On Construction of
Bivariate and Trivariate Vertex Splines
on Arbitrary Mixed Grid Partitions

(August 1989)

Ming-Jun Lai, B.S., Hangzhou University, China
Chair of Advisory Committee: Dr. Charles K. Chui

ABSTRACT

The procedures for constructing vertex splines in various spline spaces $S_d^r(\triangle)$ in the bivariate and trivariate settings are described and approximation formulas based on these vertex splines are constructed in this thesis. These vertex splines span a super spline subspace of $S_d^r(\triangle)$ and the optimal approximation order of $S_d^r(\triangle)$ is attained by using these approximation formulas. Here, $S_d^r(\triangle)$ stands for the following space of all piecewise polynomial functions of degree d and of smoothness order r on a given grid partition $\triangle$:

(i) $r \geq 1$, $d \geq 3r+2$, and $\triangle$ consists of triangles and parallelograms in the bivariate setting;

(ii) $r = 1$, $d = 7$, and $\triangle$ consists of tetrahedra and satisfies that the number of tetrahedra around each nonsingular edge is odd in the trivariate setting;

(iii) $r \geq 1$, $d \geq 6r+3$, and $\triangle$ consists of tetrahedra in the trivariate setting; and

(iv) $r \geq 1$, $d \geq 8r+1$, and $\triangle$ consists of tetrahedra, prisms and parallelepipeds in the trivariate setting.

ACKNOWLEDGMENTS

I would like to take this opportunity to express my sincere thanks to Professor Charles K. Chui, my thesis advisor, for his constant encouragement and significant guidance in various areas in mathematics during the last five years. Also, I am grateful to the members of my committee Professors Guy L. Curry, Larry L. Schumaker, and Joseph D. Ward for their endurance and patience in helping me to achieve my goal. In addition, I am thankful for Dept. of Mathematics, Texas A&M University to provide me an opportunity to pursue my Ph. D. degree. Finally, I would like to thank my wife Lingyun Ma for her love and constant support.

TABLE OF CONTENTS

LIST OF FIGURES

1. INTRODUCTION

One of the most important problems in multivariate spline approximation (MSA) is derivation of effective constructive schemes of piecewise polynomial functions with certain smoothness which are good approximations to a target function with only partial information known. For instance, fitting a surface to given discrete data is one of the aspects to deal with. This problem has not only theoretical interest, but also a lot of applications in various sciences and engineering research areas, such as computer graphics, computer-aided geometric design, large-scale integrated circuit design, fitting a wind field over complex terrains, analyzing electromagnetic fields of an optical waveguide, petroleum exploration, heart and brain potential measurements, etc.. Indeed, almost all problems in the real world are multivariable or multiparameteric in nature, and only partial known data are usually available.

Though some individual research works on this subject were scattered in the literature, a systematical study with emphasis on locally supported splines in the multivariate setting did not start until the later part of the 70's. First, simplicial B-splines and geometrical interpretation of univariate B-splines in multivariate setting were studied. (cf. [51, 52, 56, 57, 76, 92].) Later, box splines were introduced and studied extensively in both theoretical and computational aspects. (cf. [7, 9, 12–18, 20–23, 32, 38, 41, 42, 49, 50, 54, 55, 58–66, 77, 81–87, 94, 98, 99, 117].) A special feature of box splines is that they are supported on uniformly spaced triangular partitions. For an arbitrary but given triangulation, the notion of bivariate vertex splines was introduced in [36]. Later, they were generalized to any higher dimension (cf. [39, 40]). A systematic study of this subject of multivariate polynomial splines is given in the recent monograph [28].

Given a partition consisting of patches (triangles, parallelograms, or simplices, parallelepipeds), vertex splines are piecewise polynomial functions with preassigned order of smoothness supported only on a part of the union of all patches sharing at most one common vertex. In any Euclidean space $\mathbb{R}^s$, for any smoothness requirement $r \geq 1$, vertex splines may be constructed as long as the degree d of the polynomials used in the construction is at least $2^s r + 1$ where $s \geq 3$ and at least $3r + 2$ where $s = 2$. Even for a mixed partition consisting of triangles and parallelograms in $\mathbb{R}^2$, vertex splines may also be constructed when $d \geq 4r + 1$. These results can be found in [37, 39, 40].

This manuscript follows the style of the *SIAM Journal on Numerical Analysis.*

However, by a result in [36] or [4], any vertex spline in $S_4^1(\triangle)$ in $\mathbb{R}^2$ for some triangulation must vanish at all vertices; and by a result in [87] the full approximation order may not be realized by $S_{3r+1}^r(\triangle)$ where $\triangle$ is a three direction mesh. Thus, in the bivariate setting, the minimal degree of $3r + 2$ is necessary in general to give a useful spline space on which useful locally supported splines may be constructed to realize the full approximation order.

On the one hand, the computation of vertex spline surfaces is fairly easy. In addition, only B-nets of polynomial pieces of vertex splines are stored in a computer, manipulated in various operations of arithmetic including differentiation and integration. (cf. [8, 10, 11, 28, 53, 70, 71, 103–106].) Also evaluation of vertex spline surfaces is easily implemented by using their B-nets and de Casteljau's algorithm or other polynomial evaluators (cf. [10, 53, 112]). On the other hand, the subjects of computer-aided geometric design (CAGD) and the finite elements methods (FEM) are closely related to the subject of multivariate spline approximation (MSA). The powerful tool of Bézier representation of polynomials over triangular patches in CAGD has been adopted and developed as a useful tool in MSA. Also, the construction of vertex splines is seen to be intimately related to FEM. For instance, imposition of extra smoothness conditions at vertices in construction of vertex splines is similar to that in FEM. An improvement of vertex splines over FEM is that they can be constructed when $d \geq 3r+2$ in the bivariate case (cf. [37]) and $d \geq 6r+3$ in the trivariate case (cf. Section 3.5 of this dissertation), but FEM only applies to $d \geq 4r + 1$ and $8r + 1$ respectively (cf. [25, 88–90, 118–120]). Hence, vertex splines of lower degrees can be used as trial functions in the variational formulation in finite elements analysis. Also, the solution of a partial differential equation may be represented by using vertex splines. In this case, the inner product of vertex splines can be found efficiently by using their B-nets and no mapping to the standard triangle is necessary. Similarly, due to the fact that the supports of vertex splines are local, vertex splines can also be used in CAGD. Indeed, a change in the given data alters the spline space, a linear combination of vertex splines, only in a small region around those points where the change take places and evaluation of the spline space is largely independent of the amount of data. These properties will be helpful in surface design. In summary, MSA benefits both FEM and CAGD.

Therefore, vertex splines seem to be very promising in both theoretical and practical purposes and will find many applications. Thus, we would like to study the theory of multivariate spline approximation and continue our efforts to develop the theory on vertex splines in this dissertation. In the following cases, we shall describe the procedure of constructing vertex splines in $S_d^r(\triangle)$ and find a linear projection which can be used to realize the full approximation order of $S_d^r(\triangle)$:

1. (1) $\triangle$ is a mixed partition consisting of triangles and parallelograms and $d \geq 3r+2$ in the bivariate setting;
2. (2) $\triangle$ is a partition consisting of tetrahedra with an additional constraint, $r = 1$ and $d = 7$ in the trivariate setting;
3. (3) $\triangle$ is a partition consisting of tetrahedra only, and $r \geq 1$ and $d \geq 6r + 3$ in the trivariate setting;
4. (4) $\triangle$ is an arbitrary partition consisting of tetrahedra, prisms, and parallelepipeds, $r \geq 1$ and $d \geq 8r + 1$ in the trivariate setting.

In this dissertation, barycentric coordinates will be adopted instead of rectangular coordinates so that polynomial pieces of a vertex spline can be represented in B-form (cf. [11], [28] or [71]). Thus, smoothness conditions of polynomials over two adjacent geometric configurations (triangles, parallelograms; simplices, prisms, parallelepipeds) will be expressed in a nice and symmetric form, independent of coordinates. These smoothness conditions provide some useful applications which will be used in the construction of vertex splines. The technique of "disentangling the rings" in [19] will be generalized to those cases where $\triangle$ consists of both triangles and parallelograms in the bivariate setting and $\triangle$ consists of tetrahedra, prisms and parallelepipeds in the trivariate setting. Hence, vertex splines in the spline spaces mentioned above can be possibly constructed. Approximation formulas based on these vertex spines will also be considered and studied to some details. The results obtained in this dissertation can be summarized as follows.

- (i) In the bivariate case, for any mixed partition consisting of both triangles and parallelograms, fundamental vertex splines of smoothness r and degree $d \geq 3r + 2$ are constructed and the full approximation order is realized by using these vertex splines. These results generalize the ones in [37] to mixed partition regions.
- (ii) In the trivariate case, when $r = 1$ and $d = 7$, vertex splines can be constructed when the simplicial partition satisfied an additional constraint, which is that each of interior edges is either singular edge or an edge sharing by odd numbers tetrahedra. If the partition $\triangle$ satisfies this additional constraint, the full approximation order can be realized by using these vertex splines in $S_7^1(\triangle)$.

(iii) Again in the trivariate case, for any smoothness requirement r, vertex splines can be constructed on any given simplicial partition when degree $d \geq 6r+3$ and the full approximation order will be realized by using these vertex splines. This improves a result of [88–90] that Hermite elements may be constructed when $d \geq 8r+1$.

(iv) Again in the trivariate case, for any mixed partition consisting of tetrahedra, prisms, and parallelepipeds, vertex splines can be constructed when degree $d \geq 8r+1$ and it seems that the degree cannot be reduced.

The layout of this dissertation is as follows: in Sections 2.1–2.5, bivariate vertex splines on mixed partitions are studies; Sections 3.1–3.6 consists of the study on trivariate vertex splines. In Section 2.1–2.5, we first start with preliminary materials: polynomial representations, polynomial interpolation, and smoothness conditions and applications. Then the construction of vertex splines and a linear projection are outlined, and the verification that the linear projection realizes the full order is provided. In Sections 3.1–3.6, after introducing polynomial representations and polynomial interpolation of trivariate polynomials based on tetrahedron, prism, and parallelepiped, we present smoothness conditions of trivariate polynomials over adjacent patches and their applications. Then we study the cases (2), (3), and (4) mentioned above separately and in some details. We leave the discussion on the application aspects of vertex splines as well as other comments on vertex splines to Section 4. Pictures of some vertex splines in $\widehat{S}_5^1$ in $\mathbb{R}^2$ are included in the appendix. An extensive list of references on the theory of multivariate splines is also included in this dissertation.

2. BIVARIATE VERTEX SPLINES

2.1 Polynomial Representations

Grid partitions of a given region $R \subset \mathrm{I\!R}^2$ to be studied throughout this part consists of both triangles and parallelograms. In order to construct vertex splines on such grid partitions, we will use both polynomials of total degree and of coordinate degree. On a triangle, we use polynomials of total degree, and on a parallelogram, polynomials of coordinate degree corresponding to the parallelogram. Each polynomial piece of a spline on the grid partition will take on Bézier or Bernstein representations, in short, B-forms. Let us introduce these B-forms and we will use barycentric coordinates to do so.

For a triangle $T_1 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle = \left\{ \sum\limits_{i=1}^{3} \lambda_i \mathbf{x}_i : \sum\limits_{i=1}^{3} \lambda_i = 1, \lambda_i \geq 0 \right\}$, where $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3 \in \mathrm{I\!R}^2$, any $\mathbf{x} = (x_1, x_2) \in \mathrm{I\!R}^2$ can be identified by the 3-tuple $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ satisfying

$$\mathbf{x} = \sum_{i=1}^{3} \lambda_i \mathbf{x}_i \tag{2.1.1}$$

and

$$\sum_{i=1}^{3} \lambda_i = 1. \tag{2.1.2}$$

This 3-tuple is called the barycentric coordinate of $\mathbf{x}$ with respect to the triangle T_1. For any $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{Z}_+^3$ with $|\beta| = \beta_1 + \beta_2 + \beta_3$, we denote

$$\Phi_\beta(\lambda) = \frac{|\beta|!}{\beta!} \lambda^\beta = \frac{|\beta|!}{\beta_1! \beta_2! \beta_3!} (\lambda_1)^{\beta_1} (\lambda_2)^{\beta_2} (\lambda_3)^{\beta_3}.$$

Knowing from (2.1.1) and (2.1.2) that λ is a linear function of $\mathbf{x}$, $\Phi_\beta(\lambda)$ is a polynomial of total degree $|\beta|$ of $\mathbf{x}$. It is also known that $\{\Phi_\beta(\lambda) : |\beta| = n\}$ is a basis of the polynomial space π_n, the space of all polynomials of total degree n. Hence, we may express a polynomial $P_n(\mathbf{x})$ of total degree n by using the following representation

$$P_n(\mathbf{x}) = \sum_{|\beta|=n} a_\beta \Phi_\beta(\lambda) \tag{2.1.3}$$

which is called Bézier representation, in short, B-form of polynomial P_n with respect to the triangle T_1. We also denote by $\pi_n(T_1)$ the space of all polynomials of total degree n in B-form (2.1.3) with respect to T_1. In addition, the set

$$\left\{ \left(\frac{\beta_1}{n}\mathbf{x}_1 + \frac{\beta_2}{n}\mathbf{x}_2 + \frac{\beta_3}{n}\mathbf{x}_3, a_\beta \right) : |\beta| = n \right\}$$

is called the Bézier net of P_n on T_1, in short, the B-net of P_n, and $a_\beta, |\beta| = n$ are called the B-coefficients of P_n which may be simply shown as in Figure 2.1 where $n = 5$.

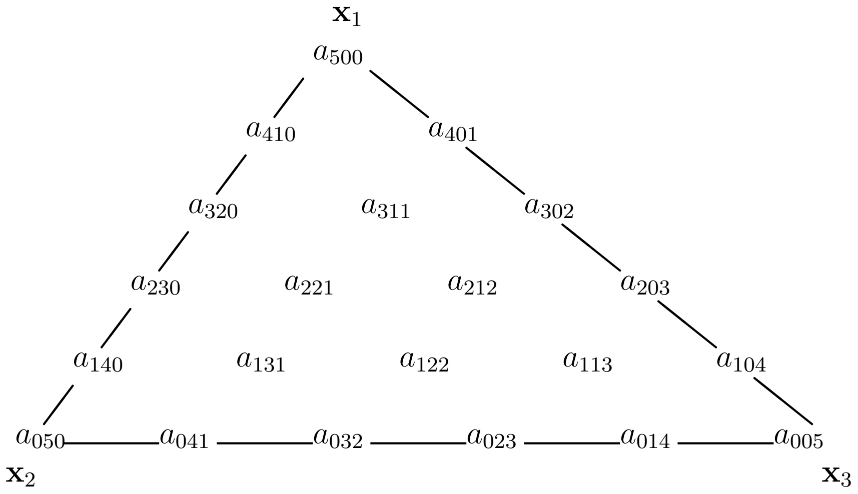


Fig. 2.1 The B-coefficients of P_5

Next, for a parallelogram $T_2 = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4 \rangle$, where $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4 \in \mathbb{R}^2$ are its four vertices, we assume that $\langle \mathbf{y}_1, \mathbf{y}_2 \rangle || \langle \mathbf{y}_3, \mathbf{y}_4 \rangle$and $\langle \mathbf{y}_1, \mathbf{y}_3 \rangle || \langle \mathbf{y}_2, \mathbf{y}_4 \rangle$ without loss of generality. For each $\mathbf{x} \in \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4 \rangle$, it is clear that $\mathbf{x}$ may be uniquely expressed as

$$\mathbf{x} = \mathbf{y}_1 + \mu_1(\mathbf{y}_2 - \mathbf{y}_1) + \mu_2(\mathbf{y}_3 - \mathbf{y}_1)$$

where μ_1, μ_2 are two nonnegative numbers. Set $\mu = (\mu_1, \mu_2)$ which is called barycentric coordinate of $\mathbf{x}$ with respect to T_2. For any $\sigma = (\sigma_1, \sigma_2) \in \mathbb{Z}_+^2$ and $\alpha = (\alpha_1, \alpha_2) \leq (\sigma_1, \sigma_2)$, denote

$$\tilde{\Phi}_\alpha^\sigma(\mu) = \binom{\sigma_1}{\alpha_1}\binom{\sigma_2}{\alpha_2}(\mu_1)^{\alpha_1}(1-\mu_1)^{\sigma_1-\alpha_1}(\mu_2)^{\alpha_2}(1-\mu_2)^{\sigma_2-\alpha_2}.$$

Then, $\tilde{\Phi}_\alpha^\sigma$ is a polynomial of $\mathbf{x}$. We denote by $\pi_\sigma(T_2)$ the space of all polynomials in the form

$$\tilde{P}_\sigma(\mathbf{x}) = \sum_{\alpha \leq \sigma} \tilde{a}_\alpha \tilde{\Phi}_\alpha^\sigma(\mu). \tag{2.1.4}$$

Here, $\tilde{P}_\sigma$ is called Bernstein representation, in short, B-form of polynomial of coordinate degree σ with respect to T_2.

If $\sigma = (n, n)$, we simply write $\pi_n(T_2), \tilde{P}_n(\mathbf{x}), \tilde{\Phi}^n_\alpha$ for $\pi_\sigma(T_2), \tilde{P}_\sigma, \tilde{\Phi}^\sigma_\alpha$, respectively. In addition, the set

$$\{(\mathbf{y}_1 + \frac{\alpha_1}{\sigma_1}(\mathbf{y}_2 - \mathbf{y}_1) + \frac{\alpha_2}{\sigma_2}(\mathbf{y}_3 - \mathbf{y}_1), \tilde{a}_\alpha) : (\alpha_1, \alpha_2) \le (\sigma_1, \sigma_2)\}$$

is called the B-net of $\tilde{P}_\sigma$ on T_2 and $\tilde{a}_\alpha, \alpha \le \sigma$ are called the B-coefficients of $\tilde{P}_\sigma$ which may be simply shown as in Figure 2.2 where $\sigma = (5, 5)$.

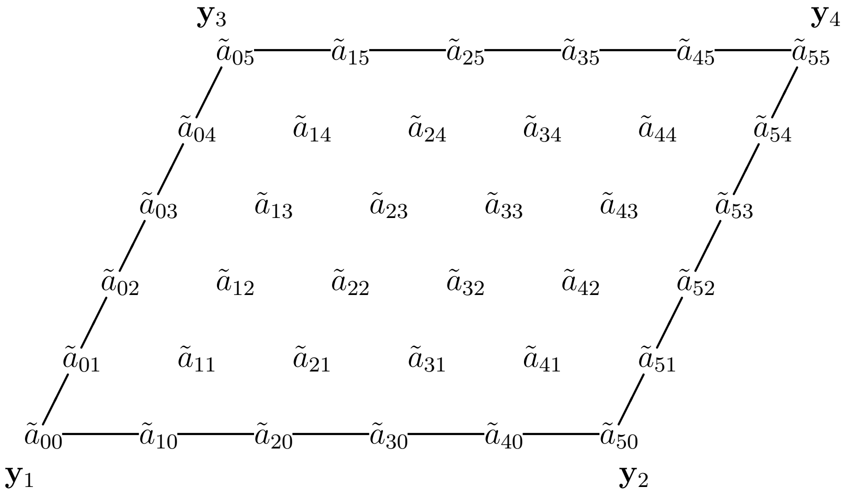


Figure 2.2 The B-coefficients of $\tilde{P}_5$

2.2. Polynomial Interpolation

It can be easily understood that B-coefficients of P_n (resp. $\tilde{P}_\sigma$) are closely related to interpolation conditions at vertices of the triangle (resp. parallelogram). Let us explore their relations.

First of all, let us introduce some necessary notations and definitions.

A subset $M \subset \mathbb{Z}_+^2$ is called a lower set if $\beta \in M$ and $\gamma \leq \beta$ imply $\gamma \in M$. Let $\Gamma_n = \{\beta \in \mathbb{Z}_+^2 : |\beta| \leq n\}$ and $\Lambda_n = \{\alpha \in \mathbb{Z}_+^3 : |\alpha| = n\}$. We say that the subsets M_1, M_2, M_3 of Γ_n induce a partition of Λ_n if they satisfy:

(i) $A_i^n M_i \cap A_j^n M_j = \emptyset$ for $i \neq j$, and

(ii) $\cup_{i=1}^3 A_i^n M_i = \Lambda_n$,

where A_i^n is a map: $\mathbb{Z}_+^2 \longrightarrow \mathbb{Z}_+^3$ defined by

$$\begin{aligned} A_1^n\beta &= (n-\beta_1-\beta_2, \beta_1, \beta_2), \\ A_2^n\beta &= (\beta_1, n-\beta_1-\beta_2, \beta_2), \\ A_3^n\beta &= (\beta_1, \beta_2, n-\beta_1-\beta_2), \end{aligned}$$

for $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^2$.

We will use the following inversion formula: Let $M \subset \mathbb{Z}_+^2$ be a lower set and

$$f(\beta) = \sum_{0\leq\alpha\leq\beta} \binom{\beta}{\alpha} (-1)^{|\alpha|} g(\alpha), \qquad \forall \beta \in M.$$

Then

$$g(\alpha) = \sum_{0\leq\gamma\leq\alpha} \binom{\alpha}{\gamma} (-1)^{|\gamma|} f(\gamma), \qquad \forall \alpha \in M.$$

Fix a triangle $T_1 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$. We denote by

$$\begin{aligned} D_1^\beta &= (D_{\mathbf{x}_3-\mathbf{x}_1})^{\beta_1} (D_{\mathbf{x}_2-\mathbf{x}_1})^{\beta_2}, \\ D_2^\beta &= (D_{\mathbf{x}_1-\mathbf{x}_2})^{\beta_1} (D_{\mathbf{x}_3-\mathbf{x}_2})^{\beta_2}, \\ D_3^\beta &= (D_{\mathbf{x}_1-\mathbf{x}_3})^{\beta_1} (D_{\mathbf{x}_2-\mathbf{x}_3})^{\beta_2}, \end{aligned}$$

for any $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^2$.

Also, we denote by e^i the standard unit vector in $\mathbb{R}^3, i = 1, 2, 3$ as usual and let Δ_{ij} be a difference operator defined by

$$\Delta_{ij} c_\beta = c_{\beta+e^i} - c_{\beta+e^j}, \ \forall i, j = 1, 2, 3, \beta \in \mathbb{Z}_+^3$$

We are now ready to establish several propositions.

PROPOSITION 2.1. *Suppose that* M_1, M_2, M_3 *are all lower subsets of* Γ_n *that induce a partition of* Λ_n. *Then for any given data* $\{f_{i\beta} : \beta \in M_i, i = 1, 2, 3\}$, *there exists a unique polynomial* $p_n(\mathbf{x})$ *of total degree* n *satisfying*

$$D_i^{\beta} p_n(\mathbf{x}_i) = f_{i\beta}, \qquad \beta \in M_i, i = 1, 2, 3. \tag{2.2.1}$$

Moreover, $p_n(\mathbf{x})$ *may be expressed as follows:*

$$p_n(\mathbf{x}) = \sum_{i=1}^{3} \sum_{\beta \in M_i} \left\{ \sum_{0 \le \gamma \le \beta} \binom{\beta}{\gamma} \frac{(n-\gamma_1-\gamma_2)!}{n!} f_{i\gamma} \right\} \Phi_{A_i^n \beta}(\lambda). \tag{2.2.2}$$

Proof. Since M_1, M_2, M_3 induce a partition of Λ_n, any polynomial $p_n(\mathbf{x})$ of total degree n can be written in the form of

$$\begin{aligned} p_n(\mathbf{x}) &= \sum_{|\alpha|=n} a_\alpha \Phi_\alpha(\lambda) \\ &= \sum_{i=1}^{3} \sum_{\beta \in M_i} a_{A_i^n \beta} \Phi_{A_i^n \beta}(\lambda). \end{aligned}$$

For $\gamma = (\gamma_1, \gamma_2) \in M_1$,

$$D_1^{\gamma} p_n(\mathbf{x}_1) = \frac{n!}{(n-\gamma_1-\gamma_2)!} \Delta_{21}^{\gamma_1} \Delta_{31}^{\gamma_2} a_{(n-\gamma_1-\gamma_2,0,0)}$$

or

$$\begin{aligned} & (-1)^{|\gamma|} \frac{(n-\gamma_1-\gamma_2)!}{n!} D_1^{\gamma} p_n(\mathbf{x}_1) \\ = \; & (-1)^{|\gamma|} \Delta_{21}^{\gamma_1} \Delta_{31}^{\gamma_2} a_{(n-\gamma_1-\gamma_2,0,0)} \\ = \; & \sum_{\beta \le \gamma} \binom{\gamma}{\beta} (-1)^{|\beta|} a_{(n-\beta_1-\beta_2,\beta_1,\beta_2)} \\ = \; & \sum_{\beta \le \gamma} \binom{\gamma}{\beta} (-1)^{|\beta|} a_{A_1^n \beta}. \end{aligned}$$

Thus, by using inversion formula, we obtain

$$a_{A_1^n \beta} = \sum_{0 \le \alpha \le \beta} \binom{\beta}{\alpha} (-1)^{|\alpha|} (-1)^{|\alpha|} \frac{(n-\alpha_1-\alpha_2)!}{n!} D_1^{\alpha} p_n(\mathbf{x}_1)$$

for $\beta \in M_1$. Similarly,

$$a_{A_2^n \beta} = \sum_{0 \le \alpha \le \beta} \binom{\beta}{\alpha} \frac{(n-\alpha_1-\alpha_2)!}{n!} D_2^{\alpha} p_n(\mathbf{x}_2), \quad \beta \in M_2$$

and

$$a_{A_3^n\beta} = \sum_{0\le\alpha\le\beta} \binom{\beta}{\alpha} \frac{(n-\alpha_1-\alpha_2)!}{n!} D_3^{\alpha} p_n(\mathbf{x}_3), \quad \beta \in M_3.$$

Therefore, the polynomial $p_n(\mathbf{x})$ satisfying the interpolation condition (2.2.1) can be expressed as in (2.2.2) and this polynomial $p_n(\mathbf{x})$ is unique because M_1, M_2, M_3 induce a partition of Λ_n. Thus, we have established the proposition.

Actually, we may slightly reduce the requirements on sets M_1, M_2, M_3. Namely, we have the following

PROPOSITION 2.2. *Suppose $M_1, M_2, M_3 \subset \Gamma_n$ induce a partition of Λ_n. Further, suppose that*

(a) M_1 is a lower set;

(b) the union of M_2 and some elements of $\{(\beta_1, \beta_2) : (\beta_1, n-\beta_1-\beta_2, \beta_2) \in A_1^n M_1\}$ is a lower set; and

(c) the union of M_3 and some elements of $\{(\gamma_1, \gamma_2) : (\gamma_1, \gamma_2, n-\gamma_1-\gamma_2) \in A_1^n M_1 \cup A_2^n M_2\}$ is a lower set.

Then there exists a unique polynomial p_n of total degree n satisfying the interpolation condition (2.2.1) for any given data $\{f_{i\beta} : \beta \in M_i, i = 1, 2, 3\}$.

The proof of this result is similar to that of proposition 2.1 if we note that we may use the previous information in determining later part of B-coefficients of $p_n(\mathbf{x})$. We omit the detail.

Example 2.1. Let $n = 6$. We choose the sets $M_1 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (2,1), (3,0), (3,1)\}$, $M_2 = \{ (0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (2,1), (3,0), (3,1)\}$, and $M_3 = \{ (0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (1,2), (0,3), (1,3), (2,2)\}$. We may determine the interpolation polynomial p_6 that satisfies the conditions

$$D_i^{\beta} p_6(\mathbf{x}_i) = f_{i\beta}, \quad \beta \in M_i, \quad i = 1, 2, 3$$

for any given data $\{f_{i\beta} : \beta \in M_i^2, i = 1, 2, 3\}$ by using the above proposition.

Next, let $T_2 = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4 \rangle$ be a parallelogram. Consider a polynomial $\tilde{p}_n$ of "degree" (n, n) with respect to T_2 in the form

$$\tilde{p}_n(\mathbf{x}) = \sum_{\alpha\le(n,n)} \tilde{a}_\alpha \tilde{\Phi}_\alpha^n(\mu)$$

where $\mu = \mu(\mathbf{x}) = (\mu_1, \mu_2)$, $\mathbf{x} = \mathbf{y}_1 + \mu_1(\mathbf{y}_2 - \mathbf{y}_1) + \mu_2(\mathbf{y}_3 - \mathbf{y}_1)$. Let $\eta^1 = (1, 1), \eta^2 = (-1, 1), \eta^3 = (1, -1)$ and $\eta^4 = (-1, -1)$. Denote $\tilde{\Gamma}_n = \{\beta \in \mathbb{Z}_+^2, \beta \le (n, n)\}$ and define a one-to-one map $B_i^n, i = 1,2,3,4$ from $\tilde{\Gamma}_n$ to itself by

$$B_i^n\beta = (\beta_1\eta_1^i, \beta_2\eta_2^i) + \left(\frac{1-\eta_1^i}{2}n, \frac{1-\eta_2^i}{2}n\right), \qquad \forall\beta \in \Gamma_n,$$

where $(\eta_1^i, \eta_2^i) = \eta^i, i = 1,2,3,4$. We say that subsets N_1, N_2, N_3, N_4 of $\tilde{\Gamma}_n$ induce a partition of $\tilde{\Gamma}_n$ if they satisfy

(i) $B_i^n N_i \cap B_j^n N_j = \emptyset$ for $i \neq j$, and

(ii) $\cup_{i=1}^4 B_i^n N_i = \tilde{\Gamma}_n$.

Also, we define difference operators Δ_1 and Δ_2 by

$$\Delta_1 b_{ij} = b_{i+1,j} - b_{ij}$$

and

$$\Delta_2 b_{ij} = b_{i,j+1} - b_{ij}.$$

Then we have the following proposition.

PROPOSITION 2.3. *Suppose that* $N_i \subset \tilde{\Gamma}_n, i = 1,2,3,4$ *are lower sets that induce a partition of* $\tilde{\Gamma}_n$*. Then for any given data* $\{f_{i\beta} : \beta \in N_i, i = 1,2,3,4\}$*, there exists a unique interpolation polynomial* $p_n \in \pi_i(T_2)$ *satisfying*

$$(D_{\mathbf{y}_2-\mathbf{y}_1})^{\beta_1}(D_{\mathbf{y}_3-\mathbf{y}_1})^{\beta_2} p_n(\mathbf{y}_i) = f_{i\beta}, \quad \beta = (\beta_1, \beta_2) \in N_i, \tag{2.2.3}$$

for $i = 1,2,3,4$*. Moreover,* p_n *may be expressed as follows:*

$$p_n(\mathbf{x}) = \sum_{i=1}^4 \sum_{\beta \in N_i} \left[\sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \frac{(n-\gamma_1)!(n-\gamma_2)!}{n!n!} (\eta^i)^\gamma f_{i\gamma}\right] \tilde{\Phi}^n_{B_i^n\beta}(\mu). \tag{2.2.4}$$

Proof. Write any $p_n \in \pi_n(T_2)$ in the form of

$$\begin{aligned} p_n(\mathbf{x}) &= \sum_{\alpha \leq (n,n)} \tilde{a}_\alpha \tilde{\Phi}^n_\alpha(\mu) \\ &= \sum_{i=1}^4 \sum_{\beta \in N_i} \tilde{a}_{B_i^n\beta} \tilde{\Phi}^n_{B_i^n\beta}(\mu)\,. \end{aligned}$$

This is possible, since $\{\tilde{a}_\alpha : \alpha \in B_i^n N_i\}, i = 1,2,3,4$, are mutually disjoint and induce a partition of $\tilde{\Gamma}_n$ according to the assumption. Since

$$D^{\beta_1}_{\mathbf{y}_2-\mathbf{y}_1} D^{\beta_2}_{\mathbf{y}_3-\mathbf{y}_1} p_n(\mathbf{y}_1) = \frac{n!}{(n-\beta_1)!} \frac{n!}{(n-\beta_2)!} \Delta_1^{\beta_1} \Delta_2^{\beta_2} \tilde{a}_{(0,0)},$$

or

$$(-1)^{\beta_1+\beta_2}\frac{(n-\beta_1)!(n-\beta_2)!}{n!n!}D^{\beta_1}_{\mathbf{y}_2-\mathbf{y}_1}D^{\beta_2}_{\mathbf{y}_3-\mathbf{y}_1}p_n(\mathbf{y}_1)=\sum_{\gamma\le(\beta_1,\beta_2)}\binom{\beta}{\gamma}(-1)^{|\gamma|}\tilde{a}_\gamma,$$

for $\beta\in N_1=B^n_1N_1$, we have, by using the inversion formula,

$$\tilde{a}_\beta=\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\frac{(n-\gamma_1)!(n-\gamma_2)!}{n!n!}D^{\gamma_1}_{\mathbf{y}_2-\mathbf{y}_1}D^{\gamma_2}_{\mathbf{y}_3-\mathbf{y}_1}p_n(\mathbf{x}_1),\beta\in N_1,$$

since N_1 is a lower set. Thus,

$$\tilde{a}_\beta=\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\frac{(n-\gamma_1)!(n-\gamma_2)!}{n!n!}f_{1\gamma},\beta\in N_1$$

if p_n satisfies (2.2.3). Similarly, $\{\tilde{a}_\gamma,\gamma\in B^n_iN_i\},i\ge 2$ are uniquely determined by $\{f_{i\gamma}:\gamma\in N_i\},i\ge 2$. The existence and uniqueness of an interpolation polynomial p_n satisfying (2.2.3) follow, if we choose $\tilde{a}_\beta$ to be

$$\tilde{a}_{B^n_i\beta}=\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\frac{(n-\gamma_1)!(n-\gamma_2)!}{n!n!}(\eta^i)^\gamma f_{i\gamma},\beta\in N_i,i=1,2,3,4.$$

Thus we have established the result.

We may slightly reduce the requirement on $N_i,i=1,2,3,4$ in Proposition 2.3 above so that the result is more applicable. That is, we have the following

PROPOSITION 2.4. *Suppose that* $N_i\subset\bar{\Gamma}_n,i=1,2,3,4$, *induce a partition of* $\bar{\Gamma}_n$ *and suppose further that*

(a) N_1 *is a lower set; and*

(b) the union of N_i *and some elements of*

$$(B^n_i)^{-1}(\cup_{j=0}^{i-1}B^n_jN_j)$$

is a lower set for $i=2,3,4$. *Then for any given data* $\{f_{i\beta};\beta\in N_i,i=1,2,3,4\}$, *there exists a unique polynomial* $p_n\in\pi_n(T_2)$ *satisfying (2.2.3).*

The proof is similar to that of Proposition 2.3 if we note that we may use the previous information to determine later $\tilde{a}_\beta$'s. We omit the details.

Example 2.2. Suppose that $N_1=\{(0,0),(1,0),(2,0),(3,0),(0,1),(1,1),(2,1),(3,1),(0,2),(1,2),(2,2)\}$, $N_2=\{(0,0),(1,0),(0,1),(1,1),(0,2),(1,2)\}$, $N_3=\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(3,0)\}$ and $N_4=\{(0,0),(1,0),(0,1),(1,1),(2,1),(3,1),(0,2),(1,2),(2,2),(3,2),(2,3)\}$. The above proposition implies that for any given data $\{f_{i\beta}:\beta\in N_i,i=1,2,3,4\}$, there is a unique polynomial $p_{(5,5)}$ interpolating the given data, although N_4 is not a lower set.

2.3. Smoothness Conditions and Their Applications

In this section, we are going to derive the conditions to ensure that two polynomials pieces P_n and Q_n defined on two adjacent patches (triangles or parallelograms) are joined smoothly. There are three possibilities of two adjacent patches: two triangles, one triangle and the other parallelogram, and two parallelograms. We will study these cases separately and in some details.

1° Suppose that P_n and Q_n are defined on two adjacent triangles $T_1 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ and $T_2 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4 \rangle$ which share a common edge $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$. More precisely, let

$$P_n(\mathbf{x}) = \sum_{|\beta|=n} a_\beta \Phi_\beta(\lambda) \qquad \text{and} \qquad Q_n(\mathbf{x}) = \sum_{|\beta|=n} b_\beta \Phi_\beta(\mu),$$

where $\mathbf{x} = \sum_{i=1}^{3} \lambda_i \mathbf{x}_i = \sum_{i=1}^{2} \mu_i \mathbf{x}_i + \mu_3 \mathbf{x}_4, \sum_{i=1}^{3} \lambda_i = \sum_{i=1}^{3} \mu_i = 1$. See Figure 2.3 for reference of the B-nets of P_n and Q_n where $n = 5$.

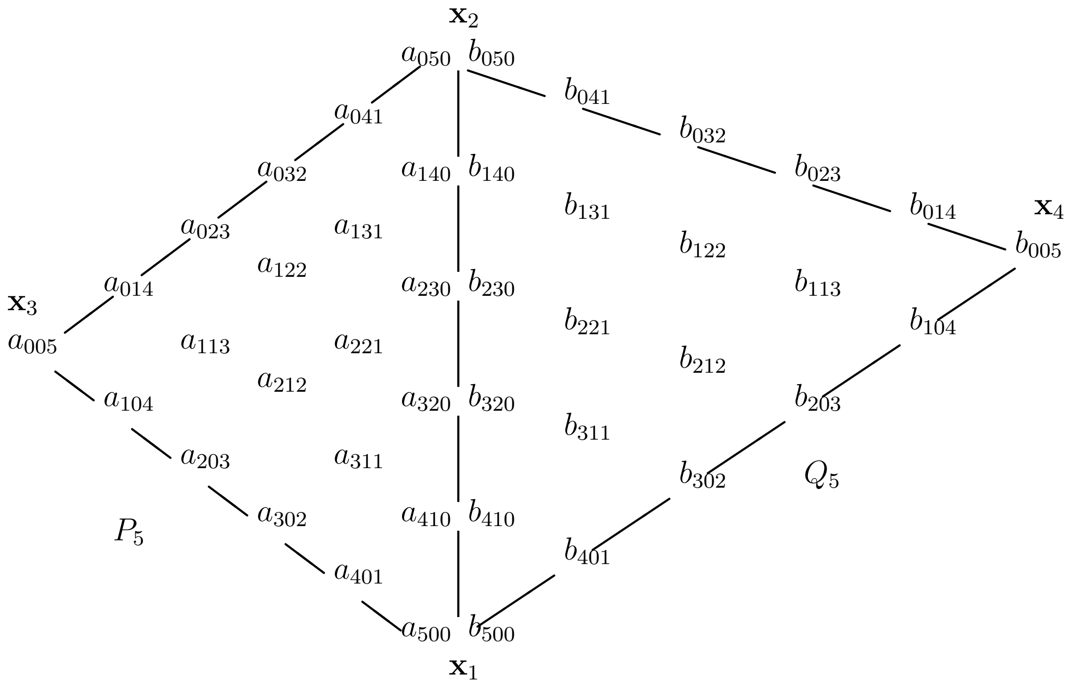


Figure 2.3 The B-nets of P_5 and Q_5

Write $\mathbf{x}_4 = \lambda_1^0 \mathbf{x}_1 + \lambda_2^0 \mathbf{x}_2 + \lambda_3^0 \mathbf{x}_3$ with $\lambda_1^0 + \lambda_2^0 + \lambda_3^0 = 1$. We denote $\lambda^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0)$ and define by

$$D_{\mathbf{x}_4 - \mathbf{x}_1} = \lambda_2^0 D_{\mathbf{x}_2 - \mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_3 - \mathbf{x}_1}$$

the directional derivative operator along $\langle \mathbf{x}_1, \mathbf{x}_4 \rangle$. Let F be a piecewise polynomial function defined as follows:

$$F(\mathbf{x}) = \begin{cases} P_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_1 \\ Q_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_2. \end{cases}$$

Then, $F \in C^r(T_1 \cup T_2)$ if and only if

$$(D_{\mathbf{x}_4 - \mathbf{x}_1})^k Q_n \mid_{T_1 \cap T_2} = (\lambda_2^0 D_{\mathbf{x}_2 - \mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_3 - \mathbf{x}_1})^k P_n \mid_{T_1 \cap T_2} \tag{2.3.1}$$

for $0 \le k \le r$. Then the smoothness conditions between P_n and Q_n to be stated in Lemma 2.1 and Lemma 2.2 follow easily from (2.3.1)

LEMMA 2.1. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$\Delta_{31}^k b_{ij0} = (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31})^k a_{ij0}, i + j = n - k \tag{2.3.2}$$

for $0 \le k \le r$.

The proof of this lemma may be found in [36]. By using the inversion formula in the previous section, we will reach the following:

LEMMA 2.2. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$b_{ijk} = \sum_{|\beta|=k} a_{(ij0)+\beta} \Phi_\beta(\lambda^0), \qquad 0 \le k \le r. \tag{2.3.3}$$

This lemma was earlier proved in [70] by a different method. Refer to [39] for details of a proof of this lemma.

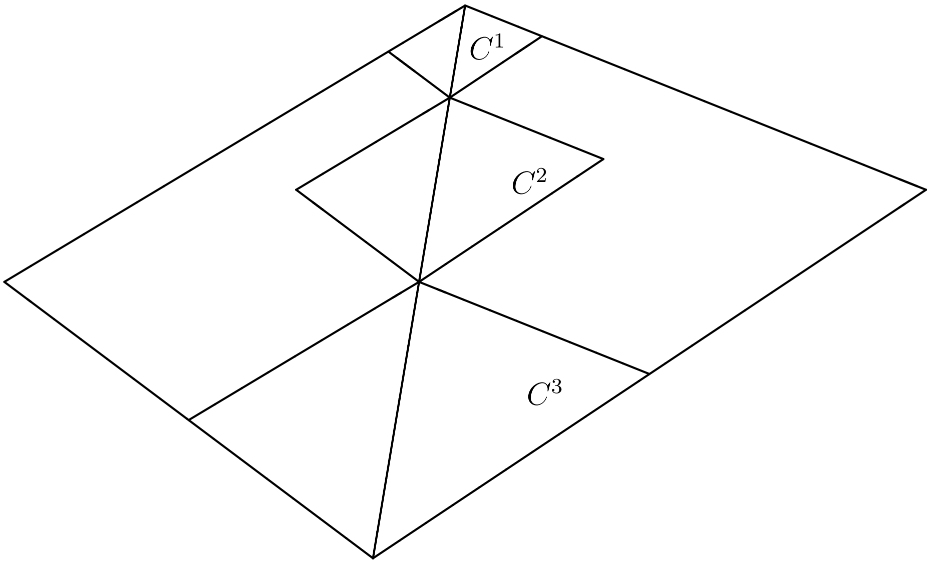


Figure 2.4 The supports of C^1, C^2 and C^3 smoothness conditions

The supports of these smoothness conditions (2.3.2) or (2.3.3) are as shown as in Figure 2.4 above. The geometric interpolation of these conditions may be found elsewhere. (see, e.g., [39, 53].)

