i(\mathbf{v}) \text{ is neither a one-sided nor two-sided singular triangle relative to } \mathbf{v}; \\ C_2 \cup \bar{C}_4 & \text{if } t_i(\mathbf{v}) \text{ is a one-sided singular triangle relative to } \mathbf{v}; \\ C_2 \cup C_4 \cup \bar{C}_4 & \text{if } \mathbf{v} \text{ is a singular or near-singular vertex and } t_i(\mathbf{v}) = T_{\mathbf{v},1}, \text{ or} \\ & \text{if } \mathbf{v} \text{ is not a singular nor near-singular vertex but } t_i(\mathbf{v}) \text{ is a two-sided singular triangle relative to } \mathbf{v}; \\ C_4 \cup \bar{C}_4 & \text{if } \mathbf{v} \text{ is a singular vertex and } t_i(\mathbf{v}) = T_{\mathbf{v},3}; \\ \bar{C}_4 & \text{if } \mathbf{v} \text{ is a near-singular vertex and } t_i(\mathbf{v}) = T_{\mathbf{v},4}. \end{cases}$$

In the following we outline the procedure for constructing the vertex splines in $\hat{S}_{3r+2}^r$. In general, we consider three types of vertex splines of interest. They are required to satisfy the following specifications of interpolation parameters:

(I) For any $\mathbf{v} \in \mathcal{V}$ and $\gamma \in C_1$, let $V_{\mathbf{v}}^{\gamma}$ be a piecewise polynomial function satisfying:

(I.1) $$D^{\alpha} V_{\mathbf{v}}^{\gamma}(\mathbf{u}) = \delta_{\alpha,\gamma} \delta_{\mathbf{v},\mathbf{u}}, \qquad \alpha \in C_1, \quad \mathbf{u} \in \mathcal{V};$$

(I.2) $$D_{t_j(\mathbf{u})}^{\alpha} V_{\mathbf{v}}^{\alpha}|_{t_j(\mathbf{u})}(\mathbf{u}) = 0, \qquad \alpha \in I_{\mathbf{u},j}, \quad t_j(\mathbf{u}) \in \mathcal{T};$$

(I.3) $$D_e^{\alpha} V_{\mathbf{v}}^{\alpha}|_{\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3} \rangle}(\mathbf{x}_{e,1}) = 0, \qquad \alpha \in I_e, \quad e \in \mathcal{E};$$

(I.4) $$V_{\mathbf{v}}^{\gamma} \in C^r(\mathbf{R}^2).$$

Here and throughout, as usual, the symbol $\delta_{\alpha,\gamma}$ or $\delta_{\mathbf{v},\mathbf{u}}$ denotes the Kronecker delta.

(II) For $t_i(\mathbf{v}) \in \mathcal{T}$ and for $\gamma \in I_{\mathbf{v},i}$, let $V^{\gamma}_{t_i(\mathbf{v})}$ be a piecewise polynomial of total degree $3r+2$ satisfying the following interpolation and smoothness conditions:

(II.1) $$D^{\alpha} V^{\gamma}_{t_i(\mathbf{v})}(\mathbf{u}) = 0, \qquad \alpha \in C_1, \quad \mathbf{u} \in \mathcal{V};$$

(II.2) $$D^{\alpha}_{t_j(\mathbf{u})} V^{\gamma}_{t_i(\mathbf{v})}|_{t_i(\mathbf{u})}(\mathbf{u}) = \delta_{\alpha,\gamma}\delta_{t_j(\mathbf{u}),t_i(\mathbf{v})}, \qquad \alpha \in I_{\mathbf{u},j}, \quad t_j(\mathbf{u}) \in \mathcal{T};$$

(II.3) $$D^{\alpha}_{d} V^{\gamma}_{t_i(\mathbf{v})}|_{\langle \mathbf{x}_{d,1},\mathbf{x}_{d,2},\mathbf{x}_{d,3}\rangle}(\mathbf{x}_{d,1}) = 0, \qquad \alpha \in I_d, \quad d \in \mathcal{E};$$

