Construction of Multivariate Compactly Supported Orthonormal Wavelets

Ming-Jun Lai *
Department of Mathematics
The University of Georgia
Athens, GA 30602

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Dedicated to Professor Charles A. Micchelli
on the occasion of his 60th birthday

Abstract

We propose a constructive method to find compactly supported orthonormal wavelets for any given compactly supported scaling function ϕ in the multivariate setting. For simplicity, we start with a standard dilation matrix $2I_{2\times 2}$ in the bivariate setting and show how to construct compactly supported functions $\psi_1, \cdots, \psi_n$ with $n > 3$ such that $\{2^k\psi_j(2^kx-\ell, 2^ky-m), k, \ell, m \in Z, j = 1, \ldots, n\}$ is an orthonormal basis for $L_2(\mathbf{R}^2)$. Here, n is dependent on the size of the support of ϕ. With parallel processes in modern computer, it is possible to use these orthonormal wavelets for applications. Furthermore, the constructive method can be extended to construct compactly supported multi-wavelets for any given compactly supported orthonormal multi-scaling vector. Finally, we mention that the constructions can be generalized to the multivariate setting.

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*mjlai@math.uga.edu

1 Introduction

Since the well-known construction of univariate compactly supported orthonormal wavelets in 1988 (cf. [Daubechies'92]), great efforts have been spent on constructing multivariate compactly supported orthonormal wavelets (cf. [J. Kovačević and M. Vetterli'92], [Cohen and Daubechies'93], [J. Kovačević and M. Vetterli'95], [Stanhill and Zeevi'96], [He and Lai'97], [Maass'97], [Stanhill and Zeevi'98], [Ayache'99], [Belogay and Wang'99]). Although many special multivariate nonseparable wavelets have been constructed, it is still an open problem how to construct multivariate compactly supported orthonormal wavelets for any given compactly supported scaling function. The purpose of this paper is to give this problem a partial answer.

Our construction of compactly supported orthonormal wavelets in the multivariate setting is based on a standard multi-resolution analysis (MRA). For simplicity, we first consider dilation matrix $2I_{2\times 2}$ in the bivariate setting. Let us assume that ϕ is a given scaling function which generates an MRA. That is, ϕ is refinable and orthonormal. In this paper, we assume that ϕ is compactly supported. Writing

$$\hat{\phi}(\xi,\eta) = m(\xi/2,\eta/2)\hat{\phi}(\xi/2,\eta/2),$$

$m(\xi,\eta)$ is a trigonometric polynomial in $e^{i\xi}$ and $e^{i\eta}$. Let

$$V_k = \mathrm{span}_{L_2}\{\phi(2^k x-\ell, 2^k y - m), \ell, m \in Z\}$$

for $k \in Z$. Let W_k be the orthogonal complement of V_k in V_{k+1}. That is,

$$V_{k+1} = V_k \oplus W_k.$$

We need to find compactly supported functions (so-called wavelets) $\psi_1,\cdots,\psi_n$ such that $V_1 = V_0 \oplus W_0$ with

$$W_0 = \mathrm{span}_{L_2}\{\psi_j(x-\ell, y-m), \ell, m \in Z, j = 1,\ldots,n\}$$

and

$$\int_{\mathbf{R^2}} \psi_j(x-\ell, y-m)\psi_k(x,y)dxdy = 0$$

for any $j \neq k, j,k = 1,\ldots,n$ with $\ell, m \in Z$ and for $j = k, k = 1,\cdots,n$ with $\ell^2+m^2 \neq 0, \ell, m \in Z$.

Usually, we expect $n = 3$ (cf. [Meyer'90]). This requires that we find trigonometric polynomials m_1, m_2, m_3 such that

$$\begin{pmatrix} m(\xi,\eta) & m(\xi+\pi,\eta) & m(\xi,\eta+\pi) & m(\xi+\pi,\eta+\pi) \\ m_1(\xi,\eta) & m_1(\xi+\pi,\eta) & m_1(\xi,\eta+\pi) & m_1(\xi+\pi,\eta+\pi) \\ m_2(\xi,\eta) & m_2(\xi+\pi,\eta) & m_2(\xi,\eta+\pi) & m_2(\xi+\pi,\eta+\pi) \\ m_3(\xi,\eta) & m_3(\xi+\pi,\eta) & m_3(\xi,\eta+\pi) & m_3(\xi+\pi,\eta+\pi) \end{pmatrix} \quad (1.1)$$

is unitary. Such unitary extension problem is still open in general.

In this paper, we propose to construct wavelets $\psi_j, j = 1, \ldots, n$ with $n > 3$. In the d-variate setting, for dilation matrix $2I_{d\times d}$, one expects to find $2^d - 1$ wavelets (cf. [Meyer'90]). Our approach will yield $n(> 2^d - 1)$ compactly supported orthonormal wavelets. Although n should be as small as possible in practice, a reasonable number n larger than 3 does not cause any serious technical and/or computational challenge since we can use parallel processes in modern computer. Indeed, it is easy to see that a wavelet decomposition and reconstruction procedure can be done in parallel. As we will point out later, the number n is dependent on the size of the support of ϕ. If the size of the support of ϕ is not very large, n will not be very large and hence we will be able to enjoy the advantages and properties that orthonormal wavelets possess.