When two polynomials P_n and Q_n are joined smoothly, certain directional derivatives of P_n and Q_n at vertex $\mathbf{x}_1$ must match. Actually, we may know more from the following lemma.

LEMMA 2.3. *Let* $M_{n,r} = \{\alpha \in \mathbb{Z}_+^3 : \alpha_3 \leq r, |\alpha| = n\}$ *and* $M_1, M_2 \subset \mathbb{Z}_+^2$ *be two lower subsets satisfying* $A_1^n M_1 \cap A_2^n M_2 = \emptyset$ *and* $A_1^n M_1 \cup A_2^n M_2 = M_{n,r}$*. Then* $F \in C^r(T_1 \cup T_2)$ *if and only if* F *satisfies the following matching conditions*

$$(2.3.4)_a \qquad (D_{\mathbf{x}_4-\mathbf{x}_1})^i (D_{\mathbf{x}_2-\mathbf{x}_1})^j Q_n(\mathbf{x}_1) = (\lambda_2^0 D_{\mathbf{x}_2-\mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_3-\mathbf{x}_1})^i (D_{\mathbf{x}_2-\mathbf{x}_1})^j P_n(\mathbf{x}_1)$$

for $(i,j) \in M_1$ *and*

$$(2.3.4)_b \qquad (D_{\mathbf{x}_4-\mathbf{x}_2})^i (D_{\mathbf{x}_1-\mathbf{x}_2})^j Q_n(\mathbf{x}_2) = (\lambda_1^0 D_{\mathbf{x}_1-\mathbf{x}_2} + \lambda_3^0 D_{\mathbf{x}_3-\mathbf{x}_2})^i (D_{\mathbf{x}_1-\mathbf{x}_2})^j P_n(\mathbf{x}_2)$$

for $(i,j) \in M_2$.

The proof and more general results along this line may be found in [39].

Further we apply smoothness conditions (2.3.1) or (2.3.2) to make F smooth across edge $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$ when its partial B-coefficients are given. We have the following two lemmas (cf.[37]).

LEMMA 2.4. *Assume that* $\mathbf{x}_2 \notin [\mathbf{x}_3, \mathbf{x}_4]$ *and* $l \leq \frac{n-2}{2}$ *is an integer. Suppose that the following B-coefficients of* P_n *and* Q_n

$$\{a_\beta, b_\beta : \beta_2 \geq 1\}$$

and

$$\{a_\beta, b_\beta : \beta_2 = 0 \text{ and } 0 \leq \beta_3 \leq n - 2l - 2\}$$

are given, and that $\{a_\beta : |\beta| = n\}$ *and* $\{b_\beta : |\beta| = n\}$ *satisfy the smoothness conditions (2.3.1) of order* $n-2l-2$*. If* $\{a_\beta : \beta_2 \geq 1\}$ *and* $\{b_\beta : \beta_2 \geq 1\}$ *also satisfy the smoothness conditions (2.3.1) of order* $n-1$*, then for any given* $\{a_\beta, b_\beta : \beta_2 = 0 \text{ and } 0 \leq \beta_1 \leq l\}$*, there exists a unique set of coefficients* $\{a_\beta, b_\beta : \beta_2 = 0 \text{ and } l+1 \leq \beta_1 \leq 2l+1\}$ *such that* $\{a_\beta : |\beta| = n\}$ *and* $\{b_\beta : |\beta| = n\}$ *satisfy the smoothness conditions (2.3.1) of order* n.

LEMMA 2.5. *Assume* $\mathbf{x}_2 \in [\mathbf{x}_3, \mathbf{x}_4]$*. Suppose that the B-coefficients* $\{a_\beta : \beta_2 \geq 1\}$ *and* $\{b_\beta : \beta_2 \geq 1\}$ *of* P_n *and* Q_n *are given and satisfy the smoothness conditions (2.3.1) up to order* $n-1$*. Furthermore, suppose that* $\{a_\beta : \beta_2 = 0 \text{ and } 0 \leq \beta_3 \leq l\}$ *and* $\{b_\beta : \beta_2 = 0 \text{ and } 0 \leq \beta_3 \leq l\}$ *are given and satisfy the smoothness conditions*

(2.3.1) of order l, where $l < n$. Then for any $\{a_\beta : \beta_2 = 0,$ and $0 \le \beta_1 \le n - l - 1\}$, there exists a unique set of coefficients $\{b_\beta : \beta_2 = 0,$ and $0 \le \beta_1 \le n - l - 1\}$ such that $\{a_\beta : |\beta| = n\}$ and $\{b_\beta : |\beta| = n\}$ satisfy the smoothness conditions (2.3.1) of order n.

We refer to [37] for the proofs of Lemma 2.4 and Lemma 2.5.

Remark. The solution set $\{a_\beta, b_\beta\}$ in Lemma 2.4 actually depends on the geometry of the triangles $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ and $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4 \rangle$. More precisely, each a_β or b_β depends on certain powers of $(\lambda_1^0)^{-1}$ and $(\lambda_3^0)^{-1}$. Thus, if the area $|\langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle|$ of the triangle $\langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle$ is very small so that λ_1^0 is very close to zero, then the magnitude of a_β and b_β would be very large. For this reason, we need the notion of "near-singularity".

For a given mixed partition $\triangle$, let $[\mathbf{x}_1, \mathbf{x}_2]$ be an interior edge of $\triangle$ shared by two patches (two triangles or one triangle and one parallelogram or two parallelograms). Denote by $\mathbf{x}_3$ and $\mathbf{x}_4$ the vertices for which $[\mathbf{x}_2, \mathbf{x}_3]$ and $[\mathbf{x}_2, \mathbf{x}_4]$ are two edges of the patches. Then the edge $[\mathbf{x}_1, \mathbf{x}_2]$ is called a *near-singular edge at* $\mathbf{x}_2$ if $|\langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle| > 0$ is near zero; e.g., $0 < \lambda_1^0 << a$, where $a = \max\{\lambda_1^0, (\lambda_1^0)^{-1}\}$. If $|\langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle| = 0$, then the edge $[\mathbf{x}_1, \mathbf{x}_2]$ is called a *singular edge at* $\mathbf{x}_2$. Also, an interior vertex $\mathbf{v}$ is said to be a *singular vertex* if it is the point of intersection of four edges with only two distinct slopes (cf. Figure 2.5, 2.8, 2.9, 2.13, 2.14, 2.15). An interior vertex $\mathbf{v}$ is said to be a *near-singular vertex* if it is the intersection point of four near-singular edges at $\mathbf{v}$ with at least three distinct slopes.

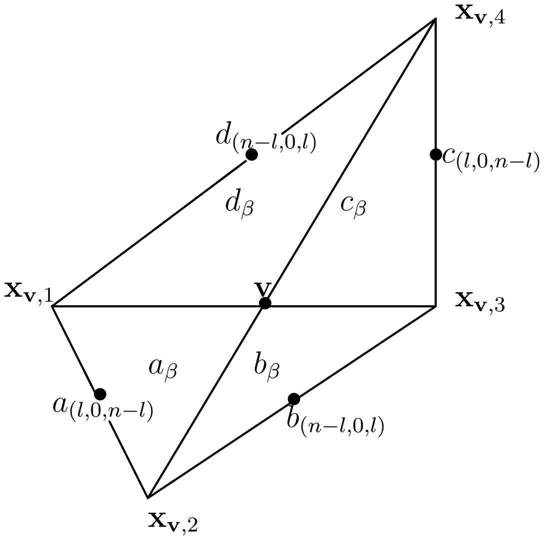


Figure 2.5 Four triangles attach at $\mathbf{v}$

The following lemma is another application of smoothness conditions (2.3.1).

LEMMA 2.6. *Let* $\mathbf{v}$ *be a single vertex such that four patches attached at* $\mathbf{v}$ *are all triangles as shown in Figure 2.5 above. Assume that the B-coefficients* $\{a_\beta, b_\beta, c_\beta, d_\beta : \beta_2 \geq 1\}$ *on the four triangles are given and satisfy the smoothness conditions of order* $n-1$ *(cf. Figure 2.5). Then for any given* $a_{(l,0,n-l)}$*, there exists a unique set of coefficients* $b_{(n-l,0,l)}, c_{(l,0,n-l)}$ *and* $d_{(n-l,0,l)}$ *that satisfy the smoothness conditions (2.3.1) of order* n*, where* $0 \leq l \leq n$.

The proof of this lemma may also be found in [37].

2° Suppose that P_n and $\tilde{Q}_n$ are defined on a triangle $T_1 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_5 \rangle$ and a parallelogram $T_2 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle$, respectively, where T_1 and T_2 share a common edge $[\mathbf{x}_1, \mathbf{x}_2]$. Write

$$P_n(\mathbf{x}) = \sum_{|\beta|=n} a_\beta \Phi_\beta(\lambda)$$

and

$$\tilde{Q}_n(\mathbf{x}) = \sum_{\alpha \leq (n,n)} b_\alpha \tilde{\Phi}^n_\alpha(\mu)$$

where $\mathbf{x} = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_5$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\mathbf{x} = \mathbf{x}_1 + \mu_1(\mathbf{x}_2 - \mathbf{x}_1) + \mu_2(\mathbf{x}_3 - \mathbf{x}_1)$. See Figure 2.6 for the B-coefficients of P_n and $\tilde{Q}_n$.

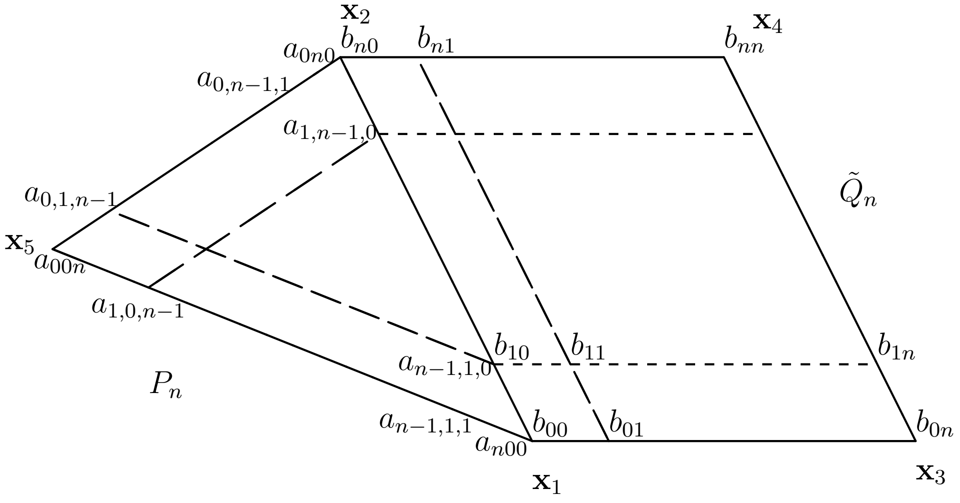


Figure 2.6 The B-nets of P_n and $\tilde{Q}_n$

Write $\mathbf{x}_3 = \lambda^0_1 \mathbf{x}_1 + \lambda^0_2 \mathbf{x}_2 + \lambda^0_3 \mathbf{x}_5$ with $\lambda^0_1 + \lambda^0_2 + \lambda^0_3 = 1$. It is clear that $D_{\mathbf{x}_3 - \mathbf{x}_1} = \lambda^0_2 D_{\mathbf{x}_2 - \mathbf{x}_1} + \lambda^0_3 D_{\mathbf{x}_5 - \mathbf{x}_1}$.

Let F be a piecewise polynomial defined by

$$F = \begin{cases} P_n & \text{if } \mathbf{x} \in T_1 \\ \tilde{Q}_n & \text{if } \mathbf{x} \in T_2. \end{cases}$$

Clearly, $F \in C^r(T_1 \cup T_2)$ if and only if

$$(D_{\mathbf{x}_3-\mathbf{x}_1})^k \tilde{Q}_n \mid_{T_1\cap T_2} = (\lambda_2^0 D_{\mathbf{x}_2-\mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_5-\mathbf{x}_1})^k P_n \mid_{T_1\cap T_2}$$

for $0 \le k \le r$. The following result is then an easy consequence.

LEMMA 2.7. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$\Delta_2^k b_{i0} = (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31})^k \mathbf{R}^k a_{n-i,i,0}, \qquad 0 \le i \le n \tag{2.3.5}$$

for $0 \le k \le r$, where $\mathbf{R}$ is a degree raising operator defined by

$$\mathbf{R} a_{i,j,k} = \frac{1}{i+j+k}(i a_{i-1,j,k} + j a_{i,j-1,k} + k a_{i,j,k-1}).$$

Proof. Indeed,

$$(D_{\mathbf{x}_3-\mathbf{x}_1})^k Q_n \mid_{T_1\cap T_2} = \frac{n!}{(n-k)!} \sum_{\alpha \le (n,0)} \Delta_2^k b_\alpha \tilde{\Phi}_\alpha^{(n,0)}(\mu_1, 0)$$

and

$$\begin{aligned}
&(\lambda_2^0 D_{\mathbf{x}_2-\mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_5-\mathbf{x}_1})^k P_n \mid_{T_1\cap T_2} \\
= &\left(\sum_{i+j=k} (\lambda_2^0)^i (\lambda_3^0)^j \frac{k!}{i!j!} D^i_{\mathbf{x}_2-\mathbf{x}_1} D^j_{\mathbf{x}_5-\mathbf{x}_1} P_n \right) \mid_{T_1\cap T_2} \\
= &\frac{n!}{(n-k)!} \sum_{i+j=k} (\lambda_2^0)^i (\lambda_3^0)^j \frac{k!}{i!j!} \sum_{|\beta|=n-k} \Delta_{21}^i \Delta_{31}^j a_\beta \Phi_\beta(\lambda_1, \lambda_2, 0) \\
= &\frac{n!}{(n-k)!} \sum_{\substack{|\beta|=n-k \\ \beta=(\beta_1,\beta_2,0)}} \sum_{i+j=k} \frac{k!}{i!j!} (\lambda_2^0)^i (\lambda_3^0)^j \Delta_{21}^i \Delta_{31}^j a_\beta \Phi_\beta(\lambda_1, \lambda_2, 0) \\
= &\frac{n!}{(n-k)!} \sum_{\substack{|\beta|=n-k \\ \beta=(\beta_1,\beta_2,0)}} (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31})^k a_\beta \Phi_\beta(\lambda_1, \lambda_2, 0) \\
= &\frac{n!}{(n-k)!} (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31})^k \sum_{\substack{|\beta|=n \\ \beta=(\beta_1,\beta_2,0)}} \mathbf{R}^k a_\beta \Phi_\beta(\lambda_1, \lambda_2, 0).
\end{aligned}$$

Since

$$\tilde{\Phi}^{(n,0)}_{(i,0)}(\mu_1, 0) = \Phi_{(n-i,i,0)}(1-\mu_1, \mu_1, 0), \qquad i = 0, 1, \cdots, n,$$

we have established the lemma.

Since $\sum_{i=1}^{3} \lambda_i = 1$ and

$$\left(\sum_{i=1}^{3} \lambda_i \right)^k \sum_{|\beta|=l} a_\beta \Phi_\beta(\lambda_1, \lambda_2, \lambda_3)$$

$$
\begin{aligned}
&= \sum_{|\beta|=l} a_\beta \left(\sum_{i=1}^{3}\lambda_i\right)^k \Phi_\beta(\lambda_1,\lambda_2,\lambda_3) \\
&= \sum_{|\beta|=l} a_\beta \sum_{|\alpha|=k} \frac{k!}{\alpha!}\lambda^\alpha \frac{l!}{\beta!}\lambda^\beta \\
&= \sum_{|\gamma|=l+k} \sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|=k,|\beta|=l}} a_\beta \frac{k!l!}{(l+k)!}\frac{\gamma!}{\alpha!\beta!}\Phi_\gamma(\lambda),
\end{aligned}
$$

we have

$$
\mathbf{R}^k a_\gamma = \sum_{\substack{\beta\le\gamma \\ |\beta|=l}} a_\beta \frac{\binom{\gamma}{\beta}}{\binom{l+k}{k}} \tag{2.3.6}
$$

which may also be found in [71]. Hence, we may rewrite Lemma 2.7 as follows.

LEMMA 2.7′ $F \in C^r(T_1 \cup T_2)$ *if and only if*

$$
\Delta_2^k b_{i0} = \sum_{\substack{\beta\le(n-i,i,0) \\ |\beta|=n-k}} (\lambda_2^0\Delta_{21} + \lambda_3^0\Delta_{31})^k a_\beta \frac{\binom{n-i}{\beta_1}\binom{i}{\beta_2}}{\binom{n}{k}}, \qquad 0 \le i \le n \tag{2.3.7}
$$

for $0 \le k \le r$.

The supports of the C^1 and C^2 smoothness conditions (2.3.7) are as shown in Figure 2.7.

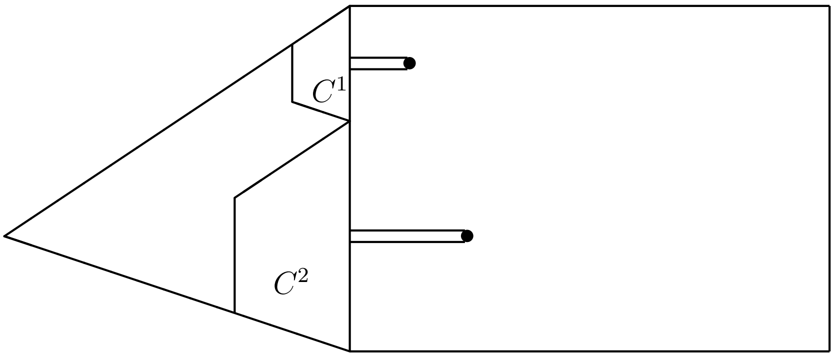


Figure 2.7 The supports of the C^1 and C^2 smoothness conditions

Example 2.3.

$$
\begin{aligned}
C^0: \quad & b_{i0} = a_{n-i,i,0} \qquad 0 \le i \le n \\
C^1: \quad & b_{i1} - b_{i0} = \lambda_2^0(1 - \frac{i}{n})(a_{n-i-1,i+1,0} - a_{n-i,i,0})
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \lambda_2^0 \frac{i}{n}(a_{n-i,i,0} - a_{n-i+1,i-1,0}) \\
&+ \quad \lambda_3^0 (1 - \frac{i}{n})(a_{n-i-1,i,1} - a_{n-i,i,0}) \\
&+ \quad \lambda_3^0 \frac{i}{n}(a_{n-i,i-1,1} - a_{n-i+1,i-1,0}), \quad 0 \le i \le n,
\end{aligned}
$$

etc..

Also, we can prove the following lemma

LEMMA 2.8. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$
(D_{\mathbf{x}_2-\mathbf{x}_1})^i (D_{\mathbf{x}_3-\mathbf{x}_1})^j \tilde{Q}_n(\mathbf{x}_1) = (D_{\mathbf{x}_2-\mathbf{x}_1})^i (\lambda_2^0 D_{\mathbf{x}_2-\mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_5-\mathbf{x}_1})^j P_n(\mathbf{x}_1) \tag{2.3.8}
$$

for $0 \le i \le n, 0 \le j \le r$.

Proof. Clearly, $F \in C^r(T_1 \cup T_2)$ implies (2.3.8). On the other hand,

$$
\begin{aligned}
& (D_{\mathbf{x}_2-\mathbf{x}_1})^i (D_{\mathbf{x}_3-\mathbf{x}_1})^j \tilde{Q}_n(\mathbf{x}_1) \\
= \quad & \frac{n!}{(n-i)!} \frac{n!}{(n-j)!} \Delta_1^i \Delta_2^j b_{00} \\
= \quad & \frac{n!}{(n-i)!} \frac{n!}{(n-j)!} \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} \Delta_2^j b_{k0}
\end{aligned}
$$

and

$$
\begin{aligned}
& (D_{\mathbf{x}_2-\mathbf{x}_1})^i (\lambda_2^0 D_{\mathbf{x}_2-\mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_5-\mathbf{x}_1})^j P_n(\mathbf{x}_1) \\
= \quad & (D_{\mathbf{x}_2-\mathbf{x}_1})^i \frac{n!}{(n-j)!} \sum_{|\beta|=n} (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31})^j \mathbf{R}^j a_\beta \Phi_\beta(\lambda) \,|_{\mathbf{x}_1} \\
= \quad & \frac{n!}{(n-i)!} \frac{n!}{(n-j)!} \Delta_{21}^i (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31})^j \mathbf{R}^j a_{n-i,0,0}.
\end{aligned}
$$

Now we use the inversion formula in §2.2 to invert $\Delta_2^j b_{k0}$ from the first equation and use the second one to deduce

$$
\begin{aligned}
\Delta_2^j b_{k0} \quad &= \quad \sum_{l=0}^{k} \binom{k}{l} (-1)^l (-1)^l \left(\frac{n!}{(n-l)!} \frac{n!}{(n-j)!} \right)^{-1} (D_{\mathbf{x}_2-\mathbf{x}_1})^l (D_{\mathbf{x}_3-\mathbf{x}_1})^j \tilde{Q}_n(\mathbf{x}_1) \\
&= \quad \sum_{l=0}^{k} \binom{k}{l} \Delta_{21}^l (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31})^j \mathbf{R}^j a_{n-l,0,0} \\
&= \quad (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31})^j \mathbf{R}^j a_{n-k,k,0}.
\end{aligned}
$$

for $0 \le k \le n$, and $0 \le j \le r$. We use Lemma 2.7 to complete the proof of this lemma.

Further, we apply the smoothness conditions (2.3.5) to make F smooth across the edge $[\mathbf{x}_1, \mathbf{x}_2]$ when some of B-coefficients of P_n and $\tilde{Q}_n$ are known. We have the follow lemmas.

LEMMA 2.9. *Assume that* $\mathbf{x}_1 \notin [\mathbf{x}_3, \mathbf{x}_5]$. *Suppose the following part of B-coefficients of* P_n *and* Q_n *are given:* a_α *with* $\alpha_1 \geq 1$ *and* $|\alpha| = n$, *and* $b_{\beta_1\beta_2}, \beta_1 + \beta_2 \leq n-1$ *as well as* $a_{(0,n-k,k)}, b_{n-k,k}, 0 \leq k \leq n-2l-2$, *where* $l \leq \frac{n-2}{2}$. *(See Figure 2.6 for the reference of the orientation of B-nets on triangle and parallelogram.) Further, suppose that* a_α, *with* $\alpha_1 \geq 1$ *and* $|\alpha| = n-1$ *and* $b_{\beta_1\beta_2}, \beta_1 + \beta_2 \leq n-1$ *satisfy the smoothness conditions up to order* $n-1$ *and* $b_{(n-k,k)}, a_{(0,n-k,k)}, 0 \leq k \leq n-2l-2$, *together with other given* a_α*'s satisfy the smoothness conditions of order* $n-2l-2$. *Then given any* $a_{(0,k,n-k)}, b_{(k,n-k)}, 0 \leq k \leq l$, *there exists a unique set of coefficients* $a_{(0,l+k,n-l-k)}, b_{(k+l,n-k-l)}, 1 \leq k \leq l+1$ *such that* $a_\alpha, |\alpha| = n$ *and* $b_{(i,j)}, i+j \leq n$ *satisfy the smoothness conditions up to order* n.

Proof. We only need to prove that there exists a unique solution set $\{a_{(0,l+k,n-l-k)}, b_{(l+k,n-l-k)} : 1 \leq k \leq l+1\}$ such that the following smoothness conditions

$$\Delta_2^k b_{n-k,0} = \sum_{\substack{\beta \leq (k,n-k,0) \\ |\beta| = n-k}} (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31})^k a_\beta \frac{\binom{n-k}{\beta_2}\binom{k}{\beta_1}}{\binom{n}{k}}, \qquad n-2l-1 \leq k \leq n$$

hold. Thus we have $2l+2$ equations and $2l+2$ unknowns $\{a_{(0,l+k,n-l-k)}, b_{(l+k,n-l-k)} : 1 \leq k \leq l+1\}$. The linear system may be decomposed into two smaller linear subsystems:

$$\Delta_2^{n-i} b_{i0} = \sum_{\substack{\beta \leq (n-i,i,0) \\ |\beta| = i}} (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31})^{n-i} a_\beta \frac{\binom{n-i}{\beta_1}\binom{i}{\beta_2}}{\binom{n}{i}}, \qquad l+1 \leq i \leq 2l+1$$

and

$$\Delta_2^k b_{i0} = (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31})^k \left(\frac{a_{(0,n-k,0)}}{\binom{n}{k}} + \sum_{\substack{\beta \leq (k,n-k,0) \\ |\beta| = n-k \\ \beta \neq (0,n-k,0)}} a_\beta \frac{\binom{k}{\beta_1}\binom{n-k}{\beta_2}}{\binom{n}{k}} \right)$$

where $n-l \leq k \leq n$, which may be rewritten as

$$\sum_{i=0}^{k} \binom{k}{i} (\lambda_2^0)^i (\lambda_3^0)^{k-i} a_{(0,n-k+i,k-i)} \frac{1}{\binom{n}{k}} = c_k, \qquad n-l \leq k \leq n$$

where $c_{n-l}, \cdots, c_n$ are certain constants involving the given a_α's and b_β's.

Further, it may be rewritten as

$$\sum_{i=0}^{n-k} \binom{n-k}{i} (\lambda_2^0)^i (\lambda_3^0)^{n-k-i} a_{(0,k+i,n-k-i)} \frac{1}{\binom{n}{k}} = c_{n-k}, \qquad 0 \le k \le l$$

or

$$\sum_{i=l+1-k}^{2l+1-k} \binom{n-k}{i} (\lambda_2^0)^i (\lambda_3^0)^{n-k-i} a_{(0,k+i,n-k-i)} \frac{1}{\binom{n}{k}} = \tilde{c}_{n-k}, \qquad 0 \le k \le l$$

or

$$\sum_{j=l+1}^{2l+1} \binom{n-k}{j-k} (\lambda_2^0)^{j-k} (\lambda_3^0)^{n-j} a_{(0,j,n-j)} \frac{1}{\binom{n}{k}} = \tilde{c}_{n-k}, \qquad 0 \le k \le l.$$

The above linear system has a unique solution $\{a_{(0,l+k,n-l-k)} : 1 \le k \le l+1\}$ because the determinant of its coefficients matrix may be simplified to be

$$\det \begin{bmatrix} \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{(l+1)!} \\ \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(l+2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(l+1)!} & \frac{1}{(l+2)!} & \cdots & \frac{1}{(2l+1)!} \end{bmatrix} = \frac{\Pi_{i=1}^{l+1}(l+1-i)!}{\Pi_{i=1}^{l+1}(2l+2-i)!} \ne 0.$$

Then substituting the values $\{a_{(0,l+k,n-l-k)} : 1 \le k \le l+1\}$ into the first subsystem, we also uniquely determine $\{b_{(k+l,n-k-l)} : 1 \le k \le l+1\}$. This completes the proof.

LEMMA 2.10. *Assume that* $\mathbf{x}_1 \in \langle \mathbf{x}_3, \mathbf{x}_5 \rangle$. *Suppose that the B-coefficients* $\{a_\beta : \beta_1 \ge 1\}$ *and* $\{b_{(\beta_1 \beta_2)} : \beta_1 + \beta_2 \le n-1\}$ *are given and satisfy the smoothness conditions (2.3.5) up to order* $n-1$. *Furthermore, suppose that* $\{a_\beta : \beta_1 = 0$ *and* $0 \le \beta_3 \le l\}$ *and* $\{b_{(j,n-j)} : 0 \le j \le l\}$ *are given and satisfy the smoothness conditions (2.3.5) of order* l, *where* $l < n$. *Then for any* $\{a_\beta : \beta_2 = 0$ *and* $0 \le \beta_2 \le n-l-1\}$, *there exists a unique set of coefficients and* $\{b_{(n-j,j)} : 0 \le j \le n-l-1\}$ *such that* $\{a_\beta : |\beta| = n\}$ *and* $\{b_{(i,j)} : i+j \le n\}$ *satisfy the smoothness conditions (2.3.5).*

Proof. This result is a simple consequence of Lemma 2.7.

Here, we have two other applications of the smoothness conditions (2.3.5).

LEMMA 2.11. *Let* $\mathbf{v}$ *be an interior and single vertex such that four patches attached at* $\mathbf{v}$ *are three triangles and one parallelogram as shown in Figure 2.8. Assume that the following B-coefficients* $\{a_\beta : \beta_1 \ge 1\}, \{b_{ij} : i+j \le n-1\}, \{c_\beta : \beta_1 \ge 1\}$, *and* $\{d_\beta : \beta_1 \ge 1\}$ *on these four patches, respectively are given and satisfy the smoothness conditions (2.3.1) and (2.3.5) up to order* $n-1$ *(cf. Figure 2.8). Then for any given* $d_{(0,n-l,l)}$, *there exists a unique set of coefficients* $a_{(0,l,n-l)}, b_{l,n-l}$, *and* $c_{(0,l,n-l)}$ *that satisfy the smoothness conditions (2.3.5), where* $0 \le l \le n$.

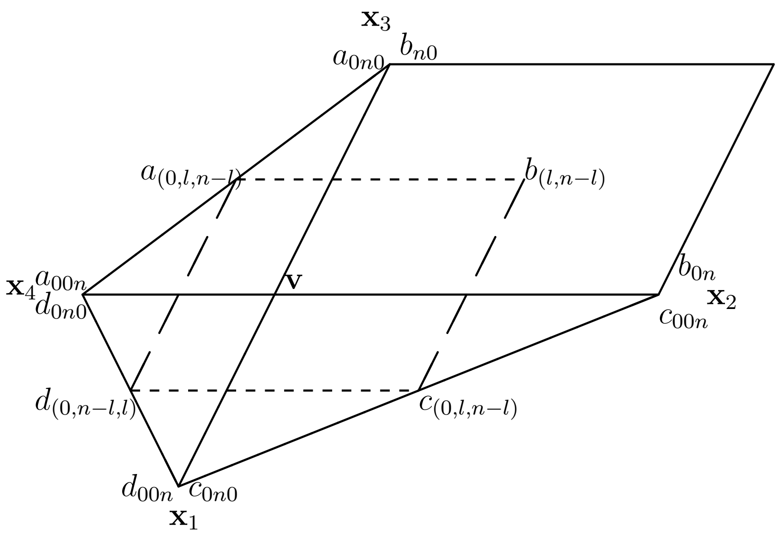


Figure 2.8 Three triangles and one parallelogram attach at $\mathbf{v}$

Proof. By using Lemma 2.1 and Lemma 2.7, for any given $d_{(0,n-l,l)}$, the values $a_{(0,l,n-l)}$, $b_{(l,n-l)}$ are consecutively determined and $c_{(0,l,n-l)}$ is also determined by (2.3.1) from $d_{(0,n-l,l)}$. To show that $b_{(l,n-l)}$ and $c_{(0,l,n-l)}$ satisfy the smoothness condition connecting them, we may assume without loss of generality that the given B-coefficients a_β's, b_{ij}'s, c_β's, and d_β's are equal to zero and obtain

$$a_{(0,l,n-l)} = \left(\frac{|\mathbf{x}_3 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_1|}\right)^l d_{(0,n-l,l)}, \tag{2.1}$$

$$\begin{aligned} b_{(l,n-l)} &= \left(\frac{|\mathbf{x}_2 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_4|}\right)^{n-l} \sum_{\substack{\gamma \le (n-l,l,0) \\ |\gamma| = l}} \Delta_{31}^{n-l} a_\gamma \frac{\binom{n-l}{\gamma_1}\binom{l}{\gamma_2}}{\binom{n}{n-l}} \\ &= \left(\frac{|\mathbf{x}_2 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_4|}\right)^{n-l} a_{(0,l,n-l)} \frac{1}{\binom{n}{n-l}}, \end{aligned} \tag{2.2}$$

and

$$c_{(0,l,n-l)} = \left(\frac{|\mathbf{x}_2 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_4|}\right)^{n-l} d_{(0,n-l,l)}. \tag{2.3}$$

Also, we need to establish that the following relation:

$$b_{(l,n-l)} = \left(\frac{|\mathbf{x}_3 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_1|}\right)^l \sum_{\substack{\gamma \le (l,0,n-l) \\ |\gamma| = n-l}} \Delta_{21}^{l} c_\gamma \frac{\binom{l}{\gamma_1}\binom{n-l}{\gamma_3}}{\binom{n}{l}}$$

$$= \left(\frac{|\mathbf{x}_3 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_1|}\right)^l c_{(0,l,n-l)} \frac{1}{\binom{n}{l}},$$

which is the smoothness condition connecting $b_{(l,n-l)}$ and $c_{(0,l,n-l)}$. This can be done after solving $d_{(0,n-l,l)}$ from (2.3) and substituting the resulting $d_{(0,n-l,l)}$ to (2.1) and then (2.1) to (2.2). Thus, we have completed the proof of this lemma.

LEMMA 2.12. *Let* $\mathbf{v}$ *be an interior and singular vertex such that four patches attached at* $\mathbf{v}$ *are two triangles and two parallelograms as shown in Figure 2.9. Assume that the following B-coefficients* $\{a_\beta : \beta_1 \geq 1\}, \{b_{ij} : i + j \leq n - 1\}, \{c_\beta : \beta_1 \geq 1\}$, *and* $\{d_{ij} : i + j \leq n - 1\}$ *on the four patches respectively are given and satisfy the smoothness conditions (2.3.1) and (2.3.5) up to order* $n - 1$. *Then for any given* $a_{(0,l,n-l)}$, *there exists a unique set of coefficients* $b_{(l,n-l)}, c_{(0,l,n-l)}, d_{(l,n-l)}$ *that satisfy the smoothness conditions (2.3.1) or (2.3.5) connecting them, where* $0 \leq l \leq n$.

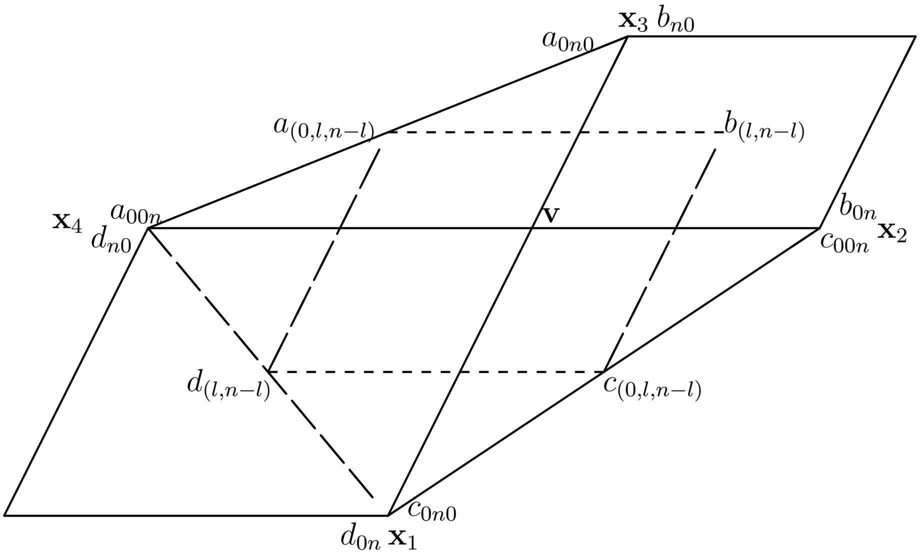


Figure 2.9 Two triangles and two parallelograms attach at $\mathbf{v}$

Proof. By Lemma 2.1 and Lemma 2.7, for any given $a_{(0,l,n-l)}$ the values $b_{(l,n-l)}, d_{(l,n-l)}$ are determined from $a_{(0,l,n-l)}$ by using the smoothness conditions (2.3.5). Also $c_{(0,l,n-l)}$ can be obtained either from $d_{(l,n-l)}$ or $b_{(l,n-l)}$. To show that $a_{(0,l,n-l)}, b_{l,n-l}, c_{(0,l,n-l)}$, and $d_{(l,n-l)}$ satisfy the smoothness conditions (2.3.1) and (2.3.5) among them, we may assume without loss of generality that the given B-coefficients a_β's, b_{ij}'s, c_β's, d_{ij}'s are equal to zero. Then we obtain

$$b_{(l,n-l)} = \left(\frac{|\mathbf{x}_2 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_4|}\right)^{n-l} \sum_{\substack{\gamma \leq (n-l,l,0) \\ |\gamma| = l}} \Delta_{31}^{n-l} a_\gamma \frac{\binom{n-l}{\gamma_1}\binom{l}{\gamma_2}}{\binom{n}{n-l}} \tag{2.4}$$

$$= \left(\frac{|\mathbf{x}_2 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_4|}\right)^{n-l} a_{(0,l,n-l)} \frac{1}{\binom{n}{l}},$$

$$d_{(l,n-l)} = \left(\frac{|\mathbf{x}_1 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_3|}\right)^{l} \sum_{\substack{\gamma \le (l,0,n-l) \\ |\gamma| = n-l}} \Delta_{21}^{l} a_\gamma \frac{\binom{l}{\gamma_1}\binom{n-l}{\gamma_2}}{\binom{n}{l}} \tag{2.5}$$

$$= \left(\frac{|\mathbf{x}_1 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_3|}\right)^{l} a_{(0,l,n-l)} \frac{1}{\binom{n}{l}},$$

and

$$b_{(l,n-l)} = \left(\frac{|\mathbf{x}_3 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_1|}\right)^{l} \sum_{\substack{\gamma \le (l,0,n-l) \\ |\gamma| = n-l}} \Delta_{21}^{l} c_\gamma \frac{\binom{l}{\gamma_1}\binom{n-l}{\gamma_3}}{\binom{n}{l}} \tag{2.6}$$

$$= \left(\frac{|\mathbf{x}_3 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_1|}\right)^{l} c_{(0,l,n-l)} \frac{1}{\binom{n}{l}}.$$

We need to prove that $d_{l,n-l}$ and $c_{(0,l,n-l)}$ satisfy the relation

$$d_{(l,n-l)} = \left(\frac{|\mathbf{x}_4 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_2|}\right)^{n-l} \sum_{\substack{\gamma \le (n-l,l,0) \\ |\gamma| = l}} \Delta_{31}^{n-l} c_\gamma \frac{\binom{n-l}{\gamma_1}\binom{l}{\gamma_2}}{\binom{n}{n-l}} \tag{2.7}$$

$$= \left(\frac{|\mathbf{x}_4 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_2|}\right)^{n-l} c_{(0,l,n-l)} \frac{1}{\binom{n}{l}}.$$

Indeed, solving $a_{(0,l,n-l)}$ from (2.4) and then substituting it into (2.5) and (2.6) into the resulting equation, we arrive at (2.7). Thus, we have established this lemma.

3°. Suppose that $\tilde{P}_n$ and $\tilde{Q}_n$ are defined on two adjacent parallelograms $T_1 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle$ and $T_2 = \langle \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6 \rangle$ which share a common edge $\langle \mathbf{x}_3, \mathbf{x}_4 \rangle$. More precisely, let

$$\tilde{P}_n(\mathbf{x}) = \sum_{\beta \le (n,n)} a_\beta \tilde{\Phi}_\beta^n(\lambda), \qquad \mathbf{x} = \mathbf{x}_3 + \lambda_1(\mathbf{x}_1 - \mathbf{x}_3) + \lambda_2(\mathbf{x}_4 - \mathbf{x}_3),$$

and

$$\tilde{Q}_n(\mathbf{x}) = \sum_{\beta \le (n,n)} b_\beta \tilde{\Phi}_\beta^n(\mu), \qquad \mathbf{x} = \mathbf{x}_3 + \mu_1(\mathbf{x}_5 - \mathbf{x}_3) + \mu_2(\mathbf{x}_4 - \mathbf{x}_3)$$

See Figure 2.10 for reference of the B-nets of $\tilde{P}_n$ and $\tilde{Q}_n$.