(II.4) $$V^{\gamma}_{t_i(\mathbf{v})} \in C^r(\mathbf{R}^2).$$

(III) For $e \in \mathcal{E}$ and $\gamma \in I_e$, let V^{γ}_e be a piecewise polynomial function satisfying the following interpolation and smoothness conditions:

(III.1) $$D^{\alpha} V^{\gamma}_e(\mathbf{u}) = 0, \qquad \alpha \in C_1, \quad \mathbf{u} \in \mathcal{V};$$

(III.2) $$D^{\alpha}_{t_j(\mathbf{u})} V^{\gamma}_e|_{t_j(\mathbf{u})}(\mathbf{u}) = 0, \qquad \alpha \in I_{\mathbf{u},j}, \quad t_j(\mathbf{u}) \in \mathcal{T};$$

(III.3) $$D^{\alpha}_{d} V^{\gamma}_e|_{\langle \mathbf{x}_{d,1},\mathbf{x}_{d,2},\mathbf{x}_{d,3}\rangle}(\mathbf{x}_{d,1}) = \delta_{e,d}\delta_{\alpha,\gamma}, \qquad \alpha \in I_d, \quad d \in \mathcal{E};$$

(III.4) $$V^{\gamma}_e \in C^r(\mathbf{R}^2).$$

The construction procedure of these vertex splines can be described as follows. Let V be one of the above vertex splines corresponding to a vertex $\mathbf{v}$ and let $\delta = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\rangle$ be an arbitrary triangle in Δ.

(1) Determination of Bézier Nets with Indices in Part I

The Bézier coefficients of $V|_{\delta}$ indexed in $A^{3r+2}_i C_1$ are simply zero when V is required to satisfy $D^{\alpha}V(\mathbf{x}_i) = 0$. When V is required to satisfy the interpolation conditions $D^{\alpha}V(\mathbf{x}_i) = \delta_{\alpha,\gamma}$, we first convert the partial derivatives D^{α} at $\mathbf{x}_i$ into derivatives relative to the triangle δ at $\mathbf{x}_i$, and then use the values $D^{\beta}_{\delta(\mathbf{x}_i)} V|_{\delta}(\mathbf{x}_i)$ to determine the Bézier coefficients of $V|_{\delta}$ with underlying indices in $A^{3r+2}_i C_1$.

(2) Determination of Bézier Nets with Indices in Part II

Case 1. Assume that δ is not one-sided singular nor two-sided singular at $\mathbf{x}_i$. Then we directly apply (I.2), (II.2), or (III.2) to obtain the portion of the Bézier net of $V|_{\delta}$ indexed in $A^{3r+2}_i C_2$.

Case 2. Assume that $[\mathbf{x}_i, \mathbf{x}_k]$ is singular or near-singular at $\mathbf{x}_i$ but $[\mathbf{x}_i, \mathbf{x}_j]$ is not, where $\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k\}$ is a rearrangement of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in the counterclockwise orientation, or assume that $\mathbf{x}_i$ is a singular or near-singular vertex such that $\delta \neq T_{\mathbf{x}_i,1}$. We obtain the portion of the Bézier net of $V|_{\delta}$ with indices in $A^{3r+2}_i C_2$ by using the smoothness conditions, Lemma 1, or Lemma 5 from the corresponding part of the Bézier coefficients of $V|_{\delta'}$, where δ' is the neighboring triangle of δ with $[\mathbf{x}_i, \mathbf{x}_k]$ as the common edge.

Case 3. Assume that δ is two-sided singular at $\mathbf{x}_i$, or $\mathbf{x}_i$ is a singular vertex and $\delta = T_{\mathbf{x}_i,1}$, or assume that $\mathbf{x}_i$ is a near-singular vertex and $\delta = T_{\mathbf{x}_i,1}$. In this case, we directly apply (I.2), (II.2), or (III.2) to obtain the portion of the Bézier net of $V|_\delta$ with indices in $A_i^{3r+2}C_2$.