One of our main ideas for the construction is to use multi-wavelets. To explain the idea, let us consider an MRA in the bivariate setting and use a standard dilation matrix $2I_{2\times 2}$. We further assume that

$$m(\xi, \eta) = \sum_{0\le j\le 5, 0\le k\le 5} c_{j,k} e^{i(j\xi + k\eta)} \tag{1.2}$$

just for simplicity.

For the given scaling function ϕ associated with mask $m(\xi, \eta)$, we let Φ be a multi-scaling vector

$$\Phi(x, y) = \begin{bmatrix} 2\phi(2x, 2y) \\ 2\phi(2x-1, 2y) \\ 2\phi(2x, 2y-1) \\ 2\phi(2x-1, 2y-1) \end{bmatrix}.$$

Since ϕ is orthonormal, so is $\Phi(x, y)$, i.e.,

$$\int_{\mathbf{R^2}} \Phi(x-\ell, y-m)\Phi(x, y)^T dxdy = I_{4\times 4}\delta_{0,\ell}\delta_{0,m}. \tag{1.3}$$

Writing $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$, we let

$$\tilde{V}_k = \text{span}_{L_2}\{\phi_j(2^k x - \ell, 2^k y - m), \ell, m \in Z, j = 1, 2, 3, 4\}. \tag{1.4}$$

Thus, $\tilde{V}_0 = V_1$. It is clear that $\Phi(x, y)$ is a refinable vector (detailed explanation will be given later). It follows that $\{\tilde{V}_k, k \in Z\}$ forms an MRA and hence Φ generates the same MRA of $L_2(\mathbf{R}^2)$ as that by ϕ.

Since $\Phi(x, y)$ is refinable, in terms of Fourier transform, we have

$$\hat{\Phi}(\xi, \eta) = \sum_{\ell,m} \frac{1}{4} M_{\ell,m} e^{i(\ell\xi + m\eta)/2} \hat{\Phi}(\xi/2, \eta/2)$$

$$= \quad M(\xi/2,\eta/2)\hat{\Phi}(\xi/2,\eta/2)$$

for a matrix symbol $M(\xi,\eta)$ with trigonometric polynomial entries in $e^{i\xi}$ and $e^{i\eta}$.

It follows from (1.3) that

$$\sum_{\ell,m\in Z}\hat{\Phi}(\xi+2\ell\pi,\eta+2m\pi)\hat{\Phi}(\xi+2\ell\pi,\eta+2m\pi)^{*}=I_{4\times 4}$$

and hence,

$$\begin{aligned} &M(\xi,\eta)M(\xi,\eta)^{*}+M(\xi+\pi,\eta)M(\xi+\pi,\eta)^{*}+M(\xi,\eta+\pi)M(\xi,\eta+\pi)^{*}+ \\ &M(\xi+\pi,\eta+\pi)M(\xi+\pi,\eta+\pi)^{*}=I_{4\times 4}. \end{aligned} \tag{1.5}$$

Let $\tilde{W}_k$ be the orthogonal complement of $\tilde{V}_k$ in $\tilde{V}_{k+1}$. We will construct three compactly supported orthonormal multi-wavelet vectors $\Psi_1,\Psi_2,\Psi_3\in\tilde{W}_0$ with $\Psi_j=(\Psi_{j1},\Psi_{j2},\Psi_{j3},\Psi_{j4})^T$, $j=1,2,3$ and

$$\tilde{W}_0=\mathrm{span}_{L_2}\{\Psi_{j,k}(x-\ell,y-m),\ell,m\in Z,j=1,2,3,k=1,\cdots,4\}$$

such that

$$\int_{\mathbf{R}^2}\Psi_j(x-\ell,y-m)\Psi_k(x,y)^T dxdy=\begin{cases} I_{4\times 4}, & \text{if } j=k \text{ and } \ell=m=0\\ 0, & \text{otherwise}\end{cases} \tag{1.6}$$

for all $j,k=1,2,3,\ell,m\in\mathbf{Z}$ and

$$\tilde{V}_1=\tilde{V}_0\oplus\tilde{W}_0.$$

Writing

$$\hat{\Psi}_j(\xi,\eta)=M_j(\xi/2,\eta/2)\hat{\Phi}(\xi/2,\eta/2), j=1,2,3, \tag{1.7}$$

we need to find matrices $M_j(\xi,\eta)$ with trigonometric polynomial entries in $(e^{i\xi},e^{i\eta})$ such that the following matrix

$$\begin{bmatrix} M(\xi,\eta) & M(\xi+\pi,\eta) & M(\xi,\eta+\pi) & M(\xi+\pi,\eta+\pi)\\ M_1(\xi,\eta) & M_1(\xi+\pi,\eta) & M_1(\xi,\eta+\pi) & M_1(\xi+\pi,\eta+\pi)\\ M_2(\xi,\eta) & M_2(\xi+\pi,\eta) & M_2(\xi,\eta+\pi) & M_2(\xi+\pi,\eta+\pi)\\ M_3(\xi,\eta) & M_3(\xi+\pi,\eta) & M_3(\xi,\eta+\pi) & M_3(\xi+\pi,\eta+\pi)\end{bmatrix} \text{ is unitary.} \tag{1.8}$$

By the properties (1.3), (1.4), (1.7), and (1.8) and the fact that Φ generates an MRA, we know that $\{\Psi_{jk}, j=1,2,3,k=1,2,3,4\}$ are compactly supported functions and the translates and dilates of them generate an orthonormal basis for $L_2(\mathbf{R}^2)$. Hence, they are compactly supported orthonormal wavelets associated with the given scaling function ϕ.