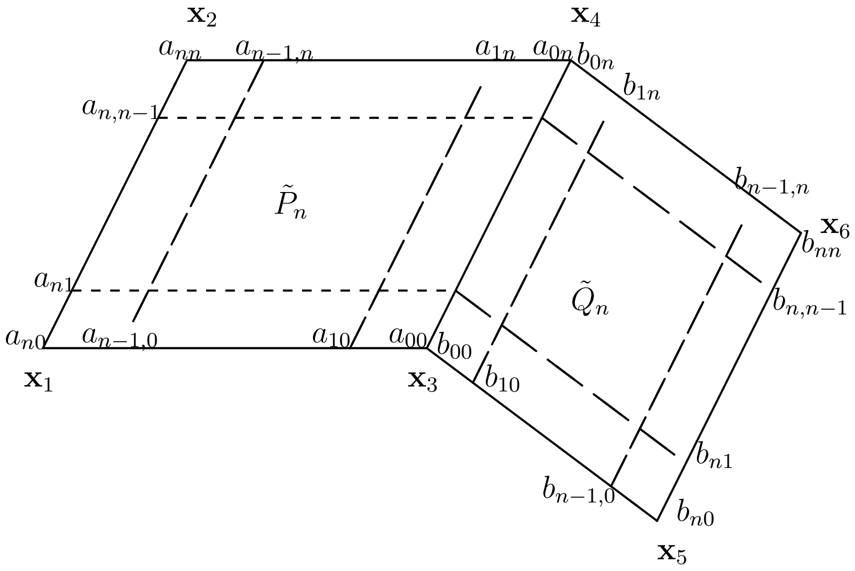


Figure 2.10 The B-nets of $\tilde{P}_n$ and $\tilde{Q}_n$

Write $\mathbf{x}_5 - \mathbf{x}_3 = \lambda_1^0(\mathbf{x}_1 - \mathbf{x}_3) + \lambda_2^0(\mathbf{x}_4 - \mathbf{x}_3)$. Then,

$$D_{\mathbf{x}_5-\mathbf{x}_3} = \lambda_1^0 D_{\mathbf{x}_1-\mathbf{x}_3} + \lambda_2^0 D_{\mathbf{x}_4-\mathbf{x}_3}.$$

Let F be a function defined as follows:

$$F(\mathbf{x}) = \begin{cases} \tilde{P}_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_1 \\ \tilde{Q}_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_2. \end{cases}$$

Clearly, $F \in C^r(T_1 \cup T_2)$ if and only if

$$(D_{\mathbf{x}_5-\mathbf{x}_3})^k Q_n \mid_{T_1 \cap T_2} = (\lambda_1^0 D_{\mathbf{x}_1-\mathbf{x}_3} + \lambda_2^0 D_{\mathbf{x}_4-\mathbf{x}_3})^k P_n \mid_{T_1 \cap T_2}$$

for $0 \le k \le r$. Define the so-called degree raising operators $\bar{\mathbf{R}}_2^k$ by

$$\bar{\mathbf{R}}_2^k a_{ij} = \sum_{\nu=0}^{j} \frac{\binom{j}{\nu}\binom{n-j}{n-k-\nu}}{\binom{n}{k}} a_{i\nu}, \qquad k \ge 0.$$

Then the smoothness conditions between $\tilde{P}_n$ and $\tilde{Q}_n$ easily follow.

LEMMA 2.13. $F \in C^r(T_1 \cup T_2)$ *if and only if*

$$\Delta_1^k b_{(0,l)} = \sum_{i+j=k} \binom{k}{i} (\lambda_1^0)^i (\lambda_2^0)^j \frac{n!(n-k)!}{(n-i)!(n-j)!} \Delta_1^i \Delta_2^j \bar{\mathbf{R}}_2^j a_{0,l}, \tag{2.3.9}$$

for $0 \le k \le r,$ *and* $0 \le l \le n.$

Proof. Clearly,

$$(D_{\mathbf{x}_5-\mathbf{x}_3})^k Q_n \,|_{T_1\cap T_2} = \frac{n!}{(n-k)!}\sum_{l=0}^{n}\Delta_1^k b_{(0l)}\tilde{\Phi}_{(0,l)}^{(0,n)}(0,\mu_2)$$

and

$$\begin{aligned}
&(\lambda_1^0 D_{\mathbf{x}_1-\mathbf{x}_3}+\lambda_2^0 D_{\mathbf{x}_4-\mathbf{x}_3})^k P_n \,|_{T_1\cap T_2}\\
= \;& \sum_{i+j=k}\binom{k}{i}(\lambda_1^0 D_{\mathbf{x}_1-\mathbf{x}_3})^i + (\lambda_2^0 D_{\mathbf{x}_4-\mathbf{x}_2})^j P_n \,|_{T_1\cap T_2}\\
= \;& \sum_{i+j=k}\sum_{m=0}^{n-j}\frac{n!}{(n-j)!}\frac{n!}{(n-i)!}(\lambda_1^0\Delta_1)^i(\lambda_2^0\Delta_2)^j a_{(0m)}\tilde{\Phi}_{(0,m)}^{(0,n-j)}(0,\lambda_2)\\
= \;& \sum_{l=0}^{n}\sum_{i+j=k}\binom{k}{i}\frac{n!}{(n-j)!}\frac{n!}{(n-i)!}(\lambda_1^0)^i(\lambda_2^0)^j\bar{\mathbf{R}}_2^j a_{(0l)}\tilde{\Phi}_{(0,l)}^{(0,n)}(0,\lambda_2).
\end{aligned}$$

Thus, we have established this lemma.

Computation of the degree raising operator $\bar{\mathbf{R}}_2^j$ is carried out as follows:

$$\begin{aligned}
&(\lambda_2+(1-\lambda_2))^j\sum_{m=0}^{n-j}a_{(0m)}\tilde{\Phi}_{(0,m)}^{(0,n-j)}\\
= \;& \sum_{m=0}^{n-j}a_{(0m)}\sum_{i=0}^{j}\binom{j}{i}\lambda_2^i(1-\lambda_2)^{j-i}\binom{n-j}{m}\lambda_2^m(1-\lambda_2)^{n-j-m}\\
= \;& \sum_{m=0}^{n-j}\sum_{i=0}^{j}\binom{j}{i}a_{(0m)}\binom{n-j}{m}\lambda_2^{m+i}(1-\lambda_2)^{n-m-i}\\
= \;& \sum_{k=0}^{n}\sum_{\substack{m+i=k\\ m\le n-j\\ i\le j}}\frac{\binom{j}{i}\binom{n-j}{m}}{\binom{n}{k}}a_{(0m)}\tilde{\Phi}_{(0,k)}^{(0,n)}(0,\lambda_2)\\
= \;& \sum_{k=0}^{n}\sum_{m+i=k}\frac{\binom{k}{m}\binom{n-k}{n-j-m}}{\binom{n}{j}}a_{(0m)}\tilde{\Phi}_{(0,k)}^{(0,n)}(0,\lambda_2).
\end{aligned}$$

Also, we have the following

LEMMA 2.14. $F\in C^r(T_1\cup T_2)$ *if and only if*

$$(D_{\mathbf{x}_5-\mathbf{x}_3})^i(D_{\mathbf{x}_4-\mathbf{x}_3})^j\tilde{Q}_n(\mathbf{x}_3) \tag{2.3.10}$$

$$= (\lambda_1^0 D_{\mathbf{x}_1-\mathbf{x}_3}+\lambda_2^0 D_{\mathbf{x}_4-\mathbf{x}_3})^i(D_{\mathbf{x}_4-\mathbf{x}_3})^j\tilde{P}_n(\mathbf{x}_3)$$

for $0\le i\le r, 0\le j\le n$.

The proof of this lemma is the same as that of Lemma 2.8. and we omit the details.

It is therefore suggestive to determine the supports of smoothness conditions (2.3.9). The following figure 2.11 shows the supports of the C^1 and C^2 smoothness conditions.

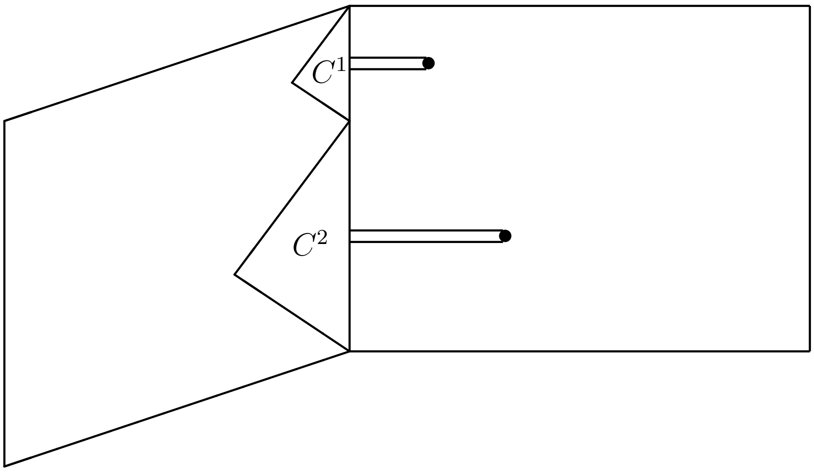


Figure 2.11 The supports of the smoothness conditions over two parallelograms

Further, when certain B-coefficients of F are given, we apply the smoothness conditions (2.3.9) to make F smooth across edge $[\mathbf{x}_3, \mathbf{x}_4]$. We have following two lemmas.

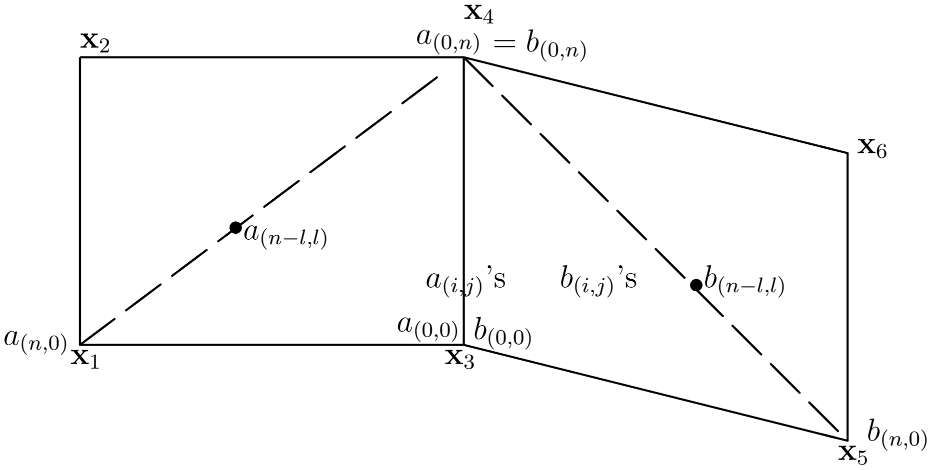


Figure 2.12 The orientation of the vertices of two parallelograms

LEMMA 2.15. *Assume that* $\mathbf{x}_4 \notin [\mathbf{x}_2, \mathbf{x}_6]$. *Suppose that the following B-coefficients of* $\tilde{P}_n$ *and* $\tilde{Q}_n$ *are given:* $a_{(i,j)}, b_{(i,j)}, i+j \leq n-1$ *as well as* $a_{(k,n-k)}, b_{(k,n-k)}, k = 0, \cdots, n-2l-2$. *(See Figure 2.12 above for reference of orientation of these two parallelograms.)Further, suppose that* $a_{(ij)}, b_{(ij)}, i+j \leq n-1$ *satisfy the smoothness conditions up to order* $n-1$ *and* $a_{(k,n-k)}, b_{(k,n-k)}$ *with some other* $a_{(ij)}$*'s,* $b_{(ij)}$*'s satisfy the* *smoothness conditions up to order* $n-2l-2$. *Then for any given* $a_{(n-k,k)}, b_{(n-k,k)}, 0 \le k \le l$, *there exists a unique set of* $a_{(n-l-k,k+l)}, b_{(n-l-k,k+l)}, 1 \le k \le l+1$, *such that* $a_{(ij)}, b_{(ij)}, i+j \le n$ *satisfy the smoothness conditions up to order* n.

Proof. We only need to prove that there exists a unique solution $\{a_{(l+k,n-l-k)}, b_{(l+k,n-l-k)} : 1 \le k \le l+1\}$ such that the following smoothness conditions

$$\begin{aligned}\Delta_1^{n-k} b_{0k} &= \sum_{i+j=n-k} \binom{n-k}{i} \lambda_1^i \lambda_2^j \frac{n!}{(n-i)!} \frac{k!}{(n-j)!} \Delta_1^i \Delta_2^j \bar{R}_2^j a_{0k} \\ &= \sum_{i+j=n-k} \binom{n-k}{i} \lambda_1^i \lambda_2^j \frac{n!}{(n-i)!} \frac{k!}{(n-j)!} \sum_{m=0}^{k} \frac{\binom{k}{m}\binom{n-k}{n-m-j}}{\binom{n}{j}} \Delta_1^i \Delta_2^j a_{0m},\end{aligned}$$

hold for $0 \le k \le 2l+1$. Thus, we have $2l+2$ equations and $2l+2$ unknown $\{a_{(n-l-k,l+k)}, b_{(n-l-k,l+k)} : 1 \le k \le l+1\}$. The linear system may be decomposed into two smaller linear subsystems:

$$\Delta_1^{n-k} b_{0k} = \sum_{i+j=n-k} \binom{n-k}{i} \lambda_1^i \lambda_2^j \frac{n!}{(n-i)!} \frac{k!}{(n-j)!} \sum_{m=0}^{k} \frac{\binom{k}{m}\binom{n-k}{n-m-j}}{\binom{n}{j}} \Delta_1^i \Delta_2^j a_{0m}$$

for $l+1 \le k \le 2l+1$ and

$$\sum_{i+j=n-k} \binom{n-k}{i} \lambda_1^i \lambda_2^j \frac{n!}{(n-i)!} \frac{k!}{(n-j)!} \sum_{m=0}^{k} \frac{\binom{k}{m}\binom{n-k}{n-m-j}}{\binom{n}{j}} \Delta_1^i \Delta_2^j a_{(0m)} = \Delta_1^{n-k} b_{(0k)}$$

where $0 \le k \le l$, which may be simplified as follows:

$$\sum_{i+j=n-k} \binom{n-k}{i} \lambda_1^i \lambda_2^j \frac{n!}{(n-i)!} \frac{k!}{(n-j)!} \frac{\binom{n-k}{n-j-k}}{\binom{n}{j}} a_{(i,j+k)} = c_k, \qquad 0 \le k \le l.$$

Further, we have

$$\sum_{i+j=n-k} \lambda_1^i \lambda_2^j \frac{\binom{n-k}{i}\binom{n}{i}}{\binom{n}{k}} a_{(i,j+k)} = c_k \qquad 0 \le k \le l,$$

or

$$\sum_{j=l+1}^{2l+1} \lambda_1^{n-j} \lambda_2^{j-k} \frac{\binom{n-k}{n-j}\binom{n}{n-j}}{\binom{n}{k}} a_{(n-j,j)} = \tilde{c}_k, \qquad 0 \le k \le l.$$

The above linear system has a unique solution $\{a_{(n-j,j)} : l+1 \le j \le 2l+1\}$ because the determinant of its coefficients matrix may be simplified to be

$$\det \begin{bmatrix} \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{(l+1)!} \\ \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(l+2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(l+1)!} & \frac{1}{(l+2)!} & \cdots & \frac{1}{(2l+1)!} \end{bmatrix} = \frac{\Pi_{i=1}^{l+1}(l+1-i)!}{\Pi_{i=1}^{l+1}(2l+2-i)!} \ne 0.$$

Then substituting the values $\{a_{(n-l-k,l+k)} : 1 \leq k \leq l+1\}$ into the first subsystem, we also have a unique solution $\{b_{(n-k-l,l+k)} : 1 \leq k \leq l+1\}$. This completes the proof of the lemma.

LEMMA 2.16. *Assume that* $\mathbf{x}_4 \in [\mathbf{x}_2, \mathbf{x}_6]$. *Suppose that the B-coefficients* $\{a_{(ij)} : i+j \leq n-1\}$ *and* $\{b_{(ij)} : i+j \leq n-1\}$ *satisfy the smoothness conditions (2.3.9) up to order* $n-1$. *Furthermore, suppose that* $\{a_{(k,n-k)}, b_{(k,n-k)} : 0 \leq k \leq l\}$ *are given and satisfy the smoothness conditions (2.3.9) up to order* l, *where* $l < n$. *Then for any* $\{a_{(k,n-k)} : l+1 \leq k \leq n\}$, *there exists a unique set of coefficients* $\{b_{(k,n-k)} : l+1 \leq k \leq n\}$ *such that* $\{a_{(ij)}, i+j \leq n\}$ *and* $\{b_{(ij)} : i+j \leq n\}$ *satisfy the smoothness conditions (2.3.9).*

Proof. This result is a simple consequence of Lemma 2.13.

LEMMA 2.17. *Let* $\mathbf{v}$ *be an interior and singular vertex such that the four patches attached at* $\mathbf{v}$ *consist one triangle and three parallelograms as shown as in Figure 2.13. Assume that the following B-coefficients* $\{a_\beta : \beta_1 \geq 1\}, \{b_{(ij)} : i+j \leq n-1\}, \{c_{(ij)} : i+j \leq n-1\}, \{d_{(ij)} : i+j \leq n-1\}$ *on these four patches respectively are given and satisfy the smoothness conditions (2.3.5) and (2.3.9) up to order* $n-1$. *Then for any given* $a_{(0,l,n-l)}$, *there exists a unique set of coefficients* $b_{(n-l,l)}, c_{(l,n-l)}, d_{(n-l,l)}$ *that satisfy the smoothness conditions (2.3.5) or (2.3.9), where* $0 \leq l \leq n$.

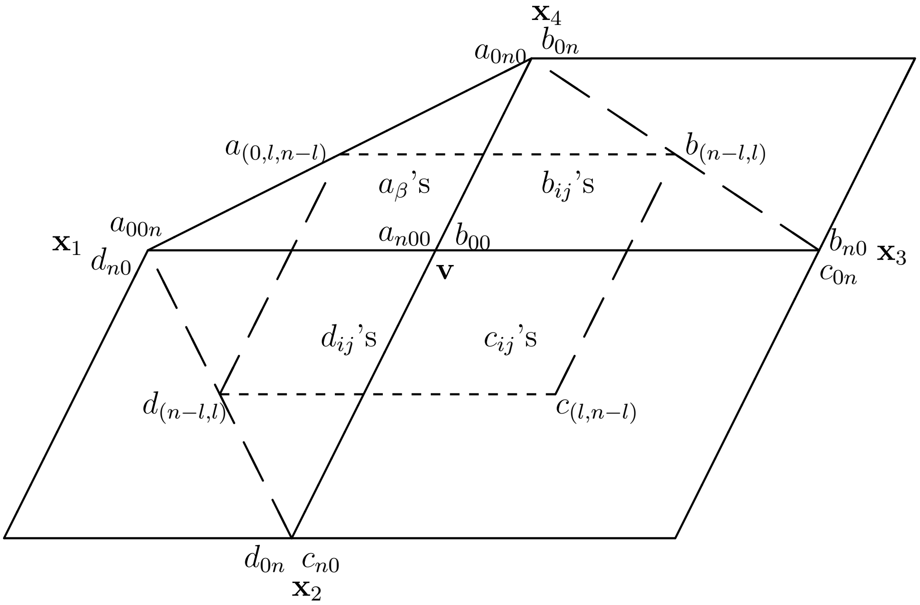


Figure 2.13 One triangle and three parallelograms attach at $\mathbf{v}$

Proof. By Lemma 2.7, for any given $a_{(0,l,n-l)}$, we can find $b_{(l,n-l)}$ and $d_{(n-l,l)}$. Then $c_{(l,n-l)}$ may be obtained from $b_{(ij)}$, say by using Lemma 2.13. We need to prove that $c_{(l,n-l)}$ and $d_{(n-l,l)}$ satisfy the smoothness conditions (2.3.9). To do this, we may assume that the given B-coefficients $\{a_\beta, \beta_1 \geq 1\}, \{b_{(ij)} : i+j \leq n-1\}, \{c_{(ij)} : i+j \leq n-1\},$ and $\{d_{(ij)} : i+j \leq n-1\}$ are equal to zeros and obtain

$$b_{(n-l,l)} = \left(\frac{|\mathbf{x}_3 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_1|}\right)^{n-l} \sum_{\substack{\gamma \leq (n-l,l,0) \\ |\gamma| = l}} \Delta_{31}^{n-l} a_\gamma \frac{\binom{n-l}{\gamma_1}\binom{l}{\gamma_2}}{\binom{n}{l}}$$

$$= \left(\frac{|\mathbf{x}_3 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_1|}\right)^{n-l} a_{(0,l,n-l)} \frac{1}{\binom{n}{l}}, \tag{2.8}$$

$$d_{(n-l,l)} = \left(\frac{|\mathbf{x}_2 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_4|}\right)^{l} \sum_{\substack{\gamma \leq (l,0,n-l) \\ |\gamma| = l}} \Delta_{21}^{l} a_\gamma \frac{\binom{l}{\gamma_1}\binom{n-l}{\gamma_3}}{\binom{n}{l}}$$

$$= \left(\frac{|\mathbf{x}_2 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_4|}\right)^{l} \frac{a_{(0,l,n-l)}}{\binom{n}{l}}, \tag{2.9}$$

$$c_{(l,n-l)} = \left(\frac{|\mathbf{x}_2 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_4|}\right)^{l} b_{(n-l,l)}. \tag{2.10}$$

We need to prove that

$$c_{(l,n-l)} = \left(\frac{|\mathbf{x}_3 - \mathbf{v}|}{|\mathbf{v} - \mathbf{x}_1|}\right)^{n-l} d_{(n-l,l)}. \tag{2.11}$$

Indeed, solving $a_{(0,l,n-l)}$ from (2.8) and substituting into (2.9) and substituting the resulting $b_{(n-l,l)}$ into (2.10), we have (2.11). Thus, this completes the proof of this lemma.

The following two lemmas can be proved similarly and will be stated without proof.

LEMMA 2.18. *Let* $\mathbf{v}$ *be an interior and singular vertex such that the four patches attached at* $\mathbf{v}$ *consist two triangles and two parallelograms as shown as in Figure 2.14. Assume that the following B-coefficients* $\{a_\beta : \beta_1 \geq 1\}, \{b_{(ij)} : i+j \leq n-1\}, \{c_{(ij)} : i+j \leq 1\},$ *and* $\{d_\beta : \beta_1 \geq 1\}$ *on the four patches are given and satisfy the smoothness conditions (2.3.1) ,(2.3.5) and (2.3.9) up to order* $n-1$ *(cf. Figure 2.14). Then for any given* $a_{(0,l,n-l)}$, *there exists a unique set of* $b_{(l,n-l)}, c_{(l,n-l)},$ *and* $d_{(0,n-l,l)}$ *that satisfy the smoothness conditions (2.3.1), (2.3.5), (2.3.9), where* $0 \leq l \leq n$.

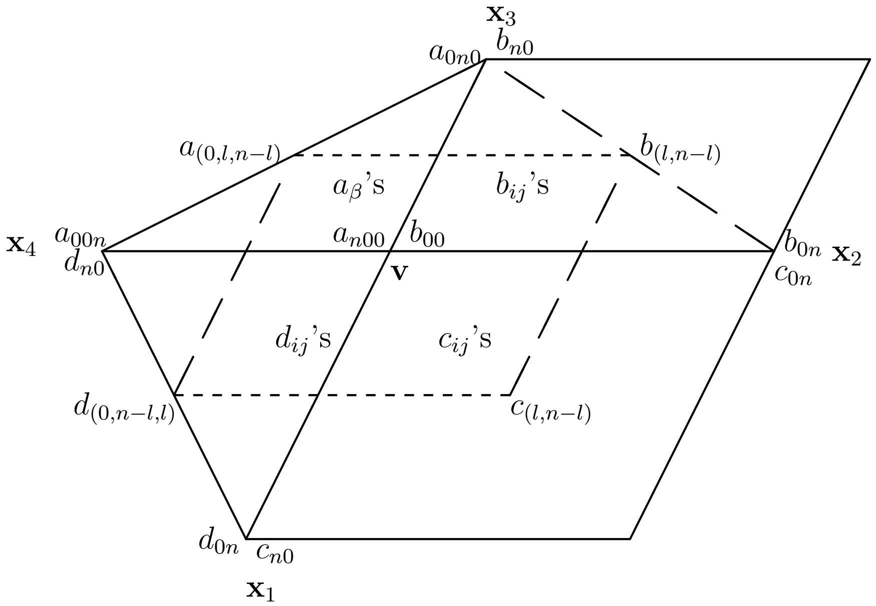


Figure 2.14 Two parallelograms and two triangles attach at $\mathbf{v}$

LEMMA 2.19. *Let* $\mathbf{v}$ *be an interior and singular vertex such that the four patches attached at* $\mathbf{v}$ *consist four triangles as shown as in Figure 2.15. Assume that the following B-coefficients* $\{a_{(ij)}, b_{(ij)}, c_{(ij)}, d_{(ij)} : i+j \leq n-1\}$ *on these patches respectively are given and satisfy the smoothness conditions (2.3.9) up to order* $n-1$*. Then for any given* $a_{(l,n-l)}$*, there exists a unique set of* $b_{(n-l,l)}, c_{(l,n-l)}$*, and* $d_{(n-l,l)}$ *such that these four B-coefficients satisfy the smoothness conditions (2.3.9), where* $0 \leq l \leq n$*.*

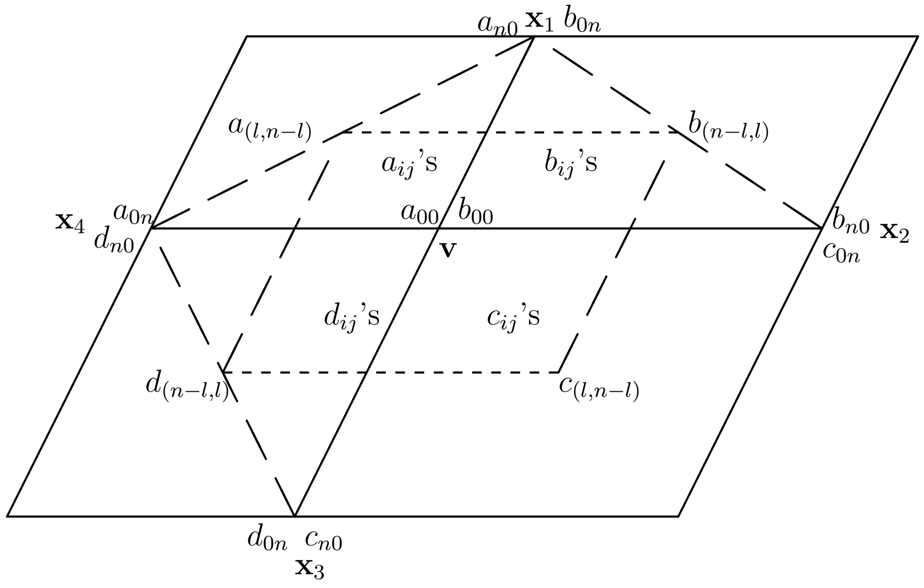


Figure 2.15 Four parallelograms attach at $\mathbf{v}$

2.4. Construction of Fundamental Vertex Splines

First of all, we need to give a precise definition of mixed grid partitions.

DEFINITION 2.1. $\triangle = \{t_i : i = 1, \cdots, L\}$ is a mixed grid partition of a region R if

(i) $R = \cup_{i=1}^{L} t_i$;

(ii) each t_i is either a triangle or a parallelogram;

(iii) $\mathrm{int}(t_i) \cap \mathrm{int}(t_j) = \emptyset$, if $i \neq j$; and

(iv) either $t_i \cap t_j = \emptyset$ or $t_i \cap t_j$ is the common edge of t_i and t_j or the common vertex of t_i and t_j.

A triangulation $\triangle$ of R satisfying (i)–(iv) is a mixed grid partition. On a mixed grid partition $\triangle$, we define the spline space $S_d^r(\triangle)$ of smoothness order r and "degree" n as follows:

$$S_d^r(\triangle) = \{f \in C^r(R) : f\Big|_{t_i} \in \pi_d(t_i), \forall i\}$$

More precisely, $f \in S_d^r(\triangle)$ means $f \in C^r(R)$ and $f\Big|_{t_i}$ is a polynomial of total degree d if t_i is a triangle or a polynomial of coordinate degree (d, d) with respect to t_i if t_i is a parallelogram.

Then we identify subspaces of $S_d^r(\triangle)$ which are called super spline spaces in [40, 111], where $d \geq 3r + 2$.

DEFINITION 2.2. For $r \geq 0$ and $d \geq 3r + 2$, and $l, r < l \leq r + [\frac{d-2r-1}{2}]$, the subspace

$$S_d^{r,l}(\triangle) = \{f : f \in S_d^r(\triangle) \text{ and } f \in C^l(\mathbf{v}) \text{ at each vertex of } \triangle\}$$

of $S_d^r(\triangle)$ is called super spline space. We will use the notation $\widehat{S}_d^r := S_d^{r,r+[\frac{d-2r-1}{2}]}(\triangle)$.

DEFINITION 2.3. A spline function $f \in S_d^{r,l}(\triangle)$ is called a vertex spline if its support is a part of the union of all patches (triangles or parallelepipeds) sharing at most one vertex.

In this section, we are going to outline a procedure for constructing a basis of the super spline space $\widehat{S}_d^r$ consisting of vertex splines. These vertex splines, called fundamental vertex splines, will be required to satisfy certain specification of interpolatory parameters at the corresponding vertex and match some directional derivatives related to the edges and patches (which will be defined below.) In the construction of each polynomial piece of a vertex spline, we subdivide the indices of the B-coefficients of this polynomial into several parts as indicated by I, II, III, IV, and V in Figures 2.16 and 2.17 based on the idea of "disentangling the rings" in [19]. The B-coefficients with indices in I and V are either zero or will be determined by the interpolatory parameters, those with indices in II will be determined by the directional derivatives related to the patch, those with indices in III by the directional derivatives related to edges, and those with indices in IV by using Lemmas 2.14, 2.9, and 2.15 or Lemmas 2.5, 2.10, and 2.16. We need first to introduce the necessary definitions and notations and then specify the interpolation parameters of these vertex splines. We will only discuss the special and most important case where $d = 3r + 2$, since it will be clear that our construction procedure is also valid for $d > 3r + 2$.

Let us subdivide the underlying index set $\{\beta \in \mathbb{Z}_+^3 : |\beta| = 3r + 2\}$ of the B-net of a polynomial of degree $3r + 2$ on a triangle into four parts. (Refer to Figure 2.16 for case $r = 5$ and $d = 17$.) Fix a triangle $\delta = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$.

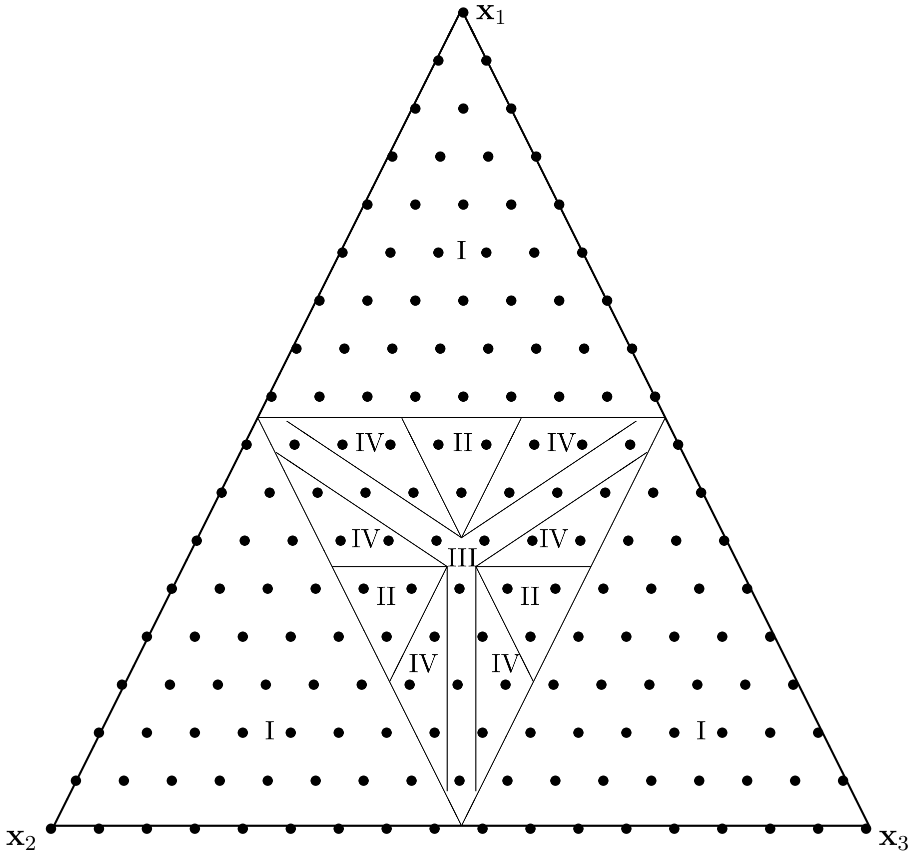


Figure 2.16 Four parts of the underlying index set of the B-net of a polynomial of degree 17

Part I is the union of the collections $A_i^{3r+2}J_1$, $i = 1, 2, 3$, where $J_1 = \{(l, m) \in \mathbb{Z}_+^2 : l + m \leq r + [r + 1/2]\}$.

Part II is the union of the collections $A_i^{3r+2}J_2$, $i = 1, 2, 3$, where $J_2 = \{(l, m) \in \mathbb{Z}_+^2 : l + m \geq r + [r + 1/2] + 1 \text{ and } l, m \leq r\}$.

Part III is the union of the collections $A_i^{3r+2}J_3, i = 1, 2, 3$, where $J_3 = \{(r-2m, r+1+m) \in \mathbb{Z}_+^2 : m = 0, \cdots, [r/2]\}$.

Part IV consists of the remaining Bézier coefficients on δ; *i.e.*, the union of the collection $A_i^{3r+2}J_4 \cup A_i^{3r+2}J_4^*, i = 1, 2, 3$, where $J_4 = \cup_{i=1}^{[r/2]}\{(r+1, r-i), \cdots, (r+1+i-1, r-i-(i-1))\}$ and $J_4^* = \{(l, m) : (m, l) \in J_4\}$.

We then subdivide the underlying index $\{\beta \in \mathbb{Z}_+^2 : \beta \leq (3r+2, 3r+2)\}$ of the B-net of a polynomial of "degree" $3r+2$ on a parallelogram into five parts. (Refer to Figure 2.17 for case $r = 5$ and $d = 17$.)

Fix a parallelogram $\delta = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\rangle$. Let $\eta^1 = (1, 1), \eta^2 = (-1, 1), \eta^3 = (1, -1)$ and $\eta^4 = (-1, -1)$ be indices associated with $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ respectively.

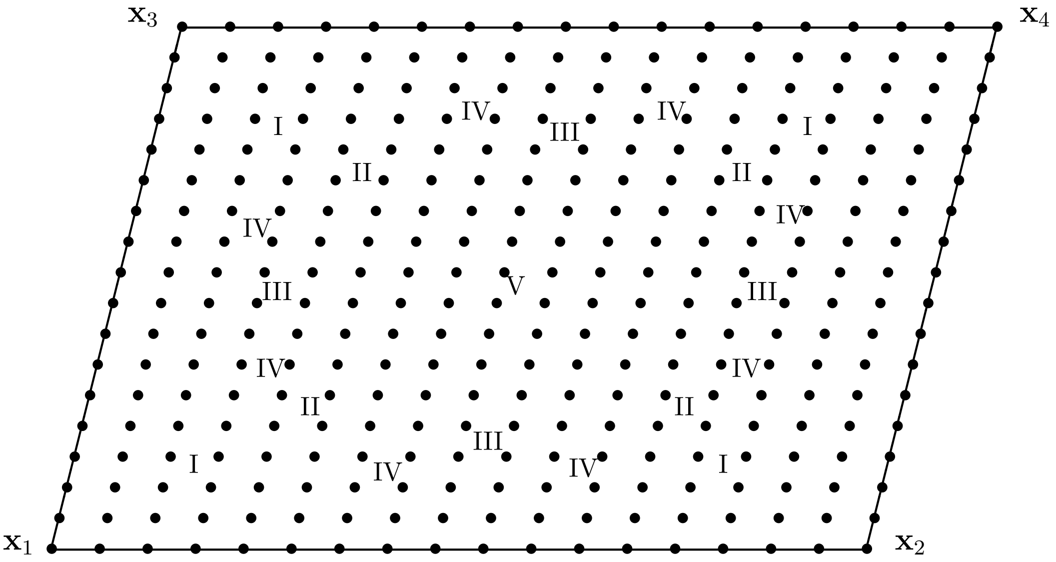


Figure 2.17 Five parts of the underlying index set of the B-net of a polynomial of "degree" 17

Part I is the union of collections $B_i^{3r+2}J_1, i = 1, 2, 3, 4$, where B_i^{3r+2} is a map from $\mathbb{Z}_+^2$ into itself defined in §2.2.

Part II is the union of collections $B_i^{3r+2}J_2, i = 1, 2, 3, 4$.

Part III is the union of collections $B_i^{3r+2}\tilde{J}_3, i = 1, 2, 3, 4$, where $\tilde{J}_3 = \{(i, j) : i = r+m, r-2m+2 \leq j \leq r, m = 1, \cdots, [\frac{r+1}{2}]\}$ if r is odd and $\tilde{J}_3 = \{(i, j) : i =$

$r+m, r-2m+2 \leq j \leq r, m = 1, \cdots, [\frac{r+1}{2}]\} \cup \{(i,j) : i = r + [\frac{r+1}{2}], 0 \leq j \leq r\}$ if r is even.

Part IV is the union of collections $B_i^{3r+2}(J_4 \cup J_4^*)$, $i = 1,2,3,4$, where $J_4 = \cup_{l=1}^{[r/2]}\{(r+1, r-l), \cdots, (r+1+l-1, r-l-(l-1))\}$.

Part V is the collection $B_1^{3r+2}J_5$, where $J_5 = \{(i,j) : r+1 \leq i, j \leq 2r+1\}$.

We now define directional derivatives related to edges, triangles, and parallelograms as follows. For an edge $e = [\mathbf{x}_{e,1}, \mathbf{x}_{e,2}]$ and a triangle $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3}\rangle$ or a parallelogram $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3}, \mathbf{x}_{e,4}\rangle$, *the directional derivatives relative to the edge* e are defined by

$$D_e^\alpha = (D_{\mathbf{x}_{e,2}-\mathbf{x}_{e,1}})^{\alpha_1}(D_{\mathbf{x}_{e,3}-\mathbf{x}_{e,1}})^{\alpha_2}, \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2,$$

where the directional derivatives are taken from inside the triangle or the parallelogram. For a triangle $t = \langle \mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k\rangle$, the *directional derivatives relative to* t *at* $\mathbf{x_i}$ are defined by

$$D_{t(\mathbf{x}_i)}^\alpha = (D_{\mathbf{x}_j - \mathbf{x}_i})^{\alpha_1}(D_{\mathbf{x}_k - \mathbf{x}_i})^{\alpha_2}, \ \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2,$$

where the derivatives are taken from inside the triangle. For a parallelogram $p = \langle \mathbf{x}_{p,1}, \mathbf{x}_{p,2}, \mathbf{x}_{p,3}, \mathbf{x}_{p,4}\rangle$ with $\langle \mathbf{x}_{p,1}, \mathbf{x}_{p,2}\rangle // \langle \mathbf{x}_{p,3}, \mathbf{x}_{p,4}\rangle$, the *directional derivatives relative to the parallelogram* p are defined by

$$D_p^\alpha = (D_{\mathbf{x}_{p,2}-\mathbf{x}_{p,4}})^{\alpha_1}(D_{\mathbf{x}_{p,3}-\mathbf{x}_{p,4}})^{\alpha_2}, \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2,$$

where the derivatives are taken from inside the parallelogram.

Assume that $\triangle$ is a mixed partition of a given region R and denote by $\mathcal{V}$ and $\mathcal{E}$, the collection of all vertices, and edges of $\triangle$, respectively. Let $\mathcal{E}_1$ denote the collection of those edges which is the common edge of two parallelograms or is a boundary edge of a parallelogram. And let $\mathcal{E}_2 = \mathcal{E} \backslash \mathcal{E}_1$ which is the collection of all the other edges of $\triangle$. Further, denote by $\mathcal{P}$ the collection of all parallelograms of $\triangle$. We numerate these vertices of $\triangle$ by $\mathcal{V} = \{\mathbf{v}_1, \cdots, \mathbf{v}_N\}$. For each edge $e = [\mathbf{v}_i, \mathbf{v}_j]$, we rewrite it as $e = [\mathbf{x}_{e,1}, \mathbf{x}_{e,2}]$ where $\mathbf{x}_{e,1} = \mathbf{v}_l, l = \min\{i,j\}$, and $\mathbf{x}_{e,2} = \mathbf{v}_m, m = \max\{i,j\}$. If $e = [\mathbf{x}_{e,1}, \mathbf{x}_{e,2}]$ is a boundary edge, e may be an edge of a triangle $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3}\rangle$ or an edge of a parallelogram $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3}, \mathbf{x}_{e,4}\rangle$ of $\triangle$. If $e \in \mathcal{E}_2$ is an interior edge, $e = [\mathbf{x}_{e,1}, \mathbf{x}_{e,2}]$ may be the common edge of two triangles $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,3}, \mathbf{x}_{e,2}\rangle$ and $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,4}\rangle$ or the common edge of one triangle $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3}\rangle$ and the other parallelogram $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,4}, \mathbf{x}_{e,5}\rangle$ or the common edge of two parallelograms $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3}, \mathbf{x}_{e,4}\rangle$ and $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,5}, \mathbf{x}_{e,6}\rangle$.

