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CONSTRUCTION OF REAL-TIME SPLINE QUASI-INTERPOLATION SCHEMES

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Abstract

Spline quasi-interpolants of the form

$$Q(f,x)=\sum_i f(t_i)M_i'(x)$$

are constructed, where $M_i'(x)$ are locally supported spline functions with arbitrary knot sequences nodes which may be different from the nodes. Our results show that they approximate sufficiently smooth functions with the optimal order of accuracy. It is important to point out that the structure of these spline quasi-interpolants have computational advantages and provide a useful and efficient tool in realtime applications.

1. Introduction

As is well known, one of the important features of spline quasiinterpolation is obtaining the optimal approximation order. For instance, de Boor and Fix in [2] using both functional and derivative values gave a B-spline quasi-interpolant. Later, a B-spline quasi-interpolant using only functional values in a divided difference scheme was given by Lyche and Schumaker in [4]. As to the generalized splines, some quasi-interpolants have also been constructed by Scherer and Schumaker [6] and Jia [5]. The reader is referred to de Boor [1] and Schumaker [8] for more details.

In this paper, we will give a new spline quasi-interpolation scheme which takes on the following expression:

$$Q(f,\ x)=\sum_i f(t_i)M_i'(x),$$

where $\{t_i\}$ is a strictly increasing sequence of real numbers which may be different from the knot sequence, and $M_i'(x)$ is a certain linear combination of the mth order B-splines or Hermitian splines. Our new approximation scheme has the following important features:

(1) Only functional values are used, and no divided difference or any equivalent procedure is involved. From the numerical point of view, this structure is sometimes advantageous.

(2) All basic functions $M_i^s(x)$, which are merely linear combinations of B-splines or Hermitian splines, can be uniquely determined and stored in the computer hardware before data processing, so that when a new sampling data $f(t_i)$ comes in, only a new term $f(t_i)M_i^s(x)$ is to be added into the previous summation $\sum_{i=-\infty}^{j-1} f(t_i)M_i^s(x)$. This makes the new structure very useful in real-time applications.

(3) It approximates smooth function f with the optimal order of accuracy. Moreover, since the sampling points $\{t_i\}$ are in some sense independent of the spline knots, it is even possible to choose some suitable $\{t_i\}$ so that the optimal approximation order maintains for those x values that locate beyond the spline knots. This ensures the new structure be very efficient for real-time predictions.

2. Construction of Real-Time Quasi-Interpolation Scheme: Using B-Splines

In this section we will construct a quasi-interpolant with the expression in the following form:

$$Q(f,x):=\sum_i f(t_i)M_i^s(x), \tag{2.1}$$

where $M_i^s(x)$ consists of (S + 1) B-splines of order m, which reproduces all polynomials of degree $s(0 \leq s < m)$. An important feature of this quasi-interpolant is that all basic fuctions $M_i^s(x)$, which will be called *molecules* later, can be computed and stored in the computer hardware before data processing, so that when a new data $f(t_j)$ comes in, we need only adding a term $f(t_i)M_i^s(x)$ to the summation. Hence, this quasi-interpolation scheme is very suitable for real-time applications. Besides, as has been mentioned above, it achieves the best approximation order.

To describe our quasi-interpolant (2.1) more precisely, we first need the following notation.

Let $\triangle:=\{y_i\}$ and $T:=\{t_i\}$ be respectively a nondecreasing and a strictly increasing sequences of real numbers, namely:

$$\cdots \leq y_{-1} \leq y_0 \leq y_1 \leq y_2 \leq \cdots, \tag{2.2}$$

with $y_{i+m}-y_i>0$ for a fixed positive integer m and all integers i, and

$$\cdots < t_{-1} < t_0 < t_1 < t_2 < \cdots. \tag{2.3}$$

As usual, denote the mth order B-spline associated with knots $\{y_i,\cdots,y_{i+m}\}$ by

$$B_i^m(x)=(-1)^m(y_{i+m}-y_i)[y_i,\ \cdots,\ y_{i+m}](x-\cdot)_+^{m-1},\ x\in R^1,$$

where $[y_i, \cdots, y_{i+m}]$ is the mth-order divided-difference operator. Set

$$S_m(\triangle) = \{\sum_i c_i B_i^m(x):\ c_i \in R^1\}.$$

Moreover, denote by $D(t_i, \cdots, t_{i+s})$ the Vandermonde determinant of $t_i, \cdots, t_{i+s}$; that is,

$$D(t_i, \cdots, t_{i+s}) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_i & t_{i+1} & \cdots & t_{i+s} \\ \vdots & \vdots & & \vdots \\ t_i^s & t_{i+1}^s & \cdots & t_{i+s}^s \end{pmatrix}.$$

Let $D(t_i, \cdots, t_{i+k-1}, \xi_k, t_{i+k+1}, \cdots, t_{i+s})$ be obtained from $D(t_i, \cdots, t_{i+s})$ by replacing its $(k+1)st$ column with the vector $\xi_k := [\xi^0(k,m), \cdots, \xi^s(k,m)]^T$, where

$$\begin{cases} \xi^0(k,m) = 1, \\ \xi^j(k,m) = \sigma^j(y_{k+1}, \cdots, y_{k+m-1}) \Big/ \binom{m-1}{j}, \ j = 1,2,\cdots,s, \end{cases} \tag{2.4}$$

and $\sigma^j(r_1, \cdots, r_n)$ is the classical symmetric function defined by

$$\begin{cases} \sigma^0(r_1, \cdots r_n) = 1, \\ \sigma^j(r_1, \cdots, r_n) = \sum\limits_{1 \le l_1 < l_2 < \cdots < l_j \le n} r_{l_1} r_{l_2} \cdots r_{l_j}, \quad j = 1,2,\cdots,s. \end{cases}$$

Definition **2.1.** The function $M_i^s(x)$ in $S_m(\triangle)$ defined by

$$M_i^s(x) \sum_{\mu=0}^{s} a_{i\mu} B_{i+\mu}^m(x), \tag{2.5}$$

where $0 \le s < m$, and

$$a_{i\mu} = \frac{D(t_{i+\mu-s}, \cdots, t_{i-1}, \xi_{i+\mu}, t_{i+1}, \cdots, t_{i+\mu})}{D(t_{i+\mu-s}, \cdots, t_{i+\mu})} \tag{2.6}$$

is called an mth-order *molecule* of *B*-splines.