Therefore, we need to show how to do (1.8) which is given in §2. We then consider the scaling function which has a larger support. The construction of the associated wavelets is similar and will be outlined there. Next we discuss how to construct multi-wavelets for any given multi-scaling functions with small support in §3. We remark that our construction can be generalized to any dimensional setting in §4. The construction of wavelets in the multivariate setting will be briefly given to indicate how to do the generalization. Finally, we make some remarks on the regularity of the wavelets so constructed, the vanishing moments of the wavelets, the number of the wavelets and general dilation matrices.

2 Constructing 2D Nonseparable Compactly Supported Wavelets

In this section we shall prove the following

Theorem 2.1. *Suppose that $\phi(x,y) \in L_2(\mathbf{R}^2)$ is a scaling function associated with dilation matrix $2I_{2\times 2}$ whose mask is*

$$m(\xi,\eta) = \frac{1}{4} \sum_{\substack{0\le j\le 5\\ 0\le k\le 5}} c_{jk} e^{i(j\xi+k\eta)}.$$

Then there exist 12 compactly supported orthonormal wavelets $\psi_{j,k}$, $j = 1,2,3, k = 1,2,3,4$ associated with ϕ in the sense that they are linear combinations of finitely many $\phi(4x-m, 4y-n)$'s with $m,n \in \mathbf{Z}$ such that the integer translates and dilates of these functions $\psi_{j,k}$'s form an orthonormal basis for $L_2(\mathbf{R}^2)$. That is,

$$\{2^\ell \psi_{j,k}(2^\ell x - m, 2^\ell y - n), \ell, m, n \in \mathbf{Z}, j = 1,2,3, k = 1,2,3,4\}$$

is an orthonormal basis for $L_2(\mathbf{R}^2)$.

In order to do so, we need the following lemma which is a generalization of the constructive procedure of bivariate compactly supported orthonormal wavelets in [He and Lai'97]. Note that the wavelets constructed there are associated with a scaling function which is supported in $[0,3]\times[0,3]$.

Lemma 2.1. *Let $[A,B,C,D]$ be a matrix of size 16×4 whose entries are trigonometric polynomials of coordinate degrees $\le (1,1)$. Suppose that $[A,B,C,D]$ is unitary, that is,*

$$[A,B,C,D]^*[A,B,C,D] = I_{4\times 4}.$$

Then there exists a unitary matrix H of size 16×16 with trigonometric polynomial entries such that

$$H[A,B,C,D] = [I_{4\times 4}, 0_{4\times 12}]^T.$$

Proof: We write

$$A = A_{16\times 4}[1, x, y, xy]^T,$$

where $A_{16\times 4}$ consists of scalar entries. Similar for B, C, D. Let

$$\mathcal{A} = [A_{16\times 4}, B_{16\times 4}, C_{16\times 4}, D_{16\times 4}]$$

be a matrix of size 16×16. Then there exist a unitary matrix H_1 such that $H_1\mathcal{A}$ is lower triangular. Note that

$$\begin{aligned}
H_1 D &= [\underbrace{0, 0, \cdots, 0}_{12}, d_1, d_2, d_3, d_4]^T \\
H_1 C &= [\underbrace{0, \cdots, 0}_{8}, c_1, \cdots, c_8]^T \\
H_1 B &= [\underbrace{0, \cdots, 0}_{4}, b_1, \cdots, b_{12}]^T \\
H_1 A &= [a_1, \cdots, a_{16}]^T
\end{aligned}$$

with a_1, b_1, c_1, and d_1 being constants independent of x and y.

Since $H_1[A, B, C, D]$ is unitary, $\sum_{i=1}^{4} |d_i|^2 = \sum_{i=1}^{8} |c_i|^2 = \sum_{i=1}^{12} |b_i|^2 = \sum_{i=1}^{16} |a_i|^2 = 1$. Let $h_2 = H(\mathbf{v})$ be a Householder matrix of size 4×4 with $\mathbf{v} = [d_1, d_2, d_3, d_4]^T - [1, 0, 0, 0]^T$. By the definition of Householder matrix, $H(\mathbf{v}) = I_{4\times 4} - \frac{2}{\mathbf{v}^*\mathbf{v}}\mathbf{v}\mathbf{v}^*$ with

$$\begin{aligned}
\mathbf{v}^T\mathbf{v} &= |d_1 - 1|^2 + \sum_{j=2}^{4} |d_j|^2 = 1 - 2d_1 + \sum_{j=1}^{4} |d_j|^2 \\
&= 2 - 2d_1 = 2(1 - d_1)
\end{aligned}$$

which is a nonzero real number in general. That is, $H(\mathbf{v})$ is a unitary matrix with trigonometric polynomial entries. Thus,

$$\begin{bmatrix} I_{4\times 4} & 0 & 0 & 0 \\ 0 & I_{4\times 4} & 0 & 0 \\ 0 & 0 & I_{4\times 4} & 0 \\ 0 & 0 & 0 & h_2 \end{bmatrix} H_1 D = [\underbrace{0, \cdots, 0}_{12}, 1, 0, 0, 0]^T.$$