(3) Determination of Bézier Nets with Indices in Part III and IV

Case 1. Assume that $[\mathbf{x}_i, \mathbf{x}_j]$ is a boundary edge. Then the Bézier coefficients of $V|_\delta$ with indices in the one-third portion of parts III and IV closest to $[\mathbf{x}_i, \mathbf{x}_j]$ (as shown in Fig. 5 for the case $r = 5$ and $d = 17$ and edge $[\mathbf{x}_1, \mathbf{x}_3]$ on the triangle $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$) are obtained by applying the specifications in (I.3) or (II.3) or (III.3).

Case 2. Assume that the edge $[\mathbf{x}_i, \mathbf{x}_k]$ is singular or near-singular at $\mathbf{x}_i$ but $[\mathbf{x}_i, \mathbf{x}_j]$ is not, where $\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k\}$ is a rearrangement of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in the counterclockwise orientation, or assume that $\mathbf{x}_i$ is a singular or near-singular vertex such that $\delta \neq T_{\mathbf{x}_i,1}$. Then we determine the one-half portion of the Bézier coefficients of $V|_\delta$ with indices in $A_i^{3r+2}C_4 \cup A_i^{3r+2}\bar{C}_4$ closest to $[\mathbf{x}_i, \mathbf{x}_k]$(e.g., $a_{(8,3,6)}$, $a_{(8,2,7)}$, $a_{(7,4,6)}$ for the case $n = 5$ and $d = 17$ in Fig. 5) by using the smoothness conditions, Lemma 1, or Lemma 4 from the corresponding portion of the Bézier coefficients of $V|_{\delta'}$, where δ' is the neighboring triangle of δ with $[\mathbf{x}_i, \mathbf{x}_k]$ as the common edge. The other half-portion is determined in Case 5.

Case 3. Assume that $[\mathbf{x}_i, \mathbf{x}_j]$ is singular or near-singular at $\mathbf{x}_i$ but $[\mathbf{x}_i, \mathbf{x}_k]$ is not, where $\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k\}$ is a rearrangement of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in the counterclockwise orientation, or assume that $\mathbf{x}_i$ is a singular or near-singular vertex such that $\delta \neq T_{\mathbf{x}_i,1}$.

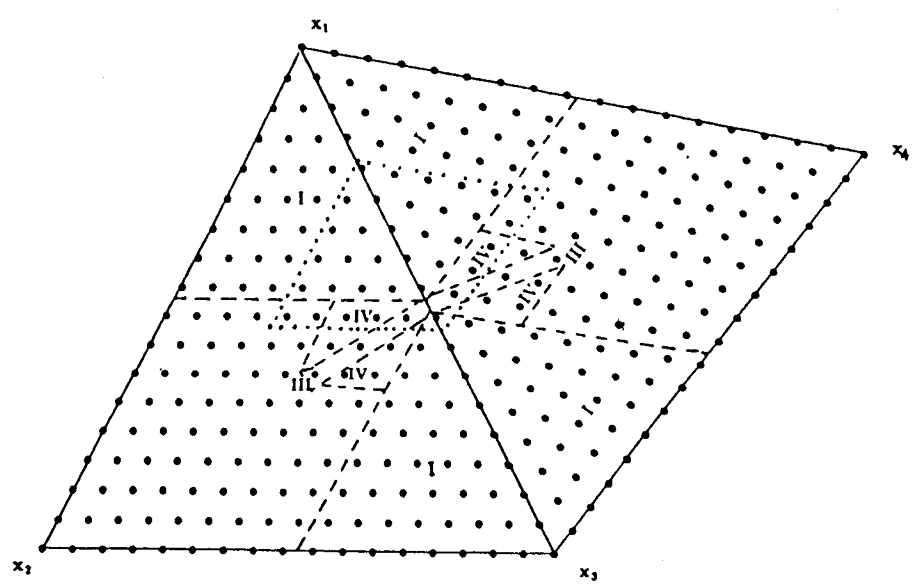

Fig. 5

Then we may directly apply the specifications (I.2), (II.2), or (III.2) to obtain the one-half portion of the Bézier coefficients of $V|_\delta$ with indices in $A_i^{3r+2}C_4 \cup A_i^{3r+2}\bar{C}_4$ closest to $[\mathbf{x}_i, \mathbf{x}_j]$. The other one-half portion is again determined in Case 5.