Note that in the definiton of the molecule $M_i^s(x)$, the partition $T = \{t_i\}$ is independent of the knots $\{y_i\}$.

We are now ready to prove the following result.

Theorem 2.1. For any given strictly increasing sequence $T = \{t_i\}$ of real numbers, there exists a unique quasi-interpolant $Q(f,x)$ of the form

$$Q(f,x) = \sum_i f(t_i) M_i^s(x), \quad M_i^s(x) \in S_m(\triangle), \tag{2.7}$$

such that

$$Q(p,x) = p(x), \quad x \in R^1, \tag{(2.8)}$$

for all polynomials p of degree s with $0 \le s < m$.

Proof. Let $Q(f,x)$ be defined by (2.7) with $M_i^s(x)$ given by (2.5), where allthe constants $a_{i\mu}, \mu = 0,1,\cdots,s$, are to be determined. We will achieve this by requiring $Q(f,x)$ to satisfy the condition (2.8). More precisely, we will determine all $a_{i\mu}$ by solving the equations

$$Q(x^j, x) = x^j, \ j = 0,1,\cdots,s, 0 \le s < m,$$

for all integers i. In doing so, let s be fixed. By the Marsden identity (cf.

[6]), we first have

$$x^j = \sum_i \xi^j(i,m) B_i^m(x), \; j = 0,1,\cdots,s, \; x \in R^1,$$

where $\xi^j(i,m)$ are defined as in (2.4). On the other hand, we have

$$\begin{aligned} Q(f,x) &= \sum_i f(t_i) \sum_{\nu=0}^{s} a_{i\nu} B_{i+\nu}^m(x) \\ &= \sum_j \left(\sum_{\substack{i+\nu=j \\ \nu=0,\cdots,s}} a_{i\nu} f(t_i) \right) B_j^m(x) \\ &= \sum_j \left(\sum_{\nu=0}^{s} a_{j-\nu,\nu} f(t_{j-\nu}) \right) B_j^m(x). \end{aligned}$$

Hence, it follows from the conditions $Q(x^j,x) = x^j$ that

$$\sum_{\nu=0}^{s} a_{i-\nu,\nu} t_{i-\nu}^j = \xi^j(i,m), \; j = 0,1,\cdots, s.$$

For each fixed integer i, the constants $a_{i-\nu,\nu}, \nu = 0,1,\cdots,s$, are uniquely determ ined by the Cramer rule as shown in (2.6). This completes the proof of the theorem.

Our next goal is to show that the quasi-interpolant $Q(f,x)$ defined in (2.7) is a bounded linear operator, To do so, it is sufficient to show that for each integer i, all coefficients $a_{i\mu}, \mu = 0,1,\cdots,s$, are bounded. We first need the following:

Lemma 2.1. Let $\{y_i\}$ and $\{t_i\}$ be two sequences of nondecreasing and strictly increasing real numbers as defined in (2.2) and (2.3), respectively. Let m and s be two integers such that $0 \leq s \leq m-1$. Then we haue

$$\frac{\sigma^s(y_1,\cdots,y_{m-1})}{\binom{m-1}{s}} - \sigma^1(t_0,\cdots,t_{s-1}) \frac{\sigma^{s-1}(y_1,\cdots,y_{m-1})}{\binom{m-1}{s-1}} + \cdots + (-1)^s \sigma^s(t^0,\cdots,t_{s-1})$$
$$= \frac{(m-s-1)!}{(m-1)} \left\{ \sum_{\substack{1 \leq i_0,\cdots,i_{s-1} \leq m=1 \\ i_l \neq i_k \text{ if } l \neq k}} \prod_{\nu=0}^{s-1} (y_{i_\nu} - t_\nu) \right\} \tag{2.9}$$

Proof. It can be directly verified that these identities hold for $m \leq 3$ and $s \leq 1$. Assume that we have the identities for $m \leq n-1$ and $s \leq k-1$. We need to show that the identity also holds for both cases of $m = n$, $s = k-1$, and $m = n-1$, $s = k$. For the case of $m = n$ and $s = k-1$, we have

$$\frac{(n-k)!}{(n-1)!} \left\{ \sum_{1 \leq i_0,\cdots,i_{k-2} \leq n-1} \prod_{v=0}^{k-2} (y_{i_\nu} - t_\nu) \right\}$$

$$= \frac{(n-k)!}{(n-1)!} \left\{ \sum_{1 \leq i_0,\cdots,i_{k-2} \leq n-2} \prod_{v=0}^{k-2} (y_{i_\nu} - t_\nu) \right.$$

$$+\sum_{\substack{1\le i_1,\cdots,i_{k-2}\le n-2\\ i_0=n-1}}\prod_{\nu=0}^{k-2}(y_{i_\nu}-t_\nu)+$$

$$+\sum_{\substack{1\le i_0,i_2,\cdots,i_{k-2}\le n-2\\ i_1=\mu-1}}\prod_{\nu=0}^{k-2}(y_{i_\nu}-t_\nu)+\cdots$$

$$\left.+\sum_{\substack{1\le i_0,\cdots,i_{k-3}\le n-2\\ i_{k-2}=n-1}}\prod_{\nu=0}^{k-2}(y_{i_\nu}-t_\nu)\right\}$$