If $2(1 - d_1)$ is zero, we have already had

$$H_1 D = [\underbrace{0, \cdots, 0}_{12}, 1, 0, 0, 0]^T.$$

Combining with an elementary row exchange, we let H_2 denote the unitary matrix such that $H_2 H_1 D = [\underbrace{0, \cdots, 0}_{15}, 1]^T =: e_{16}$. For simplicity, we write

$$H_2 H_1 [A, B, C, D] = [\tilde{A}, \tilde{B}, \tilde{C}, e_{16}].$$

Note that by the orthonormality, the last component of $\tilde{A}$ is zero and so are $\tilde{B}$ and $\tilde{C}$. In this case,

$$\begin{aligned} \tilde{C} &= [\underbrace{0,\cdots,0}_{8}, c_1,\cdots,c_4,\tilde{c}_5,\cdots,\tilde{c}_7,0]^T \\ \tilde{B} &= [\underbrace{0,\cdots,0}_{4}, b_1,\cdots,b_8,\tilde{b}_9,\cdots,\tilde{b}_{11},0]^T \\ \tilde{A} &= [a_1,\cdots,a_{12},\tilde{a}_{13},\cdots,\tilde{a}_{15},0]^T, \end{aligned}$$

where $c_1,\cdots,c_4$ are the same components in vector H_1C while $\tilde{c}_5,\tilde{c}_6,\tilde{c}_7$ are updated components. Similar for $b_1,\cdots,b_8$ and $\tilde{b}_9,\cdots,\tilde{b}_{11}$ and etc..

If $c_1 \neq 1$, let $h_3 = H(\mathbf{u})$ be a Householder matrix of size 7×7 with

$$\mathbf{u} = [c_1,\cdots,c_4,\tilde{c}_5,\cdots,\tilde{c}_7]^T - [1,\underbrace{0,\cdots,0}_{6}]^T.$$

Then we know

$$\begin{bmatrix} I_{8\times 8} & 0 & 0 \\ 0 & h_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{C} = [\underbrace{0,\cdots,0}_{8}, 1, \underbrace{0,\cdots,0}_{6}, 0]^T.$$

Here we note that h_3 is a unitary matrix with trigonometric polynomial entries. We again combine with a row exchange with the above unitary matrix into H_3 such that $H_3\tilde{C} = [\underbrace{0,\cdots,0}_{14}, 1, 0]^T := e_{15}$. If $c_1 = 1$, then

$$\tilde{C} = [\underbrace{0,\cdots,0}_{8}, 1, \underbrace{0,\cdots,0}_{6}, 0]^T.$$

We directly apply H_3 to $\tilde{C}$ to have $H_3\tilde{C} = e_{16}$. It follows that $H_3H_2H_1D = e_1$ and $H_3\tilde{A}$ has zeros in the last two components and so does $H_3\tilde{B}$. That is,

$$\begin{aligned} H_3\tilde{B} &= [\underbrace{0,\cdots,0}_{4}, b_1,\cdots,b_4,\hat{b}_5,\cdots,\hat{b}_{10},0,0]^T, \\ H_3\tilde{A} &= [a_1,\cdots,a_8,\hat{a}_9,\cdots,\hat{a}_{14},0,0]^T. \end{aligned}$$

Repeating the same argument as above, we can find unitary matrices H_4 and H_5 with trigonometric polynomial entries such that

$$H_5H_4H_3H_2H_1[A,B,C,D] = [0_{4\times 4}, 0_{4\times 4}, 0_{4\times 4}, I_{4\times 4}]^T.$$

By a row exchange matrix H_6, we obtain the desirable unitary matrix $H = H_6 \cdots H_1$ such that

$$H[A, B, C, D] = [I_{4\times4}, 0_{4\times4}, 0_{4\times4}, 0_{4\times4}]^T.$$

This completes the proof. ■

For the given scaling function ϕ associated with mask $m(\xi, \eta)$, we recall that Φ is a multi-scaling vector

$$\Phi(x, y) = \begin{bmatrix} 2\phi(2x, 2y) \\ 2\phi(2x-1, 2y) \\ 2\phi(2x, 2y-1) \\ 2\phi(2x-1, 2y-1) \end{bmatrix}. \tag{2.1}$$

Then $\Phi(x, y)$ is refinable, i.e.,

$$\Phi(x, y) = \sum_{\ell,m\in Z} M_{\ell,m}\Phi(2x-\ell, 2y-m)$$

with

$$M_{\ell,m} = 2\begin{bmatrix} c_{2\ell,2m} & c_{2\ell+1,2m} & c_{2\ell,2m+1} & c_{2\ell+1,2m+1} \\ c_{2\ell-2,2m} & c_{2\ell-1,2m} & c_{2\ell-2,2m+1} & c_{2\ell-1,2m-1} \\ c_{2\ell,2m-2} & c_{2\ell+1,2m-2} & c_{2\ell,2m-1} & c_{2\ell+1,2m-1} \\ c_{2\ell-2,2m-2} & c_{2\ell-1,2m-2} & c_{2\ell-2,2m-1} & c_{2\ell-1,2m-1} \end{bmatrix}. \tag{2.2}$$

Let $M(\xi, \eta) = \frac{1}{4}\sum_{\ell,m} M_{\ell,m}e^{i(\ell\xi+m\eta)}$ be the matrix mask associated with Φ which is of size 4×4 with trigonometric polynomial entries in $e^{i\xi}$ and $e^{i\eta}$. Note that each entry of $M(\xi, \eta)$ is a trigonometric polynomial of coordinate degrees $\leq (3, 3)$. Because of the orthonormality of Φ (see (1.3)), we know

$$\begin{aligned} &M(\xi, \eta)M(\xi, \eta)^* + M(\xi+\pi, \eta)M(\xi+\pi, \eta)^* + M(\xi, \eta+\pi)M(\xi, \eta+\pi)^* + \\ &M(\xi+\pi, \eta+\pi)M(\xi+\pi, \eta+\pi)^* = I_{4\times4}. \end{aligned} \tag{2.3}$$