For each vertex $\mathbf{v} \in \mathcal{V}$, we denote the patches (triangles and parallelograms of $\triangle$) that share $\mathbf{v}$ as their common vertex by $T_{\mathbf{v},i}, i = 1, \cdots, l(\mathbf{v})$ and let $[\mathbf{v}, \mathbf{x}_{\mathbf{v},i}] \in$

$T_{\mathbf{v},i}, i = 1, \cdots, l(\mathbf{v})$ be all the edges emanating from $\mathbf{v}$. We call $T_{\mathbf{v},i}$ a one-sided singular patch relative to $\mathbf{v}$ if $[\mathbf{v}, \mathbf{x}_{\mathbf{v},i}]$ is a singular or near-singular edge relative to $\mathbf{v}$ but not both, and $T_{\mathbf{v},i}$ a two-sided singular patch relative to $\mathbf{v}$ if both $[\mathbf{v}, \mathbf{x}_{\mathbf{v},i}]$ and $[\mathbf{v}, \mathbf{x}_{\mathbf{v},i+1}] \subset T_{\mathbf{v},i}$ are singular or near-singular edges. We relabel $T_{\mathbf{v},i}, i = 1, \cdots, l(\mathbf{v})$ to be $t_i(\mathbf{v}), i = 1, \cdots, m(\mathbf{v})$ as follows:

(1) Let $\mathbf{v}$ be a singular or near-singular vertex. If the four patches attached at $\mathbf{v}$ are four triangles as in Fig.2.5, or four parallelograms as in Fig.2.15, or two triangles and two parallelograms as in Fig.2.14, let $t_1(\mathbf{v}) = T_{\mathbf{v},1}$ and $t_2(\mathbf{v}) = T_{\mathbf{v},3}$; if four patches attached at $\mathbf{v}$ are as shown in Fig.2.8 and Fig.2.9, let $t_1(\mathbf{v}) = T_{\mathbf{v},1}$ and $t_2(\mathbf{v}) = T_{\mathbf{v},3}$ or $t_1(\mathbf{v}) = T_{\mathbf{v},2}$ and $t_2(\mathbf{v}) = T_{\mathbf{v},4}$ so that $t_1(\mathbf{v}), t_2(\mathbf{v})$ are triangles; If four patches attached at $\mathbf{v}$ are as shown in Fig.2.13, let $t_1(\mathbf{v}) = T_{\mathbf{v},1}$ and $t_2(\mathbf{v}) = T_{\mathbf{v},3}$ or $t_1(\mathbf{v}) = T_{\mathbf{v},2}$ and $t_2(\mathbf{v}) = T_{\mathbf{v},4}$ so that either $t_1(\mathbf{v})$ or $t_2(\mathbf{v})$ is a triangle. Hence, let $m(\mathbf{v}) = 2$.

(2) Assume that $\mathbf{v}$ is not a singular vertex. If one of these $T_{\mathbf{v},i},\ 1 \le i \le l(\mathbf{v})$, say $T_{\mathbf{v},j}$ is a two-sided singular patch relative to $\mathbf{v}$, we denote $\{T_{\mathbf{v},1}, \cdots, T_{\mathbf{v},j-2}, T_{\mathbf{v},j}, T_{\mathbf{v},j+2}, \cdots, T_{\mathbf{v},l(\mathbf{v})}\}$ by $\{t_i(\mathbf{v}), i = 1, \cdots, l(\mathbf{v}) - 2\}$. That is, let $m(\mathbf{v}) = l(\mathbf{v}) - 2$. Note that if $\mathbf{v}$ is a boundary vertex, $T_{\mathbf{v},1}, \cdots, T_{\mathbf{v},j-1}$ or $T_{\mathbf{v},j+1}, \cdots, T_{\mathbf{v},l(\mathbf{v})}$ may not exist.

(3) If none of these $T_{\mathbf{v},i}, i = 1, \cdots, l(\mathbf{v})$, is a two-sided singular patch relative to $\mathbf{v}$ and if both $T_{\mathbf{v},j}$ and $T_{\mathbf{v},j+1}$ are one-sided singular patches relative to $\mathbf{v}$, we denote $\{T_{\mathbf{v},1}, \cdots, T_{\mathbf{v},j}, T_{\mathbf{v},j+2}, \cdots, T_{\mathbf{v},l(\mathbf{v})}\}$ by $\{t_i(\mathbf{v}) : i = 1, \cdots, m(\mathbf{v})\}$, where $m(\mathbf{v}) = l(\mathbf{v}) - 1$.

(4) If none of these $T_{\mathbf{v},i}, i = 1, \cdots, l(\mathbf{v})$, is a one-sided or two-sided singular patches relative to $\mathbf{v}$, we let $t_i(\mathbf{v}) = T_{\mathbf{v},i}, i = 1, \cdots, m(\mathbf{v})$, where $m(\mathbf{v}) = l(\mathbf{v})$.

Let $\mathcal{T} = \{t_i(\mathbf{v}) : \mathbf{v} \in \mathcal{V}, i = 1, \cdots, m(\mathbf{v})\}$. Note that some patches may be accounted for three or four times in $\mathcal{T}$. For instance, a triangle $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle \in \triangle$ may have been called $t_i(\mathbf{x}_1), t_j(\mathbf{x}_2)$, or $t_k(\mathbf{x}_3)$ in $\mathcal{T}$.

Furthermore, we denote

$$J_e = \begin{cases} J_3 & \text{if } e \text{ is the common edge of two triangles or one triangle} \\ & \text{and one parallelogram} \\ \tilde{J}_3 & \text{if } e \text{ is the common edge of two parallelograms} \\ J_6 & \text{if } e \text{ is an edge of a triangles on the boundary of } \triangle \\ J_7 & \text{if } e \text{ is an edge of parallelogram on the boundary of } \triangle \end{cases}$$

where

$$J_6 = \{(l, m+n) \in \mathbb{Z}_+^2 : \begin{array}{l} m = r + 1 + [(r+1)/2] - l, \\ r + 1 \le l \le r + 1 + [r/2], 0 \le n \le [r/2]\} \end{array}$$

and

$$\begin{array}{rl} J_7 = & \{(l, m) : r + 1 + [(r+1)/2] - m \le l \le r + 1 + [r/2] + m, 0 \le m \le [r/2]\} \\ & \cup\{(l, m) : r + 1 \le l \le r + 1 + [(r+2)/2, [r/2] \le m \le r\}. \end{array}$$

Also, we denote

$$J_{\mathbf{v},i}=\begin{cases} J_2 & \text{if } t_i(\mathbf{v}) \text{ is neither one-sided nor two-sided singular} \\ & \text{patch relative to } \mathbf{v}; \\ J_2\cup J_4 & \text{if } t_i(\mathbf{v}) \text{ is a one-sided singular patch relative } \mathbf{v}; \\ J_2\cup J_4\cup J_4^* & \text{if } \mathbf{v} \text{ is a singular vertex and } t_i(\mathbf{v})=T_1(\mathbf{v}) \\ & \text{ or if } \mathbf{v} \text{ is not a singular vertex but } t_i(\mathbf{v}) \text{ is} \\ & \text{a two-sided singular patch;} \\ J_4\cup J_4^* & \text{if } \mathbf{v} \text{ is a singular vertex and } t_i(\mathbf{v})=T_2(\mathbf{v}). \end{cases}$$

In the following, we outline the procedure for constructing the fundamental vertex splines in $\widehat{S}^r_{3r+2}(\triangle)$. In general, we will consider four types of vertex splines of interest. They are required to satisfy the following specifications of interpolatory conditions.

(I)For any $\mathbf{v}\in\mathcal{V}$ and $\gamma\in J_1$, let $V^\gamma_{\mathbf{v}}$ be a piecewise polynomial function supported on $S=\cup_{i=1}^{l(\mathbf{v})}T_{\mathbf{v},i}$ and satisfying the following interpolation conditions and smoothness conditions:

$$\text{(I.1)}\qquad D^\alpha V^\gamma_{\mathbf{v}}(\mathbf{u})=\delta_{\alpha,\gamma}\delta_{\mathbf{v},\mathbf{u}},\alpha\in J_1,\ \mathbf{u}\in\mathcal{V};$$

$$\text{(I.2)}\qquad D^\alpha_{t_j(\mathbf{u})}V^\alpha_{\mathbf{v}}\Big|_{t_j(\mathbf{u})}(\mathbf{u})=0,\alpha\in J_{\mathbf{u},j},\ t_j(\mathbf{u})\in\mathcal{T};$$

$$\text{(I.3)}_1\qquad D^\alpha_e V^\alpha_{\mathbf{v}}\Big|_{\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,3},\mathbf{x}_{e,4}\rangle}(\mathbf{x}_{e,1})=0,\alpha\in J_e,e\in\mathcal{E}_1;$$

$$\text{(I.3)}_2\qquad D^\alpha_e V^\alpha_{\mathbf{v}}\Big|_{\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,3}\rangle}(\mathbf{x}_{e,1})=0,\alpha\in J_e,e\in\mathcal{E}_2;$$

$$\text{(I.4)}\qquad V^\gamma_{\mathbf{v}}\in C^r(\mathbb{R}^2);$$

$$\text{(I.5)}\qquad D^\gamma_p V^\alpha_{\mathbf{v}}\Big|_p(\mathbf{x}_{p,4})=0,\gamma\in J_5,p\in\mathcal{P}.$$

Here and throughout, as usual, the symbols $\delta_{\alpha,\gamma},\delta_{\mathbf{v},\mathbf{u}}$ denote the Kronecker delta.

(II) For each $e\in\mathcal{E}$ and $\gamma\in J_e$, let V^γ_e be a piecewise polynomial function such that the following requirements are satisfied:

$$\text{(II.1)}\qquad D^\alpha V^\gamma_e(\mathbf{u})=0,\alpha\in J_1,\mathbf{u}\in\mathcal{V};$$

$$\text{(II.2)}\qquad D^\alpha_{t_j(\mathbf{u})}V^\gamma_e\Big|_{t_j(\mathbf{u})}(\mathbf{u})=0,\alpha\in J_{\mathbf{u},j},t_j(\mathbf{u})\in\mathcal{T};$$

(II.3)$_1$ $$D_d^\alpha V_e^\gamma\Big|_{\langle \mathbf{x}_{d,1},\mathbf{x}_{d,2},\mathbf{x}_{d,3},\mathbf{x}_{d,4}\rangle}(\mathbf{x}_{d,1}) = \delta_{e,d}\delta_{\alpha,\gamma}, \alpha \in J_d, d \in \mathcal{E}_1;$$

(II.3)$_2$ $$D_d^\alpha V_e^\gamma\Big|_{\langle \mathbf{x}_{d,1},\mathbf{x}_{d,2},\mathbf{x}_{d,3}\rangle}(\mathbf{x}_{d,1}) = \delta_{e,d}\delta_{\alpha,\gamma}, \alpha \in J_d, d \in \mathcal{E}_2;$$

(II.4) $$V_e^\gamma \in C^r(\mathbb{R}^2);$$

(II.5) $$D_p^\alpha V_e^\gamma\Big|_p(\mathbf{x}_{p,1}) = 0, \gamma \in J_5, p \in \mathcal{P}.$$

(III) For $t_i(\mathbf{v}) \in \mathcal{T}$, and $\gamma \in J_{\mathbf{v},i}$, let $V_{t_i(\mathbf{v})}^\gamma$ be a piecewise polynomial function satisfying the following requirements:

(III.1) $$D^\alpha V_{t_i(\mathbf{v})}^\gamma = 0, \alpha \in J_1, \mathbf{u} \in \mathcal{V};$$

(III.2) $$D_{t_k(\mathbf{u})}^\alpha V_{t_i(\mathbf{v})}^\gamma\Big|_{t_k(\mathbf{u})}(\mathbf{u}) = \delta_{\alpha,\gamma}\delta_{t_k(\mathbf{u}),t_i(\mathbf{v})}, \alpha \in J_{\mathbf{u},k}, t_k(\mathbf{u}) \in \mathcal{T};$$

(III.3)$_1$ $$D_d^\alpha V_{t_i(\mathbf{v})}^\gamma\Big|_{\langle \mathbf{x}_{d,1},\mathbf{x}_{d,2},\mathbf{x}_{d,3},\mathbf{x}_{d,4}\rangle}(\mathbf{x}_{d,1}) = 0, \alpha \in J_d, d \in \mathcal{E}_1;$$

(III.3)$_2$ $$D_d^\alpha V_{t_i(\mathbf{v})}^\gamma\Big|_{\langle \mathbf{x}_{d,1},\mathbf{x}_{d,2},\mathbf{x}_{d,3}\rangle}(\mathbf{x}_{d,1}) = 0, \alpha \in J_d, d \in \mathcal{E}_2;$$

(III.4) $$V_{t_i(\mathbf{v})}^\gamma \in C^r(\mathbb{R}^2);$$

(III.5) $$D_p^\alpha V_{t_i(\mathbf{v})}^\gamma\Big|_p(\mathbf{x}_{p,1}) = 0, \alpha \in J_5, p \in \mathcal{P}.$$

(IV) For each $p \in \mathcal{P}, \gamma \in J_5$, let V_p^γ be a piecewise polynomial function with support p satisfying the following requirements:

(IV.1) $$D^\alpha V_p^\gamma(\mathbf{u}) = 0, \alpha \in D_1,\ \mathbf{u} \in \mathcal{V};$$

(IV.2) $$D_{t_j(\mathbf{u})}^\alpha V_p^\alpha\Big|_{t_j(\mathbf{u})}(\mathbf{u}) = 0, \alpha \in J_{\mathbf{u},j},\ t_j(\mathbf{u}) \in \mathcal{T};$$

(IV.3)$_1$ $$D_e^\alpha V_p^\alpha\Big|_{\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,3},\mathbf{x}_{e,4}\rangle}(\mathbf{x}_{e,1}) = 0, \alpha \in J_e, e \in \mathcal{E}_1;$$

(IV.3)$_2$ $$D_e^\alpha V_p^\alpha\Big|_{\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,3}\rangle}(\mathbf{x}_{e,1}) = 0, \alpha \in J_e, e \in \mathcal{E}_2;$$

(IV.4) $$V_p^\gamma \in C^r(\mathbb{R}^2);$$

(IV.5) $$D_q^\alpha V_p^\gamma(\mathbf{x}_{p,4}) = \delta_{\alpha,\gamma}\delta_{p,q}, \qquad \alpha \in J_5,\ q \in \mathcal{P}.$$

The construction procedure of these vertex splines can be described in the following four steps. Let V stand for one of the above vertex spline and $\delta = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ or $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle$ be an arbitrary patch of $\triangle$.

Step 1. Determination of B-net indexed in part I.

The B-nets of $V|_\delta$ indexed in $A_i^{3r+2}J_1$ (or $B_i^{3r+2}J_1$) are simply zero when V is required to satisfy $D^\alpha V(\mathbf{x}_i) = 0$. When $V(\mathbf{x})$ is required to satisfy $D^\alpha V(\mathbf{x}_i) = \delta_{\alpha,\gamma}$, we first convert the partial derivatives D^α at $\mathbf{x}_i$ into directional derivatives related to the patch at $\mathbf{x}_i$ and then use the resulting information $D^\beta_{\delta(\mathbf{x}_i)} V\Big|_\delta(\mathbf{x}_i)$ to find the B-net of $V|_\delta$ with underlying indices in $A_i^{3r+2}J_1$ (or $B_i^{3r+2}J_1$).

Step 2. Determination of B-net indexed in part II.

Case 1: Suppose that δ is not one-sided singular nor two-sided singular at $\mathbf{x}_i$. Then we directly apply the requirement (I.2), (II.2), (III.2), or (IV.2) to obtain the portion of the B-net of $V|_\delta$ with indices in $A_i^{3r+2}J_2$ (or $B_i^{3r+2}J_2$).

Case 2: Suppose that $[\mathbf{x}_i, \mathbf{x}_k]$ is singular or near-singular at $\mathbf{x}_i$ but $[\mathbf{x}_i, \mathbf{x}_j]$ is not, where $\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k$ is an rearrangement in counterclockwise of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, or $\mathbf{x}_i$ is a singular or near singular vertex such that $\delta \neq T_{\mathbf{x}_i,1}$. We will obtain the portion of the B-net of $V|_\delta$ with indices in $A_i^{3r+2}J_2$ (or $B_i^{3r+2}J_2$) by using the smoothness conditions, Lemmas 2.2, 2.7, or 2.13 from the corresponding portion of the B-net of $V_{\delta'}$, where δ' is the neighboring patch sharing $[\mathbf{x}_i, \mathbf{x}_k]$ with δ as the common edge.

Case 3: Suppose that δ is a two-sided singular at $\mathbf{x}_i$, or $\mathbf{x}_i$ is a singular vertex and $\delta = T_{\mathbf{x}_i,1}$. We directly apply the requirement (I.2), (II.2), (III.2) or (IV.2) to obtain the portion of B-net of $V|_\delta$ indexed in $A_i^{3r+2}J_2$ (or $B_i^{3r+2}J_2$) by using Lemmas 2.6, 2.11, 2.12, 2.17, 2.18 and 2.19.

Step 3. Determination of B-net indexed in part III & IV

Case 1: Suppose that $[\mathbf{x}_i, \mathbf{x}_j]$ is a boundary edge. Then the Bézier coefficients of $V\Big|_\delta$ with indices in the one-third portion of parts III and IV closest to $[\mathbf{x}_i, \mathbf{x}_j]$ (as shown in Figure 2.18 for case r=5 and d=17 and edge $[\mathbf{x}_1, \mathbf{x}_3]$ on the triangle $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$) are obtained by applying the requirements in (I.3), (II.3), (III.3), or (IV.3).

Case 2: Suppose that the edge $[\mathbf{x}_i, \mathbf{x}_k]$ is singular or near-singular at $\mathbf{x}_i$ but $[\mathbf{x}_i, \mathbf{x}_j]$ is not, where $\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k\}$ is a rearrangement of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in the counterclockwise orientation, or suppose that $\mathbf{x}_i$ is a singular or near singular vertex such that $\delta \neq T_{\mathbf{x}_i,1}$. Then we determine the one-half portion of the B-coefficients of $V|_\delta$ with indices in

$A_i^{3r+2}J_4 \cup A_i^{3r+2}J_4^*$ closest to $[\mathbf{x}_i, \mathbf{x}_k]$ (e.g., $a_{(8,3,6)}, a_{(8,2,7)}, a_{(7,4,6)}$ for case $r = 5$ and $d = 17$ in Figure 2.18) by using the smoothness conditions, Lemma 2.1, or Lemma 2.5 from the corresponding portion of the Bézier coefficients of $V|_{\delta'}$, where δ' is the neighboring triangle of δ with $[\mathbf{x}_i, \mathbf{x}_k]$ as the common edge. The other half-portion will be determined in Case 5.

Case 3: Suppose that $[\mathbf{x}_i, \mathbf{x}_j]$ is singular or near-singular at $\mathbf{x}_i$ but $[\mathbf{x}_i, \mathbf{x}_k]$ is not, where $\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k\}$ is a rearrangement of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in the counterclockwise orientation, or suppose that $\mathbf{x}_i$ is a singular or near singular vertex such that $\delta \neq T_{\mathbf{x}_i,1}$. Then we may directly apply the requirements in (I.2), (II.2), (III.2), or (IV.2) to obtain the one-half portion of the Bézier coefficients of $V|_\delta$ with indices in $A_i^{3r+2}J_4 \cup A_i^{3r+2}J_4^*$ (or $B_i^{3r+2}(J_4 \cup J_4^*)$) closest to $[\mathbf{x}_i, \mathbf{x}_j]$. The other one-half portion will again be determined in Case 5.

Case 4: Suppose that δ is two-sided singular at $\mathbf{x}_i$, or $\mathbf{x}_i$ is a singular vertex and $\delta = T_{\mathbf{x}_i,1}$, or suppose that $\mathbf{x}_i$ is a near-singular vertex and $\delta = T_{\mathbf{x}_i,1}$. In this case, we may directly apply (I.2), (II.2), or (III.2) or (IV.2) to obtain the portion of the Bézier coefficients of $V|_\delta$ with indices in $A_i^{3r+2}J_4 \cup A_i^{3r+2}J_4^*$ (or $B_i^{3r+2}(J_4 \cup J_4^*.)$

Case 5: This is the remaining case. To determine the remaining Bézier coefficients of $V|_\delta$ with indices in parts III and IV, we need to use all of (I.3) and (I.4), or (II.3) and (II.4), or (III.3) and (III.4), or (IV.3) and (IV.4) and apply Lemma 2.4 or Lemma 2.5, Lemma 2.9 or Lemma 2.10, Lemma 2.15, Lemma 2.16 accordingly. Let us illustrate with the following example. Consider $r = 5, d = 17$, and consider the edge $e = [\mathbf{x}_1, \mathbf{x}_3]$ and the requirements in (I.3). We only discuss the case where e is not singular nor near singular edge at either $\mathbf{x}_1$ or $\mathbf{x}_3$. Let δ' be the patch of $\triangle$ with e as the common edge of δ and δ'. When both δ and δ' are triangles, this case was illustrated in [37]. Hence, we may assume that δ' is a parallelogram. (see Figure 2.18.) Then the B-coefficients a_α of $V|_\delta$ and b_β of $V|_{\delta'}$, where $\alpha \in \{(8,1,8),(8,2,7),(7,2,8),(8,3,6),\ (7,3,7),\ (6,3,8),\ (7,4,6),\ (6,4,7),(6,5,6)\}$ and $\beta \in \{(1,8),(1,9),2,7),(2,8),(2,9),(2,10)\} \cup \{(i,j) : 3 \leq i \leq 5, 6 \leq j \leq 11\}$, are to be determined. Since the B-coefficients of $V|_\delta$ and $V|_{\delta'}$ in part I have already been determined, we may first apply one of the requirements in (I.3) to obtain $a_{(8,1,8)}$. Then, $b_{(1,8)}$ and $b_{(1,9)}$ are obtained by applying Lemma 2.7 and using the corresponding B-coefficients $a'_\alpha s$. Then we may apply Lemma 2.9 with $a_{(8,0,4)+\alpha}$, $|\alpha| = 5$ and $b_{(0,4)+\beta}$, $|\beta| = \beta_1 + \beta_2 = 5$ and $l = 1$ (cf. the B-coefficients inside the dotted quadrilateral indicated in Figure 2.18) to obtain $a_{(8,2,7)}, a_{(8,3,6)}$ and $b_{(2,7)}, b_{(3,6)}$. Also, $a_{(7,2,8)}, a_{(6,3,8)}$ and $b_{(2,10)}, b_{(3,11)}$ are obtained in a similar manner. Next we again use the requirements in (I.3) to obtain $a_{(7,3,7)}$, and then $b_{(2,8)}, b_{(2,9)}, b_{(3,7)}, b_{(3,8)}, b_{(3,9)}$, and $b_{(1,10)}$ by applying Lemma 2.7 and using the corresponding B-coefficients of $V|_\delta$. By applying Lemma 2.9 again with $a_{(7,0,5)+\alpha}$, $|\alpha| = 5$, and $b_{(0,5)+\beta}$, $|\beta| = 5$ and and $l = 0$, we may now determine $a_{(7,4,6)}$ and $b_{(4,6)}$. Similarly, $a_{(6,4,7)}$ and $b_{(4,11)}$ are obtained by using Lemma 2.9. Finally, $a_{(6,5,6)}$ is obtained by using (I.3) once more, and hence, $b_{(5,6)}, b_{(5,7)}, b_{(5,8)}, b_{(5,9)}, b_{(5,10)}$ and $b_{(5,11)}$ are determined by applying Lemma 2.7 and using the corresponding Bézier net of $V|_\delta$. Of course, when $[\mathbf{x}_1, \mathbf{x}_3]$ is a singular or near-singular edge at $\mathbf{x}_1$ or $\mathbf{x}_3$, we have to modify the above procedure accordingly by using Lemma 2.10 instead of Lemma 2.9. Similarly, where both δ and δ' are parallelograms, the B-coefficients of V with indices in parts III and IV can de determined also. This method is valid for any $r \geq 1$ in general.

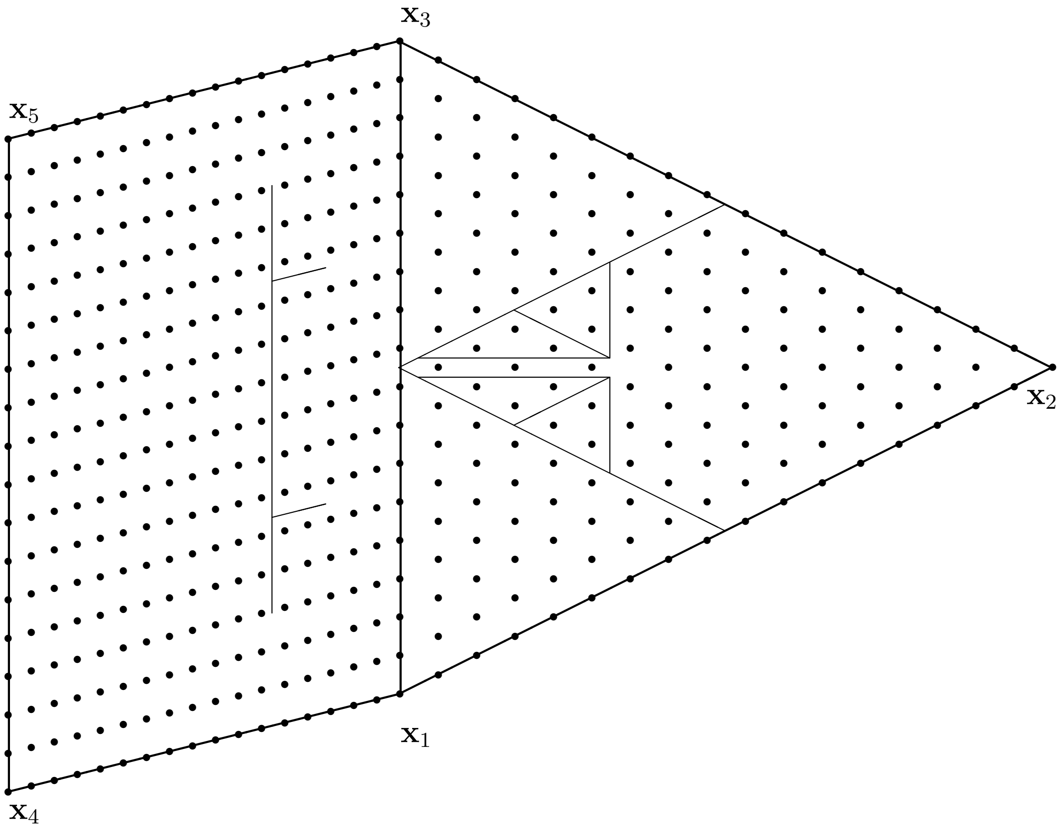


Figure 2.18 Illustration of the construction of vertex splines on a mixed partition

Step 4. Determination of B-net indexed in part V.

We use (I.5), or (II.5), or (III.5), or (IV.5) to determine B-coefficients in part V of $V(\mathbf{x})$ on each parallelogram in $\mathcal{P}$.

From their specifications and the basic construction steps above, we know that these vertex splines are in $\widehat{S}^r_{3r+2}$. And we also know that the support of $V^\gamma_{\mathbf{v}}$ is the union of all patches sharing $\mathbf{v}$, that of V^γ_e is all patches sharing e and that of V^γ_p is the parallelogram p. The support $S_{\mathbf{v},i}$ of $V^\gamma_{t_i(\mathbf{v})}$ is given as follows: Suppose $t_i(\mathbf{v}) = T_{\mathbf{v},j}$.

$$S_{\mathbf{v},i} = \begin{cases} \cup_{k=1}^4 T_{\mathbf{v},k} & \text{if } \mathbf{v} \text{ is a singular vertex and } t_i(\mathbf{v}) = T_{\mathbf{v},1}; \\ T_{\mathbf{v},j-1} \cup T_{\mathbf{v},j} \cup T_{\mathbf{v},j+1} & \text{if } t_i(\mathbf{v}) \text{ is neither a one-sided nor a two-sided} \\ & \text{singular patch relative to } \mathbf{v}; \\ \cup_{k=j-2}^{j+2} T_{\mathbf{v},k} & \text{if } t_i(\mathbf{v}) \text{ is a two-sided singular patch relative to } \mathbf{v}; \\ \cup_{k=j-1}^{j+2} T_{\mathbf{v},k} & \text{if } t_i(\mathbf{v}) \text{ is a one-sided singular patch with singular} \\ & \text{edge} \langle \mathbf{v}, \mathbf{x}_{\mathbf{v},j+1} \rangle. \end{cases}$$

From the construction procedure, we may see that with the exception of the one supported on the union of triangles with a near-singular vertex as the common vertex, all vertex splines are bounded by the constant

$$\begin{aligned} b := \; & \text{the maximum of the ratios of the areas of} \\ & \text{any two adjacent patches of } \triangle \text{ sharing a common edge.} \end{aligned}$$

But those vertex splines which are supported on the union of all triangles sharing a near-singular vertex have to be dependent on the constant η, where

$$\eta := \min \left\{ \frac{|\langle \mathbf{x}_{\mathbf{v},1}, \mathbf{v}, \mathbf{x}_{\mathbf{v},3} \rangle|}{|T_{\mathbf{v},i}|}, \frac{|\langle \mathbf{x}_{\mathbf{v},2}, \mathbf{v}, \mathbf{x}_{\mathbf{v},4} \rangle|}{|T_{\mathbf{v},i}|} : i = 1, 2, 3, 4 \right\}$$

which measures the near-singularity of $\triangle$. Here the minimum is taken over all near-singular vertices $\mathbf{v}$ and $|T_{\mathbf{v},i}|$ denotes the area of triangle $T_{\mathbf{v},i}$ and so do $|\langle \mathbf{x}_{\mathbf{v},1}, \mathbf{v}, \mathbf{x}_{\mathbf{v},3} \rangle|$ and $|\langle \mathbf{x}_{\mathbf{v},2}, \mathbf{v}, \mathbf{x}_{\mathbf{v},4} \rangle|$.

Examples of vertex splines in $\widehat{S}^1_5, \widehat{S}^2_8$ based on triangulation were already given in [36, 37, 39, 40]. Examples of vertex spline $\widehat{S}^1_5$ on a mixed grid partition will be given in Appendix as well as their graphs.

2.5. An Approximation Formula and Its Approximation Power

One of our main objectives is to construct an approximation formula, based on fundamental vertex splines we constructed in the previous section, which realizes the full approximation order of the spline space $S_d^r(\triangle)$, where $d \geq 3r+2$. In order to do so, consider the linear operator L defined as follows:

$$\begin{aligned} Lf \;=\; & \sum_{\mathbf{v}\in\mathcal{V}}\sum_{\gamma\in J_1} D^\gamma f(\mathbf{v})V_{\mathbf{v}}^\gamma + \sum_{e\in\mathcal{E}}\sum_{\gamma\in J_e} D_e^\gamma f(\mathbf{x}_{e,1})V_e^\gamma \\ & + \sum_{t_i(\mathbf{v})\in\mathcal{T}}\sum_{\gamma\in J_{\mathbf{v},i}} D^\gamma_{t_i(\mathbf{v})} f(\mathbf{v})V^\gamma_{t_i(\mathbf{v})} + \sum_{p\in\mathcal{P}}\sum_{\gamma\in J_5} D_p^\gamma f(\mathbf{x}_{p,1})V_p^\gamma \end{aligned} \tag{2.5.1}$$

where f is a sufficiently smooth function. We are now able to establish the following results.

PROPOSITION 2.5. *$Lf = f$ for any polynomial f of total degree* $\leq 3r+2$.

Proof. Let n be the number of patches in $\triangle$. We use induction on n to prove this result. For $n=1, L$ is an interpolation operator based on δ, the only patch (triangle or parallelogram) of $\triangle$. Since the interpolation conditions associated with each vertex of δ satisfy the assumptions of Propositions 2.2 or 2.4, we know that $Lf = f$ for all $f \in \pi_{3r+2}$. Suppose that the proposition holds for $m = \#\{\delta : \delta \in \triangle\}$. Let $\#\{\delta : \delta \in \triangle'\} = m+1$ with $\triangle' = \triangle \cup \{\delta_{m+1}\}$ of δ'. By relabeling if necessary, assume that δ_{m+1} is on the boundary, i.e., δ_{m+1} has at least one interior edge, and for the time being, assume that it has only one interior edge $[\mathbf{y}_1, \mathbf{y}_2]$. First, if $\delta_{m+1} = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\rangle$ is a triangle, we note that the uniqueness in Lemmas 2.4 and 2.9 and the other interpolation conditions imply $D^{\alpha_1}_{\mathbf{y}_2-\mathbf{y}_1} D^{\alpha_2}_{\mathbf{y}_3-\mathbf{y}_1} Lf(\mathbf{y}_1) = D^{\alpha_1}_{\mathbf{y}_2-\mathbf{y}_1} D^{\alpha_2}_{\mathbf{y}_3-\mathbf{y}_1} f(\mathbf{y}_1), (\alpha_1, \alpha_2) \in J_4$ and $D^{\alpha_1}_{\mathbf{y}_1-\mathbf{y}_2} D^{\alpha_2}_{\mathbf{y}_3-\mathbf{y}_2} Lf(\mathbf{y}_2) = D^{\alpha_1}_{\mathbf{y}_1-\mathbf{y}_2} D^{\alpha_2}_{\mathbf{y}_3-\mathbf{y}_2} f(\mathbf{y}_2), (\alpha_1, \alpha_2) \in J_4$ and the smoothness conditions of across the edge $\langle \mathbf{y}_1, \mathbf{y}_2\rangle$ can be rewritten as appropriate interpolation conditions (directional derivatives interpolating f at $\mathbf{y}_1$ and $\mathbf{y}_2$ by Lemmas 2.3 and 2.8.) Thus we know $L_{\triangle'} f\Big|_{\triangle} = L_\delta f, L_{\triangle'} f\Big|_{\delta_{m+1}} = L_{\delta_{m+1}} f$ where $L_\triangle f, L_{\triangle'} f$ and $L_{\delta_{m+1}} f$ are the linear operators based on mixed grid partition $\triangle, \triangle', \delta_{m+1}$, respectively. By the induction hypothesis, we have $L_{\triangle'} f\Big|_{\triangle} = f$ on $\triangle$ and $L_{\triangle'} f\Big|_{\delta_{m+1}} = f$ on δ_{m+1}. Secondly, if δ_{m+1} is a parallelogram, the argument is as same as above. Hence, $Lf = f$ on $\triangle'$. The proof is similar if δ_{m+1} contains two or three interior edges. Therefore, we have established this result.

The above result can be improved if we interpret the directional derivatives in the definition of L properly. Consider

$$
\begin{aligned}
Lf \;=\; & \sum_{\mathbf{v}\in\mathcal{V}}\sum_{\gamma\in J_1} D^{\gamma} f(\mathbf{v}) V_{\mathbf{v}}^{\gamma} + \sum_{e\in\mathcal{E}_1}\sum_{\gamma\in J_e} D_e^{\gamma} f\Big|_{\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,3},\mathbf{x}_{e,4}\rangle}(\mathbf{x}_{e,1}) V_e^{\gamma} \\
& + \sum_{e\in\mathcal{E}_2}\sum_{\gamma\in J_e} D_e^{\gamma} f\Big|_{\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,3}\rangle}(\mathbf{x}_{e,1}) V_e^{\gamma} \\
& + \sum_{t_i(\mathbf{v})\in\mathcal{T}}\sum_{\gamma\in J_{\mathbf{v},i}} D_{t_i(\mathbf{v})}^{\gamma} f\Big|_{t_i(\mathbf{v})}(\mathbf{v}) V_{t_i(\mathbf{v})}^{\gamma} \\
& + \sum_{p\in\mathcal{P}}\sum_{\gamma\in J_5} D_p^{\gamma} f\Big|_{\langle \mathbf{x}_{p,1},\mathbf{x}_{p,2},\mathbf{x}_{p,3},\mathbf{x}_{p,4}\rangle}(\mathbf{x}_{p,1}) V_p^{\gamma}.
\end{aligned}
$$

Then we have the following proposition.

PROPOSITION 2.6. *$Lf = f$ for any function $f \in \widehat{S}_{3r+2}^{r}$.*

Proof. Let $f_1 = Lf - f$. Then $f_1 \in \widehat{S}_{3r+2}^{r}$ and f_1 satisfies

$$D^{\alpha} f_1(\mathbf{v}) = 0, \qquad \alpha \in J_1, \mathbf{v} \in \mathcal{V};$$

$$D_e^{\alpha} f_1\Big|_{\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,3},\mathbf{x}_{e,4}\rangle}(\mathbf{x}_{e,1}) = 0, \alpha \in J_e, e \in \mathcal{E}_1;$$

$$D_e^{\alpha} f_1\Big|_{\langle \mathbf{x}_{e,1},\mathbf{x}_{e,2},\mathbf{x}_{e,3}\rangle}(\mathbf{x}_{e,1}) = 0, \alpha \in J_e, e \in \mathcal{E}_2;$$

$$D_{t_i(\mathbf{v})}^{\alpha} f_1\Big|_{t_i(\mathbf{v})}(\mathbf{v}) = 0, \alpha \in J_{\mathbf{v},i}, t_i(\mathbf{v}) \in \mathcal{T};$$

and

$$D_p^{\alpha} f_1(\mathbf{x}_{p,4}) = 0, \qquad \alpha \in J_5,\ p \in \mathcal{P}.$$

By using the argument in the proof of Proposition 2.5, we conclude that $f_1 \equiv 0$ on $\triangle$.

Consequently, we have

THEOREM 2.1. *The collection*

$$
\begin{aligned}
\mathcal{B} : \; & = \{V_{\mathbf{v}}^{\gamma} : \mathbf{v} \in \mathcal{V}, \gamma \in J_1\} \cup \{V_e^{\gamma} : \gamma \in J_e, e \in \mathcal{E}_1 \cup \mathcal{E}_2\} \\
& \cup \{V_{t_i(\mathbf{v})}^{\gamma} : \gamma \in J_{\mathbf{v},i}, t_i(\mathbf{v}) \in \mathcal{T}\} \cup \{V_p^{\gamma} : \gamma \in J_5, p \in \mathcal{P}\}
\end{aligned}
$$

is a basis of $\widehat{S}_{3r+2}^{r}(\triangle)$.

Proof. It is clear that $\mathcal{B} \subset \widehat{S}_{3r+2}^{r}(\triangle)$ and that $\mathcal{B}$ is a linear independent set. By Proposition 2.6 above, it also spans $\widehat{S}_{3r+2}^{r}(\triangle)$, and is therefore a basis.

We now consider the approximation power of this linear operator L. We need more notations. A space S has approximation order m if m is the largest integer such that

$$\operatorname{dist}(f, S) \leq \operatorname{const}_f |\triangle|^m,$$

with the meshsize

$$|\triangle| := sup_{\delta\in\triangle} \text{ diameter } \delta,$$

holds for all sufficiently smooth function f, where the distance between functions is measured in the uniform-norm in $G \subset \cup\{\delta : \delta \in \triangle\}$. For $f \in C^k(G)$, denote

$$||D^k f||_k = \max_{|\alpha|=k} ||D^\alpha f||_{L^\infty(G)}.$$

We also need a lemma which is borrowed from [39].

LEMMA 2.20 *There exist a constant* C *such that*

$$|||\widehat{\mathbf{v}}|||_{k+m} \le C(\sum_{j=1}^{m} |\widehat{\mathbf{v}}|_{k+j}) \qquad \textit{for all } \widehat{\mathbf{v}} \in C^{k+m}/\pi_k,$$

where $|\mathbf{v}|_{k+j} = \sum_{|\alpha|=k+j} ||D^\alpha \mathbf{v}||_\infty, ||\mathbf{v}||_{k+j} = \sum_{|\alpha|\le k+j} ||D^\alpha \mathbf{v}||_\infty$ *and* $|||\mathbf{v}|||_{k+m} = inf\{||\mathbf{v}+p||_{k+m}, p \in \pi_k\}$.

We are now ready to prove the following theorem

THEOREM 2.2. *Let* $d \ge 3r+2$. *There exist a linear operator* L *with range* $\widehat{S}_d^r$ *such that*

$$||Lf - f|| \le C_f |\triangle|^{d+1} \tag{2.5.2}$$

for all sufficiently smoothness function f, *where* C_f *is a constant independent of* $|\triangle|$. *Consequently,*

$$dist(f, \widehat{S}_k^{r,r+l}) \le C \max_{1\le j\le 2k} ||D^{d+j} f|||\triangle|^{d+1} \tag{2.5.3}$$

for $0 \le l \le [(d-2r-1)/2]$. *In particular, for* $d = 3r+2, L$ *can be chosen to be* $(*)$.

Remark. It should be emphasized that the constant C depends on the geometry of the partition $\triangle$. As a consequence of the usage of Lemmas 2.1–2.19 in the construction procedure of fundamental vertex splines, C depends on b which is the largest ratio of the areas of any two neighboring patches of $\triangle$ and also depends on the measurement η of the near-singularity of $\triangle$ when $d < 4r+1$. (cf. §2.4 for the definition of η.)

Proof. For $d \ge 4r+1$, this theorem was proved in [39]. In the following, we only consider $d = 3r+2$, since a similar argument yields the desired result for $3r+2 < d < 4r+1$. Fix a point $\mathbf{x} \in G$ and consider a linear functional

$$F(f) = Lf(\mathbf{x}) - f(\mathbf{x}).$$

It is easy to see that F satisfies the following:

(i) $|F(f)| \le K_1 \sum_{j=0}^{6r+4} ||D^j f|||\triangle|^j$ and

(ii) $F(p) = 0$ for all $p \in \pi_{3r+2}$.

Indeed, (ii) follows from Proposition 2.5. As for (i), if $|\triangle| = 1$, it follows that $|F(f)| \leq K_1 \sum_{j=0}^{6r+4} ||D^j f||$ from the construction of fundamental vertex splines in §2.4; if $|\triangle| < 1$, by simply letting $\tilde{f}(y) = f(|\triangle|y)$ and $\tilde{R} = \{y, |\triangle|y \in R\}$, we can see that the maximum of the diameters of all patches of $\tilde{R}$ induced from that of R is 1 and

$$|F(f)| = |\tilde{F}(\tilde{f})| \leq K_1 \sum_{j=0}^{6r+4} ||D^j \tilde{f}|| = K_1 \sum_{j=0}^{6r+4} ||D^j f|||\triangle|^j.$$

For $|\triangle| = 1$, clearly,

$$\begin{aligned} |F(f)| &= |F(f+p)| \leq K_1 \sum_{j=0}^{6r+4} ||D^j(f+p)|| \\ &\leq K_1 \sum_{j=0}^{6r+4} |f+p|_j \\ &= K_1 ||f+p||_{6r+4}, \end{aligned}$$

for any $p \in \pi_{3r+2}$. It follows that

$$|F(f)| \leq K_1 |||f|||_{6r+4}.$$

By Lemma 2.20,

$$\begin{aligned} |F(f)| &\leq K_1 |||f|||_{6r+4} = K_2 \sum_{j=1}^{3r+2} |f|_{3r+2+j} \\ &\leq C \max_{1 \leq j \leq 3r+2} ||D^{3r+2+j} f||_\infty. \end{aligned}$$

For $|\triangle| < 1$, we consider $\tilde{f}(y) = f(|\triangle|y)$, and $\tilde{R} = \{y : |\triangle|y \in R\}$ again and apply the same argument with $\tilde{f}$ instead of f as above. This completes the proof of (2.5.2) for $d = 3r+2$. Consequently, for $0 \leq l \leq [(r+1)/2]$

$$\mathrm{dist}(f, S_{3r+2}^{r,r+l}) \leq \mathrm{dist}(f, S_{3r+2}^{r,r+[(r+1)/2]})$$

$$\leq C \max_{1 \leq j \leq 3r+2} ||D^{3r+2+j} f||_\infty |\triangle|^{3r+3}$$

which is (2.5.3) for $d = 3r+2$.