Case 4. Assume that δ is two-sided singular at $\mathbf{x}_i$, or $\mathbf{x}_i$ is a singular vertex and $\delta = T_{\mathbf{x}_i,1}$, or assume that $\mathbf{x}_i$ is a near-singular vertex and $\delta = T_{\mathbf{x}_i,1}$. In this case, we may directly apply (I.2), (II.2), or (III.2) to obtain the portion of the Bézier coefficients of $V|_\delta$ with indices in $A_i^{3r+2}C_r \cup A_i^{3r+2}\bar{C}_4$.

Case 5. This is the remaining case. To determine the remaining Bézier coefficients of $V|_\delta$ with indices in parts III and IV, we need to use all of (I.3) and (I.4), or (II.3) and (II.4), or (III.3) and (III.4), and apply Lemma 3 or Lemma 4. Let us illustrate with the following example. Consider $r = 5$, $d = 17$, and consider the edge $e = [\mathbf{x}_1, \mathbf{x}_3]$ and the requirements in (I.3). We only discuss the case where e is not a singular or near-singular edge at either $\mathbf{x}_1$ or $\mathbf{x}_3$. Let δ' be the triangle of $\triangle$ with e as the common edge of δ and δ' (see Fig. 5). Then the Bézier coefficients a_β of $V|_\delta$ and b_β of $V|_{\delta'}$, where

$$\beta \in \{(8,1,8), (8,2,7), (7,2,8), (8,3,6), (7,3,7),$$
$$(6,3,8), (7,4,6), (6,4,7), (6,5,6)\},$$

are to be determined. Since the Bézier coefficients of $V|_\delta$ and $V|_{\delta'}$ in part I have already been determined, we may first apply one of the requirements in (I.3) to obtain $a_{(8,1,8)}$ or $b_{(8,1,8)}$ depending on whether δ or δ' is $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3}\rangle$. Without loss of generality, let us assume that $\delta = \langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3}\rangle$. Hence, $a_{(8,1,8)}$ is determined by applying one of the requirements in (I.3) and $b_{(8,1,8)}$ is obtained by applying Lemma 1 and using the corresponding Bézier coefficients a'_βs. Then we may apply Lemma 3 with $a_{(8,0,4)+\beta}$, $b_{(8,0,4)+\beta}$, $|\beta| = 5$ and $l = 1$ (cf. the Bézier coefficients inside the dotted quadrilateral indicated in Fig. 5), to obtain $a_{(8,2,7)}$, $a_{(8,3,6)}$ and $b_{(8,2,7)}$, $b_{(8,3,6)}$. Also, $a_{(7,2,8)}$, $a_{(6,3,8)}$ and $b_{(7,2,8)}$, $b_{(6,3,8)}$ are obtained in a similar manner. Next we again use the requirements in (I.3) to obtain $a_{(7,3,7)}$, and then $b_{(7,3,7)}$ by applying Lemma 1 and using the corresponding Bézier coefficients of $V|_\delta$. By applying Lemma 3 again with $a_{(7,0,5)+\beta}$, $b_{(7,0,5)+\beta}$, $|\beta| = 5$ and $l = 0$, we may now determine $a_{(7,4,6)}$ and $b_{(7,4,6)}$. Similarly, $a_{(6,4,7)}$ and $b_{(6,4,7)}$ are obtained by using Lemma 3. Finally, $a_{(6,5,6)}$ is obtained by using (I.3) once more, and hence, $b_{(6,5,6)}$ is determined by applying Lemma 1 and using the corresponding Bézier net of $V|_\delta$. Of course, when $[\mathbf{x}_1, \mathbf{x}_3]$ is a singular or near-singular edge at $\mathbf{x}_1$ or $\mathbf{x}_3$, we have to modify the above procedure accordingly by using Lemma 4 instead of Lemma 3. This method is valid for any $r \geq 1$ in general.