$$=\frac{(n-k)!}{(n-1)!}\left\{\sum_{1\le i_0,\cdots,i_{k-2}\le n-2}\prod_{\nu=0}^{k-2}(y_{i_\nu}-t_\nu)\right.$$

$$+(y_{n-1}-t_0)\sum_{1\le i_1,\cdots,i_{k-2}\le n-2}\prod_{\nu=1}^{k-2}(y_{i_\nu}-t_\nu)+\cdots$$

$$(y_{n-1}-1)\sum_{1\le i_0,i_2,\cdots,i_{k-2}\le n-2}\prod_{\substack{\mu=0\\ \nu\ne 1}}^{k-2}(y_{i_\nu}-t_\nu)+\cdots$$

$$\left.+(y_{n-1}-t_{k-2})\sum_{1\le i_0,\cdots,i_{k-3}\le n-2}\prod_{\nu=0}^{k-3}(y_{i_\nu}-t_\nu)\right\}$$

$$=\frac{n-k}{n-1}\left\{\frac{\sigma^{k-1}(y_1,\cdots,y_{n-2})}{\binom{n-2}{k-1}}-\sigma^1(t_0,\cdots,t_{k-2})\frac{\sigma^{k-2}(y_1,\cdots,y_{n-2})}{\binom{n-2}{k-2}}+\cdots+\right.$$

$$\left.(-1)^{k-1}\sigma^{k-1}(t_0,\cdots,t_{k-2})\right\}+$$

$$\frac{1}{n-1}(y_n-t_0)\left\{\frac{\sigma^{k-2}(y_1,\cdots,y_{n-2})}{\binom{n-2}{k-2}}-\sigma^1(t_1,\cdots,t_{k-2})\frac{\sigma^{k-3}(y_1,\cdots,y_{n-2})}{\binom{n-2}{k-3}}+\cdots+\right.$$

$$\left.(-1)^{k-2}\sigma^{k-2}(t_1,\cdots,t_{k-2})\right\}+$$

$$\frac{1}{n-1}(y_{n-1}-t_1)\left\{\frac{\sigma^{k-2}(y_1,\cdots,y_{n-2})}{\binom{n-2}{k-2}}-\sigma^1(t_0,t_2,\cdots t_{k-2})\frac{\sigma^{k-3}(y_1,\cdots,y_{n-2})}{\binom{n-2}{k-3}}+\cdots+\right.$$

$$\left.(-1)^{k-2}\sigma^{k-2}(t_0,t_2,\cdots,t_{k-2})\right\}+\cdots\cdots+$$

$$\frac{1}{n-1}(y_{n-1}-t_{k-2})\left\{\frac{\sigma^{k-2}(y_1,\cdots,y_{n-2})}{\binom{n-2}{k-2}}-\sigma^1(t_0,\cdots,t_{k-3})\frac{\sigma^{k-3}(y_1,\cdots,y_{n-2})}{\binom{n-2}{k-3}}+\cdots+\right.$$

$$\left.(-1)^{k-2}\sigma^{k-2}(t_0,\cdots,t_{k-3})\right\}$$

$$=\left\{\frac{n-k}{n-1}\frac{1}{\binom{n-2}{k-1}}\sigma^{k-1}(y_1,\cdots,y_{n-2})+\frac{k-1}{n-1}\frac{1}{\binom{n-2}{k-2}}y_{n-1}\sigma^{k-2}(y_1,\cdots,y_{n-2})\right\}-$$

$$\left\{\frac{n-k}{n-1}\frac{1}{\binom{n-2}{k-2}}\sigma^1(t_0,\cdots,t_{k-2})\sigma^{k-2}(y_1,\cdots,y_{n-2})+\frac{1}{n-1}\frac{1}{\binom{n-2}{k-2}}(t_0+t_1+\cdots+\right.$$

$$t_{k-2})\sigma^{k-2}(y_1,\cdots,y_{n-2})+$$

$$\frac{1}{n-1}\frac{1}{\binom{n-2}{k-3}}y_{n-1}[\sigma^1(t_1,\cdots,t_{k-2})+\sigma^1(t_0,t_2,\cdots,t_{k-2})+\cdots+$$

$$\left.\sigma^1(t_0,\cdots,t_{k-3})]\sigma^{k-3}(y_1,\cdots,y_{n-2})\right\}+\cdots+$$

$$\left\{\frac{n-k}{n-1}(-1)^{k-1}\sigma^{k-1}(t_0,\cdots,t_{k-2})+\frac{1}{n-1}(-1)^{k-1}[t_0\sigma^{k-2}(t_1,\cdots,t_{k-2})+\right.$$

$$\left.t_1\sigma^{k-2}(t_0,t_2,\cdots t_{k-2})+\cdots+t_{k-2}\sigma^{k-2}(t_0,\cdots,t_{k-3})]\right\}$$

$$=\frac{\sigma^{k-1}(y_1,\cdots,y_{n-1})}{\binom{n-1}{k-1}}-\sigma^1(t_0,\cdots,t_{k-2})\frac{\sigma^{k-2}(y_1,\cdots,y_{n-1})}{\binom{n-1}{k-2}}+\cdots+$$

$$(-1)^{k-1}\sigma^{k-1}(t_0,\cdots,t_{k-2}).$$

For the case of $m=n-1$ and $s=k$, the proof is similar, This completes the proof of the mathematical induction for the identity (2.9).

Theorem 2.2. Let $\{y_i\}$ and $\{t_i\}$ be two sequences of nondecreasing and strictly increasing real numbers as defined in (2.2) and (2.3), respectively. Suppose that

$$\inf_i\{|t_{i+1}-t_i|\}=\delta<0 \tag{2.10}$$

and

$$\sup_i\{|y_{i+m-1}-t_i|,|y_{i-m+1}-t_i|\}=b<\infty. \tag{2.11}$$

Then, the quasi-interpolant defined in (2.7) is a bounded linear operator with

$$|a_{i,\nu}|\le\frac{(m-s-1)!}{(m-1)!}\left(\frac{b}{\delta}\right)^s \tag{2.12}$$

for all i and $\nu=0,1,\cdots,s$.