3. TRIVARIATE VERTEX SPLINES

3.1. Polynomial Representations

The region $R \subset \mathbb{R}^3$ of interest is assumed to have been partitioned into patches (tetrahedra, prisms, and parallelepipeds) in this part. As in part I, we will use barycentric coordinates rather than the usual Cartesian coordinates. Thus, we will use B-forms to express each polynomial piece of any spline defined on the patches of R. The following is an introduction to B-forms of polynomials and their notations.

For a tetrahedron $T_1 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle$, where $\mathbf{x}_i \in \mathbb{R}^3, i = 1, 2, 3, 4$, any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ may be identified by

$$\mathbf{x} = \sum_{i=1}^{4} \lambda_i \mathbf{x}_i \quad \text{and} \quad \sum_{i=1}^{4} \lambda_i = 1. \tag{3.1.1}$$

In fact, if we consider the signed volume

$$\mathrm{vol}\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle = \frac{1}{3!} \begin{vmatrix} 1 & x_{11} & x_{12} & x_{13} \\ 1 & x_{21} & x_{22} & x_{23} \\ 1 & x_{31} & x_{32} & x_{33} \\ 1 & x_{41} & x_{42} & x_{43} \end{vmatrix}$$

of the convex hull $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle = \{\sum_{i=1}^{4} \lambda_i \mathbf{x}_i : \sum_{i=1}^{4} \lambda_i = 1, \lambda_i \geq 0\}$, then it is clear that

$$\lambda_i = \lambda_i(\mathbf{x}) = \frac{\mathrm{vol}\langle \mathbf{x}_1, \cdots, \mathbf{x}_{i-1}, \mathbf{x}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_4 \rangle}{\mathrm{vol}\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle}, \quad i = 1, 2, 3, 4. \tag{3.1.2}$$

This 4-tuple $\lambda = (\lambda_1(\mathbf{x}), \cdots, \lambda_4(\mathbf{x}))$ is called the barycentric coordinate of $\mathbf{x}$ relative to T_1. For any $\beta \in \mathbb{Z}_+^4$ with $|\beta| = \beta_1 + \beta_2 + \beta_3 + \beta_4$, let

$$\Phi_\beta(\lambda) := \frac{|\beta|!}{\beta!} \lambda^\beta = \frac{|\beta|!}{\beta_1! \beta_2! \beta_3! \beta_4!} (\lambda_1)^{\beta_1} (\lambda_2)^{\beta_2} (\lambda_3)^{\beta_3} (\lambda_4)^{\beta_4}. \tag{3.1.3}$$

Clearly, $\Phi_\beta(\lambda)$ is a polynomial of total degree $|\beta|$ since λ_i is a linear function of $\mathbf{x}, i = 1, 2, 3, 4$, by (3.1.2). It is well known that

$$\{\Phi_\beta(\lambda) : |\beta| = n\}$$

is a basis of the space of polynomials of total degree $\leq n$. Hence, we may uniquely express a polynomial $P_n(\mathbf{x})$ of total degree n by using following formulation

$$P_n(\mathbf{x}) = \sum_{|\beta| = n} a_\beta \Phi_\beta(\lambda) \tag{3.1.4}$$

which is called *the B-form of polynomial* $P_n(\mathbf{x})$ *with respect to* T_1. Let $\pi_n(T_1)$ denote the space all polynomials of total degree n. The set

$$\Big\{ \Big(\sum_{i=1}^{4} \frac{\beta_i}{n} \mathbf{x}_i, a_\beta \Big) : |\beta| = n \Big\} \tag{3.1.5}$$

is called *the B-net* of P_n on T_1 which may be shown in an array on T_1 as that in Figure 3.1 where a polynomial $P_5(\mathbf{x})$ of total degree 5 is considered.

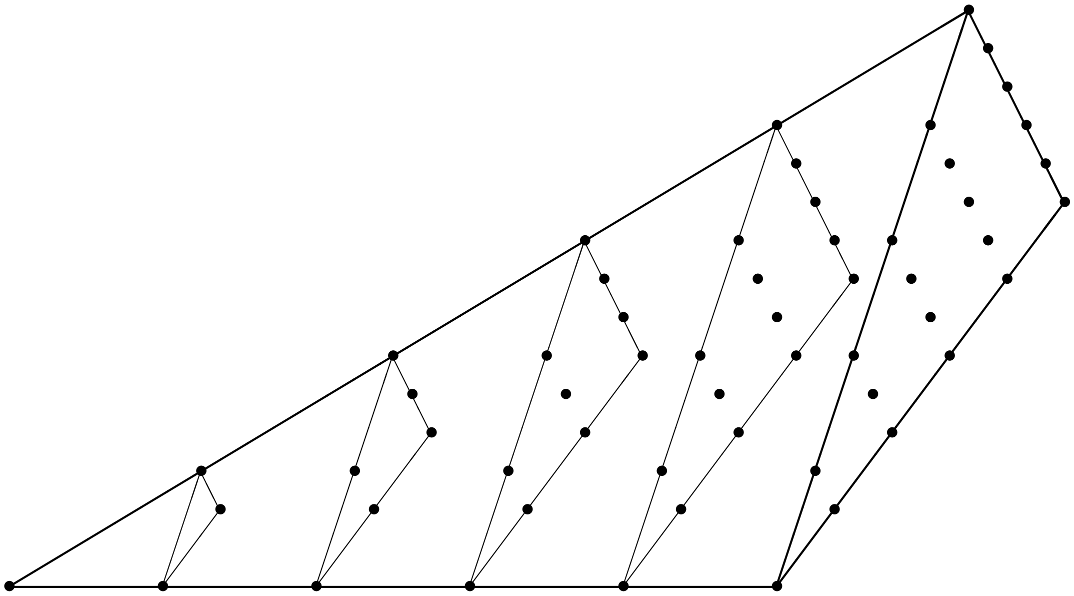


Figure 3.1 The B-net of P_5

In addition, for each vertex $\mathbf{x}_i$ of T_1, layer l of the B-net attached to $\mathbf{x}_i$ is the collection of all coefficients a_β with $\beta_i = n - l$. For an edge $e = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle$, say, layer l of the B-net around e is the collection of all coefficients a_β with $\beta_3 + \beta_4 = l$. Similarly, we may refer to layer l of the B-net around other edges of T_1. For each facet $f = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$, say, layer l of the B-net near f is the collection of all coefficients a_β with $\beta_4 = l$. We may refer to layer l of the B-net near other facets of T_1. The l^{th} core of the B-net of T_1 is the collection of coefficients a_β with $\beta_i \geq l+1, i = 1, 2, 3, 4$.

Next, let $T_2 = \langle \mathbf{y}_1, \cdots, \mathbf{y}_6 \rangle$ be a prism with vertices $\mathbf{y}_i \in \mathbb{R}^3, i = 1, \cdots, 6$. Assume that $\langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \rangle$ and $\langle \mathbf{y}_4, \mathbf{y}_5, \mathbf{y}_6 \rangle$ are two triangles in $\mathbb{R}^3$ and let $\langle \mathbf{y}_1, \mathbf{y}_4 \rangle || \langle \mathbf{y}_2, \mathbf{y}_5 \rangle || \langle \mathbf{y}_3, \mathbf{y}_6 \rangle$ without loss of generality. For each $\mathbf{x} \in T_2$, it is clear that $\mathbf{x}$ may be uniquely expressed as

$$\mathbf{x} = \nu_1 \mathbf{y}_1 + \nu_2 \mathbf{y}_2 + \nu_3 \mathbf{y}_3 + \nu_4 (\mathbf{y}_4 - \mathbf{y}_1) \tag{3.1.6}$$

with $\nu_1 + \nu_2 + \nu_3 = 1$. Set $\nu = (\nu_1(\mathbf{x}), \nu_2(\mathbf{x}), \nu_3(\mathbf{x}), \nu_4(\mathbf{x}))$. We consider a polynomial $\bar{P}_n(\mathbf{x})$ of degree (n, n) in the form of

$$\bar{P}_n(\mathbf{x}) = \sum_{\beta \in \bar{\Lambda}_n} \bar{a}_\beta \bar{\Phi}_\beta^{(n,n)}(\nu) \tag{3.1.7}$$

where

$$\bar{\Phi}_\beta^{(n,n)}(\nu) = \frac{n!}{\beta_1!\beta_2!\beta_3!} \frac{n!}{\beta_4!(n-\beta_4)!} (\nu_1)^{\beta_1} (\nu_2)^{\beta_2} (\nu_3)^{\beta_3} (\nu_4)^{\beta_4} (1-\nu_4)^{n-\beta_4} \tag{3.1.8}$$

and $\bar{\Lambda}_n = \{\beta \in \mathbb{Z}_+^4 : \beta_1 + \beta_2 + \beta_3 = n, 0 \le \beta_4 \le n\}$. Clearly, $\bar{\Phi}_\beta^{(n,n)}(\nu)$ is a polynomial since $\nu_1, \nu_2, \nu_3, \nu_4$ are linear functions of $\mathbf{x}$. Let $\pi_n(T_2)$ denote the space of all polynomials $\bar{P}_n$ in the formulation (3.1.7) which is called *the B-form of polynomial* $\bar{P}_n$ *with respect to* T_2. Also, the set

$$\left\{ \left(\sum_{i=1}^{3} \frac{\beta_i}{n} \mathbf{y}_i + \frac{\beta_4}{n} (\mathbf{y}_4 - \mathbf{y}_1) \right, \bar{a}_\beta) : \beta_1 + \beta_2 + \beta_3 = n, \beta_4 \le n \right\}$$

is called *the B-net* of $\bar{P}_n$ on T_2 which may be shown as an array as in Figure 3.2.

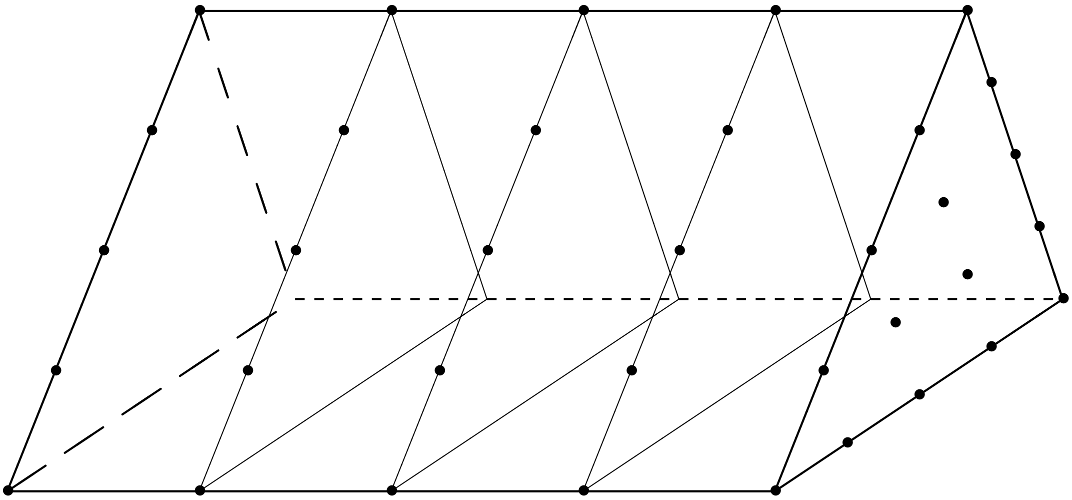


Figure 3.2 The B-net of $\bar{P}_4$

In addition, for each vertex $\mathbf{y}_1$, say, layer l of the B-net attached to $\mathbf{y}_1$ is the collection of coefficients $\bar{a}_\beta$ with $\beta_1 = n - l + \beta_4, 0 \le \beta_4 \le l$. Similarly, we may refer to layer l of the B-net of other vertices of T_2. For each edge $e = \langle \mathbf{y}_1, \mathbf{y}_4 \rangle$, say, layer l of the B-net around e is the collection of coefficients $\bar{a}_\beta$ with $\beta_1 = n - l$. For edge

$e = \langle \mathbf{y}_1, \mathbf{y}_2 \rangle$, layer l of B-net around the edge e is the collection of all coefficients $\bar{a}_\beta$ with $\beta_3 = l - \beta_4, \beta_4 = 0, \cdots, l$. Similarly, we may refer to layer l of the B-net of other edges of T_2. For a triangular facet $f = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \rangle$, say, layer l of the B-net near f is the collection of all coefficients $\bar{a}_\beta$ with $\beta_4 = l$. For a parallelogram facet $f = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_4, \mathbf{y}_5 \rangle$, say, layer l of the B-net near f is the collection of all coefficients $\bar{a}_\beta$ with $\beta_3 = l$. Similarly, we may refer to layer l of other facets of T_2. The l^{th} core of the B-net on T_2 is the collection of all coefficients $\bar{a}_\beta$ with $\beta_i \geq l+1, i = 1, 2, 3$ and $l + 1 \leq \beta_4 \leq n - l - 1$.

Now, let $T_3 = \langle \mathbf{z}_1, \cdots, \mathbf{z}_8 \rangle$ be a parallelepiped with vertices $\mathbf{z}_i \in \mathbb{R}^3, i = 1, \cdots, 8$. Without loss of generality, we may assume that $\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4 \rangle || \langle \mathbf{z}_5, \mathbf{z}_6, \mathbf{z}_7, \mathbf{z}_8 \rangle$, $\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_5, \mathbf{z}_6 \rangle ||\ \langle \mathbf{z}_3, \mathbf{z}_4, \mathbf{z}_7, \mathbf{z}_8 \rangle$, and $\langle \mathbf{z}_1, \mathbf{z}_3, \mathbf{z}_5, \mathbf{z}_7 \rangle || \langle \mathbf{z}_2, \mathbf{z}_6, \mathbf{z}_6, \mathbf{z}_8 \rangle$ where each of them is a parallelogram. For any $\mathbf{x} \in \mathbb{R}^3, \mathbf{x}$ may be uniquely expressed by

$$\mathbf{x} = \mathbf{z}_1 + \mu_1(\mathbf{z}_2 - \mathbf{z}_1) + \mu_2(\mathbf{z}_3 - \mathbf{z}_1) + \nu_3(\mathbf{z}_5 - \mathbf{z}_1). \tag{3.1.9}$$

Clearly, if $\mathbf{x} \in T_3$, then $\mu_i \geq 0, i = 1, 2, 3$. Set $\mu = (\mu_1(\mathbf{x}), \mu_2(\mathbf{x}), \mu_3(\mathbf{x}))$. Consider a polynomial $\tilde{P}_n(\mathbf{z})$ in the form of

$$\tilde{P}_n(\mathbf{x}) = \sum_{\beta \leq (n,n,n)} \tilde{a}_\beta \tilde{\Phi}_\beta^{(n,n,n)}(\mu), \tag{3.1.10}$$

where

$$\tilde{\Phi}_\beta^{(n,n,n)}(\mu) = \prod_{i=1}^{3} \frac{n!}{\beta_i!(n-\beta_i)!} (\mu_i)^{\beta_i} (1-\mu_i)^{n-\beta_i}. \tag{3.1.11}$$

The expression (3.1.10) is called *the B-form of polynomial* $\tilde{P}_n$ *with respect to* T_3. We denote by $\pi_n(T_3)$ the space of all such polynomials $\tilde{P}_n(\mathbf{x})$ in the form of (3.1.10). The set

$$\left\{ (\mathbf{z}_1 + \frac{\beta_1}{n}(\mathbf{z}_2 - \mathbf{z}_1) + \frac{\beta_2}{n}(\mathbf{z}_3 - \mathbf{z}_1) + \frac{\beta_3}{n}(\mathbf{z}_5 - \mathbf{z}_1), \tilde{a}_\beta) : \beta \leq (n, n, n) \right\}$$

is called *the B-net of* $\tilde{P}_n$ on T_3 which may be arranged as an array shown as Figure 3.3.

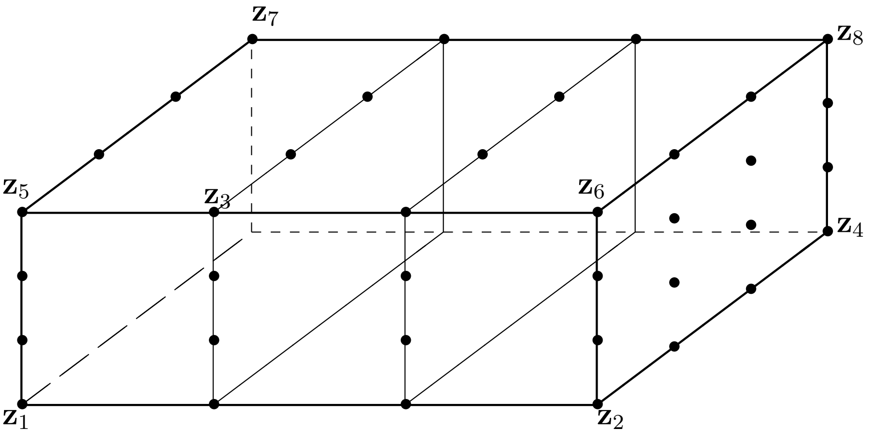


Figure 3.3 The B-net of $\tilde{P}_3$

In addition, for each vertex $\mathbf{z}_1$, say, layer l of the B-net attached to $\mathbf{z}_1$ is the collection of all coefficients $\tilde{a}_\beta$ with $\beta_1 + \beta_2 + \beta_3 = l$. Similarly, layer l of the B-net of other vertices of T_3 may be specified. For each edge $e = \langle \mathbf{z}_1, \mathbf{z}_2 \rangle$, say, layer l of the B-net around edge e is the collection of all coefficients $\tilde{a}_\beta$ with $\beta_1 + \beta_3 = l$. Similarly, layer l of the B-net around other edges of T_3 may be specified. For each facet $f = \langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4 \rangle$, say, layer l of the B-net near facet f is the collection of all coefficients $\tilde{a}_\beta$ with $\beta_3 = l$. Similarly, we may refer to layer l of the B-net near other facets of T_3. The l^{th} core of B-net on T_3 is the collection of all coefficients $\tilde{a}_\beta$ with $\beta_i \geq l + 1, i = 1, 2, 3$.

3.2. Polynomial Interpolation

By considering polynomial interpolation at the vertices at a tetrahedron, a prism, or a parallelepiped, we will understand the relationship between interpolation conditions and the B-nets of the interpolating polynomials. This will help us in constructing vertex splines in the later sections. We need more notation and definitions for discussing polynomial interpolation.

Let $\Gamma_n := \{\beta \in \mathbb{Z}_+^3 : |\beta| \leq n\}$ and $\Lambda_n := \{\beta \in \mathbb{Z}_+^4 : |\beta| = n\}$. We say that the subsets M_i of $\Gamma_n, i = 1,2,3,4$, induce a partition of Λ_n if they satisfy

1° $A_i^n M_i \cap A_j^n M_j = \emptyset$, for $i \neq j$, and

2° $\cup_{i=1}^4 A_i^n M_i = \Lambda_n$,

where A_i^n is a map from $\mathbb{Z}_+^3$ to $\mathbb{Z}_+^4$ defined by

$$A_i^n \beta = (\beta_1, \cdots, \beta_{i-1}, n - |\beta|, \beta_i, \cdots, \beta_3) \in \mathbb{Z}_+^4$$

for $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{Z}_+^3, i = 1,2,3,4$.

For a tetrahedron $T_1 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle$, we denote

$$\begin{aligned} D_1^\beta &:= D_{\mathbf{x}_2-\mathbf{x}_1}^{\beta_1} D_{\mathbf{x}_3-\mathbf{x}_1}^{\beta_2} D_{\mathbf{x}_4-\mathbf{x}_1}^{\beta_3} \\ D_2^\beta &:= D_{\mathbf{x}_1-\mathbf{x}_2}^{\beta_1} D_{\mathbf{x}_3-\mathbf{x}_2}^{\beta_2} D_{\mathbf{x}_4-\mathbf{x}_2}^{\beta_3} \\ D_3^\beta &:= D_{\mathbf{x}_1-\mathbf{x}_3}^{\beta_1} D_{\mathbf{x}_2-\mathbf{x}_3}^{\beta_2} D_{\mathbf{x}_4-\mathbf{x}_3}^{\beta_3} \end{aligned}$$

and

$$D_4^\beta := D_{\mathbf{x}_1-\mathbf{x}_4}^{\beta_1} D_{\mathbf{x}_2-\mathbf{x}_4}^{\beta_2} D_{\mathbf{x}_3-\mathbf{x}_4}^{\beta_3}$$

Also, we define a map $C_i : \mathbb{Z}_+^4 \longrightarrow \mathbb{Z}_+^3$ by

$$C_i\beta = C_i(\beta_1, \beta_2, \beta_3, \beta_4) = (\beta_1, \cdots, \beta_{i-1}, \beta_{i+1}, \cdots, \beta_4)$$

for $i = 1,2,3,4$.

We are now ready to present the following propositions which are similar to those in section 2.2.

PROPOSITION 3.1. *Suppose that* M_1, M_2, M_3, *and* M_4 *are all lower sets of* Γ_n *that induce a partition of* Λ_n. *Then for any given data* $\{f_{i,\beta} : \beta \in M_i, i = 1,2,3,4\}$, *there exists a unique polynomial* $p_n(\mathbf{x}) \in \pi_n(T_1)$ *satisfying*

$$D_i^\beta p_n(\mathbf{x}_i) = f_{i,\beta}, i = 1,2,3,4. \tag{3.2.1}$$

Moreover, $p_n(\mathbf{x})$ may be expressed as follows:

$$p_n(\mathbf{x}) = \sum_{i=1}^{4} \sum_{\beta \in M_i} \left\{ \sum_{\gamma \le \beta} \binom{\beta}{\gamma} \frac{(n-|\gamma|)!}{n!} f_{i,\gamma} \right\} \Phi_{A_i^n \beta}(\lambda)$$

Actually, we will use a more generalized version of this proposition later which can be stated as follows.

PROPOSITION 3.2. *Suppose that $M_i \subset \Gamma_n, i = 1, 2, 3, 4$ induce a partition of Λ_n. Further, suppose that*

(i) M_1 is a lower set, and

(ii) for $i = 2, 3, 4$, the union of M_i and some elements of $C_i(\cup_{j=1}^{i-1} A_j^n M_j)$ is a lower set.

Then for any given data $\{f_{i,\beta} : \beta \in M_i, i = 1, 2, 3, 4\}$, there exists a $p_n(\mathbf{x}) \in \pi_n(T_1)$ that satisfies (3.2.1).

The proofs of these two propositions may be found in [39] and we omit its details here.

Next, let $T_2 = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_5, \mathbf{y}_6 \rangle$ be a prism in $\mathbb{R}^3$. For $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{Z}_+^3$, let

$$\begin{aligned} \bar{D}_1^\beta &:= D_{\mathbf{y}_2-\mathbf{y}_1}^{\beta_1} D_{\mathbf{y}_3-\mathbf{y}_1}^{\beta_2} D_{\mathbf{y}_4-\mathbf{y}_1}^{\beta_3} \\ \bar{D}_1^\beta &:= D_{\mathbf{y}_1-\mathbf{y}_2}^{\beta_1} D_{\mathbf{y}_3-\mathbf{y}_2}^{\beta_2} D_{\mathbf{y}_5-\mathbf{y}_2}^{\beta_3} \\ \bar{D}_1^\beta &:= D_{\mathbf{y}_2-\mathbf{y}_1}^{\beta_3} D_{\mathbf{y}_2-\mathbf{y}_3}^{\beta_2} D_{\mathbf{y}_6-\mathbf{y}_3}^{\beta_3} \end{aligned}$$

and

$$\bar{D}_4^\beta = (-1)^{\beta_3} \bar{D}_1^\beta, \bar{D}_5^\beta = (-1)^{\beta_3} \bar{D}_2^\beta, \bar{D}_3^\beta = (-1)^{\beta_3} \bar{D}_3^\beta.$$

Let $\bar{\Gamma}_n = \{\beta \in \mathbb{Z}_+^3 : \beta = (\beta_1, \beta_2, \beta_3) \le (n, n, n)\}$ and $\bar{\Lambda}_n = \{\beta \in \mathbb{Z}_+^4 : \beta = (\beta_1, \beta_2, \beta_3, \beta_4), \beta_1 + \beta_2 + \beta_3 = n, 0 \le \beta_4 \le n\}$. Define a map $\bar{A}_i^n : \bar{\Gamma}_n \longrightarrow \bar{\Lambda}_n$ by

$$\begin{aligned} \bar{A}_1^n \beta &:= (n - \beta_1 - \beta_2, \beta_1, \beta_2, \beta_3) \\ \bar{A}_2^n \beta &:= (\beta_1, n - \beta_1 - \beta_2, \beta_2, \beta_3) \\ \bar{A}_3^n \beta &:= (\beta_1, \beta_2, n - \beta_1 - \beta_2, \beta_3) \\ \bar{A}_4^n \beta &:= (n - \beta_1 - \beta_2, \beta_1, \beta_2, n - \beta_3) \\ \bar{A}_5^n \beta &:= (\beta_1, n - \beta_1 - \beta_2, \beta_2, n - \beta_3) \\ \bar{A}_6^n \beta &:= (\beta_1, \beta_2, n - \beta_1 - \beta_2, n - \beta_3) \end{aligned}$$

for any $\beta = (\beta_1, \beta_2, \beta_3) \in \bar{\Gamma}_n$.

We say that the subsets $M_i, i = 1, \cdots, 6$, of $\bar{\Gamma}_n$ induce a partition of $\bar{\Lambda}_n$ if

3° $\bar{A}_i^n M_i \cap \bar{A}_j^n M_j = \emptyset$ for $i \neq j$, and

4° $\cup_{i=1}^6 \bar{A}_i^n M_i = \bar{\Lambda}_n$.

Further, difference operators Δ_{21} and Δ_{31} are defined by

$$\Delta a_\alpha = a_{\alpha+e^i} - a_{\alpha+e^j}, i, j = 1, 2, 3, \alpha \in \mathbb{Z}_+^4$$

and Δ_i is defined as usual by

$$\Delta_i a_\alpha = a_{\alpha+e^i} - a_\alpha, \alpha \in \mathbb{Z}_+^4, \quad i = 1, 2, 3, 4.$$

We have the following:

PROPOSITION 3.3. *Suppose that* $M_i \subset \bar{\Gamma}_m, i = 1, \cdots, 6$, *are lower sets that induce a partition of* $\bar{\Lambda}_n$. *Then for any given data* $\{f_{i,\beta} : \beta \in M_i, i = 1, \cdots, 6\}$, *there exists a unique polynomial* $p_n(\mathbf{x}) \in \pi_n(T_2)$ *that satisfies*

$$\bar{D}_i^\beta p_n(\mathbf{y}_i) = f_{i,\beta},\ \beta \in M_i, i = 1, \cdots, 6 \tag{3.2.3}$$

Moreover, $p_n(\mathbf{x})$ *may be expressed as*

$$p_n(\mathbf{x}) = \sum_{i=1}^6 \sum_{\beta \in M_i} \left\{ \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \frac{(n-\gamma_1-\gamma_2)!}{n!} \frac{(n-\gamma_3)!}{n!} f_{i,\gamma} \right\} \bar{\Phi}_{\bar{A}_i^n \beta}(\nu). \tag{3.2.4}$$

Proof. By the above assumption, we have a $p_n \in \pi_n(T_2)$ in the form of

$$\begin{aligned} p_n(\mathbf{x}) &= \sum_{\alpha \in \bar{\Lambda}_n} \bar{a}_\alpha \bar{\Phi}_\alpha^{(n,n)}(\nu) \\ &= \sum_{i=1}^6 \sum_{\beta \in M_i} \bar{a}_{\bar{A}_i^n \beta} \bar{\Phi}_{\bar{A}_i^n \beta}^{(n,n)}(\nu)\,. \end{aligned}$$

For any $\beta \in M_1$,

$$\bar{D}_1 p_n(\mathbf{y}_1) = \frac{n!}{(n-\beta_1-\beta_2)!} \frac{n!}{(n-\beta_3)!} \Delta_{21}^{\beta_1} \Delta_{31}^{\beta_2} \Delta_4^{\beta_3} \bar{a}_{(n-\beta_1-\beta_2,0,0,0)}.$$

Thus,

$$\begin{aligned} & (-1)^{\beta_1+\beta_2+\beta_3} \frac{(n-\beta_1-\beta_2)!}{n!} \frac{(n-\beta_3)!}{n!} \bar{D}_1^\beta p_n(\mathbf{y}_1) \\ = \; & (-1)^{\beta_1+\beta_2+\beta_3} \Delta_{21}^{\beta_1} \Delta_{31}^{\beta_2} \Delta_4^{\beta_3} \bar{a}_{(n-\beta_1-\beta_2,0,0,0)} \\ = \; & \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} (-1)^\gamma \bar{a}_{(n-\gamma_1-\gamma_2,\gamma_1,\gamma_2,\gamma_3)}. \end{aligned}$$

By using the inversion formula, we obtain

$$\bar{a}_{\bar{A}_1^n\beta} = \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \frac{(n-\gamma_1-\gamma_2)!}{n!} \frac{(n-\gamma_3)!}{n!} \bar{D}_1^{\gamma} p_n(\mathbf{y}_1)$$

for $\beta \in M_1$. Similarly,

$$\bar{a}_{\bar{A}_i^n\beta} = \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \frac{(n-\gamma_1-\gamma_2)!}{n!} \frac{(n-\gamma_3)!}{n!} \bar{D}_i^{\gamma} p_n(\mathbf{y}_i)$$

for $\beta \in M_i, i = 2, \cdots, 6$. By the interpolation condition (3.2.3) and the assumption that M_i are lower sets, we see that $\bar{a}_{\bar{A}_i^n\beta}, \beta_i \in M_i, i = 1, \cdots, 6$, are uniquely determined by the given data set $\{f_{i,\beta} : \beta \in M_i, i = 1, \cdots, 6\}$ and a polynomial p_n with these coefficients $\bar{a}_{\bar{A}_i^n\beta}$ satisfies (3.2.3) since $M_i, i = 1, \cdots, 6$, induce a partition of $\bar{\Lambda}_n$.

Actually, we may relax the requirement on $M_i, i = 1, \cdots, 6$ slightly to make it more applicable. Thus, we have

PROPOSITION 3.4. *Suppose that* $M_i \subset \bar{\Gamma}_n, i = 1, \cdots, 6$, *induce a partition of* $\bar{\Lambda}_n$. *Further, suppose that*

(i) M_1 *is a lower set, and*

(ii) for each $i = 2, \cdots, 6$, *the union of* M_i *and some elements of* $\bar{C}_i(\cup_{j=1}^{i-1} \bar{A}_j^n M_j)$ *is a lower set.*

Then for any given data $\{f_{i,\beta} : \beta \in M_i, i = 1, \cdots, 6\}$, *there exists a unique* $p_n(\mathbf{x}) \in \pi_n(T_2)$ *that satisfies (3.2.3).*

Here, $\bar{C}_i = C_i, i = 1, 2, 3$ and $\bar{C}_i, i = 4, 5, 6$ are maps defined by

$$\begin{aligned} \bar{C}_4\beta &= (\beta_2, \beta_3, n-\beta_4) \\ \bar{C}_5\beta &= (\beta_1, \beta_3, n-\beta_4) \\ \bar{C}_6\beta &= (\beta_1, \beta_2, n-\beta_4) \end{aligned}$$

for $\beta = (\beta_1, \beta_2, \beta_3, \beta_4) \in \mathbb{Z}_+^4$. We omit its proof here since we can see the B-net determined by using information $\{f_{1,\beta} : \beta \in M_1\}$ can be used to determine other part of the B-net and so on.

Now let $T_3 = \langle \mathbf{z}_1, \cdots, \mathbf{z}_8 \rangle$ be a parallelepiped. For any $\beta \in \mathbb{Z}_+^3$, we denote

$$\tilde{D}^{\beta} := D_{\mathbf{z}_2-\mathbf{z}_1}^{\beta_1} D_{\mathbf{z}_3-\mathbf{z}_1}^{\beta_2} D_{\mathbf{z}_5-\mathbf{z}_1}^{\beta_3}$$

and $\eta^1 = (1,1,1)$, $\eta^2 = (-1,1,1)$, $\eta^3 = (1,-1,1)$, $\eta^4 = (-1,-1,1)$, $\eta^5 = (1,1,-1)$, $\eta^6 = (-1,1,-1)$, $\eta^7 = (1,-1,-1)$ and $\eta^8 = (-1,-1,-1)$. We define a map $\tilde{A}_i^n$:

$\bar{\Gamma}_n \longrightarrow \bar{\Gamma}_n$ by

$$\tilde{A}_i^n \beta = (\eta_1^i \beta_1, \eta_2^i \beta_2, \eta_3^i \beta_3) + \left(\frac{1-\eta_1^i}{2} n, \frac{1-\eta_2^i}{2} n, \frac{1-\eta_3^i}{2} n \right)$$

for any $\beta = (\beta_1, \beta_2, \beta_3) \in \bar{\Gamma}_n$, where $\eta^i = (\eta_1^i, \eta_2^i, \eta_3^i), i = 1, \cdots, 8$.

We say that the subsets M_i of $\bar{\Gamma}_n, i = 1, \cdots, 8$, induce a partition of $\bar{\Gamma}_n$ if $M_i, i = 1, \cdots, 8$, satisfy

5° $\tilde{A}_i^n M_i \cap \tilde{A}_j^n M_j = \emptyset$ for $i \neq j$, and

6° $\cup_{i=1}^8 \tilde{A}_i^n M_i = \bar{\Lambda}_n$.

Then we have the following

PROPOSITION 3.5. *Suppose that $M_i, i = 1, \cdots, 8$, are lower subsets of $\bar{\Gamma}_n$ that induce a partition of $\bar{\Gamma}_n$. Then for any given data $\{f_{i,\beta} : \beta \in M_i, i = 1, \cdots, 8\}$, there exists a unique a polynomial $p_n(\mathbf{x}) \in \pi_n(T_3)$ that satisfies*

$$\tilde{D}^\beta p_n(\mathbf{z}_i) = f_{i,\beta}, \ \beta \in M_i, i = 1, \cdots, 8. \tag{3.2.5}$$

Moreover, $p_n(\mathbf{x})$ may be expressed as

$$p_n(\mathbf{x}) = \sum_{i=1}^{8} \sum_{\beta \in M_i} \left\{ \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \frac{(n-\gamma_1)!}{n!} \frac{(n-\gamma_2)!}{n!} \frac{(n-\gamma_3)!}{n!} (\eta^i)^\gamma f_{i,\gamma} \right\} \tilde{\Phi}^{(n,n,n)}_{\tilde{A}_i^n \beta}(\mu) \tag{3.2.6}$$

The proof of Proposition 3.5 may be found in [39].

We may also relax the requirement on $M_i, i = 1, \cdots, 8$, slightly so that the resulting one is more applicable. That is, we have

PROPOSITION 3.6. *Suppose $M_i \subset \bar{\Gamma}_n, i = 1, \cdots, 8$ induce a partition of $\bar{\Lambda}_n$. Further, suppose that*

(i) M_1 is a lower set, and

(ii) for each $i = 2, \cdots, 8$, the union of M_i and some elements of $\tilde{C}_i(\cup_{j=1}^{i-1} \tilde{A}_j^n M_j)$ is a lower set.

Then for any given data $\{f_{i,\beta} : \beta \in M_i, i = 1, \cdots, 8\}$, there exists a unique $p_n(\mathbf{x}) \in \pi_n(T_3)$ that satisfies (3.2.5).

Here, $\tilde{C}_i = \tilde{A}_i^{-1}$ for $i = 1, \cdots, 8$. We omit its proof here and refer the reader to [39] for more details.

3.3. Smoothness Conditions and Their Applications

We are now going to study what conditions ensure that two polynomials P_n and Q_n defined on two adjacent geometric configurations (tetrahedron, prism, or parallelepiped) are joined smoothly together. There are six possibilities of two adjacent geometric configurations we need to study: two tetrahedrons, one tetrahedron and one prism, two prisms sharing a common triangular boundary facet, two prisms sharing a common parallelogram boundary facet, one prism and one parallelepiped, and two parallelepipeds. We have to study all these cases separately.

1° Suppose that P_n and Q_n are defined on two adjacent tetrahedrons $T_1 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle$ and $T_2 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_5 \rangle$ which share a common facet $T_1 \cap T_2 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$. See Figure 3.4 for the B-nets of P_n and Q_n when $n = 4$.

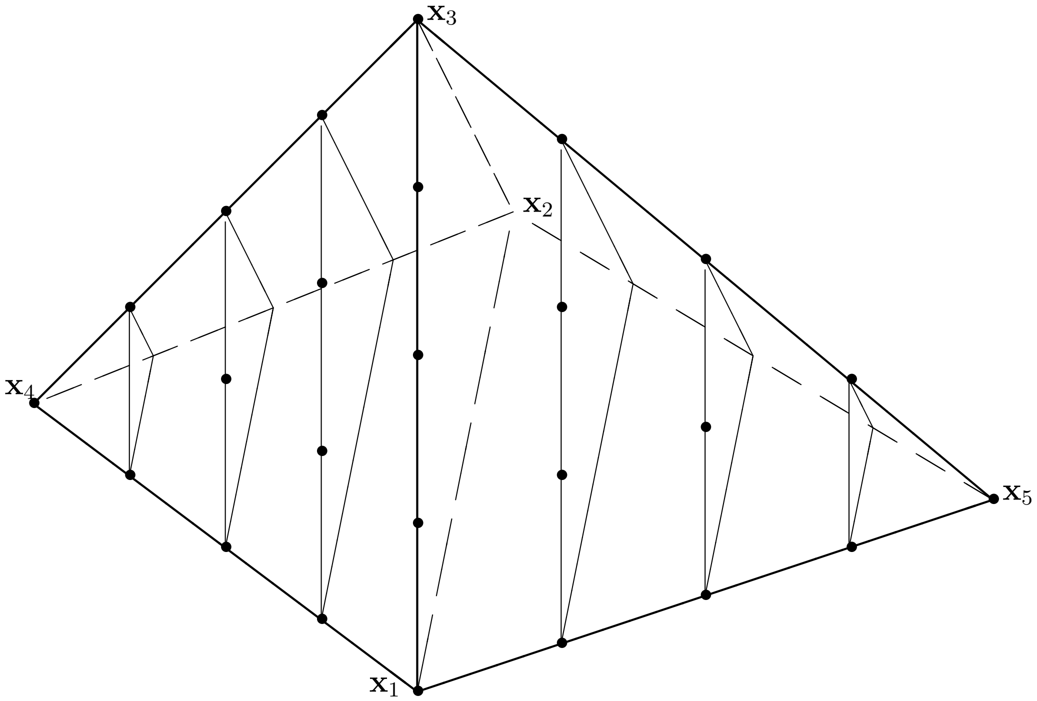


Figure 3.4 The B-nets of P_4 and Q_4

More precisely, let

$$P_n(\mathbf{x}) = \sum_{|\beta|=n} a_\beta \Phi_\beta(\lambda) \qquad \text{and} \qquad Q_n(\mathbf{x}) = \sum_{|\beta|=n} b_\beta \Phi_\beta(\mu),$$

where $\mathbf{x} = \sum_{i=1}^{4} \lambda_i \mathbf{x}_i = \sum_{i=1}^{3} \mu_i \mathbf{x}_i + \mu_4 \mathbf{x}_5$ with $\sum_{i=1}^{4} \lambda_i = \sum_{i=1}^{4} \mu_i = 1$.

Let F be a function defined by

$$F(\mathbf{x}) = \begin{cases} P_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_1 \\ Q_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_2. \end{cases}$$

Write $\mathbf{x}_5 = \sum_{i=1}^4 \lambda_i^0 \mathbf{x}_i$ with $\sum_{i=1}^4 \lambda_i^0 = 1$. Then clearly,

$$D_{\mathbf{x}_5-\mathbf{x}_1} = \lambda_2^0 D_{\mathbf{x}_2-\mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_3-\mathbf{x}_1} + \lambda_4^0 D_{\mathbf{x}_4-\mathbf{x}_1}.$$

Hence, we know that $F \in C^r(T_1 \cup T_2)$ if and only if

$$(D_{\mathbf{x}_5-\mathbf{x}_1})^k Q_n\Big|_{T_1\cap T_2} = (\lambda_2^0 D_{\mathbf{x}_2-\mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_3-\mathbf{x}_1} + \lambda_4^0 D_{\mathbf{x}_4-\mathbf{x}_1})^k P_n\Big|_{T_1\cap T_2}$$

for $0 \le k \le r$. Then the smoothness conditions between P_n and Q_n easily follow.

LEMMA 3.3.1. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$\Delta_{41}^l b_{ijk0} = (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31} + \lambda_4^0 \Delta_{41})^l a_{ijk0}, \qquad i+j+k = n-l, \tag{3.3.1}$$

for $0 \le k \le r, i, j = 1, \cdots, 4,$ and $\beta \in \mathbb{Z}_+^4$.

The proof may be found in [39]. The supports of these smoothness conditions (3.3.1) are shown in Figure 3.5a and Figure 3.5b.