From their specifications and the above construction steps, we know that the support of the vertex spline $V_\mathbf{v}^\gamma$ is the union of all triangles of $\triangle$ with $\mathbf{v}$ as the common vertex, and the support of V_e^γ is the union of all triangles with the common edge e. However, the support of $V_{t_i(\mathbf{v})}^\gamma$ is a little bit more complicated. It can be described as follows. Let $t_i(\mathbf{v}) = T_{\mathbf{v},j}$. Then the support of $V_{t_i(\mathbf{v})}^\gamma$ is given by

$$S_{\mathbf{v},i} := \begin{cases} \bigcup_{k=1}^{4} T_{\mathbf{v},k} & \text{if } \mathbf{v} \text{ is a singular or near-singular vertex and } t_i(\mathbf{v}) = T_{\mathbf{v},1}, \\ \bigcup_{k=2}^{4} T_{\mathbf{v},k} & \text{if } \mathbf{v} \text{ is a singular vertex and } t_i(\mathbf{v}) = T_{\mathbf{v},3} \text{ or if } \mathbf{v} \text{ is a near-singular vertex and } t_i(\mathbf{v}) = T_{\mathbf{v},4}, \\ T_{\mathbf{v},j-1} \cup T_{\mathbf{v},j} \cup T_{\mathbf{v},j+1} & \text{if } t_i(\mathbf{v}) \text{ is neither a one-sided nor a two-sided singular triangle relative to } \mathbf{v}, \\ \bigcup_{k=j-2}^{j+2} T_{\mathbf{v},k} & \text{if } t_i(\mathbf{v}) \text{ is a two-sided singular triangle relative to } \mathbf{v}, \\ \bigcup_{k=j-1}^{j+2} T_{\mathbf{v},k} & \text{if } t_i(\mathbf{v}) \text{ is a one-sided singular triangle with singular edge } \langle \mathbf{v}, \mathbf{x}_{\mathbf{v},i+1} \rangle. \end{cases}$$

From the construction procedure, we may see that with the exception of the one supported on the union of triangles with a near-singular vertex as the common vertex, all vertex splines are bounded by the constant

$$\text{(3.1)} \qquad b := \text{maximum of the ratios of the areas of any two adjacent triangles } \triangle \text{ sharing a common edge.}$$

For $r = 1, 2, 3$, we may use Lemmas 6-8 to ensure that those vertex splines which are supported on the union of the triangles sharing a near-singular vertex are also bounded by b. By applying Lemma 6, we may also see that any vertex spline whose support is the union of the triangles sharing a near-singular vertex attached to the edges with exactly three distinct slopes is bounded by b. But those vertex splines which are supported on the union of all triangles sharing a near-singular vertex attached to four edges with distinct slopes have to be dependent on the constant

$$\text{(3.2)} \qquad \eta := \min\left\{ \frac{|\langle \mathbf{x}_{\mathbf{v},1}, \mathbf{v}, \mathbf{x}_{\mathbf{v},3} \rangle|}{|T_{\mathbf{v},i}|}, \frac{|\langle \mathbf{X}_{\mathbf{v},2}, \mathbf{v}, \mathbf{x}_{\mathbf{v},4} \rangle|}{|T_{\mathbf{v},i}|} \right\}$$

which measures the near-singularity of $\triangle$. Here the minimum is taken over all near-singular vertices $\mathbf{v}$ associated with four edges with distinct slopes.

As an example, vertex splines in $S_8^{2,3}$ are constructed in [CL4] where their graphs are also shown. Examples of vertex splines in $S_5^{1,2}$ were already given in [C] and [CL2] and their graphs on various supports were also shown in [CL3].

4. Main Results

The main objective of this paper is to construct an approximation formula from the super-spline subspaces $\hat{S}_d^r$ of S_d^r, where $d \geq 3r+2$. In the following we only consider the special and most important case where $d = 3r+2$. The discussion for $d > 3r+2$ is similar. For $d = 3r+2$, we consider the linear operator L defined by

$$\text{(4.1)} \qquad Lf = \sum_{\mathbf{v} \in \mathcal{V}} \sum_{\gamma \in C_1} D^\gamma f(\mathbf{v}) V_{\mathbf{v}}^\gamma + \sum_{e \in \mathcal{E}} \sum_{\gamma \in I_e} D_e^\gamma f(\mathbf{x}_{e,1}) V_e^\gamma + \sum_{t_i(\mathbf{v}) \in \mathcal{T}} \sum_{\gamma \in I_{\mathbf{v},i}} D_{t_i(\mathbf{v})}^\gamma f(\mathbf{v}) V_{t_i(\mathbf{v})}^\gamma$$

for all sufficiently smooth functions f. We are now in a position to establish the main results in this paper. The following propositions are of independent interest.