We now show that there exist trigonometric polynomial matrices M_j, $j = 1, 2, 3$ such that the matrix in (1.8) is unitary. That is, we are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We first write $M(\xi, \eta)$ in polyphase form

$$\begin{aligned} M(\xi, \eta) &= N_0(2\xi, 2\eta) + e^{i\xi}N_1(2\xi, 2\eta) + e^{i\eta}N_2(2\xi, 2\eta) + e^{i(\xi+\eta)}N_3(2\xi, 2\eta) \\ &= [\, N_0(2\xi, 2\eta), N_1(2\xi, 2\eta), N_2(2\xi, 2\eta), N_3(2\xi, 2\eta) \,] \begin{bmatrix} I_{4\times4} \\ e^{i\xi}I_{4\times4} \\ e^{i\eta}I_{4\times4} \\ e^{i(\xi+\eta)}I_{4\times4} \end{bmatrix}. \end{aligned} \tag{2.4}$$

Then it follows that

$$2\,[\, N_0(\xi, \eta) \quad N_1(\xi, \eta) \quad N_2(\xi, \eta) \quad N_3(\xi, \eta) \,]$$

is unitary, that is

$$4[N_0(\xi,\eta), N_1(\xi,\eta), N_2(\xi,\eta), N_3(\xi,\eta)] \begin{bmatrix} N_0(\xi,\eta)^* \\ N_1(\xi,\eta)^* \\ N_2(\xi,\eta)^* \\ N_3(\xi,\eta)^* \end{bmatrix} = I_{4\times 4}.$$

Note that all the entries in $N_j(\xi,\eta)$ are polynomials of degree $\le (1,1)$. We write

$$[A,B,C,D] = 2 \begin{bmatrix} N_0(\xi,\eta)^* \\ N_1(\xi,\eta)^* \\ N_2(\xi,\eta)^* \\ N_3(\xi,\eta)^* \end{bmatrix}_{16\times 4} \tag{2.5}$$

and apply Lemma 2.1 to find unitary matrix H such that

$$H[A,B,C,D] = [I_{4\times 4}, 0_{4\times 4}, 0_{4\times 4}, 0_{4\times 4}]^T.$$

Partition H in the following form:

$$H = 2 \begin{bmatrix} N_0(\xi,\eta) & N_1(\xi,\eta) & N_2(\xi,\eta) & N_3(\xi,\eta) \\ N_{1,0}(\xi,\eta) & N_{1,1}(\xi,\eta) & N_{1,2}(\xi,\eta) & N_{1,3}(\xi,\eta) \\ N_{2,0}(\xi,\eta) & N_{2,1}(\xi,\eta) & N_{2,2}(\xi,\eta) & N_{2,3}(\xi,\eta) \\ N_{3,0}(\xi,\eta) & N_{3,1}(\xi,\eta) & N_{3,2}(\xi,\eta) & N_{3,3}(\xi,\eta) \end{bmatrix}. \tag{2.6}$$

We define

$$M_j(\xi,\eta) = N_{j,0}(2\xi,2\eta) + e^{i\xi}N_{j,1}(2\xi,2\eta) + e^{i\eta}N_{j,2}(2\xi,2\eta) + e^{i(\xi+\eta)}N_{j,3}(2\xi,2\eta) \tag{2.7}$$

for $j = 1,2,3$ which are desirable trigonometric polynomial matrices which, together with $M(\xi,\eta)$, satisfy the matrix orthonormal condition (1.8). We define multi-wavelet vectors Ψ_j, in terms of Fourier transform by

$$\widehat{\Psi}_j(\xi,\eta) = M_j(\xi/2,\eta/2)\widehat{\Phi}(\xi/2,\eta/2), j = 1,2,3. \tag{2.8}$$

Writing $\Psi_j = (\psi_{j,1},\psi_{j,2},\psi_{j,3},\psi_{j,4})^T$, $j = 1,2,3$, we know that $\psi_{j,k}$ are orthonormal for $k = 1,\cdots,4, j = 1,2,3$ by (1.6) and

$$\widehat{V}_1 = \widehat{V}_0 \oplus \widehat{W}_0.$$

Then the standard argument shows that $2^\ell\Psi_j(2^\ell x - m, 2^\ell y - n), \ell,m,n \in \mathbf{Z}$ form an orthonormal basis for $L_2(\mathbf{R}^2)$. In terms of the components of Ψ_j, $2^\ell\psi_{j,k}(2^\ell x - m, 2^\ell y - n), \ell,m,n \in \mathbf{Z}, j = 1,2,3$ and $k = 1,2,3,4$ form an orthonormal basis for $L_2(\mathbf{R}^2)$.