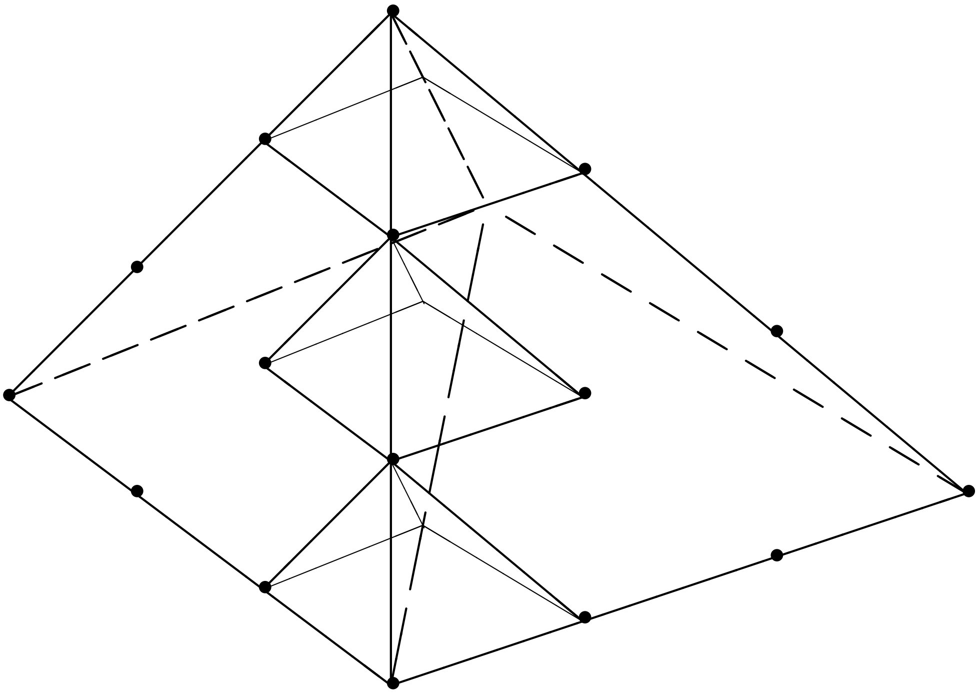


Figure 3.5a Some supports of the C^1 smoothness condition

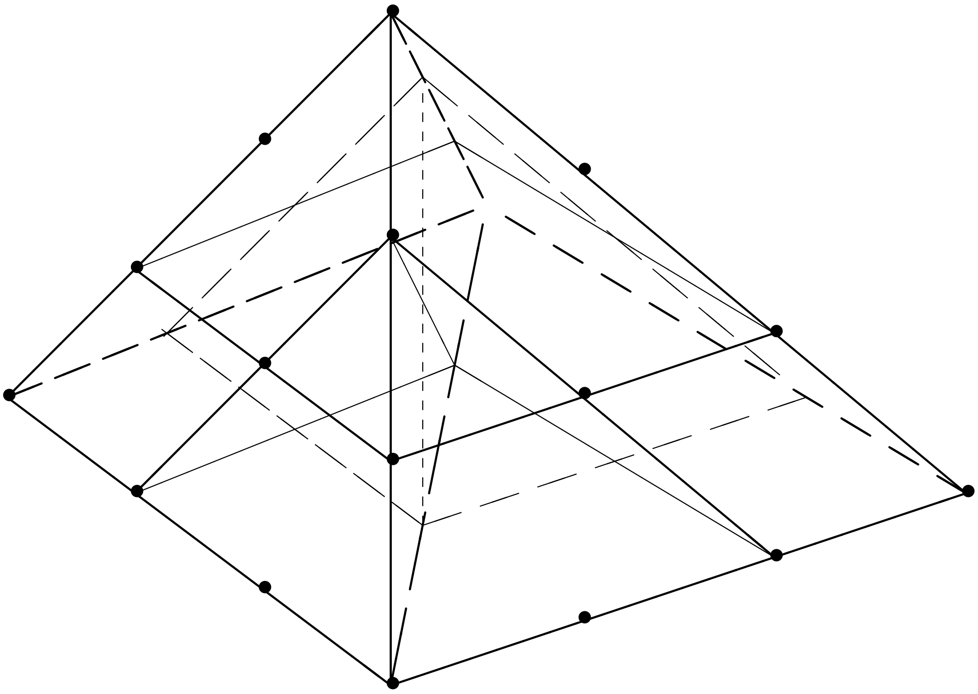


Figure 3.5b The supports of the C^2 smoothness condition

Also, we may use the inversion formula (cf. §2.2) to obtain the following

LEMMA 3.3.2. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$b_{ijkl} = \sum_{|\beta|=l} a_{(ijk0)+\beta}\Phi_\beta(\lambda^0), \qquad 0 \le l \le r, \tag{3.3.2}$$

where $\lambda^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0, \lambda_4^0)$.

This lemma was earlier proved in [11] and [71] by different methods.

Also, we may prove the following

LEMMA 3.3.3. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$\begin{aligned} &(D_{\mathbf{x}_5-\mathbf{x}_1})^i(D_{\mathbf{x}_2-\mathbf{x}_1})^j(D_{\mathbf{x}_3-\mathbf{x}_1})^k Q_n(\mathbf{x}_1) \\ &= (\lambda_2^0 D_{\mathbf{x}_2-\mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_3-\mathbf{x}_1} + \lambda_4^0 D_{\mathbf{x}_4-\mathbf{x}_1})^i (D_{\mathbf{x}_2-\mathbf{x}_1})^j (D_{\mathbf{x}_3-\mathbf{x}_1})^k P_n(\mathbf{x}_1) \end{aligned} \tag{3.3.3}$$

for $0 \le j+k \le n-i, \quad 0 \le i \le r.$

The proof may be found in [39].

Further, we may apply the smoothness conditions (3.3.1) or (3.3.2) to ensure F is smooth across the intersection surface $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ of T_1 and T_2 when the partial B-nets of F are given. We have the following

LEMMA 3.3.4. *Assume that* $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle \notin \langle \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_5 \rangle$. *(See Figure 3.6 for the reference of orientation of the vertices* $\mathbf{x}_1, \cdots, \mathbf{x}_5$.*) Suppose that the B-coefficients* $\{a_\beta : \beta_1 \geq 1\} \cup \{a_\beta : \beta_1 = 0, \beta_2 \geq 1\}$ *and* $\{b_\beta : \beta_1 \geq 1\} \cup \{a_\beta : \beta_1 = 0, \beta_2 \geq 1\}$ *of* P_n *and* Q_n*, respectively, are given and satisfy the smoothness conditions (3.3.1) up to order* $n-1$*. Furthermore, suppose that* $\{a_\beta : \beta_1 = \beta_2 = 0, \text{ and } 2l+2 \leq \beta_3 \leq n\}$ *and* $\{b_\beta : \beta_1 = \beta_2 = 0 \text{ and } n-l \leq \beta_3 \leq n\}$ *are given and satisfy the smoothness conditions (3.3.1) up to order* l*, where* $l \leq \frac{n-2}{2}$*. Then for any given* $\{a_\beta, b_\beta : \beta_1 = \beta_2 = 0 \text{ and } 0 \leq \beta_3 \leq l\}$*, there exists a unique set of coefficients* $\{a_\beta, b_\beta : \beta_1 = \beta_2 = 0, l+1 \leq \beta_3 \leq 2l+1\}$ *such that* $\{a_\beta : |\beta| = n\}$ *and* $\{b_\beta : |\beta| = n\}$ *satisfy the smoothness conditions (3.3.1) of order* n*.*

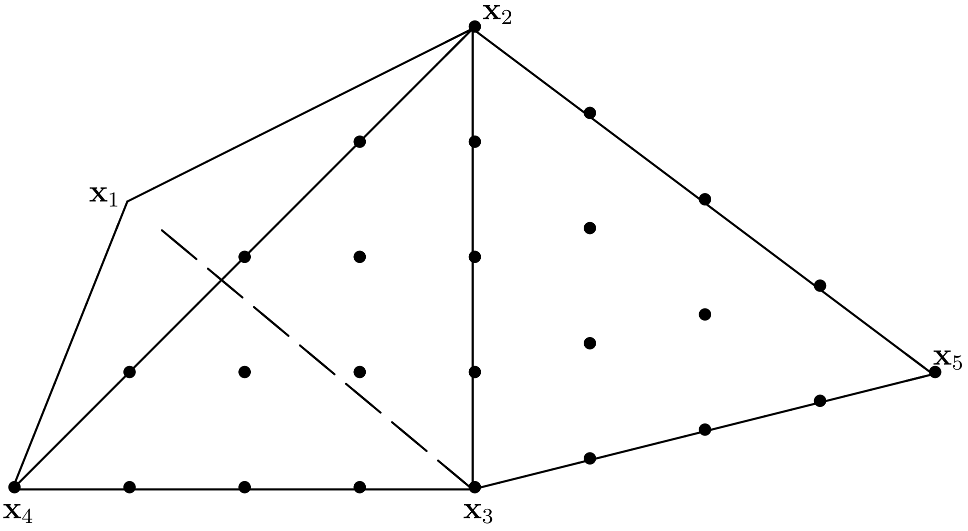


Figure 3.6 The orientation of vertices $\mathbf{x}_1, \cdots, \mathbf{x}_5$

The proof of this lemma is similar to that of its counterpart in §2.3. We may omit it here.

LEMMA 3.3.5. *Assume that* $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle \subset \langle \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_5 \rangle$. *(See Figure 3.7 for reference.) Suppose that the B-coefficients* $\{a_\beta : \beta_1 \geq 1\} \cup \{a_\beta : \beta_1 = 0, \beta_2 \geq 1\}$ *and* $\{b_\beta : \beta_1 \geq 1\} \cup \{b_\beta : \beta_1 = 0, \beta_2 \geq 1\}$ *of* P_n *and* Q_n *respectively, are given and satisfy the smoothness conditions (3.3.1) up to order* $n-1$*. Furthermore, suppose that* $\{a_\beta : \beta_1 = \beta_2 = 0 \text{ and } l+1 \leq \beta_3 \leq n\}$ *and* $\{b_\beta : \beta_1 = \beta_2 = 0 \text{ and } l+1 \leq \beta_3 \leq n\}$ *are given and satisfy the smoothness conditions (3.3.1) up to order* $n-l-1$*, where* $l < n$*. Then for any* $\{a_\beta : \beta_1 = \beta_2 = 0, \text{ and } 0 \leq \beta_3 \leq l\}$*, there exists a unique set of coefficients* $\{b_\beta : \beta_1 = \beta_2 = 0, 0 \leq \beta_3 \leq l\}$ *such that* $\{a_\beta : |\beta| = n\}$ *and* $\{b_\beta : |\beta| = n\}$ *satisfy the smoothness conditions (3.3.1) of order* n*.*

The proof of this lemma is a simple consequence of Lemma 3.1 or 3.2.

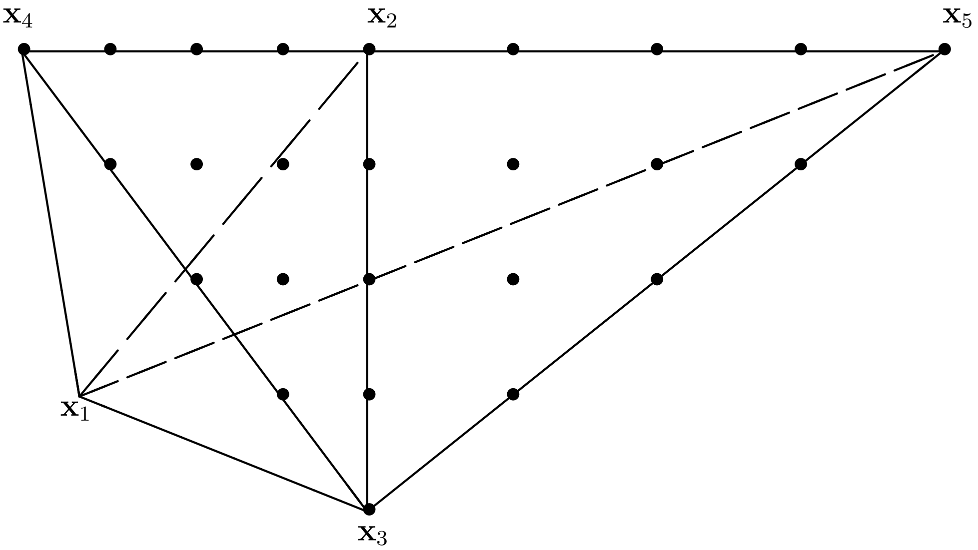


Figure 3.7 The case that $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle \subset \langle \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_5 \rangle$

2° Suppose that P_n and $\bar{Q}_n$ are defined on a tetrahedron $T_1 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle$ and a prism $T_2 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \rangle$, respectively, which are adjacent and share a common facet $T_1 \cap T_2 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$. Write

$$P_n(\mathbf{x}) = \sum_{|\beta|=n} a_\beta \Phi_\beta(\lambda) \qquad \text{and} \qquad \bar{Q}_n(\mathbf{x}) = \sum_{\beta \in \bar{\Lambda}_n} b_\beta \bar{\Phi}_\beta^{(n,n)}(\mu)$$

where $\mathbf{x} = \sum_{i=1}^4 \lambda_i \mathbf{x}_i = \sum_{i=1}^3 \mu_i \mathbf{x}_i + \mu_4(\mathbf{y}_1 - \mathbf{x}_1)$ with $\sum_{i=1}^4 \lambda_i = 1$ and $\sum_{i=1}^3 \mu_i = 1$. See Figure 3.8 for reference of the B-nets of P_n and $\bar{Q}_n$ when $n = 3$.

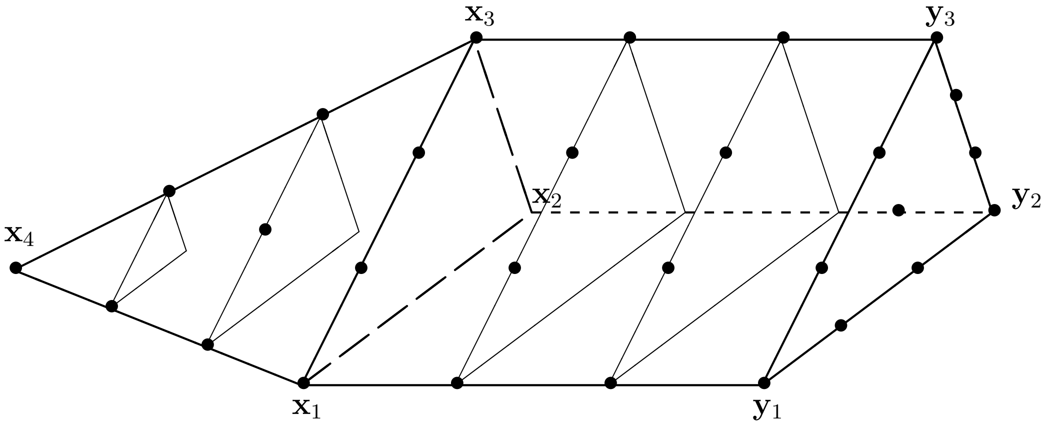


Figure 3.8 The B-nets of P_3 and $\bar{Q}_3$

Let F be a function defined by

$$F = \begin{cases} P_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_1 \\ \bar{Q}_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_2. \end{cases}$$

Write $\mathbf{y}_1 = \lambda_1^0\mathbf{x}_1 + \lambda_2^0\mathbf{x}_2 + \lambda_3^0\mathbf{x}_3 + \lambda_4^0\mathbf{x}_4$ with $\sum_{i=1}^4 \lambda_i^0 = 1$ and $\lambda^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0, \lambda_4^0)$. Then, clearly $F \in C^r(T_1 \cup T_2)$ if and only if

$$(D_{\mathbf{y}_1-\mathbf{x}_1})^l Q_n\Big|_{T_1\cap T_2} = (\lambda_2^0 D_{\mathbf{x}_2-\mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_3-\mathbf{x}_1} + \lambda_4^0 D_{\mathbf{x}_4-\mathbf{x}_1})^l P_n\Big|_{T_1\cap T_2}$$

for $0 \le l \le r$. It easily follows that

$$(D_{\mathbf{y}_3-\mathbf{x}_1})^l Q_n\Big|_{T_1\cap T_2} = \frac{n!}{(n-l)!} \sum_{\substack{|\beta|=n \\ \beta=(\beta_1,\beta_2,\beta_3,0)}} \Delta_4^l b_\beta \bar{\Phi}_\beta^{(n,0)}(\mu)$$

and

$$\begin{aligned}
&(\lambda_2^0 D_{\mathbf{x}_2-\mathbf{x}_1} + \lambda_3^0 D_{\mathbf{x}_3-\mathbf{x}_1} + \lambda_4^0 D_{\mathbf{x}_4-\mathbf{x}_1})^l P_n\Big|_{T_1\cap T_2} \\
= & \sum_{|\gamma|} \frac{l!}{\gamma!} (\lambda_2^0)^{\gamma_1} (\lambda_3^0)^{\gamma_2} (\lambda_4^0)^{\gamma_3} D_{\mathbf{x}_2-\mathbf{x}_1}^{\gamma_1} D_{\mathbf{x}_3-\mathbf{x}_1}^{\gamma_2} D_{\mathbf{x}_4-\mathbf{x}_1}^{\gamma_3} P_n\Big|_{T_1\cap T_2} \\
= & \sum_{|\gamma|} \frac{l!}{\gamma!} (\lambda_2^0)^{\gamma_1} (\lambda_3^0)^{\gamma_2} (\lambda_4^0)^{\gamma_3} \frac{n!}{(n-l)!} \sum_{i+j+k=n-l} \Delta_{21}^{\gamma_1} \Delta_{31}^{\gamma_2} \Delta_{41}^{\gamma_3} a_{ijk0} \Phi_{ijk0}(\mu) \\
= & \sum_{i+j+k=n-l} \frac{n!}{(n-l)!} (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31} + \lambda_4^0 \Delta_{41})^l a_{ijk0} \Phi_{ijk0}(\mu) \\
= & \sum_{\substack{|\beta|=n \\ \beta=(\beta_1,\beta_2,\beta_3,0)}} \frac{n!}{(n-l)!} (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31} + \lambda_4^0 \Delta_{41})^l \mathbf{R}^l a_\beta \Phi_\beta(\mu),
\end{aligned}$$

where $\mu = (\mu_1, \mu_2, \mu_3, 0)$ and

$$\mathbf{R}^l a_\beta = \sum_{\substack{\alpha\le\beta \\ |\alpha|=n-l}} a_\alpha \frac{\binom{\beta}{\alpha}}{\binom{n}{l}}.$$

Therefore, we have established the following

LEMMA 3.3.6. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$\Delta_4^l b_\beta = \frac{1}{\binom{n}{l}} \sum_{\substack{|\alpha|=n-l \\ \alpha\le\beta}} (\lambda_2^0 \Delta_{21} + \lambda_3^0 \Delta_{31} + \lambda_4^0 \Delta_{41})^l a_\alpha \binom{\beta}{\alpha}, \tag{3.3.4}$$

for $\beta = (\beta_1, \beta_2, \beta_3, 0)$ with $|\beta| = n$ and $0 \le l \le r$.

The supports of the C^1 and C^2 smoothness conditions (3.3.4) are as shown as in Figure 3.9a and Figure 3.9b.

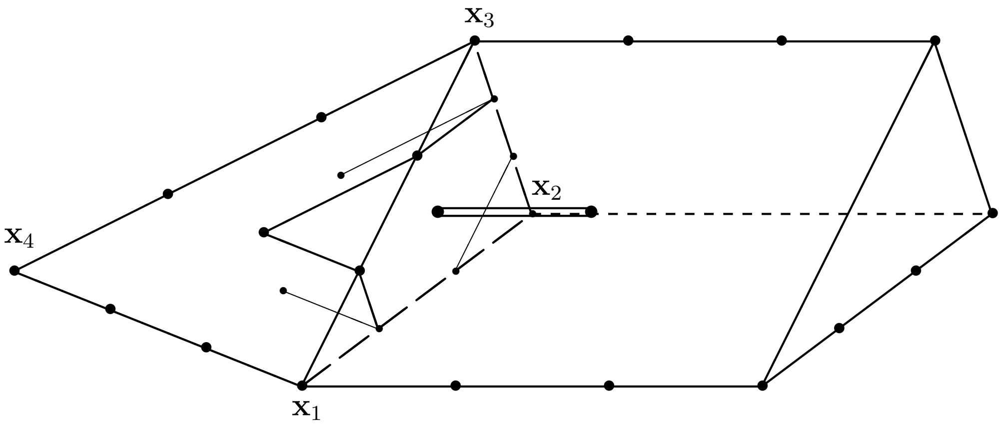


Figure 3.9a A support of the C^1 smoothness condition

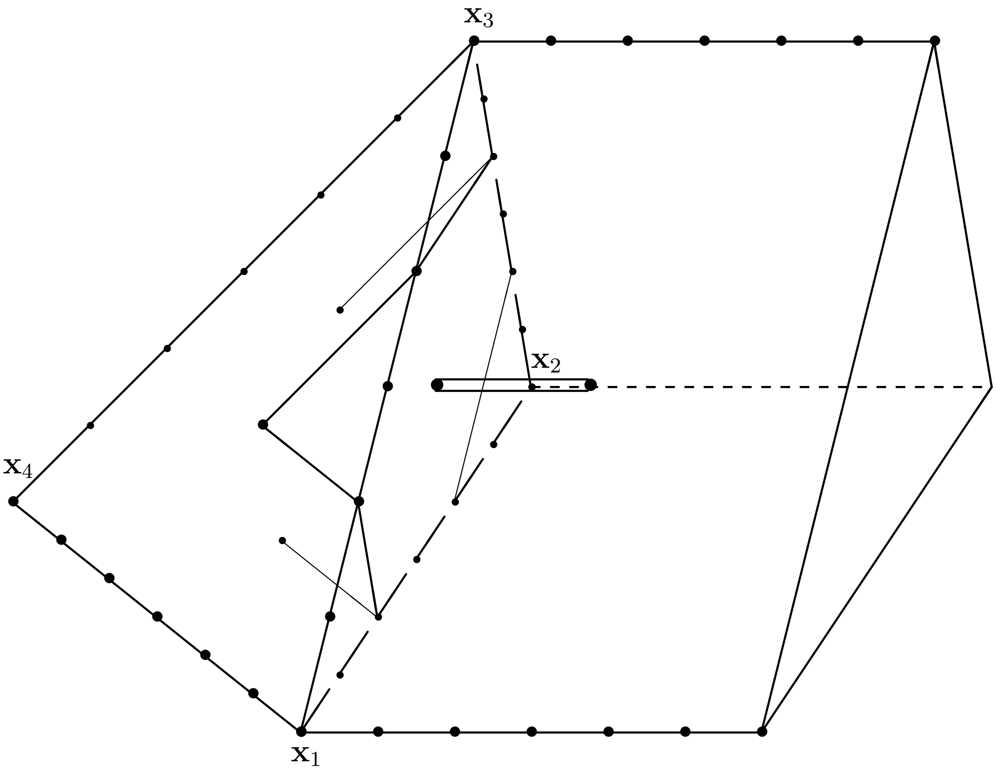


Figure 3.9b A support of the C^2 smoothness condition

Example 3.1

$$\begin{aligned} C^0: &\quad b_{(\beta_1,\beta_2,\beta_3,0)} \\ = &\quad a_{(\beta_1,\beta_2,\beta_3,0)}, \qquad\qquad \beta_1+\beta_2+\beta_3=n \\ C^1: &\quad b_{(\beta_1,\beta_2,\beta_3,1)} \end{aligned}$$

$$\begin{aligned}
&= a_{(\beta_1,\beta_2,\beta_3,0)} + \frac{1}{n}\sum_{\substack{|\alpha|=n-1\\ \alpha\le(\beta_1,\beta_2,\beta_3,0)}} (\lambda_2^0\Delta_{21}+\lambda_3^0\Delta_{31}+\lambda_4^0\Delta_{41})a_\alpha\binom{\beta}{\alpha}\\
&= a_{(\beta_1,\beta_2,\beta_3,0)} + \frac{\beta_1}{n}(\lambda_2^0\Delta_{21}+\lambda_3^0\Delta_{31}+\lambda_4^0\Delta_{41})a_{(\beta_1-1,\beta_2,\beta_3,0)}\\
&\quad+\frac{\beta_2}{n}(\lambda_2^0\Delta_{21}+\lambda_3^0\Delta_{31}+\lambda_4^0\Delta_{41})a_{(\beta_1,\beta_2-1,\beta_3,0)}\\
&\quad+\frac{\beta_3}{n}(\lambda_2^0\Delta_{21}+\lambda_3^0\Delta_{31}+\lambda_4^0\Delta_{41})a_{(\beta_1,\beta_2,\beta_3-1,0)};\quad \beta_1+\beta_2+\beta_3=n,
\end{aligned}$$

etc..

Also, the following matching conditions are easy to verify

LEMMA 3.3.7. $F \in C^r(T_1 \cup T_2)$ *if and only if*

$$(D_{\mathbf{y}_1-\mathbf{x}_1})^i(D_{\mathbf{x}_2-\mathbf{x}_1})^j(D_{\mathbf{x}_3-\mathbf{x}_1})^k\bar{Q}_n(\mathbf{x}_1) \tag{3.3.5}$$

$$= (\lambda_2^0 D_{\mathbf{x}_2-\mathbf{x}_1}+\lambda_3^0 D_{\mathbf{x}_3-\mathbf{x}_1}+\lambda_4^0 D_{\mathbf{x}_4-\mathbf{x}_1})^i(D_{\mathbf{x}_2-\mathbf{x}_1})^j(D_{\mathbf{x}_3-\mathbf{x}_1})^k P_n(\mathbf{x}_1)$$

for $0 \le j+k \le n, 0 \le i \le r.$

The proof is similar to that of Lemma 2.8. We may omit its details here.

3°. Suppose that $\bar{P}_n$ and $\bar{Q}_n$ are defined on two adjacent prisms $T_1 = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_5, \mathbf{y}_6\rangle$ and $T_2 = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\rangle$, respectively, share a common facet $T_1 \cap T_2 = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\rangle$. See Figure 3.10 for reference of the orientation of the vertices of T_1 and T_2.

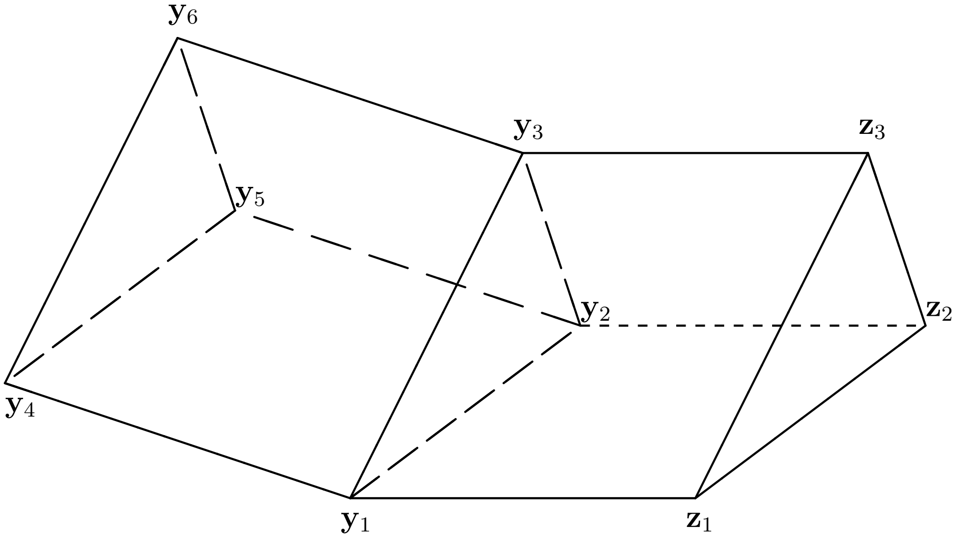


Figure 3.10 The orientation of the vertices of T_1 and T_2

More precisely, let

$$\bar{P}_n(\mathbf{x}) = \sum_{\beta\in\bar{\Lambda}_n} a_\beta \bar{\Phi}_\beta^{(n,n)}(\mu(\mathbf{x}))$$

and

$$\bar{Q}_n(\mathbf{x}) = \sum_{\beta \in \bar{\Lambda}_n} b_\beta \bar{\Phi}_\beta^{(n,n)}(\nu(\mathbf{x}))$$

where $\mathbf{x} = \mu_1\mathbf{y}_1 + \mu_2\mathbf{y}_2 + \mu_3\mathbf{y}_3 + \mu_4(\mathbf{y}_4 - \mathbf{y}_1)$ with $\mu_1 + \mu_2 + \mu_3 = 1$ and $\mathbf{x} = \nu_1\mathbf{y}_1 + \nu_2\mathbf{y}_2 + \nu_3\mathbf{y}_3 + \nu_4(\mathbf{z}_1 - \mathbf{y}_1)$ with $\nu_1 + \nu_2 + \nu_3 = 1$.

Let F be a function defined as follows:

$$F(\mathbf{x}) = \begin{cases} \bar{P}_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_1 \\ \bar{Q}_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_2. \end{cases}$$

Write $\mathbf{z}_1 = \mu_1^0\mathbf{y}_1 + \mu_2^0\mathbf{y}_2 + \mu_3^0\mathbf{y}_3 + \mu_4^0(\mathbf{y}_4 - \mathbf{y}_1)$ with $\mu_1^0 + \mu_2^0 + \mu_3^0 = 1$. Then it is clear that $F \in C^r(T_1 \cup T_2)$ if and only if

$$\begin{aligned} &(D_{\mathbf{z}_1-\mathbf{y}_1})^l \bar{Q}_n(\mathbf{x})\Big|_{T_1\cap T_2} \\ &= (\mu_2^0 D_{\mathbf{x}_2-\mathbf{x}_1} + \mu_3^0 D_{\mathbf{x}_3-\mathbf{x}_1} + \mu_4^0 D_{\mathbf{x}_4-\mathbf{x}_1})^l \bar{P}_n(\mathbf{x})\Big|_{T_1\cap T_2} \end{aligned}$$

for $0 \le l \le r$. It follows that

$$(D_{\mathbf{z}_1-\mathbf{y}_1})^l \bar{Q}_n(\mathbf{x})\Big|_{T_1\cap T_2} = \frac{n!}{(n-l)!} \sum_{\substack{\beta=(\beta_1,\beta_2,\beta_3,0)\\ |\beta|=n}} \Delta_4^l b_\beta \bar{\Phi}_\beta^{(n,n)}(\nu(\mathbf{x}))$$

where $\nu(\mathbf{x}) = (\nu_1(\mathbf{x}), \nu_2(\mathbf{x}), \nu_3(\mathbf{x}), 0)$ and

$$\begin{aligned} &(\mu_2^0 D_{\mathbf{x}_2-\mathbf{x}_1} + \mu_3^0 D_{\mathbf{x}_3-\mathbf{x}_1} + \mu_4^0 D_{\mathbf{x}_4-\mathbf{x}_1})^l \bar{P}_n(\mathbf{x})\Big|_{T_1\cap T_2} \\ = &\sum_{\substack{\gamma=(\gamma_1,\gamma_2,\gamma_3)\\ |\gamma|=l}} \frac{l!}{\gamma!} (\mu_1^0)^{\gamma_1} (\mu_2^0)^{\gamma_2} (\mu_3^0)^{\gamma_3} \times \\ &(D_{\mathbf{x}_2-\mathbf{x}_1})^{\gamma_1} (D_{\mathbf{x}_2-\mathbf{x}_1})^{\gamma_2} (D_{\mathbf{x}_4-\mathbf{x}_1})^{\gamma_3} \bar{P}_n(\mathbf{x})\Big|_{T_1\cap T_2} \\ = &\sum_{\substack{\gamma=(\gamma_1,\gamma_2,\gamma_3)\\ |\gamma|=l}} \frac{l!}{\gamma!} (\mu_1^0)^{\gamma_1} (\mu_2^0)^{\gamma_2} (\mu_3^0)^{\gamma_3} \frac{n!}{(n-\beta_1-\beta_2)!} \frac{n!}{(n-\beta_3)!} \times \\ &\sum_{\substack{\beta=(\beta_1,\beta_2,\beta_3,0)\\ |\beta|=n-\gamma_1-\gamma_2}} \Delta_{21}^{\gamma_1} \Delta_{31}^{\gamma_2} \Delta_{41}^{\gamma_3} a_\beta \bar{\Phi}_\beta^{(n-\gamma_1-\gamma_2,0)}(\mu(\mathbf{x})) \\ = &\sum_{\substack{\beta=(\beta_1,\beta_2,\beta_3)\\ |\beta|=l}} \frac{l!}{\beta!} (\mu_1^0)^{\beta_1} (\mu_2^0)^{\beta_2} (\mu_3^0)^{\beta_3} \frac{n!}{(n-\beta_1-\beta_2)!} \frac{n!}{(n-\beta_3)!} \times \\ &\sum_{\substack{\gamma=(\gamma_1,\gamma_2,\gamma_3,0)\\ |\gamma|=n}} \Delta_{21}^{\beta_1} \Delta_{31}^{\beta_2} \Delta_4^{\beta_3} \mathbf{R}^{\beta_1+\beta_2} a_\beta \bar{\Phi}_\gamma^{(n,0)}(\mu(\mathbf{x})) \end{aligned}$$

where $\mu(\mathbf{x}) = (\mu_1(\mathbf{x}), \mu_2(\mathbf{x}), \mu_3(\mathbf{x}), 0) = \nu(\mathbf{x})$ and

$$\mathbf{R}^{\beta_1+\beta_2} a_\gamma = \sum_{\substack{|\alpha|=n-\beta_1-\beta_2\\ \alpha\le\gamma}} a_\alpha \frac{\binom{\gamma}{\alpha}}{\binom{n}{\beta_1+\beta_2}}.$$

Therefore we have the following

LEMMA 3.3.8. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$\Delta_4^l b_\gamma = \sum_{\substack{\beta=(\beta_1,\beta_2,\beta_3)\\ |\beta|=l}} \frac{l!}{\beta!} \frac{n!(n-l)!}{(n-\beta_1-\beta_2)!(n-\beta_3)!} \times (\mu_2^0\Delta_{21})^{\beta_1}(\mu_3^0\Delta_{31})^{\beta_2}(\mu_4^0\Delta_4)^{\beta_3}\mathbf{R}^{\beta_1+\beta_2}a_\gamma \tag{3.3.5}$$

for $\gamma = (\gamma_1, \gamma_2, \gamma_3, 0)$ with $|\gamma| = n$ and $0 \le l \le r$.

The following figures 3.11a and 3.11b indicate the supports of the C^1 and C^2 smoothness conditions.

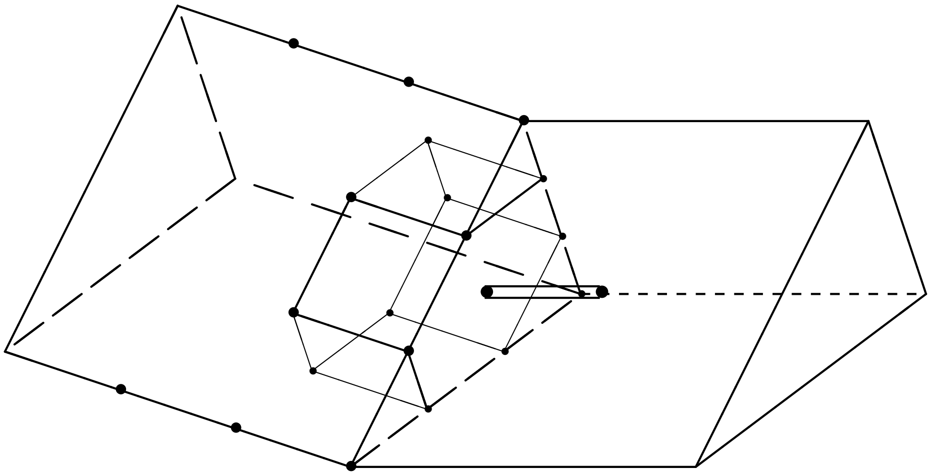


Figure 3.11a A support of the C^1 smoothness condition over two adjacent prisms

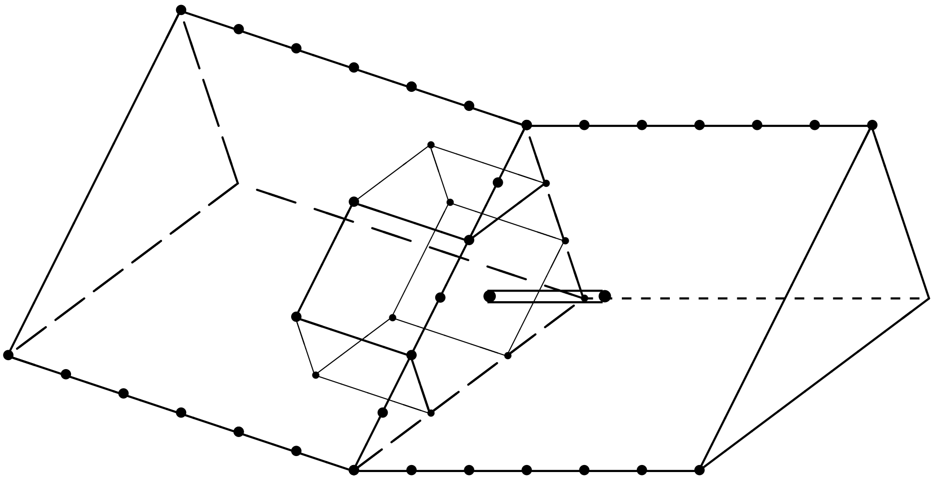


Figure 3.11b A support of the C^2 smoothness condition over two adjacent prisms

We also have the following matching conditions

LEMMA 3.3.9. $F \in C^r(T_1 \cup T_2)$ *if and only if*

$$(D_{\mathbf{z}_1-\mathbf{y}_1})^k (D_{\mathbf{y}_2-\mathbf{y}_1})^l (D_{\mathbf{y}_3-\mathbf{y}_1})^m \bar{Q}_n(\mathbf{y}_1) \tag{3.3.6}$$

$$= (\lambda_2^0 D_{\mathbf{y}_2-\mathbf{y}_1} + \lambda_3^0 D_{\mathbf{y}_3-\mathbf{y}_1} + \lambda_4^0 D_{\mathbf{y}_4-\mathbf{y}_1})^k (D_{\mathbf{y}_2-\mathbf{y}_1})^l (D_{\mathbf{y}_3-\mathbf{y}_1})^m \bar{P}_n(\mathbf{y}_1)$$

for $0 \le l+m \le n, 0 \le k \le r.$

Proof. Clearly, $F \in C^r(T_1 \cup T_2)$ implies (3.3.6). On the other hand, we have

$$\begin{aligned}
& (D_{\mathbf{z}_1-\mathbf{y}_1})^k (D_{\mathbf{y}_2-\mathbf{y}_1})^l (D_{\mathbf{y}_3-\mathbf{y}_1})^m \bar{Q}_n(\mathbf{y}_1) \\
= \; & \frac{n!}{(n-k)!} \frac{n!}{(n-l-m)!} \Delta_{21}^l \Delta_{31}^m \Delta_4^k b_{(n-j-k,0,0,0)} \\
= \; & \frac{n!}{(n-k)!} \frac{n!}{(n-l-m)!} \sum_{(\gamma_1,\gamma_2) \le (l,m)} \binom{l}{\gamma_1} \binom{m}{\gamma_2} (-1)^{l+m-\gamma_1-\gamma_2} \Delta_4^k b_{(n-\gamma_1-\gamma_2,\gamma_1,\gamma_2,0)}
\end{aligned}$$

and

$$\begin{aligned}
& (\lambda_2^0 D_{\mathbf{y}_2-\mathbf{y}_1} + \lambda_3^0 D_{\mathbf{y}_3-\mathbf{y}_1} + \lambda_4^0 D_{\mathbf{y}_4-\mathbf{y}_1})^k (D_{\mathbf{y}_2-\mathbf{y}_1})^l (D_{\mathbf{y}_3-\mathbf{y}_1})^m \bar{P}_n(\mathbf{y}_1) \\
= \; & (D_{\mathbf{y}_2-\mathbf{y}_1})^l (D_{\mathbf{y}_3-\mathbf{y}_1})^m \sum_{\substack{\gamma=(\gamma_1,\gamma_2,\gamma_3) \\ |\gamma|=k}} \frac{k!}{\gamma!} (\lambda_2^0 D_{\mathbf{y}_2-\mathbf{y}_1})^{\gamma_1} (\lambda_3^0 D_{\mathbf{y}_3-\mathbf{y}_1})^{\gamma_2} (\lambda_4^0 D_{\mathbf{y}_4-\mathbf{y}_1})^{\gamma_3} \bar{P}_n(\mathbf{y}_1) \\
= \; & (D_{\mathbf{y}_2-\mathbf{y}_1})^l (D_{\mathbf{y}_3-\mathbf{y}_1})^m \sum_{\substack{\gamma=(\gamma_1,\gamma_2,\gamma_3) \\ |\gamma|=k}} \frac{k!}{\gamma!} \frac{n!}{(n-\gamma_1-\gamma_2)!} \frac{n!}{(n-\gamma_3)!} \times \\
& \sum_{\substack{\beta=(\beta_1,\beta_2,\beta_3,0) \\ |\beta|=n}} (\lambda_2^0 \Delta_{21})^{\gamma_1} (\lambda_3^0 \Delta_{31})^{\gamma_2} (\lambda_4^0 \Delta_4)^{\gamma_3} \mathbf{R}^{\gamma_1+\gamma_2} a_\beta \bar{\Phi}_\beta^{(n,0)}(\mu(\mathbf{x})) \Big|_{\mathbf{x}=\mathbf{y}_1} \\
= \; & \frac{n!}{(n-l-m)!} \frac{n!}{(n-k)!} \Delta_{21}^l \Delta_{31}^m \Big(\sum_{\substack{\gamma=(\gamma_1,\gamma_2,\gamma_3) \\ |\gamma|=k}} \frac{k!}{\gamma!} \frac{(n-k)!}{(n-\gamma_1-\gamma_2)!} \frac{n!}{(n-\gamma_3)!} \times \\
& (\lambda_2^0 \Delta_{21})^{\gamma_1} (\lambda_3^0 \Delta_{31})^{\gamma_2} (\lambda_4^0 \Delta_4)^{\gamma_3} \mathbf{R}^{\gamma_1+\gamma_2} a_{(n-l-m,0,0,0)} \Big) \\
= \; & \frac{n!}{(n-l-m)!} \frac{n!}{(n-k)!} \sum_{(\beta_1,\beta_2) \le (l,m)} \binom{l}{\beta_1} \binom{m}{\beta_2} (-1)^{l+m-\beta_1-\beta_2} \times \\
& \sum_{\substack{\gamma=(\gamma_1,\gamma_2,\gamma_3) \\ |\gamma|=k}} \frac{k!}{\gamma!} \frac{(n-k)!}{(n-\gamma_1-\gamma_2)!} \frac{n!}{(n-\gamma_3)!} (\lambda_2^0 \Delta_{21})^{\gamma_1} (\lambda_3^0 \Delta_{31})^{\gamma_2} (\lambda_4^0 \Delta_4)^{\gamma_3} \times \\
& \mathbf{R}^{\gamma_1+\gamma_2} a_{(n-l-m,0,0,0)}.
\end{aligned}$$

Hence, we may use the inversion formula to yield (3.3.5). By Lemma 3.3.8, we have established this lemma.