Proposition 1. *$Lp = p$ for any polynomial p of total degree $\le d = 3r+2$.*

Proof. This result is proved by induction on the number of triangles in $\triangle$. If $\triangle$ consists of a single triangle δ, then L is an interpolation operator based on δ. Since the interpolation conditions associated with each vertex of δ induce a partition of Λ_{3r+2}, we see that $Lp = p$ for all $p \in \pi_{3r+2}$ by Lemma 9. Assume that the proposition holds for $m = \#\{\delta\colon \delta \in \triangle\}$. Let $\#\{\delta\colon \delta \in \triangle\} = m+1$ and set $\triangle = \{\delta_i\colon i = 1, \ldots, m+1\}$. By relabeling, if necessary, assume that $\delta_{m+1} = \langle \mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3 \rangle$ has at least one boundary edge, and for the time being, assume that it has only one interior edge $\langle \mathbf{y}^1, \mathbf{y}^2 \rangle$, say. Let $\triangle' = \{\delta_i\colon i = 1, \ldots, m\} = \triangle \backslash \delta_{m+1}$. Observing the uniqueness in Lemma 3 and applying Lemma 2, we see that the smoothness conditions of Lp across the edge $\langle \mathbf{y}^1, \mathbf{y}^2 \rangle$ can be rewritten as appropriate interpolatory matching conditions (directional derivatives interpolating p at $\mathbf{y}^1$ and $\mathbf{y}^2$) such that $L_\triangle p|_{\triangle'} = L_{\triangle'} p$, where $L_\triangle$, $L_{\triangle'}$ denote the linear operators based on $\triangle$, $\triangle'$ respectively. By the induction hypothesis, we have $L_\triangle p|_{\triangle'} = L_{\triangle'} p = p$ on $\triangle'$. Again from Lemmas 2 and 3 and the fact that $Lp = p$ on $\triangle'$, we see that $Lp|_{\delta_{m+1}}$ interpolates p, since the smoothness conditions across $\langle \mathbf{y}^1, \mathbf{y}^2 \rangle$ can be rewritten as interpolatory matching conditions and the totality of these resulting interpolatio conditions induces a partition of Λ_{3r+2}. Hence, $Lp = p$ on $\triangle$. The proof is similar if δ_{m+1} contains two interior edges. This completes the proof of the proposition. ∎

The above result can in fact be improved. To do so, the directional derivatives in the definition of L must be interpreted properly. We interpret Lf as

$$Lf = \sum_{\mathbf{v} \in \mathscr{V}} \sum_{\gamma \in C_1} D^\gamma f(\mathbf{v}) V_{\mathbf{v}}^\gamma + \sum_{e \in \mathscr{E}} \sum_{\gamma \in I_e} D_e^\gamma f|_{\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3} \rangle}(\mathbf{x}_{e,1}) V_e^\gamma$$
$$+ \sum_{t_i(\mathbf{v}) \in \mathscr{T}} \sum_{\gamma \in I_{\mathbf{v},i}} D^\gamma_{t_i(\mathbf{v})} f|_{t_i(\mathbf{v})}(\mathbf{v}) V^\gamma_{t_i}(\mathbf{v}).$$

Then we have the following result.

Proposition 2. *$Lf = f$ for any function $f \in \hat{S}^r_{3r+2}$.*

Proof. Let $f_1 = Lf - f$. Then $f_1 \in \hat{S}^r_{3r+2}$ and f_1 satisfies

$$D^\alpha f_1(\mathbf{v}) = 0, \qquad \alpha \in C_1, \quad \mathbf{v} \in \mathscr{V};$$
$$D_e^\alpha f_1|_{\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3} \rangle}(\mathbf{x}^{e,1}) = 0, \qquad \alpha \in I_e, \quad e \in \mathscr{E};$$
$$D^\gamma_{t_i(\mathbf{v})} f_1|_{t_i(\mathbf{v})}(\mathbf{v}) = 0, \qquad \alpha \in I_{\mathbf{v},i}, \quad t_i(\mathbf{v}) \in \mathscr{T}.$$

By using the argument in the proof of Proposition 1, we conclude that $f_1 \equiv 0$ on $\triangle$. ∎

As a consequence, we have the following result.