Proof. If $s=0$, then we have $a_{i,\nu}=1$ for all i and $\nu(=s=0)$. (2.12) holds. Without loss of generality, let us assume that $i=0$ and $\nu=s>0$. Then, since

$$\det\begin{pmatrix}1 & 1 & \cdots & 1 & 1\\ t_0 & t_1 & \cdots & t_{s-1} & r\\ \vdots & \vdots & & \vdots & \vdots\\ \vdots & \vdots & & \vdots & \vdots\\ t_0^s & t_1^s & \cdots & t_{s-1}^s & r^s\end{pmatrix}$$

$$= \prod_{0\le l<i\le s-1} (t_i - t_l) \prod_{j=0}^{s-1} (r - t_j)$$

$$= \prod_{0\le l<i\le s-1} (t_i - t_l)[r^s - \sigma^1(t_0,\cdots,t_{s-1})r^{s-1} + \cdots + (-1)^s\sigma^s(t_0,\cdots,t_{s-1})],$$

it follows from (2.6) that

$$a_{0\nu} = \frac{\prod_{0\le l<i\le s-1} (t_i-t_l)[\xi^s(0,m)-\sigma^1(t_0,\cdots,t_{s-1})\xi^{s-1}(0,m)+\cdots+(-1)^s\sigma^s(t_0,\cdots,t_{s-1})]}{\prod_{0\le l<i\le s}(t_i-t_l)}$$

$$= \prod_{j=0}^{s-1}(t_s-t_j)^{-1}\Big[\frac{\sigma^s(y_1,\cdots,y_{m-1})}{\binom{m-1}{s}} - \sigma^1(t_0,\cdots,t_{s-1})\frac{\sigma^{s-1}(y_1,\cdots,y_{m-1})}{\binom{m-1}{s-1}} + \cdots + (-s)^s\sigma^s(t_0,\cdots,t_{s-1})\Big]$$

$$= \prod_{j=0}^{s-1}(t_s-t_j)^{-1}\frac{(m-s-1)!}{(m-1)!}\Big\{\sum_{\substack{1\le i_0,\cdots,i_{s-1}\le m-1\\ i_l\ne i_v \text{ if } l\ne v}}\prod_{k=0}^{s-1}(y_{i_k}-t_k)\Big\},$$

where the last equality is obtained from Lemma 2.1. Hence, it follows that

$$\left|a_{i\nu}\right| \le \frac{(m-s-1)!}{(m-s)!}\left(\frac{b}{\delta}\right)^s$$

for all i and $\nu=0,1,\cdots,s$. This complets the proof of theorem.

We are now in the position to prove the following result.

Theorem 2.3. Suppose that conditions (2.10) and (2.11) are satisfied. Let

$$h=\sup_i\{|y_{i+1}-y_i|,|t_{i+1}-t_i|,|y_{i+m-1}-t_i|,|y_{i-m+1}-t_i|\}.$$

Then, for $f\in C^{s+1}(R)\cup L_\infty(R)$, we have

$$|Q(f,x)-f(x)|\le C\left(\frac{b}{\delta}\right)^s h^{s+1}\|f^{(s+1)}\|_\infty, \tag{2.15}$$

where C is a constant independent of b,δ,h, and f.

Proof. Let $x\in[y_{i_0},y_{i_0+1}]$ for some integer i_0. Define

$$\alpha=\min\{t_{i_0-m},y_{i_0}\} \quad\text{and}\quad \beta=\max\{t_{i_0},y_{i_0+1}\}.$$

The Taylor expansion of $f(y)$ at α can be written as

$$f(y)=p_s(y,\alpha)+\int_\alpha^\beta\frac{f^{(s+1)}(r)}{s!}(y-r)_+^s dr,$$

where p_s is a polynomial of degree s. It follows that

$$|f(x)-Q(f,x)|$$

$$\le|p_s(x,\alpha)-\sum_{i=i_0-m}^{i_0}p_s(t_i,\alpha)M_i^s(x)|$$

$$+\left|\int_a^\beta \frac{f^{(s+1)}(\mu)}{s!}(x-\mu)_+^s d\mu - \sum_{i=i_0-m}^{i_0}\int_a^\beta \frac{f^{(s+1)}(\mu)}{s!}(t_i-\mu)_+^s M_i^s(x)d\mu\right|$$

$$=\left|\int_a^\beta \frac{f^{(s+1)}(\mu)}{s!}\left[(s-\mu)_+^s - \sum_{i=i_0-m}^{i_0}(t_i-\mu)_+^s \sum_{\nu=0}^{s} a_{i\nu}B_{i+\nu}^m(\mu)\right]d\mu\right|$$

$$\leq C\left(\frac{b}{\delta}\right)^s h^{s+1}\|f^{(s+1)}\|_\infty,$$

where by Theorem 2.2 the constant C is independent of b, δ, h, and f. This completes the proof of the theorem.

If we denote by $\omega_{s+1}(f,h)$ the $(s+1)$st modulus of smoothness for the function f, then we also have the following:

Corollary 2.1. *Let* $f\in C(\mathrm{R})\cup L_\infty(\mathrm{R})$. *Then*

$$|f(x)-Q(f,x)|\leq \widetilde{C}\omega_{s+1}(f,h),$$

where $\widetilde{C}$ is a constant.

This can be easily verifed by using the K-functional technique.

We close this section with two examples.