4°. Suppose that $\bar{P}_n$ and $\bar{Q}_n$ are defined on two adjacent prisms $T_1 = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_5, \mathbf{y}_6 \rangle$ and $T_2 = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_4, \mathbf{y}_5, \mathbf{z}_3, \mathbf{z}_6 \rangle$, respectively, which share a common facet

$T_1 \cap T_2 = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_4, \mathbf{y}_5 \rangle$. Write

$$\bar{P}_n(\mathbf{x}) = \sum_{\beta \in \bar{\Lambda}_n} a_\beta \bar{\Phi}_\beta^{(n,n)}(\mu(\mathbf{x}))$$

and

$$\bar{Q}_n(\mathbf{x}) = \sum_{\beta \in \bar{\Lambda}_n} b_\beta \bar{\Phi}_\beta^{(n,n)}(\nu(\mathbf{x}))$$

where $\mu(\mathbf{x}) = (\mu_1(\mathbf{x}), \mu_2(\mathbf{x}), \mu_3(\mathbf{x}), \mu_4(\mathbf{x}))$ is the barycentric coordinate of $\mathbf{x}$ with respect to T_1, and $\nu(\mathbf{x}) = (\nu_1(\mathbf{x}), \nu_2(\mathbf{x}), \nu_3(\mathbf{x}), \nu_4(\mathbf{x}))$ is the barycentric coordinate of $\mathbf{x}$ with respect to T_2. The following Figure 3.12 shows the orientation of the vertices of T_1 and T_2.

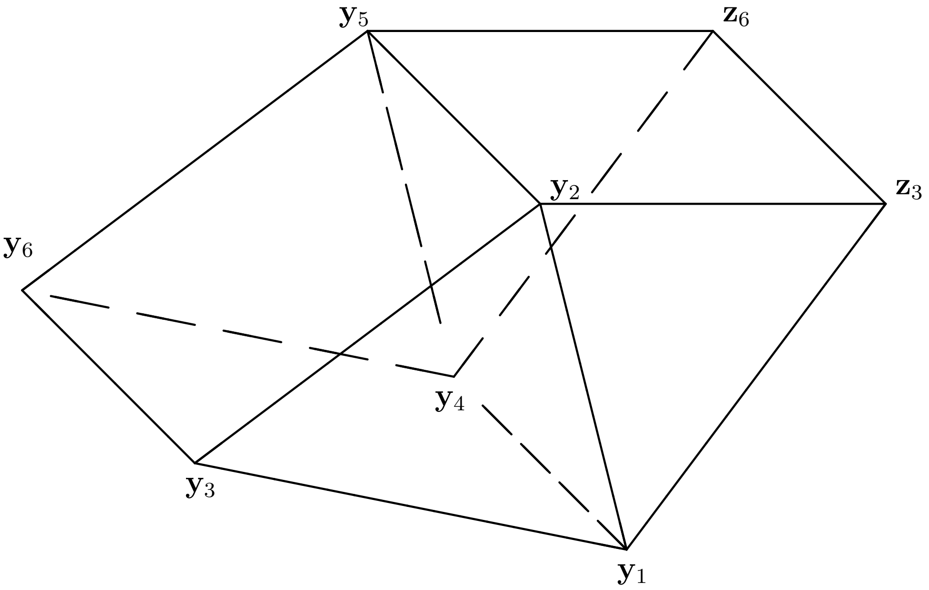


Figure 3.12 The orientation of the vertices of two prisms

Let F be a function defined by

$$F(\mathbf{x}) = \begin{cases} \bar{P}_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_1 \\ \bar{Q}_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_2. \end{cases}$$

Write $\mathbf{z}_3 = \mu_1^0 \mathbf{y}_1 + \mu_2^0 \mathbf{y}_2 + \mu_3^0 \mathbf{y}_3 + \mu_4^0 (\mathbf{y}_4 - \mathbf{y}_1)$ with $\mu_1^0 + \mu_2^0 + \mu_3^0 = 1$.

Then, clearly, $F \in C^r(T_1 \cup T_2)$ if and only if

$$(D_{\mathbf{z}_3 - \mathbf{y}_1})^l \bar{Q}_n \Big|_{T_1 \cap T_2}$$

$$= (\mu_2^0 D_{\mathbf{y}_2 - \mathbf{y}_1} + \mu_3^0 D_{\mathbf{y}_3 - \mathbf{y}_1} + \mu_4^0 D_{\mathbf{y}_4 - \mathbf{y}_1})^l \bar{P}_n \Big|_{T_1 \cap T_2}$$

It is easy to verify that

$$(D_{\mathbf{z}_3 - \mathbf{y}_1})^l \bar{Q}_n \Big|_{T_1 \cap T_2} = \frac{n!}{(n-l)!} \sum_{k=0}^{n} \sum_{i+j=n-l} \Delta_{31}^l b_{(i,j,0,k)} \bar{\Phi}_{(i,j,0,k)}^{(n-l,n)}(\mu(\mathbf{x})) \Big|_{T_1 \cap T_2}$$

and

$$
\begin{aligned}
&(\mu_2^0 D_{\mathbf{y}_2-\mathbf{y}_1} + \mu_3^0 D_{\mathbf{y}_3-\mathbf{y}_1} + \mu_4^0 D_{\mathbf{y}_4-\mathbf{y}_1})^l P_n\Big|_{T_1\cap T_2} \\
= &\sum_{\substack{\gamma=(\gamma_1,\gamma_2,\gamma_3)\\|\gamma|=l}} \frac{l!}{\gamma!}(\mu_1^0 D_{\mathbf{y}_2-\mathbf{y}_1})^{\gamma_1}(\mu_2^0 D_{\mathbf{y}_3-\mathbf{y}_1})^{\gamma_2}(\mu_3^0 D_{\mathbf{y}_4-\mathbf{y}_1})^{\gamma_3} P_n\Big|_{T_1\cap T_2} \\
= &\sum_{\substack{\beta=(\beta_1,\beta_2,\beta_3)\\|\beta|=l}} \frac{l!}{\beta!}\frac{n!}{(n-\beta_1-\beta_2)!}\frac{n!}{(n-\beta_3)!} \times \\
&\sum_{\substack{\alpha=(\alpha_1,\alpha_2,0,\alpha_4)\\ \alpha_1+\alpha_2=n-\beta_1-\beta_2\\ \alpha_4\le n-\beta_3}} (\mu_2^0\Delta_{21})^{\beta_1}(\mu_3^0\Delta_{31})^{\beta_2}(\mu_4^0\Delta_4)^{\beta_3} a_\alpha \bar{\Phi}^{(n-\beta_1-\beta_2,n-\beta_3)}(\nu(\mathbf{x}))
\end{aligned}
$$

where $\nu(\mathbf{x}) = (\nu_1,\nu_2,0,\nu_3)$ with $\nu_1+\nu_2=1$ and $0\le\nu_3\le 1$.

By recalling the degree raising operator $\mathbf{R}$ from §2.3, we have the following

LEMMA 3.3.10. *$F\in C^r(T_1\cup T_2)$ if and only if*

$$
\Delta_{31}^l b_{(i,j,0,k)} = \sum_{\substack{\beta=(\beta_1,\beta_2,\beta_3)\\|\beta|=l}} \frac{l!}{\beta!}\frac{n!(n-l)!}{(n-\beta_1-\beta_2)!(n-\beta_3)!}(\mu_2^0\Delta_{21})^{\beta_1}(\mu_3^0\Delta_{31})^{\beta_2}(\mu_4^0\Delta_4)^{\beta_3}
$$

$$
\times\sum_{m=0}^{k}\frac{\binom{k}{m}\binom{n-k}{n-\beta_3-m}}{\binom{n}{\beta_3}}\mathbf{R}^{\beta_1+\beta_2}a_{(i,j,0,m)} \tag{3.3.7}
$$

for $i+j=n-l, 0\le k\le n, 0\le l\le r$.

The supports of the C^1 and C^2 smoothness conditions (3.3.7) are shown as in Figure 3.13a and Figure 3.13b.

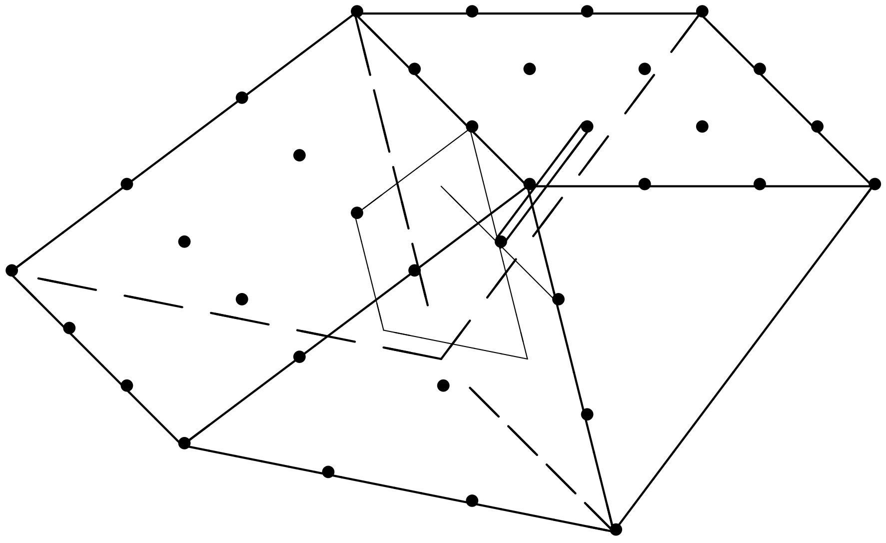


Figure 3.13a A support of the C^1 smoothness condition over neighboring prisms

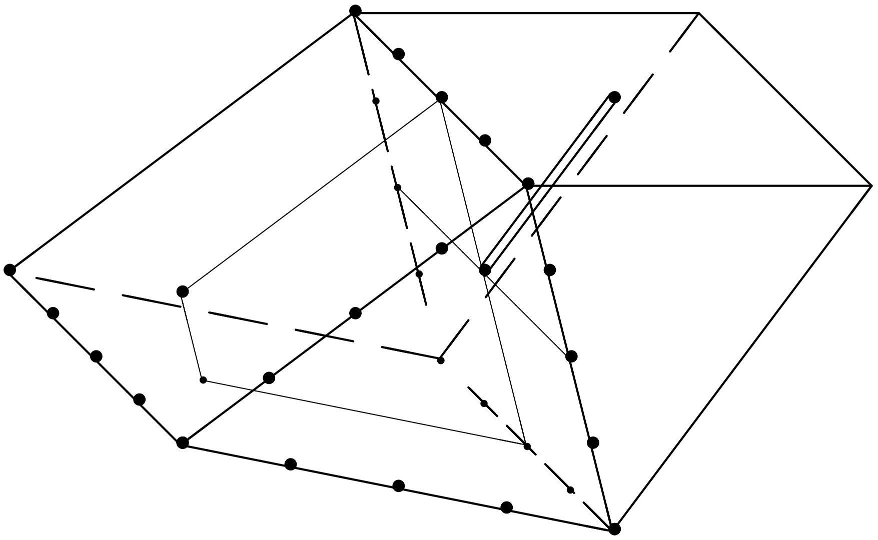


Figure 3.13b A support of the C^2 smoothness condition over neighboring prisms

We have the following matching conditions.

LEMMA 3.3.11. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$(D_{\mathbf{z}_3-\mathbf{y}_1})^k (D_{\mathbf{y}_2-\mathbf{y}_1})^l (D_{\mathbf{y}_4-\mathbf{y}_1})^m \tilde{Q}_n(\mathbf{y}_1) \tag{3.3.8}$$

$$= (\mu_2^0 D_{\mathbf{y}_2-\mathbf{y}_1} + \mu_3^0 D_{\mathbf{y}_3-\mathbf{y}_1} + \mu_4^0 D_{\mathbf{y}_4-\mathbf{y}_1})^k (D_{\mathbf{y}_2-\mathbf{y}_1})^l (D_{\mathbf{y}_4-\mathbf{y}_1})^m \bar{P}_n(\mathbf{y}_1)$$

for $0 \le m \le n, 0 \le l \le n-k, 0 \le k \le r$.

The proof is similar to the counterpart of other cases. We omit it here.

5°. Suppose that the polynomials $\bar{P}_n$ and $\tilde{Q}_n$ are defined on a prism $T_1 = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_5, \mathbf{y}_6 \rangle$ and a parallelepiped $T_2 = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_4, \mathbf{y}_5, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4 \rangle$, respectively, which share a common facet $T_1 \cap T_2 = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_4, \mathbf{y}_5 \rangle$. Write

$$\bar{P}_n(\mathbf{x}) = \sum_{\beta \in \bar{\Lambda}_n} a_\beta \bar{\Phi}_\beta^{(n,n)}(\mu(\mathbf{x}))$$

$$\tilde{Q}_n(\mathbf{x}) = \sum_{\alpha \le (n,n,n)} b_\alpha \tilde{\Phi}_\alpha^{(n,n,n)}(\nu(\mathbf{x}))$$

where $\mathbf{x} = \mu_1 \mathbf{y}_1 + \mu_2 \mathbf{y}_2 + \mu_3 \mathbf{y}_3 + \mu_4(\mathbf{y}_4 - \mathbf{y}_1)$ with $\mu_1 + \mu_2 + \mu_3 = 1$ and $\mathbf{x} = \mathbf{y}_1 + \nu_1(\mathbf{y}_2 - \mathbf{y}_1) + \nu_2(\mathbf{y}_4 - \mathbf{y}_1) + \nu_3(\mathbf{z}_1 - \mathbf{y}_1)$. The orientation of the vertices of T_1 and T_2 is shown in Figure 3.14.

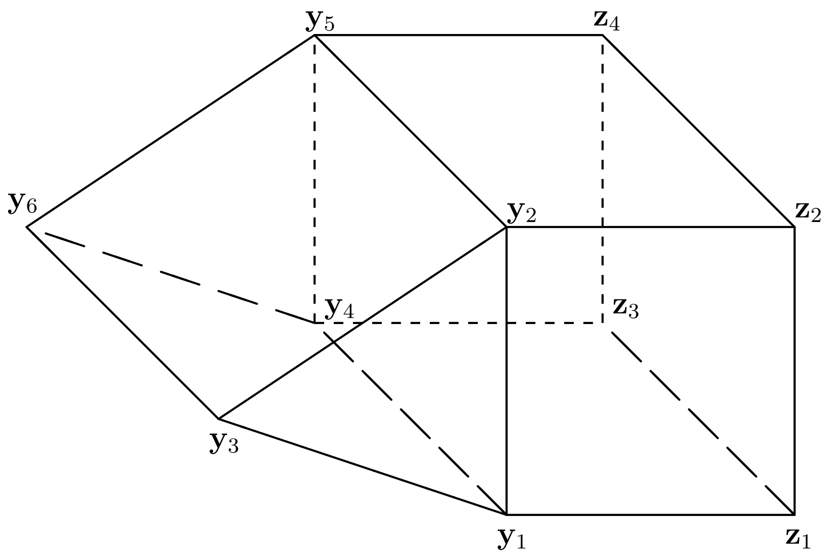


Figure 3.14 The orientation of the vertices of one prism and one parallelogram

Let F be a function defined by

$$F(\mathbf{x}) = \begin{cases} \bar{P}_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_1 \\ \tilde{Q}_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_2. \end{cases}$$

Write $\mathbf{z}_1 = \mu_1^0\mathbf{y}_1 + \mu_2^0\mathbf{y}_2 + \mu_3^0\mathbf{y}_3 + \mu_4^0(\mathbf{y}_4 - \mathbf{y}_1)$ with $\mu_1^0 + \mu_2^0 + \mu_3^0 = 1$. Then it is clear that $F \in C^r(T_1 \cup T_2)$ if and only if

$$(D_{\mathbf{z}_1-\mathbf{y}_1})^l \tilde{Q}_n\Big|_{T_1\cap T_2} = (\mu_2^0 D_{\mathbf{y}_2-\mathbf{y}_1} + \mu_3^0 D_{\mathbf{y}_3-\mathbf{y}_1} + \mu_4^0 D_{\mathbf{y}_4-\mathbf{y}_1})^l \bar{P}_n\Big|_{T_1\cap T_2}$$

for $0 \le l \le r$. Therefore, we may easily conclude the following

LEMMA 3.3.12. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$(3.3.9)\quad \Delta_2^l b_{(i,0,j)} = \sum_{\substack{\beta=(\beta_1,\beta_2,\beta_3)\\ |\beta|=l}} \frac{l!}{\beta!} \frac{n!(n-l)!}{(n-\beta_1-\beta_2)!(n-\beta_3)!} (\mu_2^0\Delta_{21})^{\beta_1} (\mu_3^0\Delta_{31})^{\beta_2} (\mu_4^0\Delta_4)^{\beta_3}$$

$$\times \sum_{m=0}^{j} \frac{\binom{j}{m}\binom{n-j}{n-\beta_3-m}}{\binom{n}{\beta_3}} \mathbf{R}^{\beta_1+\beta_2} a_{(i,n-i,0,m)}$$

for $0 \le i \le n, 0 \le j \le n, 0 \le l \le r$.

The supports of the C^1 and C^2 smoothness conditions (3.3.9) are shown as in Figure 3.15a and 3.15b.

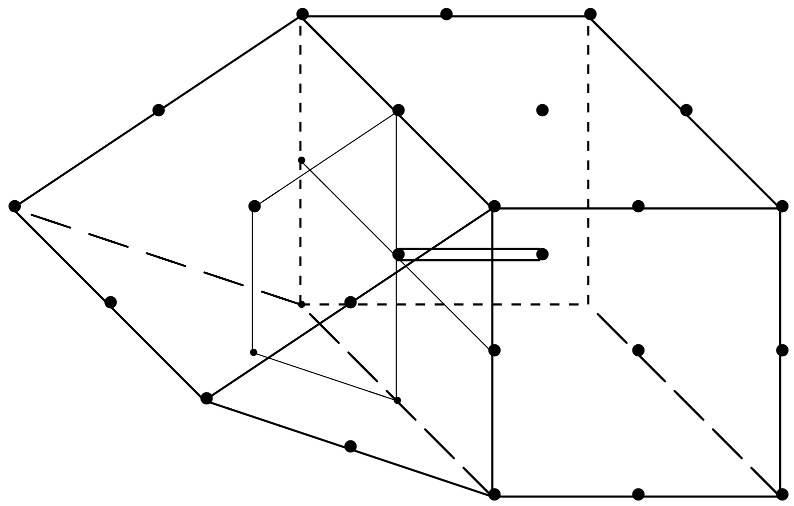


Figure 3.15a A support of the C^1 smoothness condition over two patches

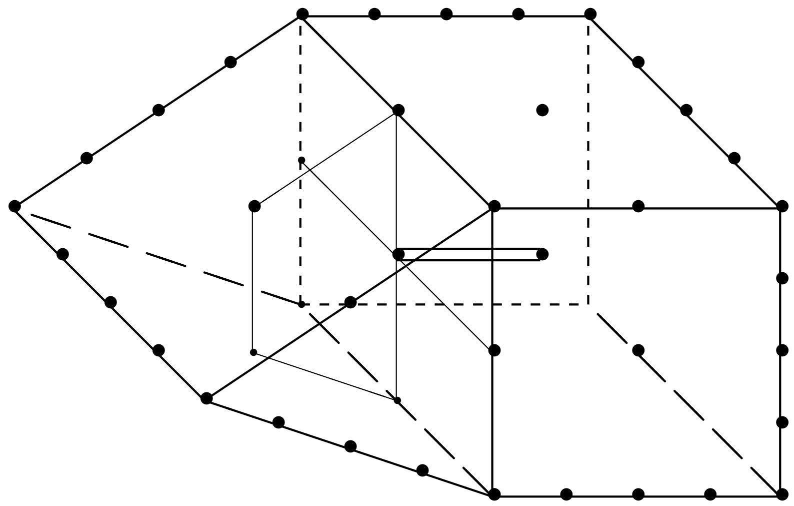


Figure 3.15b A support of the C^2 smoothness condition over two patches

Further, we have

LEMMA 3.3.13. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$(D_{\mathbf{z}_1-\mathbf{y}_1})^k (D_{\mathbf{y}_2-\mathbf{y}_1})^l (D_{\mathbf{y}_4-\mathbf{y}_1})^m \tilde{Q}_n(\mathbf{y}_1)$$

$$= (\mu_2^0 D_{\mathbf{y}_2-\mathbf{y}_1} + \mu_3^0 D_{\mathbf{y}_3-\mathbf{y}_1} + \mu_4^0 D_{\mathbf{y}_4-\mathbf{y}_1})^k (D_{\mathbf{y}_2-\mathbf{y}_1})^l (D_{\mathbf{y}_3-\mathbf{y}_1})^m \bar{P}_n(\mathbf{y}_1)$$

for $0 \le m \le n, 0 \le l \le n, 0 \le k \le r$.

We omit its proof again.

6°. Suppose that the polynomials $\tilde{P}_n$ and $\tilde{Q}_n$ are defined on two adjacent parallelepipeds $T_1 = \langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4, \mathbf{z}_5, \mathbf{z}_6, \mathbf{z}_7, \mathbf{z}_8 \rangle$, $T_2 = \langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4 \rangle$ which share a common facet $T_1 \cap T_2 = \langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4 \rangle$. Write

$$\tilde{P}_n(\mathbf{x}) = \sum_{\beta \le (n,n,n)} a_\beta \tilde{\Phi}_\beta^{(n,n,n)}(\mu(\mathbf{x}))$$

and

$$\tilde{Q}_n(\mathbf{x}) = \sum_{\alpha \le (n,n,n)} b_\alpha \tilde{\Phi}_\alpha^{(n,n,n)}(\nu(\mathbf{x}))$$

where $\mathbf{x} = \mathbf{z}_1 + \mu_1(\mathbf{z}_2 - \mathbf{z}_1) + \mu_2(\mathbf{z}_3 - \mathbf{z}_1) + \mu_3(\mathbf{z}_5 - \mathbf{z}_1)$ with $\mu(\mathbf{x}) = (\mu_1, \mu_2, \mu_3)$ and $\mathbf{x} = \mathbf{z}_1 + \nu_1(\mathbf{z}_2 - \mathbf{z}_1) + \nu_2(\mathbf{z}_3 - \mathbf{z}_1) + \nu_3(\mathbf{y}_1 - \mathbf{z}_1)$ with $\nu(\mathbf{x}) = (\nu_1, \nu_2, \nu_3)$. The orientation of the vertices of T_1 and T_2 are shown in Figure 3.16.

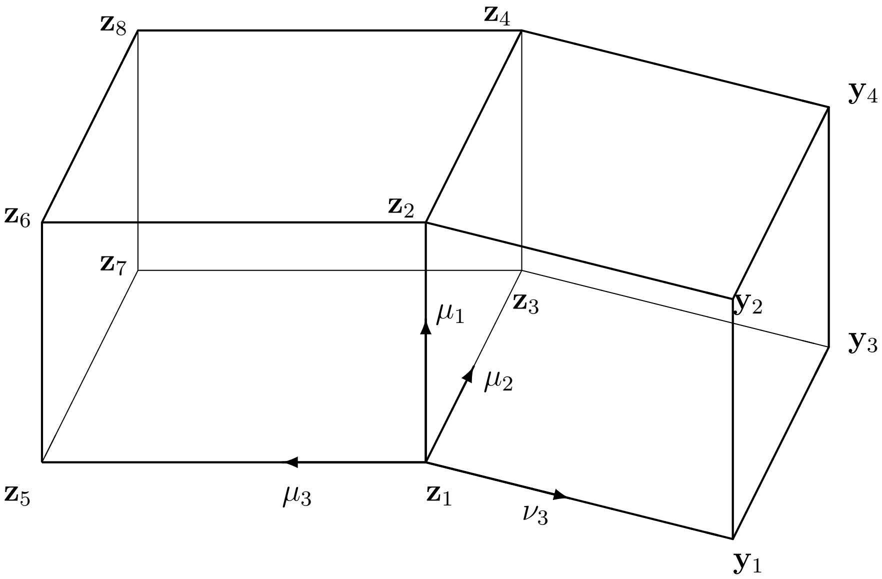


Figure 3.16 The orientation of the vertices of two parallelograms

Let F be a function defined by

$$F(\mathbf{x}) = \begin{cases} \tilde{P}_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_1 \\ \tilde{Q}_n(\mathbf{x}) & \text{if } \mathbf{x} \in T_2. \end{cases}$$

Write $\mathbf{y}_1 = \mathbf{z}_1 + \mu_1^0(\mathbf{z}_2 - \mathbf{z}_1) + \mu_2^0(\mathbf{z}_3 - \mathbf{z}_1) + \mu_3^0(\mathbf{z}_5 - \mathbf{z}_1)$. Clearly, $F \in C^r(T_1 \cup T_2)$ if and only if

$$(D_{\mathbf{y}_1 - \mathbf{z}_1})^l \tilde{Q}_n \Big|_{T_1 \cap T_2} = (\mu_1^0 D_{\mathbf{z}_2 - \mathbf{z}_1} + \mu_2^0 D_{\mathbf{z}_3 - \mathbf{z}_1} + \mu_3^0 D_{\mathbf{z}_5 - \mathbf{z}_1})^l \tilde{P}_n \Big|_{T_1 \cap T_2}$$

for $0 \le l \le r$. It follows that

$$(D_{\mathbf{y}_1-\mathbf{z}_1})^l \tilde{Q}_n\Big|_{T_1\cap T_2} = \frac{n!}{(n-l)!} \sum_{(i,j)\le(n,n)} \Delta_3^l b_{(i,j,0)} \tilde{\Phi}^{(n,n,0)}_{(i,j,0)}(\nu(\mathbf{x}))\Big|_{T_1\cap T_2}$$

and

$$\begin{aligned}
&(\mu_1^0 D_{\mathbf{z}_2-\mathbf{z}_1} + \mu_2^0 D_{\mathbf{z}_3-\mathbf{z}_1} + \mu_3^0 D_{\mathbf{z}^5-\mathbf{z}_1})^l \tilde{P}_n\Big|_{T_1\cap T_2} \\
= &\sum_{\substack{\beta=(\beta_1,\beta_2,\beta_3)\\ |\beta|=l}} \frac{l!}{\beta!} \frac{n!}{(n-\beta_1)!} \frac{n!}{(n-\beta_2)!} \frac{n!}{(n-\beta_3)!} \times \\
&\sum_{(i,j,0)\le(n-\beta_1,n-\beta_2,0)} (\mu_1^0\Delta_1)^{\beta_1}(\mu_2^0\Delta_2)^{\beta_2}(\mu_3^0\Delta_3)^{\beta_3} a_{(i,j,0)} \tilde{\Phi}^{(n-\beta_1,n-\beta_2,0)}_{(i,j,0)}(\mu(\mathbf{x})) \\
= &\sum_{\substack{\beta=(\beta_1,\beta_2,\beta_3)\\ |\beta|=l}} l!\binom{n}{\beta_1}\binom{n}{\beta_2}\binom{n}{\beta_3} \sum_{(i,j,0)\le(n,n,0)} (\mu_1^0\Delta_1)^{\beta_1}(\mu_2^0\Delta_2)^{\beta_2}(\mu_3^0\Delta_3)^{\beta_3} \\
&\times \bar{\mathbf{R}}_1^{\beta_1}\bar{\mathbf{R}}_2^{\beta_3} a_{(i,j,0)} \tilde{\Phi}^{(n,n,0)}_{(i,j,0)}(\mu(\mathbf{x}))\Big|_{T_1\cap T_2}
\end{aligned}$$

Therefore, we have the following

LEMMA 3.3.14. *$F \in C^r(T_1 \cup T_2)$ if and only if*

$$\Delta_3^l b_{(i,j,0)} = \sum_{\substack{\beta=(\beta_1,\beta_2,\beta_3)\\ |\beta|=l}} \frac{\binom{n}{\beta_1}\binom{n}{\beta_2}\binom{n}{\beta_3}}{\binom{n}{l}} (\mu_1^0\Delta_1)^{\beta_1}(\mu_2^0\Delta_2)^{\beta_2}(\mu_3^0\Delta_3)^{\beta_3}\bar{\mathbf{R}}_1^{\beta_1}\bar{\mathbf{R}}_2^{\beta_3} a_{(i,j,0)}$$

for $0 \le i,j \le n, 0 \le l \le r$, where $\bar{\mathbf{R}}_1, \bar{\mathbf{R}}_2$ are degree raising operators acting on the first and second indices of a_{ij}, respectively.

3.4. Vertex Splines with Smoothness Order One and Degree Seven

In this section, the region $R \subset \mathbb{R}^3$ of interest is assumed to be a simplicial partition (cf. [39]). Hence, the partition $\triangle$ of R consists only of tetrahedra. Let $S_7^1 := S_7^1(\triangle)$ be the space of all spline functions of smoothness order one and degree seven; i.e.,

$$S_7^1(\triangle) = \{s \in C^1(R) : s\Big|_t \in \mathbb{P}_7, t \in \triangle\}$$

where $\mathbb{P}_7$ denotes the space of all polynomials of total degree ≤ 7, and t denotes a tetrahedron of $\triangle$. In this section, we are going to construct a collection of vertex splines in S_7^1 that spans a useful subspace of S_7^1. In order to construct locally supported splines in S_7^1, the partition $\triangle$ of R has to satisfy an additional assumption which will be given below. In that case, the full approximation order of S_7^1 can be realized by using the subspace spanned by this collection of vertex splines.

Let us begin our discussion by introducing more notations and lemmas besides the auxiliary results from the previous sections.

Let $\mathcal{V}, \mathcal{E}, \mathcal{F}$, and $\mathcal{T}$ be the collections of all vertices, edges, facets, and tetrahedra of $\triangle$, respectively. We label $\mathcal{V} = \{\mathbf{v}_1, \cdots, \mathbf{v}_N\}$. Then for each edge $e \in \mathcal{E}$ with vertices $\mathbf{v}_i$ and $\mathbf{v}_j$, we assign a direction to e, say $e = \langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2} \rangle$, where

$$\mathbf{v}_{e,1} = \mathbf{v}_{min\{i,j\}} \quad \text{ and } \mathbf{v}_{e,2} = \mathbf{v}_{max\{i,j\}}.$$

Similarly, for each facet $f \in \mathcal{F}$ with vertices $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$, we rewrite it as $f = \langle \mathbf{v}_{f,1}, \mathbf{v}_{f,2}, \mathbf{v}_{f,3} \rangle$, where

$$\mathbf{v}_{f,1} = \mathbf{v}_{min\{i,j,k\}} \quad \mathbf{v}_{f,3} = \mathbf{v}_{max\{i,j,k\}}$$

and $\mathbf{v}_{f,2}$ is the remaining one.

For a vertex $\mathbf{v} \in V$, let $t_{\mathbf{v},i} \in T, i = 1, \cdots, l(\mathbf{v})$ be the tetrahedra in $\triangle$ that share the common vertex $\mathbf{v}$ and denote them by

$$t_{\mathbf{v},i} = \langle \mathbf{v}, \mathbf{x}_{\mathbf{v},i}, \mathbf{y}_{\mathbf{v},i}, \mathbf{z}_{\mathbf{v},i} \rangle, \ i = 1, \cdots, l(\mathbf{v}).$$

For each edge $e = \langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2} \rangle$, let $t_{e,i} \in T, i = 1, \cdots, l(e)$, be the tetrahedra in $\triangle$ which have e as their common edge. We rearrange $t_{e,i}$ if necessary and denote

$$t_{e,i} = \langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2}, \mathbf{v}_{e,i+2}, \mathbf{v}_{e,i+3} \rangle, i = 1, \cdots, l(e).$$

When e is an interior edge, $\mathbf{v}_{e,3+l(e)} := \mathbf{v}_{e,3}$.

For each facet $f = \langle \mathbf{v}_{f,1}, \mathbf{v}_{f,2}, \mathbf{v}_{f,3} \rangle$, let $t_{f,1} = \langle f, \mathbf{u}_f \rangle$ and $t_{f,2} = \langle f, \mathbf{w}_f \rangle$ be two tetrahedra in $\triangle$ that share f as a common facet, if f is an interior facet and let $t_{f,1} = \langle f, \mathbf{u}_f \rangle$ be the tetrahedron in $\triangle$ containing f if f is a boundary facet.

An interior edge e is said to be singular if $\langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2}, \mathbf{v}_{e,3} \rangle \ || \ \langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2}, \mathbf{v}_{e,5} \rangle$, and $\langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2}, \mathbf{v}_{e,4} \rangle \ || \ \langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2}, \mathbf{v}_{e,6} \rangle$ and $l(e) = 4$. And we say that an interior vertex $\mathbf{v}$ a singular vertex if $l(\mathbf{v}) = 8$ and only 6 edges emanate from $\mathbf{v}$ with three distinct slopes.

Our construction of the vertex splines in $S_7^1(\triangle)$ is based on an additional assumption that each interior edge of $\triangle$ satisfies one of the following two conditions:

1° $l(e)$ is odd, or

2° $l(e)$ is 4 and e is singular.

Example 3.2. The following example (cf. Figure 3.17) gives a partition $\triangle$ of the unit cube satisfying the above requirement.

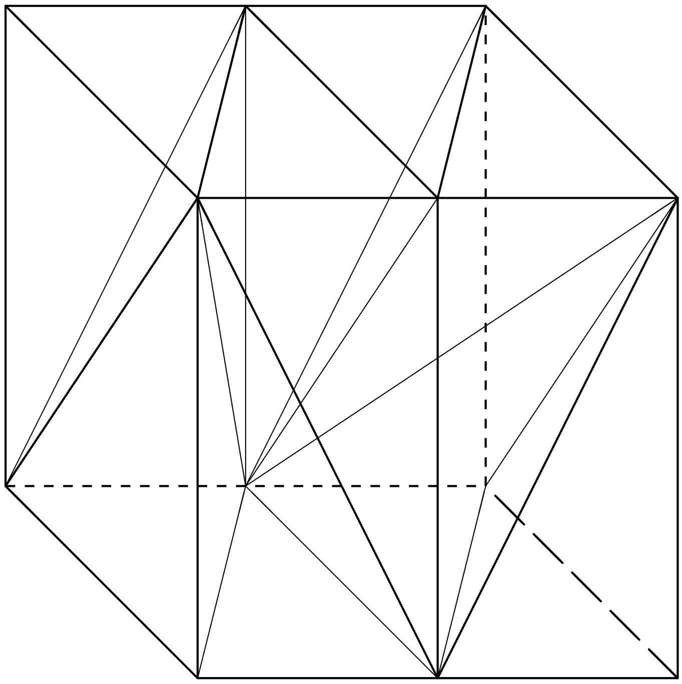


Figure 3.17 A partition of the unit cube

Remark: Although the above assumption seems to be quite strong, it is necessary to make such an assumption in order to construct locally supported spline functions in $S_7^1(\triangle)$. Otherwise, the support of some basic spline functions in $S_7^1(\triangle)$ may no longer be local in general. Indeed, let T stand for a triangulation of a hyperplane domain $D \subset \mathbb{R}^3$ and let $\mathbf{v}$ be a point in $\mathbb{R}^3$ which does not lie on D. Connecting $\mathbf{v}$ to each vertex of T, we obtain a simplicial partition region $\triangle_0$. Clearly, the fourth layer of B-nets of a locally supported spline function in $S_7^1(\triangle_0)$ attached at vertex $\mathbf{v}$ is the B-nets of a local supported spline in $S_4^1(T)$. It is known that some of basis splines of $S_4^1(T)$ are no longer of being local for arbitrary triangulation. (cf. [4, 36].)

Next, we consider piecewise polynomials of total degree 2 defined on $\cup_{i=1}^{l(e)} t_{e,i}$ around edge e. Let F be a piecewise polynomial function defined by

$$F(\mathbf{x})\Big|_{\mathbf{x}\in t_{e,i}} = \sum_{|\alpha|=2} a_\alpha^i \Phi_\alpha(\lambda)$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $\mathbf{x} = \lambda_1 \mathbf{v}_{e,1} + \lambda_2 \mathbf{v}_{e,2} + \lambda_3 \mathbf{v}_{e,i+2} + \lambda_4 \mathbf{v}_{e,i+3}$ with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$. In the following lemmas, we will always assume that

(i) a^i_α with $\alpha_1 \geq 1$ are given and satisfy the C^1 smoothness condition, $i = 1, \cdots, l(e)$, and

(ii) $a^i_{(0,0,2,0)}, a^i_{(0,0,0,2)}$ are given, $i = 1, \cdots, l(e)$ and $a^i_{(0,0,0,2)} = a^{i+1}_{(0,0,2,0)}, i = 1, \cdots, l(e)$ with $a^{l(e)+3}_{(0,0,0,2)} = a^1_{(0,0,2,0)}$ if e is an interior edge.

LEMMA 3.4.1. *Suppose e is a boundary edge of* $\triangle$ *and both (i) and (ii) hold. The for any given* $a^1_{(0,2,0,0)}, a^1_{(0,0,1,1)}$, $a^1_{(0,1,1,0)}$, *and* $a^1_{(0,1,0,1)}$, *there exists a unique set of* $\{a^i_{(0,2,0,0)}, a^i_{(0,0,1,1)}, a^i_{(0,1,1,0)}, a^i_{(0,1,0,1)} : i = 2, \cdots, l(e)\}$ *such that* $F(\mathbf{x}) \in C^1$.

Proof. Since $a^1_{(0,2,0,0)}, a^1_{(0,1,1,0)}, a^1_{(0,1,0,1)}$, and $a^1_{(1,1,0,0)}$ determine all the first partial derivatives of $F(\mathbf{x})$ at $\mathbf{v}_{e,2}$ on $t_{e,1}$, these first partial derivatives can be used to determine all the other $a^i_{(0,2,0,0)}, a^i_{(0,1,1,0)}, a^i_{(1,1,0,0)}$, and $a^i_{(1,1,0,0)}, i = 2, \cdots, l(e)$ by the method in proof of Proposition 3.1 and all these coefficients satisfy the C^1 smoothness conditions by Lemma 3.3.3. By using lemma 3.3.1 and arbitrarily fixing $a^1_{(0,0,1,1)}$, we can determine the other $a^i_{(0,0,1,1)}, i = 1, \cdots, l(e)$. Therefore, $F(\mathbf{x}) \in C^1$ which completes the proof.

LEMMA 3.4.2. *Suppose that e is an interior edge and both (i) and (ii) hold. Further, suppose* $l(e)$ *is an odd integer. Then for any given* $a^1_{(0,2,0,0)}, a^1_{(0,1,0,1)}, a^1_{(0,1,1,0)}$, *there exists a unique set of* $\{a^i_{(0,2,0,0)}, a^i_{(0,1,0,1)}, a^i_{(0,1,1,0)} : i = 2, \cdots, l(e)\} \cup \{a^i_{(0,0,1,1)} : i = 1, \cdots, l(e)\}$ *such that* $F(\mathbf{x}) \in C^1$.

Proof. Clearly, $a^1_{(0,2,0,0)}, a^1_{(0,1,0,1)}, a^1_{(0,1,1,0)}$, and $a^1_{(1,1,0,0)}$ determine all the three first derivative values of $F\Big|_{t_{e,1}}$ at $\mathbf{v}_{e,2}$. By using these derivatives and the function value of F at $\mathbf{v}_{e,2}$, we may determine all other $a^i_{(0,2,0,0)}, a^i_{(0,1,0,1)}$, and $a^i_{(0,1,1,0)}, i = 1, \cdots, l(e)$ by the method in proof of Proposition 3.1 and ensure that $F \in C^1(\mathbf{v}_{e,2})$ by Lemma 3.3.3.

In order to have $F \in C^1$, we know that $a^i_{(0,0,1,1)}, i = 1, \cdots, l(e)$ must satisfy the following conditions, by using C^1 smoothness condition (3.3.1),

$$a^{i+1}_{(0,0,1,1)} + \frac{\text{vol}(t_{e,i+1})}{\text{vol}(t_{e,i})} a^i_{(0,0,1,1)} = c_i, i = 1, \cdots, l(e).$$

Here $a^{l(e)+1}_{(0,0,1,1)} = a^1_{(0,0,1,1)}, t_{e,l(e)+1} = t_{e,1}$, and $\text{vol}(t_{e,i}), \text{vol}(t_{e,i+1})$ denote the volumes of $t_{e,i}, t_{e,i+1}$, respectively, $c_i, i = 1, \cdots, l(e)$ are certain constants involved only the known a^i_α's.