Finally, let us point out that $\psi_{j,k}$ are associated with ϕ. From (2.8), we know that $\psi_{i,k}$'s are linear combinations of $\phi(4x-m, 4y-n), m,n \in \mathbf{Z}$. This completes the proof. ∎

What happens when the support size of ϕ is bigger. Let

$$m(\xi,\eta) = \sum_{\substack{0\le j\le 9\\ 0\le k\le 9}} c_{jk} e^{i(j\xi+k\eta)}$$

be a trigonometric polynomial associated with a scaling function ϕ. We let

$$\Phi(x,y) = \begin{bmatrix} 4\phi(4x,4y) \\ 4\phi(4x+1,4y) \\ 4\phi(4x,4y+1) \\ 4\phi(4x+1,4y+1) \\ 4\phi(4x+2,4y) \\ 4\phi(4x+2,4y+1) \\ 4\phi(4x+2,4y+2) \\ 4\phi(4x+1,4y+2) \\ 4\phi(4x,4y+2) \\ 4\phi(4x+3,4y) \\ 4\phi(4x+3,4y+1) \\ 4\phi(4x+3,4y+2) \\ 4\phi(4x+3,4y+3) \\ 4\phi(4x+2,4y+3) \\ 4\phi(4x+1,4y+3) \\ 4\phi(4x,4y+3) \end{bmatrix}_{16\times 1}.$$

Then Φ is a scaling vector. Writing $\Phi = (\phi_1, \cdots, \phi_{16})^T$, let

$$\widehat{V}_0 = \text{ span }_{L_2}\{\phi_j(x-\ell, y-m), \ell, m \in \mathbf{Z}, j = 1, \cdots, 16\}.$$

Then $\widehat{V}_0 = V_2$. Thus, Φ generates a bona fide MRA. The above construction procedure can be simply extended for this Φ. Indeed, since Φ is a refinable function vector,

$$\widehat{\Phi}(\xi,\eta) = M(\xi/2,\eta/2)\widehat{\Phi}(\xi/2,\eta/2)$$

with a matrix symbol $M(\xi,\eta)$ which is of size 16×16 whose entries are trigonometric polynomials of $e^{i\xi}$ and $e^{i\eta}$. The coordinate degree of these trigonometric polynomials is $\le (3,3)$. Lemma 2.1 can be generalized to handle unitary matrix of size 64×16. Thus, we can construct three multi-wavelet vectors of size 16×1. These results 3×16 compactly supported wavelets. Details are omitted here. Similarly, we can do the same thing for any compactly supported scaling function ϕ with any support. In general, if ϕ is supported on $[0, 2^k+1]^2$, our construction yields $3\times 2^{2k-2}$ compactly supported orthonormal wavelets. Therefore, we can conclude

Theorem 2.2. *Suppose that $\phi(x,y) \in L_2(\mathbf{R}^2)$ is a scaling function associated with dilation matrix $2I_{2\times 2}$. Then there exist compactly supported orthonormal wavelets ψ_{jk}, $j = 1,2,3, k = 1,\cdots,n$ with appropriate $n = 3\times 2^{2k-2}$ if the support of ϕ in $[0,2^k+1]^2$ such that translates and dilates of these $\psi_{j,k}$'s form an orthonormal basis for $L_2(\mathbf{R}^2)$.*

3 Construction of Compactly Supported Multi-Wavelets

The construction of bivariate compactly supported orthonormal wavelets in the previous section can be generalized to the multi-wavelet setting. We start with the construction of bivariate compactly supported orthonormal multi-wavelets.

We first recall from the constructive proof of Theorem 2.1 that the same construction procedure can be used to construct multi-wavelets for any given multi-scaling vector Φ with support in $[0,3]\times[0,3]$. That is, we have

Lemma 3.1. *Given a multi-scaling function vector Φ of size $r\times 1$ with $r\geq 1$, if Φ is supported over $[0,3]^2$, then we can construct three compactly supported orthonormal multi-wavelets.*

Proof: For Φ of size $r\times 1$, let

$$\widehat{\Phi}(\xi,\eta) = M(\xi/2,\eta/2)\widehat{\Phi}(\xi/2,\eta/2)$$

with trigonometric polynomial mask $M(\xi,\eta)$ of size $r\times r$. We write $M(\xi,\eta)$ in polyphase form as (2.1):

$$M(\xi,\eta) = [\,N_0(2\xi,2\eta), N_1(2\xi,2\eta), N_2(2\xi,2\eta), N_3(2\xi,2\eta)\,] \begin{bmatrix} I_{4\times 4} \\ e^{i\xi}I_{4\times 4} \\ e^{i\eta}I_{4\times 4} \\ e^{i(\xi+\eta)}I_{4\times 4} \end{bmatrix}.$$

Then it follows that

$$2\,[\,N_0(\xi,\eta) \quad N_1(\xi,\eta) \quad N_2(\xi,\eta) \quad N_3(\xi,\eta)\,]$$

is unitary. Note that the above matrix is of size $r\times 4r$ and each entry is of trigonometric polynomials of coordinate degree $\leq (1,1)$. Let

$$[A,B,C,D] = 2\,[\,N_0(\xi,\eta) \quad N_1(\xi,\eta) \quad N_2(\xi,\eta) \quad N_3(\xi,\eta)\,]^*$$

be a matrix of size $4r \times r$. Using the similar arguments to the proof of Lemma 2.1, we can find a unitary matrix H of size $4r \times 4r$ with trigonometric polynomial entries such that

$$H[A, B, C, D] = [I_{r\times r}, 0_{r\times 3r}]^T.$$

Partition H into $r \times r$ blocks similar to (2.2) and let $M_j, j = 1, 2, 3$ be the matrices similar to (2.3). Define Ψ_j of size $r \times 1$, $j = 1, 2, 3$ as (2.4). Then Ψ_j's are multi-wavelets. We have thus completed the proof. ■