Example 2.1. Let $s=1$, $m>1$, and choose for all i

$$t_i=\frac{1}{m+1}(y_{i+1}+\cdots+y_{i+m-1}).$$

Then it follows from (2.6) that $a_{i0}=1$ and $a_{i1}=0$ for all i. Hence,

$$Q(f,x) = \sum_i f(t_i)B_i^m(x) = \sum_i f\left(\frac{y_{i+1}+\cdots+y_{i+m-1}}{m-1}\right)B_i^m(x),$$

which is precisely the well-known variation-dimishing spline.

Example 2.2. Let $y_i=t_{i+s}=ih$ with $h>0$. Then, from (2.6) we have

$$a_{i\nu} = \prod_{\substack{\mu=1;\ \mu\neq l\\ \mu\neq 0;\ l\neq 0}} (t_{i+l}-t_{i+\mu})\{\textstyle\sum_{j=0}^{s}(-1)^{s-j}\sigma^{s-j}(t_{i+\nu-s},\cdots,t_{i-1},t_{i+1},\cdots,t_{i+\nu})\cdot$$

$$\xi^j(i+\nu,i+\nu+m)\}\Big/\prod_{\nu\leq\mu<l\leq\nu}(t_{i+l}-t_{i+\nu})$$

$$=\prod_{l=1}^{\nu}(t_{i+l}-t_i)^{-1}\prod_{l=\nu-s}^{-1}(t_i-t_{i+l})^{-1}\frac{(m-s-1)!}{(m-1)!}\cdot$$

$$\left\{\sum_{\substack{i+\nu\leq i_0,\cdots,i_{\nu-1},i_{\nu+1},\cdots,i_s\leq i+\nu+m\\ i_l\neq i_k \text{ if } l\neq k}}\ \prod_{\substack{l=0\\ l\neq\nu}}^{s}(y_{i_l}-t_{i+\nu-l})\right\}$$

$$=\frac{1}{\nu!(s-\nu)!h^s}\frac{(m-s-1)!}{(m-1)!}\sum_{\substack{i+\nu\leq i_l\leq i+\nu+m\\ l=0,\cdots,\nu-1,\nu+1,\cdots,s}}\ \prod_{\substack{l=0\\ l\neq\nu}}^{s}(i_l-(i+\nu-l-s))h^s$$

$$=\frac{1}{\nu!(s-\nu)!\nu}\frac{(m-s-1)!}{(m-1)!}\sum_{\substack{0\leq i_l\leq m\\ l=0,\cdots,\nu-1,\nu+1,\cdots,s}}\ \prod_{\substack{l=0\\ l\neq\nu}}(i_l+l+s),$$

where $\nu=0,1\cdots,s$. Hence, $a_{i\nu}$ is a constant independent of h and i. Let

$c_\nu := v_{i\nu}, \nu = 0,1,\cdots,s$. Then, for each i,

$$M_i^s(x) = \sum_{\nu=0}^{s} a_{i\nu} B_{i+\nu}^m(x) = \sum_{\nu=0}^{s} c_\nu N^m\left(\frac{x}{h} - i - \nu\right),$$

where $N^m(x)$ is the equally spaced and normalized B-spline of order m. Furthermore, set

$$\widehat{M}^s(x) = M_i^s(h(x+i)) = \sum_{\nu=0}^{s} c_\nu N^m(x-\nu).$$

Then, we have

$$Q(f,x) = \sum_i f((i-s)h)\, \widehat{M}^s\left(\frac{x}{h} - i\right). \tag{2.14}$$

This implies that evaluating $Q(f,x)$ at a point $x \in [i_0 h, (i_0+1)h]$ only requires the previous $(m+s)$ bits of data, $\{f((i_0 - m - 2s + 1)h), \cdots, f((i_0 - s)h)\}$, and it will maintain the heighest approximation order given by (2.13). Hence, the quasi-interpolant (2.14) gives an efficient recursive scheme for real-time predictions.

3. Construction of Real-Time Quasi-Interpolation Scheme: Using Hermitian Splines

Let $\triangle := \{y_i\}$ and $T := \{t_i\}$ be two strictly increasing sequences of real numbers. Define the Hermitian splines of order $2n$ associated with knots $\{y_{i-1}, y_i, y_{i+1}\}$ by

$$h^j(x,i) = \begin{cases} \dfrac{(x-y_i)^j}{j!}\left(\dfrac{y_{i+1}-x}{y_{i+1}-y_i}\right)^n \displaystyle\sum_{k=0}^{n-1-j} \left\langle {n \atop k} \right\rangle \left(\dfrac{x-y_i}{y_{i+1}-y_i}\right)^k, & x \in [y_i, y_{i+1}], \\ \dfrac{(x-y_i)^j}{j!}\left(\dfrac{x-y_{i-1}}{y_i-y_{i-1}}\right)^n \displaystyle\sum_{k=0}^{n-1-j} \left\langle {n \atop k} \right\rangle \left(\dfrac{y_i-x}{y_i-y_{i-1}}\right)^k, & x \in [y_{i-1}, y_i], \\ 0, & x \notin [y_{i-1}, y_{i+1}], \end{cases} \tag{3.1}$$

for $j = 0,1,\cdots,n-1$, and all i, where

$$\left\langle {n \atop k} \right\rangle = \begin{cases} n(n+1)\cdots(n+k-1)/k! , & k = 1,2,\cdots,n-1, \\ 1 , & k = 0. \end{cases}$$

It can be easily verified that

$$\left(\frac{d}{dx}\right)^k h^j(x,i) \Big|_{x=y_l} = \delta_{il}\delta_{jk}, \quad k,j = 0,1,\cdots,n-1,$$

where δ_{ij} is the Kronecker delta. Set

$$\widetilde{S}_{2n}(\triangle) = \left\{ \sum_i \sum_{j=0}^{n-1} c_{ij} h^j(x,i) : c_{ij} \in R^1 \right\}.$$