Theorem 1. *The collection*

$$\mathcal{B} := \{V_{\mathbf{v}}^{\gamma}: \mathbf{v} \in \mathcal{V}, \gamma \in C_1\} \cup \{V_e^{\gamma}: e \in \mathcal{E}, \gamma \in I_e\} \cup \{V_{t_i(\mathbf{v})}^{\gamma}: t_i(\mathbf{v}) \in \mathcal{T}, \gamma \in I_{\mathbf{v},i}\}$$

is a basis of $\hat{S}_{3r+2}^r$.

Proof. It is clear that $\mathcal{B} \subset \hat{S}_{3r+2}^r$ and that $\mathcal{B}$ is a linearly independent set. By Proposition 2, it also spans $\hat{S}_{3r+2}^r$ and is therefore a basis. ■

Corollary. *Let* $r \geq 1$. *Then*

$$\dim \hat{S}_{3r+2}^r = \binom{r + \lfloor (r+1)/2 \rfloor + 2}{2} \Delta_{\mathrm{v}} + 3\binom{\lfloor r/2 \rfloor + 1}{2} \Delta_{\mathrm{t}}$$
$$+ \binom{[r/2]+1}{2} \Delta_{\mathrm{s}} + (\lfloor r/2 \rfloor + 1)\Delta_{\mathrm{e_i}} + (\lfloor r/2 \rfloor + 1)^2 \Delta_{\mathrm{e_b}},$$

where Δ_{v}, Δ_{t}, Δ_{s}, $\Delta_{\mathrm{e_i}}$, *and* $\Delta_{\mathrm{e_b}}$ *denote the numbers of vertices, triangles, singular vertices, interior edges, and boundary edges of* Δ, *respectively.*

Proof. This result is a simple consequence of Theorem 4.1 by determining the cardinality of the basis $\mathcal{B}$. ■

Consider a domain G in $\bigcup\{\delta: \delta \in \Delta\}$. For $f \in C^k(G)$, denote

$$\|D^k f\| = \max_{|\alpha|=k} \|D^\alpha f\|_{L^\infty(G)}$$

and

$$\operatorname{dist}(f, \mathcal{S}) = \inf_{s \in \mathcal{S}} \|f - s\|.$$

For the given Δ, let $|\Delta|$ denote the maximum of the diameters of the triangles in Δ. We consider two situations in terms of the order of smoothness r and the partition Δ. In the first situation, we consider $r \leq 4$ and an arbitrary triangulation Δ, or $r \geq 4$ and Δ which does not contain any near-singular vertex associated with four distinct slopes. The remaining situation is considered next. Let b and η be as defined in (3.1) and (3.2), respectively. One of the main results in this paper is the following.

Theorem 2. *Let* $d \geq 3r+2$. *There exists a linear operator* L *with range* $\hat{S}_d^r$ *such that*

$$\|Lf - f\| \leq C\|D^{d+1}f\| \, |\Delta|^{d+1} \tag{4.2}$$

for all sufficiently smooth functions f, *where* C *is a constant independent of* f *and* $|\Delta|$. *Consequently,*

$$\operatorname{dist}(f, S_d^{r,r+l}) \leq C\|D^{d+1}f\| \, |\Delta|^{d+1} \tag{4.3}$$

for $r \leq l \leq \rho(r, d)$. *In particular, for* $d = 3r+2$, L *can be chosen to be the operator defined in* (4.1).

Remark. It should be emphasized that the constant C must depend on the geometry of the triangulation Δ. In [dBH2], the constant C in (4.3) depends on the smallest angle of triangles in Δ. Here, as a consequence of the construction process, it only depends on b which is the largest ratio of the areas of any two neighboring triangles Δ in the first situation, and, in addition, also depends on the measurement η of the near-singularity of Δ in the remaining situation. The estimate (4.3) can be established by using the argument in [dBH2]. However, our result given in (4.2) is more constructive.