When a multi-scaling vector $\Phi = [\phi_1, \cdots, \phi_r]^T$ has a larger support, say, Φ is supported on $[0, 5] \times [0, 5]$, we let

$$\widetilde{\Phi}(x, y) = \begin{bmatrix} 2\phi_1(2x, 2y) \\ 2\phi_1(2x-1, 2y) \\ 2\phi_1(2x, 2y-1) \\ 2\phi_1(2x-1, 2y-1) \\ \vdots \\ \vdots \\ 2\phi_r(2x, 2y) \\ 2\phi_r(2x-1, 2y) \\ 2\phi_r(2x, 2y-1) \\ 2\phi_r(2x-1, 2y-1) \end{bmatrix}.$$

Then it is easy to see that $\widetilde{\Phi}$ is a new multi-scaling vector. The matrix symbol $M(\xi, \eta)$ is of size $4r \times 4r$ with trigonometric polynomial entries and the degree of all these trigonometric polynomials is $\leq (3, 3)$. Then the above Theorem 2.3 can be applied so that we can construct three compactly supported orthonormal multi-wavelets of size $4r \times 1$. Therefore, we conclude the following

Theorem 3.2. *Let Φ be a multi-scaling function vector of size $r \times 1$ with $r \geq 2$. Suppose that Φ is of compact support. Then there exist three compactly supported orthonormal multi-wavelets Ψ_j's of size $n \times 1$ with integer $n \geq r$ dependent on the size of the support of Φ such that the translates and dilates of these Ψ_j's form an orthonormal basis for $L_2(\mathbf{R}^2)$.*

4 Construction of Multidimensional Compactly Supported Multi-Wavelets

Next we generalize the above construction procedures in §2 and §3 to the general multivariate setting.

Indeed, let $\phi \in L^2(\mathbf{R}^d)$ be a multivariate scaling function associated with trigonometric polynomial mask $m_0(\omega)$ with $\omega := (\omega_1, \cdots, \omega_d)$ and $d \geq 2$. To construct compactly supported orthonormal wavelets associated with ϕ, we first consider the case that ϕ is supported in $[0,3]^d$. That is,

$$m_0(\omega) = \sum_{0 \leq n_1, \cdots, n_d \leq 3} c_{n_1, \cdots, n_d} e^{i(n \cdot \omega)}$$

with $n \cdot \omega := n_1\omega_1 + \cdots + n_d\omega_d$.

We need to compute $m_j(\omega), j = 1, \ldots, 2^d - 1$ such that the following matrix

$$[m_j(\omega + \pi\xi_k)]_{0 \leq j,k \leq 2^d-1}$$

is unitary, where $\{\xi_k, k = 0, \ldots, 2^d - 1\} = \{0,1\}^d$.

To this end, we use polyphase form $(f_0, f_1, \ldots, f_{2^d-1})$ of $m_0(\omega)$, that is,

$$m_0(\omega) = \ell(\omega)(f_0(x_1^2, \ldots, x_s^2), f_1(x_1^2, \ldots, x_d^2), \ldots, f_{2^d-1}(x_1^2, \ldots, x_d^2))^T,$$

where $x_k = e^{i\omega_k}, k = 1, \cdots, d$ and $\ell(\omega) := (1, x_1, \ldots, x_s, x_1x_2, \ldots, x_1x_2 \cdots x_d)^T$ is a vector of size $2^d \times 1$ consisting of all basis trigonometric polynomials of coordinate degree $\leq (\underbrace{1, \ldots, 1}_{d})$ as its entries. Note that all $f_i(x_1, \ldots, x_d)$'s are trigonometric polynomials of degree $\leq (\underbrace{1, \ldots, 1}_{d})$. Thus, we can write

$$\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{2^d-1} \end{bmatrix} = L\ell(\omega),$$

where L is a scalar matrix of size $2^d \times 2^d$. We first have

Lemma 4.1. *There exists a unitary matrix $U(x_1, \ldots, x_d)$ with trigonometric polynomial entries such that*

$$U(x_1, \ldots, x_d) \begin{bmatrix} f_0(x_1, \ldots, x_d) \\ f_1(x_1, \ldots, x_d) \\ \vdots \\ f_{2^d-1}(x_1, \ldots, x_d) \end{bmatrix} = \begin{bmatrix} \frac{1}{2^{d/2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The proof is the multivariate version of the arguments given in [He and Lai'97]. Indeed, let H be a unitary scalar matrix which converts L into a lower triangular matrix. Note that $H(f_0, f_1, \ldots, f_{2^d-1})^T = (g_0, g_1, \ldots, g_{2^d-1})^T$ with g_0 being a constant and g_j trigonometric polynomials, $j = 1, \ldots, 2^d - 1$. Since H is orthonormal,

$$\frac{1}{2^{d/2}} = \sum_{j=0}^{2^d-1} |f_j|^2 = \sum_{j=0}^{2^d-1} |g_j|^2 = |g_0|^2 + \sum_{j=1}^{2^d-1} |g_j|^2,$$

we know that $\sum_{j=1}^{2^d-1} |g_j|^2$ is constant. We now apply the following Householder matrix

$$H(v) = I_{2^d\times 2^d} - \frac{2vv^T}{v^T v}$$

with $v = (g_0, g_1, \ldots, g_{2^d-1})^T - \left(\frac{1}{2^{d/2}}, 0\ldots 0\right)^T$. Then $H(v)$ is unitary such that

$$H(v)H(f_0, f_1, \ldots, f_{2^d-1})^T = \left(\frac{1}{2^{d/2}}, 0, \ldots, 0\right)^T.$$