Moreover, define

$$D_0(t_0,\cdots t_{s-1},r)=\det\begin{pmatrix}1 & 1 & \cdots & 1 & 1\\ t_0 & t_1 & & t_{s-1} & r\\ \vdots & \vdots & & \vdots & \vdots\\ t_0^s & t_1^s & \cdots & t_{s-1}^s & r^s\end{pmatrix}$$

and

$$D_k(t_0,\cdots,t_{s-1},r)=\left(\frac{d}{dr}\right)^k D_0(t_0,\cdots,t_{s-1},r), \qquad j=1,2\cdots.$$

Definition 3.1. The function $\widetilde{M}_i^s(x)$ in $\widetilde{S}_{2n}(\triangle)$ defined by

$$\widetilde{M}_i^s(x)=\sum_{\mu=0}^{s}\sum_{j=0}^{n-1} b_{i,j,\mu}h^j(x,i+\mu), \tag{3.2}$$

where $0\le s\le 2n-1$, and

$$b_{i,j,\nu}=(-1)^\nu\frac{D_j(t_{i+\nu-s},\cdots,t_{i-1},r,t_{i+1},\cdots,t_{i+\nu})|_{r=y_i}}{D_0(t_{i+\nu-s},\cdots,t_{i+\nu})} \tag{3.3}$$

is called a $(2n)$th order *molecule* of Hermitian splines.

Note, again, that in the definition of the molecule $\widetilde{M}_i^s(x)$, the partition $T=\{t_i\}$ is independent of the knots $\{y_i\}$.

We have the following result.

Theorem 3.1. For any given strictly increasing sequence $T=\{t_i\}$ of real numbers, there exists a unique quasi-interpolant $\widetilde{Q}(f,x)$ of the form

$$\widetilde{Q}(f,x)=\sum_i f(t_i)\ \widetilde{M}_i^s(x), \qquad \widetilde{M}_i^s(x)\in S_{2n}(\triangle), \tag{3.4}$$

such that

$$\widetilde{Q}(p,x)=p(x), \qquad x\in R^1, \tag{3.5}$$

for all palynomials p of degree s with $0\le s\le 2n-1$.

Proof. Let $\varphi_\mu(x)=x^\mu,\mu=0,1,\cdots,s$. To prove the theorem, it is equivalent to show that $\widetilde{Q}(f,x)$ can be uniquely determined in the form of (3.4) such that $\widetilde{Q}(\varphi_\mu,x)=\varphi_\mu,\mu=0,1,\cdots,s$.

On one hand, it is clear that

$$\varphi_\mu(x)=\sum_i\sum_{j=0}^{n-1}\left(\frac{d}{dx}\right)^j\varphi_\mu(x)\Big|_{x=y_i}h^j(x,i), \quad \mu=0,1,\cdots,s.$$

On the other hand, $\widetilde{Q}(f,x)$, defined in (3.4), can be formally rewritten as

$$\begin{aligned}\widetilde{Q}(f,x)&=\sum_i f(t_i)\sum_{\mu=0}^{s}\sum_{j=0}^{n-1} b_{i,j,\mu}h^j(x,i+\mu)\\ &=\sum_i\sum_{j=0}^{n-1}\Big(\sum_{\mu=0}^{s} f(t_{i-\mu})b_{i-\mu,j,\mu}\Big)h^j(x,i).\end{aligned}$$

Hence, the required equalities $\widetilde{Q}(\varphi_\mu,x)=\varphi_\mu$, $\mu=0,1,\cdots,s$, become

$$\sum_i\sum_{j=0}^{n-1}\left(\sum_{\nu=0}^{j}\varphi_\mu(t_{i-\nu})b_{i-\nu,j,\nu}\right)h^j(x,i)=\sum_i\sum_{j=0}^{n-1}\left(\frac{d}{dx}\right)^j\varphi_\mu(y_i)h^j(x,i),$$

$\mu=0,1,\cdots,s$. Since $\{h^j(x,i)\}$ are linearly independent, these yield

$$\sum_{\nu=0}^{j}\varphi_\mu(t_{i-\nu})b_{i-\nu,j,\nu}=\left(\frac{d}{dx}\right)^j\varphi_\mu(y_i),\quad \mu=0,1,\cdots,s,$$

for all i and j.

Now, observe that $D_0(t_{i-s},t_{i-s+1},\cdots,t_i)$ is a Vandermonde determinant which is nonzero for the distinct points $t_{i-s},t_{i-s+1},\cdots,t_i$. We can solve the above system of linear equations and obtain

$$b_{i-\nu,j,\nu}=\frac{D_j(t_{i-s},\cdots,t_{i-\nu-1},r,t_{i-\nu+1},\cdots,t_i)|_{r=y_i}}{D_0(t_{i-s},t_{i-s+1},\cdots,t_i)},\quad \mu=0,1,\cdots,s,$$

which are uniquely determined by the given strictly increasing sequence $T=\{t_i\}$ of real numbers. This completes the proof of the theorem.

Next, we will show that the quasi-interpolant $\widetilde{Q}(f,x)$ defined by (3.4) is a bounded linear operator from $C^{s+1}(R)\cup L_\infty(R)$ to $S_{2n}(\triangle)$. To do so, we first need the following lemmas.

Lemma 3.1. For $k=0,1,\cdots,n-1$, and for all i and j,

$$\left|\left(\frac{d}{dx}\right)^k h^j(x,i)\right|\le C\max\{|y_{i+1}-y_i|^{j-k},\ |y_i-y_{i-1}|^{j-k}\},\quad (3.6)$$

where C is a constant.