Proof. For $d \geq 4r+1$, this theorem is well known. In the following, we only consider $d=3r+2$, since a similar argument yields the desired result for $3r+2<d<4r+1$. Fix a point $\mathbf{x} \in G$ and consider the linear functional

$$F(f)=Lf(\mathbf{x})-f(\mathbf{x}),$$

where L is given in (4.1). It is easy to see that F satisfies the following:

(i) $|F(f)| \leq K_1 \sum_{j=0}^{3r+2} \|D^j f\| |\Delta|^j$ and
(ii) $F(p)=0$ for all $p \in \pi_{3r+2}$.

Indeed, (ii) follows from Proposition 1. As for (i), if $|\Delta|=1$, it is easy to see that $|F(f)| \leq K_i \sum_{j=0}^{3r+2} \|D^j f\|$ from the construction of the vertex splines in Section 3, where the constant K_1 is dependent only on b in the first situation by Lemmas 6-8, and, in addition, is dependent on $(\eta)^{-1}$ in the remaining situation as discussed in the previous section. If $|\Delta|<1$, by letting $\tilde{f}(\mathbf{y})=f(|\Delta|\mathbf{y})$, we see that

$$\begin{aligned}|F(f)| &= |F(\tilde{f})| \\ &\leq K_1 \sum_{j=0}^{3r+2} \|D^\alpha \tilde{f}\|_{\tilde{G}} \\ &= K_1 \sum_{j=0}^{3r+2} \|D^\alpha f\| |\Delta|^j.\end{aligned}$$

By Lemma 10, there exists a constant C independent of $\mathbf{x}$ and f such that

$$|Lf(\mathbf{x})-f(\mathbf{x})| \leq C \|D^{3r+3}\| |\Delta|^{3r+3}.$$

This completes the proof of (4.2) for $d=3r+2$. Consequently, for $r \leq l \leq \rho(r,d)$

$$\operatorname{dist}(f, S_{3r+2}^{r,l}) \leq \operatorname{dist}(f, S_{3r+2}^{r,\rho(r)}) \leq C \|D^{3r+2} f\| |\Delta|^{3r+3}$$

which yields (4.3) for $d=3r+2$. ∎

We conclude our discussion on super splines by comparing the dimensions of the spline space with its super-spline subspace. In general the super-spline subspace $S_d^{r,l}$ is a proper subspace of the spline space S_d^r, $r<l\leq \rho(r,d)$. In fact, the following example gives an exact comparison for $l=\rho(r,d)$.

Example. Comparing the above dimension formula with that of S_{3r+2}^r in [H] which coincides with the lower bound given in [S1], we note that their difference is

$$\dim S^r_{3r+2} - \dim \hat{S}^r_{3r+2} = 2\binom{\lfloor (r+1)/2 \rfloor + 2}{2}\Delta_{e_i} - \left((r+1)\left\lfloor \frac{r+1}{2} \right\rfloor + 2\binom{\lfloor (r+1)/2 \rfloor + 2}{2}\right)\Delta_{v_i},$$

where Δ_{v_i} denotes the number of interior vertices of Δ and we have assumed for simplicity that Δ does not contain a singular vertex. Since $\Delta_{e_i} \geq 3\Delta_{v_i}$ and $\lfloor (r+1)/2 \rfloor \geq r/2$, we have

$$\dim S^r_{3r+2} - \dim \hat{S}^r_{3r+2} \geq \frac{1}{2}\left\lfloor \frac{r+1}{2} \right\rfloor \left(3 + \left\lfloor \frac{r+1}{2} \right\rfloor\right)\Delta_{v_i}$$

which is positive when Δ contains at least one interior vertex.

Remark. For further information on the dimensions of other super-spline spaces, see [CH] and [S2].

Acknowledgments. This research was supported by NSF under Grant No. DMS 8602337 and SDIO/IST managed by the U.S. Army Research Office under Contract No. DAAL03-87-K-0025. We would like to thank Carl de Boor for his many valuable comments that helped improve the presentation of this paper.

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