We note that $v^T v = \left(g_0 - \frac{1}{2^{d/2}}\right)^2 + \sum_{j=1}^{2^d-1} |g_j|^2$ is a constant and hence $H(v)$ is a matrix with trigonometric polynomial entries. It thus follows that

$$U(x_1, \ldots, x_d) = H(v)H$$

is a desirable matrix. This completes the proof of Lemma 4.1. ■

With this unitary matrix U, we have the following

Theorem 4.1. *Suppose that m_0 is a trigonometric polynomial of coordinate degree $\le (\underbrace{3, \ldots, 3}_{d})$ satisfying $m_0(0, 0, \ldots, 0) = 1$ and*

$$\sum_{k=0}^{2^d-1} |m_0(\omega + \pi\xi_k)|^2 = 1,$$

where $\{\xi_k, k = 0, \cdots, 2^d - 1\}$ is an ordered list of multi-integer set $\{0,1\}^d \subset \mathbf{Z}^d$. Then one can construct trigonometric polynomials $m_j(\omega), j = 1, \ldots, 2^d - 1$ such that

$$[m_j(\omega + \pi\xi_k)]_{0\le j,k\le 2^d-1}$$

is unitary.

Proof: Recall U in Lemma 4.1. Let $A = U^*$ be the transpose and conjugate of U. Then A is also a unitary matrix with trigonometric polynomial entries. Define $m_j, j = 1, \ldots, 2^d - 1$ by

$$[m_0, m_1, \ldots, m_{2^d-1}] = \frac{1}{2^{d/2}} \ell(\omega) A(x_1^2, x_2^2, \ldots, x_d^2).$$

We claim that the $[m_j(\omega + \pi\xi_k)]_{0\le j,k\le 2^d-1}$ is unitary. Indeed, we have

$$[m_j(\omega + \pi\xi_k)]_{0\le j,k\le 2^d-1} = \frac{1}{2^{d/2}} [\ell(\omega + \pi\xi_k)]_{0\le k\le 2^d-1} A(x_1^2, x_2^2, \ldots, x_d^2).$$

Note that $\frac{1}{2^{d/2}} [\ell(\omega + \pi\xi_k)]_{0\le k\le 2^d-1}$ is unitary. Since A is unitary, so is $[m_j(\omega + \pi\xi_k)]_{0\le j,k\le 2^d-1}$. This completes the proof of Theorem 4.1. ■

When the scaling function ϕ has a larger support, we shall use the same technique introduced in §2. Suppose that ϕ is supported in $[0,5]^d$. Then its mask m_0 is a trigonometric polynomial of coordinate degree $\le (\underbrace{5,\cdots,5}_{d})$. We consider a multi-scaling function vector

$$\Phi(x) = \left[\phi(2x-\xi_k), k=0,\ldots,2^d-1\right]^T$$

of size $2^d \times 1$. Then it can be easily seen that Φ is a refinable vector and

$$\widehat{\Phi}(\omega) = \mathcal{M}(\omega/2)\widehat{\Phi}(\omega)$$

with trigonometric polynomial matrix $\mathcal{M}$ of size $2^d \times 2^d$. Note that each entry of $\mathcal{M}$ is a trigonometric polynomial of coordinate degree $\le (\underbrace{3,\ldots,3}_{d})$. We can use the same arguments for Theorem 2.1 to find 2^d-1 multi-wavelet vectors Ψ_j's of size $2^d \times 1$ associated with Φ such that the translates and dilates of all the components $\psi_{jk}, k = 1,\ldots,2^d$ of Ψ_j for $j=1,\ldots,2^d-1$ form an orthonormal basis for $L_2(\mathbf{R}^d)$. That is, we have $(2^d-1)\times 2^d$ wavelet functions.

When ϕ has a larger support, we consider a multi-scaling function vector Φ of larger size as in §2 and construct more wavelet functions. Therefore, we can conclude

Theorem 4.1. *Let ϕ be a compactly supported scaling function in $L_2(\mathbf{R}^d)$. Then one can find $n(>> 2^d-1)$ compactly supported orthonormal wavelets $\psi_j, j=1,\cdots,n$ associated with ϕ in the sense that ψ_j's are linear combinations of $\phi(2^\ell x - m), m \in \mathbf{Z}^d$ with appropriate integer ℓ which is dependent on the support of ϕ such that their translates and dilates form an orthonormal basis for $L_2(\mathbf{R}^d)$. That is,*

$$\{2^{kd/2}\psi_j(2^k\cdot-\ell), \ell\in\mathbf{Z}^d, k\in\mathbf{Z}, j=1,\cdots,n\}$$

is an orthonormal basis for $L_2(\mathbf{R}^d)$.

Similarly we can generalize the construction in §3 to the multivariate multi-wavelet setting. The detail is omitted here.

5 Remarks

We have the following remarks in order.

1. It is easy to see from (1.5) that the regularity of the wavelets are the same as the scaling functions.
2. However, we do not know the property of the vanishing moments of the wavelets so constructed in this paper.

3. When the support of a scaling function gets larger, the number of compactly supported orthonormal wavelets gets bigger. It is desirable to have a fixed number of wavelets independent of the size of the scaling functions. It remains open how to construct $2^d - 1$ compactly supported wavelets associated with a compactly supported scaling function $\phi \in L_2(\mathbf{R}^d)$ for $d \geq 2$.
4. It is possible to expand the construction in the paper to the case with general dilation matrices. We leave the details to a future paper.

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