This result can be easily verified by using (3.1). We also have the following:

Lemma 3.2. Suppose that the given strictly increasing sequences $\triangle:=\{y_i\}$ and $T:=\{t_i\}$ satisfy the following conditions

$$\inf_i\{|t_{i+1}-t_i|\}\ge\delta>0 \qquad (3.7)$$

and

$$\sup_i\{|y_{i+1}-y_i|,\ |y_i-t_{i-s}|,\ |y_i-t_{i+s}|\}\le b<\infty, \qquad (3.8)$$

where $0\le s\le 2n-1$. Then for $\nu=0,1,\cdots,s$, and for all i,

$$|b_{i,j,\nu}|\begin{cases}\le C\dfrac{b^{s-j}}{\delta^s}, & j=0,1,\cdots,s,\\ =0, & j=s+1,\cdots,\cdots n-1,\end{cases}$$

where C is constant independent of δ and b.

Proof. Since

$$D_0(t_{i+\nu-s},\cdots,t_{i+\nu})=\prod_{\nu-s\le k<l\le\nu}(t_{i+l}-t_{i+k})$$

and

$$D_0(t_{i+\nu-s},\cdots,t_{i-1},r,t_{i+1},\cdots t_{i+\nu})$$

$$= \prod_{\substack{\nu-s\le n<l\le \nu\\ n\ne i,l\ne i}} (t_{i+l}-t_{i+n}) \prod_{l=\nu-s}^{-1}(r-t_{i+l}) \prod_{l=1}^{\nu}(t_{i+l}-r),$$

we have

$$b_{i,j,\nu}=\left(\frac{d}{dr}\right)^j\left\{\prod_{l=\nu-s}^{-1}\frac{(r-t_{i+l})}{(t_i-t_{i+l})}\prod_{l=1}^{\nu}\frac{(t_{i+l}-r)}{(t_{i+l}-t_i)}\right\}\Bigg|_{r=y_i}$$

$$\begin{cases} \le C\delta^{-s} \max\limits_{\substack{\nu-s\le l\le \nu\\ l\ne 0}} \{|y_i-t_{i+l}|^{s-j}\}, & j=0,1,\cdots,s,\\ =0, & j=s+1,\cdots,n-1.\end{cases}$$

This yields the result and completes the proof of the lemma.

We are now in the position to prove the following result.

Theorem 3.2. Let $\triangle: =\{y_i\}$ aed $T: =\{t_i\}$ be two strictly increasing sequences satisfying conditions (3.7) and (3.8), and let

$$h=\sup_i\{|t_i-y_{i+\nu+1}|,|t_{i+\nu+1}-t_i|,|y_{i+1}-y_i|\}.$$

Then, for $f\in C^{s+1}(R)\cup L_\infty(R)$, we have

$$|\widetilde{Q}(f,x)-f(x)|\le C\left(\frac{b}{\delta}\right)^s h^{s+1}\|f^{(s+1)}\|_\infty, \tag{3.9}$$

where C is a constant independent of b,δ,h, and f.

Proof. Fix $x\in[y_{i_0},y_{i_0+1}]$. Let $\alpha=\min\{t_{i_0-s},y_{i_0}\}$ and $\beta=\max\{t_{i_0+1},y_{i_0+1}\}$. Since $f\in C^{s+1}(R)\cup L_\infty(R)$, we have

$$f(y)=p_s(y,\alpha)+\int_\alpha^\beta\frac{f^{(s+1)}(r)}{s!}(y-r)_+^s dr,\quad y\in(\alpha,\beta),$$

where p_s is a polynomial of degree s. It follows that

$$|f(x)-\widetilde{Q}(f,x)|$$

$$\le|p_s(x,\alpha)-\sum_i p_s(t_i,\alpha)\widetilde{M}_i^s(x)|$$

$$+\left|\int_\alpha^\beta\frac{f^{(s+1)}(r)}{s!}(x-r)_+^s dr-\sum_{i=i_0-s}^{i_0+1}\int_\alpha^\beta\frac{f^{(s+1)}(r)}{s!}(t_i-r)_+^s\widetilde{M}_i^s(x)dr\right|$$

$$=\left|\int_\alpha^\beta\frac{f^{(s+1)}(r)}{s!}\left[(x-r)_+^s-\sum_{i=i_0-s}^{i_0+1}\sum_{j=0}^{n-1}\sum_{\nu=0}^{s}b_{i,j,\nu}(t_i-r)_+^j h^j(x,i)\right]dr\right|$$

$$\le C\left(\frac{b}{\delta}\right)^s h^{s+1}\|f^{(s+1)}\|_\infty$$

by Lemmas 3.1 and 3.2. This completes the proof of the theorem.

4. Construction of Real-Time Quasi-Interpolation Schemes: Tensor-Product Cases

Let R^r be the r-dimensional Euclidean space and

$$N^r=\{i:i=(i_1,\cdots,i_r),i_l \text{ an integer}, l=1,\cdots,r\},$$

the set of all multi-integers of R^r. Denote by N_+^r the subset of N^r in which all integers $i_l, l=1,\cdots,r$, are nonnegative. By convention, $i=(i_1,\cdots,i_r)\le j=(j_1,\cdots,j_r)$ if and only if $i_l\le j_l$ for all $l, l=1,\cdots,r$. For $k\in N_+^r, L_k=\{i\in N_+^r: i\le k\}$ is called the *lower set* of k. Further, let

$$\Delta^r=\{y^i:y^i=(y_1^{i_1},\cdots,y_r^{i_r})\in R^r, i\in N^r\}$$

be a partition of R^r such that $y_l^{i_l}<y_l^{i_l+1}$ for all i, and $l=1,\cdots,r$. Define the mth order tensor-product molecule of B-splines by

$$M_i^s(x)=\prod_{l=1}^{r} M_{i_l}^{s_l}(x_l), \tag{4.1}$$

where $i=(i_1,\cdots,i_r)\in N^r$, $x=(x_1,\cdots,x_r)\in R^r$, $s=(s_1,\cdots,s_r)\in L_{m-1}$, and

$$M_{i_l}^{s_l}(x)=\sum_{\nu_l=0}^{s_l} a_{l_l\mu_l} B_{i_l+\nu_l}^{m}(x_l;y_l^{i_l},\cdots,y_l^{i_l+m_l}),$$

with $m=(m_1,\cdots,m_r)\in N_+^r\backslash\{0\}$, and $B_{i_l+\nu_l}^{m}(x_l,y_l^{i_l},\cdots,y_l^{i_l+m_l})$ being the normalized m_lth order B-spline associated with knots $\{y_l^{i_l},\cdots,y_l^{i_l+m_l}\}$, $l=1,\cdots,r$. Similary, define the 2nth order tensor-product molecule of Hermitian splines by

$$\widetilde{M}_i^s(x)=\prod_{l=1}^{r}\widetilde{M}_{i_l}^{s_l}(x_l),$$

where $i=(i_1,\cdots,i_r)\in N_r$, $x=(x_1,\cdots,x_r)\in R^r$, $s=(s_1,\cdots,s_r)\in L_{n-1}$, and

$$\widetilde{M}_{i_l}^{s_l}(x_l)=\sum_{\nu=0}^{s_l}\sum_{j=0}^{n_l-1} b_{l_l,j_l,\nu_l} h^{j_l}(x_l,i_l+\nu_l)$$

with $n=(n_1,\cdots,n_r)\in N_+^r\backslash\{0\}$, and $h^{j_l}(x_l,i_l+\nu_\nu)$ being the Hermitian spline of order $2n_l$ associated with knots $\{y_l^{i_l+\nu_l-1},y_l^{i_l+\nu_l},y_l^{i_l+\nu_l+1}\}$ defined by (3.1), and $j_l\le n_l-1, l=1,\cdots,r$. Furthermore, let

$$T=\{t^i:t^i=(t_1^{i_1},\cdots,t_r^{i_r})\in R^r,\ i\in N^r\}$$

be a set of real numbers such that $t_l^{i_l}<t_l^{i_l+1}$ for all i_l and $l=1,\cdots,r$, and suppose that for $l=1,\cdots,r$,

$$\inf_{i_l}\{|t_l^{i_l+1}-t_l^{i_l}|\}\ge\delta_l>0,$$

$$\sup_{i_l}\{|y_l^{i_l+m_l-1}-t_l^{i_l}|,|y_l^{i_l+m_l+1}-t_{i_l}|\}\le b_l<\infty,$$

and

$$\sup_{i}\{|y_l^{i_l+1}-y_l^{i_l}|,|y_l^{i_l}-t_l^{i_l-1}|,|y_l^{i_l}-t_l^{i_l+1}|\}\le\sigma_l<\infty.$$

Then, we have the following results.

Theorem 4.1. Let $f\in L_\infty(R^r)$. Then there exists a unique tensor-produot quasi-interpolant $Q(f,x)$ of the form

$$Q(f,x)=\sum_i f(t^i)M_i^s(x)$$

satisfying

$$Q(p,x)=p(x)$$

for all polynomials $p(x)\in\{\sum_{0\le j\le s} a_j x^j : a_j=a_{j_1\cdots j_r}, x^j=x_1^{j_1}\cdots x_r^{j_r}\}$, where the tensor-product molecules $M_i^s(x)$ are obtained using (2.5)-(2.6). Furthermore, if $f\in C^{s+1}(R^r)\cup L_\infty(R^r)$, then

$$|Q(f,x)-f(x)|\le C\frac{b^s}{\delta^s}\sum_{j\in L_{s+1}\setminus L_s}\{\|D^jf\|_{L_\infty(R^r)}h^j\},$$

$b^s=b_1^{s_1}\cdots b_r^{s_r}$, $\delta^s=\delta_1^{s_1}\cdots\delta_r^{s_r}$, $h^j=h_1^{j_1}\cdots h_r^{j_r}$, *with*

$$h_l^{j_l}=\sup_i\{|y_l^{i_l+1}-y_l^{i_l}|,|t_l^{i_l+1}-t_l^{i_l}|,|y_l^{i_l+m_l-1}-t_l^{i_l}|,|y_l^{i_l+m_l+1}-t_l^{i_l}|\},$$

$l=1,\cdots,r$, and C is a constant independent of b,δ,h, and f.

Theorem 4.2. Let $f\in L_\infty(R^r)$. Then there exists a unique tensor-product quasi-interpolant $\widetilde{Q}(f,x)$ of the form

$$\widetilde{Q}(f,x)=\sum_i f(t^i)\widetilde{M}_i^s(x)$$

satisfying

$$\widetilde{Q}(p,x)=p(x)$$

for all polynomials $p(x)\in\{\sum_{0\le j\le s} a_j x^j : a_j=a_{j_1\cdots j_r}, x^j=x_1^{j_1}\cdots x_r^{j_r}\}$, wheret the tensor-product molecules $\widetilde{M}_i^s(x)$ are obtainedbyusing (3.2)-(3.3). Furthermore, if $f\in C^{s+1}(R^r)\cup L_\infty(R^r)$, then

$$|\widetilde{Q}(f,x)-f(x)|\le C\frac{\sigma^s}{\delta^s}\sum_{j\in L_{s+1}\setminus L_s}\{\|D^jf\|_{L_\infty(R^r)}h^j\}$$

where $\sigma^s=\sigma_1^{s_1}\cdots\sigma_r^{s_r}$, $\delta^s=\delta_1^{s_1}\cdots\delta_r^{s_r}$, $h^j=h_1^{j_1}\cdots h_r^{j_r}$, with

$$h_l^{j_l}=\sup_i\{|t_l^{i_l}-y_l^{i_l+s_l+1}|,|t_l^{i_l+s_l+1}-t_l^{i_l}|,|y_l^{i_l+1}-y_l^{i_l}|\}$$

$l=1,\cdots,r$,and C is a constant independent of σ,δ,h, and f.

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