We can easily verify that the determinant of the coefficient matrix of the system of linear equations above is 2. Therefore, we have a unique solution set $\{a^i_{(0,0,1,1)} : i = 1, \cdots, l(e)\}$. Hence, we have established the above lemma.

LEMMA 3.4.3. *Suppose that e is singular and both (i) and (ii) hold. Then for any given* $a^1_{(0,2,0,0)}$,$a^1_{(0,1,0,1)}$, $a^1_{(0,1,1,0)}$, *and* $a^1_{(0,0,1,1)}$, *there exists a unique set of* $\{a^i_{(0,2,0,0)}, a^i_{(0,1,0,1)}, a^i_{(0,1,1,0)}, a^i_{(0,0,1,1)} : i = 1, \cdots, 4\}$ *such that* $F(\mathbf{x}) \in C^1$.

Proof. First, $a^i_{(0,2,0,0)}$, $a^i_{(0,1,0,1)}$, and $a^i_{(0,1,1,0)}$ may be determined by using $\{a^1_{(0,2,0,0)}, a^1_{(0,1,0,1)}$, and $a^1_{(0,1,1,0)}, i = 2, 3, 4\}$ as before. Then, we can determine $a^i_{(0,0,1,1)}, i = 2, 3, 4$, by using the smoothness conditions and $a^1_{(0,0,1,1)}$. To verify that $a^i_{(0,0,1,1)}, i = 1, 2, 3, 4$ satisfy the C^1 smoothness conditions (3.3.1), we may assume that $a^i_{(0,2,0,0)} = 0$, $a^i_{(0,1,0,1)} = 0$, and $a^i_{(0,1,1,0)} = 0$, $i = 1, 2, 3, 4$. Thus, we only need to verify the following equations that relate $a^i_{(0,0,1,1)}, i = 1, 2, 3, 4$ (cf. Fig. 3.18)

$$\begin{aligned}
a^2_{(0,0,1,1)} &= \beta_1 a^1_{(1,1,0,0)} + \theta_1 a^1_{(0,0,1,1)} \\
a^3_{(0,0,1,1)} &= \beta_2 a^2_{(1,1,0,0)} + \theta_2 a^2_{(0,0,1,1)} \\
a^4_{(0,0,1,1)} &= -\frac{\beta_1}{\theta_1} a^3_{(1,1,0,0)} + \frac{1}{\theta_1} a^4_{(0,0,1,1)} \\
a^1_{(0,0,1,1)} &= -\frac{\beta_2}{\theta_2} a^4_{(1,1,0,0)} + \frac{1}{\theta_2} a^4_{(0,0,1,1)}
\end{aligned}$$

are consistent, where $\beta_i, i = 1, 2$ and $\theta_i, i = 1, 2$ are certain constants which may be dependent on the geometry of these four tetrahedra (cf. the smoothness conditions (3.3.1)). Since the following two equations

$$\begin{aligned}
a^1_{(0,0,1,1)} &= \beta_1 a_{(2,0,0,0)} + \theta_1 a^4_{(1,1,0,0)} \\
a^3_{(1,1,0,0)} &= \beta_2 a_{(2,0,0,0)} + \theta_2 a^1_{(1,1,0,0)}
\end{aligned}$$

also hold, we can easily verify the consistence of the above four equations. Thus, we have established this lemma.

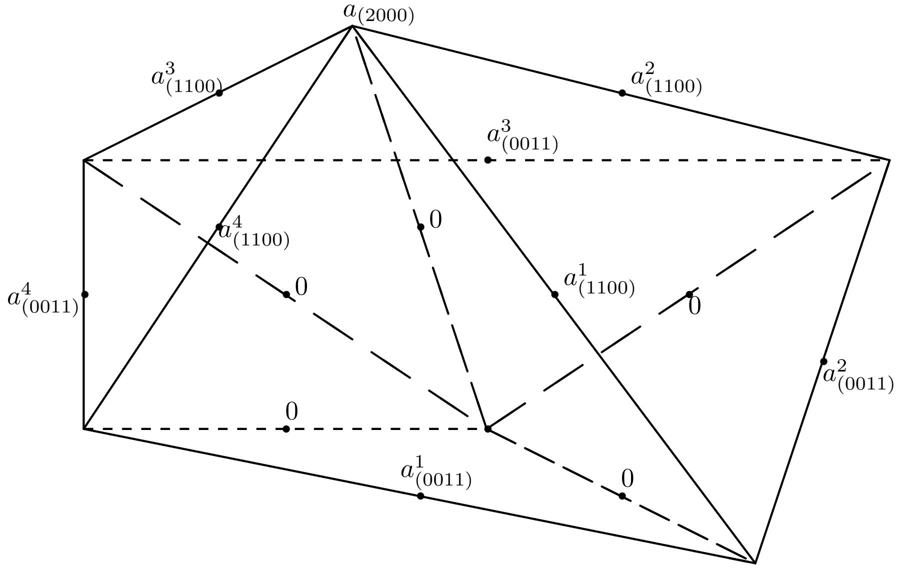


Figure 3.18 Some B-nets of F on $\cup_{i=1}^4 t_{e,i}$ with singular edge e

LEMMA 3.4.4. *Suppose that e is a nonsingular interior edge with $l(e) = 4$. And suppose that (i) and (ii) are satisfied. For any given $a^1_{(0,2,0,0)}$, $a^1_{(0,1,1,0)}$, $a^1_{(0,0,1,1)}$, there exists unique set $\{a^i_{(0,2,0,0)}, a^i_{(0,1,0,1)}, a^i_{(0,1,1,0)}, a^i_{(0,0,1,1)} : i = 2,3,4\} \cup \{a^1_{(0,1,0,1)}\}$ such that $F(\mathbf{x}) \in C^1$.*

Proof. Since F is required to be in $C(\cup_{i=1}^4 t_{e,i})$, we may use $a_1, \cdots, a_6$ to represent the B-coefficients of F to be determined and $b_1, b_2, b_3, c_1, \cdots, c_4, d_1, \cdots, d_6$ are the given B-coefficients of F as shown in Figure 3.19.

By using the smoothness conditions (3.3.1), we obtain the following relations among the a's, b's, c's, and d's.

$$\begin{aligned}
a_1 &= \alpha_1 a_5 + \beta_1 b_2 + \gamma_1 b_1 + \theta_1 d_5 \\
a_2 &= \alpha_2 b_3 + \beta_2 c_2 + \gamma_2 a_1 + \theta_1 d_1 \\
a_3 &= \alpha_2 b_2 + \beta_2 a_1 + \gamma_2 b_1 + \theta_1 d_5 \\
a_4 &= \alpha_3 a_2 + \beta_3 c_3 + \gamma_3 a_3 + \theta_3 d_2 \\
a_5 &= \alpha_3 a_1 + \beta_3 a_3 + \gamma_3 b_1 + \theta_3 d_5 \\
a_6 &= \alpha_0 a_4 + \beta_0 c_4 + \gamma_0 a_5 + \theta_0 d_3
\end{aligned}$$

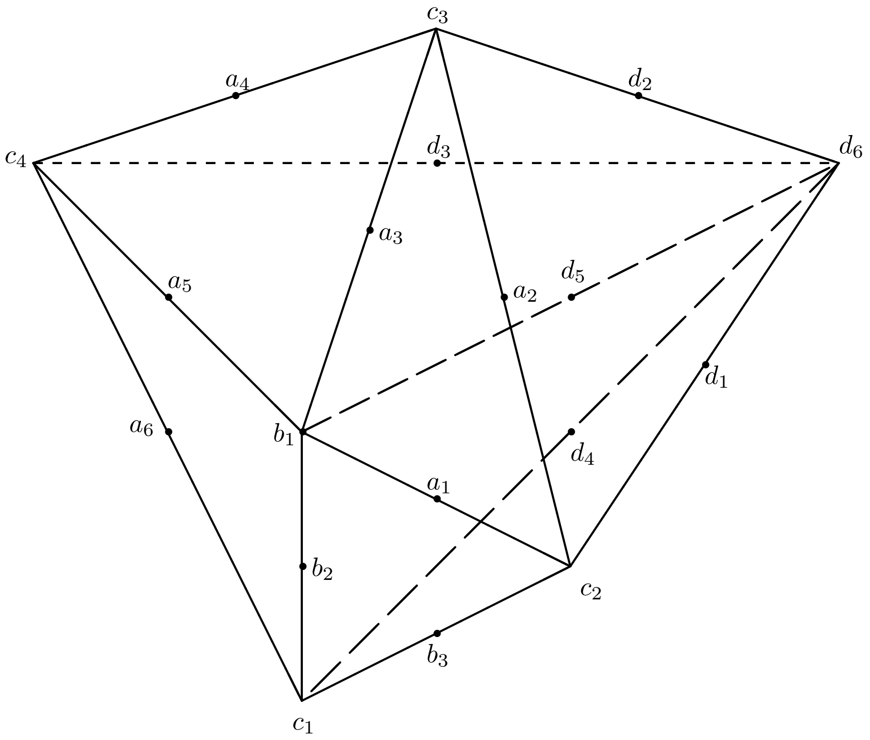


Figure 3.19 The B-nets of F on $\cup_{i=1}^{4} t_{e,i}$ with nonsingular edge e

$$\begin{aligned}
b_2 &= \alpha_0 a_3 + \beta_0 a_5 + \gamma_0 b_1 + \theta_0 d_5 \\
b_3 &= \alpha_1 a_6 + \beta_1 c_1 + \gamma_1 b_2 + \theta_1 d_4 \\
d_1 &= \alpha_1 d_3 + \beta_1 d_4 + \gamma_1 d_5 + \theta_1 d_6 \\
d_2 &= \alpha_2 d_4 + \beta_2 d_1 + \gamma_2 d_5 + \theta_2 d_6 \\
d_3 &= \alpha_3 d_1 + \beta_3 d_2 + \gamma_3 d_5 + \theta_3 d_6 \\
d_4 &= \alpha_0 d_2 + \beta_0 d_3 + \gamma_0 d_5 + \theta_0 d_6
\end{aligned}$$

where the α's, β's, γ's, and θ's are defined in the following

$$\begin{aligned}
\mathbf{u} &= \alpha_0 \mathbf{w} + \beta_0 \mathbf{x} + \theta_0 \mathbf{y} \\
\mathbf{v} &= \alpha_1 \mathbf{x} + \beta_1 \mathbf{u} + \theta_1 \mathbf{y} \\
\mathbf{w} &= \alpha_2 \mathbf{u} + \beta_2 \mathbf{v} + \theta_2 \mathbf{y} \\
\mathbf{x} &= \alpha_3 \mathbf{v} + \beta_3 \mathbf{w} + \theta_3 \mathbf{y}.
\end{aligned}$$

Actually, the α's, β's, γ's, and θ's satisfy the following:

$$\alpha_0\alpha_1\alpha_2\alpha_3 = 1; \quad \beta_0 = -\frac{\beta_2}{\alpha_2\alpha_3}, \quad \beta_1 = -\frac{\beta_3}{\alpha_0\alpha_3};$$

$$\alpha_0 = \frac{\alpha_3 + \beta_2\alpha_3}{\alpha_2\alpha_3},\ \alpha_1 = \frac{\alpha_0 + \beta_0\alpha_3}{\alpha_0\alpha_3},\ \alpha_2 = \frac{\alpha_1 + \beta_0\alpha_1}{\alpha_0\alpha_1},\ \alpha_3 = \frac{\alpha_2 + \beta_1\alpha_2}{\alpha_1\alpha_2};$$

$$\gamma_0 = \frac{\beta_2\gamma_3 - \gamma_2\alpha_3}{\alpha_2\alpha_3},\ \gamma_1 = \frac{\beta_3\gamma_0 - \gamma_3\alpha_0}{\alpha_0\alpha_3},\ \gamma_2 = \frac{\beta_0\gamma_1 - \gamma_0\alpha_1}{\alpha_0\alpha_1},\ \gamma_3 = \frac{\beta_1\gamma_2 - \gamma_1\alpha_2}{\alpha_1\alpha_2};$$

and

$$\theta_0 = \frac{\beta_2\theta_3 - \theta_2\alpha_3}{\alpha_2\alpha_3},\ \theta_1 = \frac{\beta_3\theta_0 - \theta_3\alpha_0}{\alpha_0\alpha_3},\ \theta_2 = \frac{\beta_0\theta_1 - \theta_0\alpha_1}{\alpha_0\alpha_1},\ \theta_3 = \frac{\beta_1\theta_2 - \theta_1\alpha_2}{\alpha_1\alpha_2}.$$

We know that the last four equations hold because the assumption (i). We can easily prove that the first eight linear equations that involve the unknowns $a_1, \cdots, a_6$ have a unique solution by using the relations among the α's, β's, γ's, and θ's. Hence, the proof is complete.

As before, for each edge $e = \langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2} \rangle$, we denote *the directional derivatives relative to e* by

$$D_e^\alpha := D_{\mathbf{v}_{e,2}-\mathbf{v}_{e,1}}^{\alpha_1} D_{\mathbf{v}_{e,3}-\mathbf{v}_{e,1}}^{\alpha_2} D_{\mathbf{v}_{e,4}-\mathbf{v}_{e,1}}^{\alpha_3}, \alpha \in \mathbb{Z}_+^3$$

where the derivatives are taken from inside the tetrahedron.

For each facet $f = \langle \mathbf{v}_{f,1}, \mathbf{v}_{f,2}, \mathbf{v}_{f,3} \rangle$, we denote *the directional derivatives relative to f* by

$$\begin{aligned} D_{f,1}^\beta &:= D_{\mathbf{v}_{f,2}-\mathbf{v}_{f,1}}^{\beta_1} D_{\mathbf{v}_{f,3}-\mathbf{v}_{f,1}}^{\beta_2} \\ D_{f,2}^\beta &:= D_{\mathbf{v}_{f,1}-\mathbf{v}_{f,2}}^{\beta_1} D_{\mathbf{v}_{f,3}-\mathbf{v}_{f,2}}^{\beta_2} \\ D_{f,3}^\beta &:= D_{\mathbf{v}_{f,1}-\mathbf{v}_{f,3}}^{\beta_1} D_{\mathbf{v}_{f,2}-\mathbf{v}_{f,3}}^{\beta_2} \end{aligned}$$

for any $\beta \in \mathbb{Z}_+^2$, where the derivatives are taken from inside the facet and

$$D_{f,0}^\alpha = D_{\mathbf{v}_{f,2}-\mathbf{v}_{f,1}}^{\alpha_1} D_{\mathbf{v}_{f,3}-\mathbf{v}_{f,1}}^{\alpha_2} D_{\mathbf{u}_f-\mathbf{v}_{f,1}}^{\alpha_3}$$

for any $\alpha \in \mathbb{Z}_+^3$, where the derivatives are taken inside the tetrahedron. For each tetrahedron $t = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \rangle$, we denote *the directional derivatives relative to t at $\mathbf{v}_i$* by

$$D_{t,i}^\alpha = D_{\mathbf{v}_j-\mathbf{v}_i}^{\alpha_1} D_{\mathbf{v}_k-\mathbf{v}_i}^{\alpha_2} D_{\mathbf{v}_l-\mathbf{v}_i}^{\alpha_3}$$

for any $\alpha \in \mathbb{Z}_+^3$, $i = 1, 2, 3, 4$, where $\{i, j, k, l\}$ is a permutation of $\{1, 2, 3, 4\}$ and the derivatives are taken inside t.

Also, denote

$$I_e = \begin{cases} \{(3,1,0),(3,0,1)\} & \text{if } e \text{ is an interior edge,} \\ \{(3,1,0),(3,0,1),(2,1,1),(3,1,1)\} & \begin{array}{c}\text{if } e \text{ is a boundary edge, or} \\ e \text{ is a singular edge.}\end{array} \end{cases}$$

and $I_{f,1} = \{(2,2)\}$ and $I_{f,0} = \{(2,2,1)\}$.

We are now ready to outline the construction procedure of three types of vertex splines in $S_7^1(\triangle)$ of interest. They satisfy the following specifications of interpolatory parameters and smoothness conditions.

(I) For each vertex $\mathbf{v}$ of $\mathcal{V}$ and $\gamma \in \mathbb{Z}_+^3$ with $|\gamma| \le 3$, let $V_{\mathbf{v}}^{\gamma}$ be a piecewise polynomial function of degree 7 on $\triangle$ satisfying the following:

$$D^{\alpha} V_{\mathbf{v}}^{\gamma}(\mathbf{u}) = \delta_{\alpha,\gamma}\delta_{\mathbf{v},\mathbf{u}}, |\alpha| \le 3, \mathbf{u} \in \mathcal{V}; \tag{I.1}$$

$$D_e^{\alpha} V_{\mathbf{v}}^{\gamma}\Big|_{t_{e,1}}(\mathbf{v}_{e,1}) = 0, \alpha \in I_e, e \in \mathcal{E}; \tag{I.2}$$

$$D_{f,i}^{\beta} V_{\mathbf{v}}^{\gamma}\Big|_f(\mathbf{v}_{f,i}) = 0, i = 1,2,3,\ \beta \in I_{f,1}, f \in \mathcal{F}; \tag{I.3}$$

$$D_{f,0}^{\alpha} V_{\mathbf{v}}^{\gamma}\Big|_{t_{f,1}}(\mathbf{v}_{f,1}) = 0, \alpha \in I_{f,0}, f \in \mathcal{F}; \tag{I.4}$$

(I.5) $V_{\mathbf{v}}^{\gamma}$ satisfies the C^1 smoothness condition across each facet f of $\mathcal{F}$.

Here and throughout, as usual, the symbol $\delta_{a,b}$ is the Kronecker delta.

(II) For each edge e and $\gamma \in I_e$, let V_e^{γ} be a piecewise polynomial function of degree 7 on $\triangle$ that satisfies the following:

$$D^{\alpha} V_e^{\gamma}(\mathbf{u}) = 0, |\alpha| \le 3, \mathbf{u} \in \mathcal{V}; \tag{II.1}$$

$$D_d^{\alpha} V_e^{\gamma}\Big|_{t_{d,1}}(\mathbf{v}_{d,1}) = \delta_{\alpha,\gamma}\delta_{e,d}, \alpha \in I_e, d \in \mathcal{E}; \tag{II.2}$$

$$D_{f,i}^{\beta} V_e^{\gamma}\Big|_f(\mathbf{v}_{f,i}) = 0, i = 1,2,3,\ \beta \in I_{f,1}, f \in \mathcal{F}; \tag{II.3}$$

$$D_{f,0}^{\alpha} V_e^{\gamma}\Big|_{t_{f,1}}(\mathbf{v}^{f,1}) = 0, \alpha \in I_{f,0}, f \in \mathcal{F}; \tag{II.4}$$

(II.5) V_e^{γ} satisfies the C^1 smoothness condition across each facet f of $\mathcal{F}$.

(III) For each $f \in \mathcal{F}$ and $i = 0,1,2,3$, let $V_{f,i}$ be a piecewise polynomial function of degree 7 on $\triangle$ that satisfies the following:

$$D^{\alpha} V_f^{i}(\mathbf{u}) = 0, |\alpha| \le 3, \mathbf{u} \in \mathcal{V}; \tag{III.1}$$

(III.2) $$D_e^{\alpha} V_f^i\Big|_{t_{e,1}}(\mathbf{v}_{e,1}) = 0, \alpha \in I_e, e \in \mathcal{E};$$

(III.3) $$D_{g,j}^{\beta} V_f^i\Big|_{g}(\mathbf{v}_{g,i}) = \delta_{g,f}\delta_{i,j}, j = 1, 2, 3,\ \beta \in I_{f,1}, g \in \mathcal{F};$$

(III.4) $$D_{g,0}^{\alpha} V_f^i\Big|_{t_{g,1}}(\mathbf{v}_{g,1}) = \delta_{g,f}\delta_{i,0}, \alpha \in I_{f,0}, \quad g \in \mathcal{F};$$

(III.5) V_f^i satisfies the C^1 smoothness condition across each facet f of $\mathcal{F}$.

The outline of constructing vertex spline $V_{\mathbf{v}}^{\gamma}$ is as follows. Let $t = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4 \rangle$ be a tetrahedron of $\triangle$.

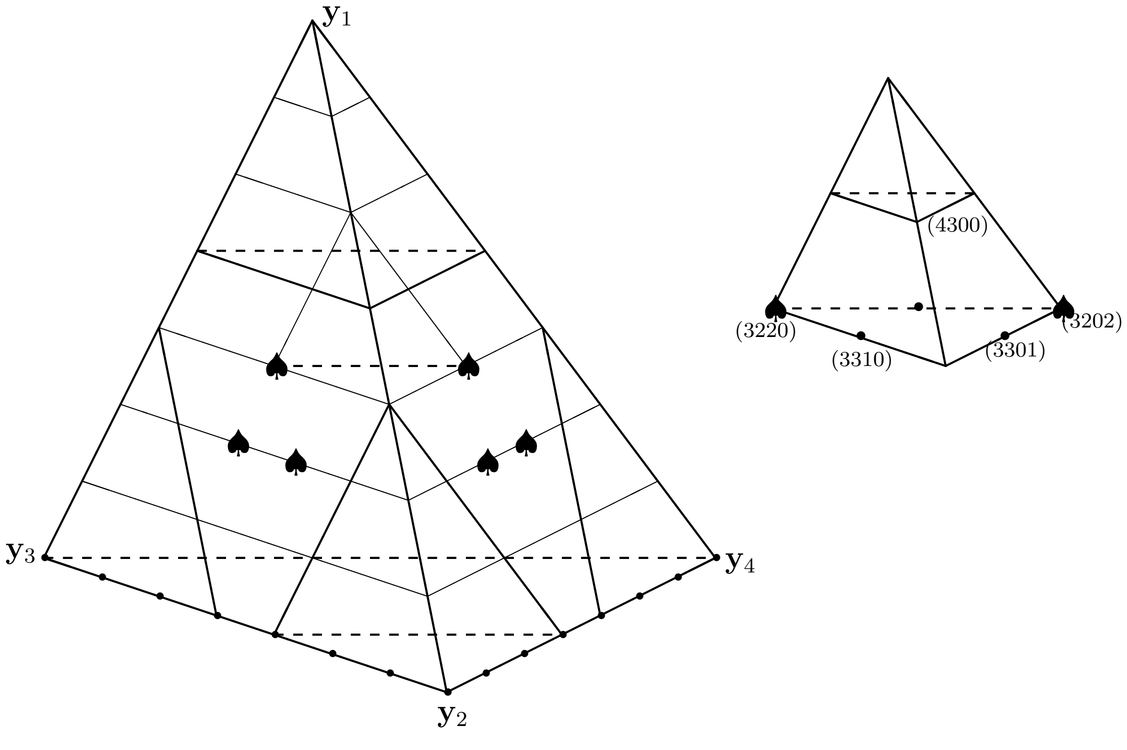


Figure 3.20 Illustration of constructing vertex spline in $S_7^1(\triangle)$

The B-coefficients of $V_{\mathbf{v}}^{\gamma}|_t$ on layer l attached to $\mathbf{y}_i$ can be determined by the requirements in (I.1), $0 \le l \le 3$ and $i = 1, 2, 3, 4$. Indeed, we first convert the partial derivatives D^{α} at $\mathbf{y}_i$ to the directional derivatives relative to t at $\mathbf{y}_i$ and then use the resulting $D_{t,i}^{\alpha}$'s at $\mathbf{y}_i$ to find the B-coefficients of $V_{\mathbf{v}}^{\gamma}|_t$ on layer l attached to $\mathbf{y}_i$, $0 \le l \le 3$. By using the requirements in (I.2) and (I.3) and Lemmas 3.4.1–3.4.4, whichever applies, we can choose the B-coefficients of $V_{\mathbf{v}}^{\gamma}$ on layer l, $l = 0, 1, 2,$

around e to satisfy the C^1 smoothness conditions. Let us use the following example to illustrate what we mean. Consider $e = \langle \mathbf{y}_1, \mathbf{y}_2 \rangle$.(cf. Figure 3.20.) By using the requirements in (I.2), we obtain $a_{(3,3,1,0)}$ and $a_{(3,3,0,1)}$ of $V_{\mathbf{v}}^{\gamma}|_t$ if $t = t_{e,1}$. Otherwise, they can be obtained in terms of the B-coefficients $a_{(3,3,1,0)}$, $a_{(3,3,0,1)}$, $a_{(4,3,0,0)}$ and $a_{(3,4,0,0)}$ of $V_{\mathbf{v}}^{\gamma}|_{t_{e,1}}$ by using the C^1 smoothness condition. By using the requirements in (I.3), $a_{(3,2,2,0)}, a_{(3,2,0,2)}, a_{(2,3,2,0)}$ and $a_{(2,3,0,2)}$ are obtained and we may apply Lemmas 3.4.1–3.4.4, whichever applies, to determine $a_{(3,2,1,1)}$ and $a_{(2,3,1,1)}$ if e is an interior edge or if e is a boundary edge but $t \neq t_{e,1}$ or if e is a singular edge but $t \neq t_{e,1}$. If e is a boundary edge or a singular edge and $t = t_{e,1}$, then $a_{(3,2,1,1)}$ and $a_{(2,3,1,1)}$ of $V_{\mathbf{v}}^{\gamma}$ are obtained by using the requirements in (I.2). Similarly, we can determine the remaining coefficients of $V_{\mathbf{v}}^{\gamma}$ on layer l around other edges of t, $0 \leq l \leq 2$. Finally, we may use the requirements in (I.4) and (I.5) to find $a_{(2,2,2,1)}, a_{(2,2,1,2)}, a_{(2,1,2,2)}$ and $a_{(1,2,2,2)}$ of $V_{\mathbf{v}}^{\gamma}$. Indeed, suppose that $f = \langle \mathbf{y_1}, \mathbf{y}_2, \mathbf{y}_3 \rangle$. If $t = t_{f,1}$, then $a_{(2,2,2,1)}$ can be obtained directly by using the requirements in (I.4). If $t \neq t_{f,1}$, $a_{(2,2,2,1)}$ of $V_{\mathbf{v}}^{\gamma}$ is determined by using the C^1 smoothness conditions and the corresponding coefficients of $V_{\mathbf{v}}^{\gamma}|_{t'}$, where t' is the tetrahedron of $\triangle$ which has a common facet f with t. The other $a_{(2,2,1,2)}, a_{(2,1,2,2)}$ and $a_{(1,2,2,2)}$ can be determined in a similar manner.

All these steps assume that $V_{\mathbf{v}}^{\gamma} \in C^1(R)$ and $V_{\mathbf{v}}^{\gamma} \in C^3$ at each vertex of $\triangle$ since it interpolates the data $\delta_{\mathbf{u},\mathbf{v}}\delta_{\alpha,\gamma}, |\alpha| \leq 3$. Therefore, $V_{\mathbf{v}}^{\gamma} \in S_7^1(\triangle)$ for any γ with $|\gamma| \leq 3$.

From the above construction steps, we conclude that the support $S_{\mathbf{v}}$ of $V_{\mathbf{v}}^{\gamma}$ is the union of the tetrahedra of $\triangle$ with $\mathbf{v}$ as the only common vertex.

Similarly, we can see as above that the requirements in (II.1)-(II.6) uniquely determine a piecewise polynomial function $V_e^{\gamma} \in S_7^1(\triangle)$ whose support is the union of all tetrahedra of $\triangle$ sharing e.

To prove that the requirements in (III.1)–(III.6) uniquely determine $V_{f,i}$ follows along the lines as the proof for $V_{\mathbf{v}}^{\gamma}$. Hence, the support of $V_{f,i}$ is the union of all tetrahedra of $\triangle$ sharing f.

We now consider the space

$$\begin{aligned}\widehat{S}_7^1(\triangle) &= \operatorname{span}\{ \quad V_{\mathbf{v}}^{\gamma} : |\gamma| \leq 3,\, \mathbf{v} \in \mathcal{V}\} \cup \{V_e^{\gamma} : \gamma \in I_e, e \in \mathcal{E}\} \\ &\qquad\qquad \cup \{V_{f,i} : i = 0,1,2,3, f \in \mathcal{F}\}.\end{aligned}$$

Clearly, $\widehat{S}_7^1(\triangle)$ is a subspace of $S_7^1(\triangle)$.

For each sufficiently smooth function g, we define

$$Lg(\mathbf{x}) = \sum_{\mathbf{v}\in\mathcal{V}} \sum_{|\gamma|\leq 3} D^{\gamma}g(\mathbf{v})V_{\mathbf{v}}^{\gamma}(\mathbf{x}) + \sum_{e\in\mathcal{E}} \sum_{\gamma\in I_e} D_e^{\gamma}g(\mathbf{v}_{e,1})V_e^{\gamma}(\mathbf{x}) + \sum_{f\in\mathcal{F}} \sum_{i=0}^{3} D_f^{\alpha(i)}g(\mathbf{v}_{f,i})V_{f,i}(\mathbf{x}),$$

where $\alpha(0) \in I_{f,0}$ and $\alpha(i) \in I_{f,1}, i = 1, 2, 3$.

We are now ready to derive some properties of the super spline space $\widehat{S}_7^1(\triangle)$.

LEMMA 3.4.5. *$Lp = p$ for any polynomial p of total degree ≤ 7.*

Proof. We use mathematical induction on the number of tetrahedra in $\triangle$ to prove this lemma. For $n = 1$, L is an interpolatory operator based on t which is the only tetrahedron of $\triangle$. Since the sets of interpolation conditions associated with each vertex of t are lower sets and induce a partition of Λ_7, we see that $Lp = p$ for all p of total degree 7 by Proposition 3.1. Suppose now that the result holds for $m = \#\{t : t \in \triangle\}$. Let $\#\{t : t \in \triangle\} = m + 1$ and set $\triangle = \{t_i : i = 1, \cdots, m + 1\}$. By relabeling if necessary, assume that $t_{m+1} = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle$ has at least one boundary facet, and for the time being, assume that it has only one interior facet $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$, say. Let $\triangle' = \{t_i : i = 1, \cdots, m\} = \triangle \backslash \{t_{m+1}\}$. Observing the uniqueness in Lemma 3.4.1 and applying Theorem 4.1.3 in [39], we can see that the smoothness of Lp across $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ may be rewritten as appropriate interpolation conditions (directional derivatives related to the edges and related to the facet) such that $L_\triangle p\Big|_{\triangle'} = L_{\triangle'} p$ and $L_\triangle p\Big|_{t_{m+1}} = L_{t_{m+1}} p$, where $L_\triangle, L_{\triangle'}, L_{t_{m+1}}$ are linear operators L based on $\triangle, \triangle', t_{m+1}$, respectively. By the induction hypothesis, we have $L_\triangle p\Big|_{\triangle'} = p$ and $L_\triangle p\Big|_{t_{m+1}} = p$. Hence, $Lp = p$ on $\triangle$. The proof is similar if t_{m+1} contains two or three interior facets. This completes the proof.

If Lg is interpreted as

$$\begin{aligned} Lg(\mathbf{x}) \quad = \quad & \sum_{\mathbf{v} \in \mathcal{V}} \sum_{|\gamma| \leq 3} D^\gamma g(\mathbf{v}) V_\mathbf{v}^\gamma(\mathbf{x}) + \sum_{e \in \mathcal{E}} \sum_{\gamma \in I_e} D_e^\gamma g\Big|_{t_{e,1}} (\mathbf{v}_{e,1}) V_e^\gamma(\mathbf{x}) \\ & + \sum_{f \in \mathcal{F}} \sum_{i=0}^{3} D_{f,i}^{\alpha(i)} g\Big|_{t_{t,1}} (\mathbf{v}_{f,i}) V_{f,i}(\mathbf{x}), \end{aligned}$$

then, by the same argument as above, we can prove the following lemma.

LEMMA 3.4.6. *$Lg = g$ for any $g \in \widehat{S}_7^1(\triangle)$.*

Let $S = \{s \in S_7^1 : s \in C^3$at each vertex of $\triangle\}$. Then S is called a super spline space because each spline in S has extra smoothness at each vertex of $\triangle$. We have the following consequence of Lemma 3.4.6.

THEOREM 3.4.1. *The collection*

$$\mathcal{B} := \{V_\mathbf{v}^\gamma : \mathbf{v} \in \mathcal{V}, |\gamma| \leq 3\} \cup \{V_e^\gamma : \gamma \in I_e, e \in \mathcal{E}\} \cup \{V_{f,i}, i = 0, 1, 2, 3, f \in \mathcal{F}\}$$

is a basis of S. Therefore $S = \widehat{S}_7^1(\triangle)$.

Let $G \subset \sup\{t : t \in \triangle\}$ and for $g \in C^k(G)$, denote

$$\|D^k g\| = \max_{|\alpha|=k} \|D^\alpha g\|_{C(G)}$$

and

$$\mathrm{dist}(f, S) = \inf_{s\in S} \|f - s\|.$$

For the given $\triangle$, let $|\triangle|$ denote the maximum of the diameter of all $t \in \triangle$. Our main result in this section is the following:

THEOREM 3.4.2. *For any* $g \in C^8(G)$,

$$\|Lg - g\| \le K\|D^8 g\||\triangle|^8$$

where K *is a constant independent of* g *and* $|\triangle|$. *Consequently,*

$$\mathrm{dist}(g, \widehat{S}_7^1) \le K\|D^8 g\||\triangle|^8.$$

Proof. Fix a point $\mathbf{x} \in G$ and consider a linear functional

$$F(g) = Lg(\mathbf{x}) - g(\mathbf{x}).$$

It is easy to see that F satisfies the following:

(i) $|F(g)| \le K_1 \sum_{j=0}^{8} \|D^j g\||\triangle|^j$

(ii) $F(p) = 0$ for all $p \in \mathbb{P}_7$.

By a result of Bramble and Hilbert [24], there exists a constant K independent of g, $\mathbf{x}$, and $|\triangle|$ such that

$$|Lg(\mathbf{x}) - g(\mathbf{x})| \le K\|D^8 g\||\triangle|^8.$$

Therefore, we have established the theorem.

As we know that $\widehat{S}_7^1$ is a proper subspace of S_7^1, the exact dimension of $\widehat{S}_7^1$ is given in the following which is a consequence of Theorem 3.4.1.

THEOREM 3.4.3. *Suppose that* $\triangle$ *satisfies the additional assumption mentioned before. Then*

$$\dim\ \widehat{S}_7^1 = 15N_v + 2N_e + 2N_b + 2N_s + 4N_f$$

where N_v, N_e, N_b, N_s, N_f *denote the numbers of vertices, edges, boundary edges, singular edges, and facets of* $\triangle$, *respectively.*

3.5. Vertex Splines with Smoothness Order r and Degree $d \geq 6r+3$

In this section, the partition $\triangle$ of the region $R \subset \mathbb{R}^3$ of interest is again assumed to be a simplicial partition (cf. [39] for the definition of a simplicial partition). From the previous section, we have some feeling that fundamental locally supported spline functions in $S_7^1(\triangle)$ do not exit on an arbitrarily simplicial partition. In general, we conjecture that we cannot construct nontrivial locally supported splines functions in $S_{4r+3}^r(\triangle) = \{s \in C^r : s\Big|_t \in \mathbb{P}_{4r+3}, t \in \triangle\}$ which serve as fundamental functions that the give the full approximation order of $(4r+3)+1$. On the other hand, it was conjectured in [119] and proved in [88–90] that piecewise polynomial functions of smoothness r could be constructed when the degree of polynomials is $\geq 8r+1$. In the following, we will outline the construction procedure of fundamental vertex splines of smoothness order r and degree $d \geq 6r+3$ which have local supports and span a super spline subspace of

$$S_d^r(\triangle) = \{s : s \in C^r(R) \text{ and } s|_t \in \mathbb{P}_d, t \in \triangle\}.$$

Moreover, we will construct an approximation formula based on these fundamental vertex splines to realize the full approximation order of $d+1$ from $S_d^r(\triangle), d \geq 6r+3$.

We will only consider the special and most important case where $d = 6r+3$ and $r \geq 1$. The discussion for $d > 6r+3$ is similar.

For a given arbitrarily simplicial partition $\triangle$, we denote the collections of all vertices, edges, facets, and tetrahedra of $\triangle$ by $\mathcal{V}, \mathcal{E}, \mathcal{F}$, and $\mathcal{T}$, respectively. Let $\mathcal{V} = \{\mathbf{v}_1, \cdots, \mathbf{v}_N\}$. As §3.4, for each edge $e \in \mathcal{E}$, we rewrite $e = \langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2} \rangle$ and let $t_{e,i}, i = 1, \cdots, l(e)$ be all tetrahedra in $\mathcal{T}$ which share e. Denote $t_{e,1} = \langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2}, \mathbf{v}_{e,3}, \mathbf{v}_{e,4} \rangle$. For a facet $f \in F$, we also rewrite $f = \langle \mathbf{v}_{f,1}, \mathbf{v}_{f,2}, \mathbf{v}_{f,3} \rangle$ and let $t_{f,i}, i = 1, \cdots, l(f)$ be all the tetrahedra in $\mathcal{T}$ which share f, where $l(f) = 1$ or 2 according to whether f is a boundary facet or an interior facet. And we write $t_{f,1} = \langle f, \mathbf{u} \rangle$ and $t_{f,2} = \langle f, \mathbf{w} \rangle$. Rewrite $t_{f,1} = \langle \mathbf{v}_{f,1}, \mathbf{v}_{f,2}, \mathbf{v}_{f,3}, \mathbf{v}_{f,4} \rangle$. For each $t \in \mathcal{T}$, we again rewrite $t = \langle \mathbf{v}_{t,1}, \cdots, \mathbf{v}_{t,4} \rangle$ as §3.4.

An interior edge e is said to be singular at $\mathbf{v}_{e,1}$ if $\langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2}, \mathbf{v}_{e,3} \rangle || \langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2}, \mathbf{v}_{e,5} \rangle$, $\langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2}, \mathbf{v}_{e,4} \rangle || \langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2}, \mathbf{v}_{e,6} \rangle$ and $l(e) = 4$. And we say that an interior vertex $\mathbf{v}$ is a singular vertex if $l(\mathbf{v}) = 8$ and only 6 edge emanating from $\mathbf{v}$ with three distinct slopes. An interior facet $f = \langle \mathbf{v}_{f,1}, \mathbf{v}_{f,2}, \mathbf{v}_{f,3} \rangle$ is said to be singular at $\langle \mathbf{v}_{f,1}, \mathbf{v}_{f,2} \rangle$ if $\langle \mathbf{v}_{f,1}, \mathbf{v}_{f,2}, \mathbf{u} \rangle || \langle \mathbf{v}_{f,1}, \mathbf{v}_{f,2}, \mathbf{w} \rangle$.

For each edge $e \in \mathcal{E}$, *the derivatives relative to the edge* e are defined by

$$D_e^\alpha = (D_{\mathbf{v}_{e,2}-\mathbf{v}_{e,1}})^{\alpha_1} (D_{\mathbf{v}_{e,3}-\mathbf{v}_{e,1}})^{\alpha_2} (D_{\mathbf{v}_{e,4}-\mathbf{v}_{e,1}})^{\alpha_3}$$

for $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^2_+$, where the derivatives are taken from inside the tetrahedron $t_{e,1}$.

For each facet $f \in \mathcal{F}$, *the derivatives relative to* f are defined by

$$\begin{aligned} D^{\alpha}_{f,1} &= (D_{\mathbf{v}_{f,2}-\mathbf{v}_{f,1}})^{\alpha_1}(D_{\mathbf{v}_{f,3}-\mathbf{v}_{f,1}})^{\alpha_2}(D_{\mathbf{v}_{f,4}-\mathbf{v}_{f,1}})^{\alpha_3} \\ D^{\beta}_{f,2} &= (D_{\mathbf{v}_{f,1}-\mathbf{v}_{f,2}})^{\beta_1}(D_{\mathbf{v}_{f,3}-\mathbf{v}_{f,2}})^{\beta_2}(D_{\mathbf{v}_{f,4}-\mathbf{v}_{f,2}})^{\beta_3} \\ D^{\gamma}_{f,3} &= (D_{\mathbf{v}_{f,1}-\mathbf{v}_{f,3}})^{\gamma_1}(D_{\mathbf{v}_{f,2}-\mathbf{v}_{f,3}})^{\gamma_2}(D_{\mathbf{v}_{f,4}-\mathbf{v}_{f,3}})^{\beta_3} \end{aligned}$$

for $\alpha, \beta, \gamma \in \mathbb{Z}^3_+$, where the derivatives are taken from inside the tetrahedron $t_{f,1}$.

For each tetrahedron $t \in \mathcal{T}$, *the derivatives relative to* t are defined by

$$D^{\alpha}_{t} = (D_{\mathbf{v}_{t,1}-\mathbf{v}_{t,4}})^{\alpha_1}(D_{\mathbf{v}_{t,2}-\mathbf{v}_{t,4}})^{\alpha_2}(D_{\mathbf{v}_{t,3}-\mathbf{v}_{t,4}})^{\alpha_3}$$

for $\alpha \in \mathbb{Z}^3_+$ and

$$D^{\alpha}_{t,ij} g(\mathbf{v}_{t,i}) = (D_{\mathbf{v}_{t,j}-\mathbf{v}_{t,i}})^{\alpha_1}(D_{\mathbf{v}_{t,k}-\mathbf{v}_{t,i}})^{\alpha_2}(D_{\mathbf{v}_{t,l}-\mathbf{v}_{t,i}})^{\alpha_3} g(\mathbf{v}_{t,i})$$

for $\alpha \in \mathbb{Z}^3_+$ and $\{l, k\} \in \{1, 2, 3, 4\} \backslash \{i, j\}$, where the derivatives are taken from inside t.

Let us divide the underlying index set $\{\beta \in \mathbb{Z}^4_+ : |\beta| = 6r + 3\}$ of the B-net on t into six parts as follows. The subdivision is based on the idea of "disentangling of the rings" in the trivariate setting.

For simplicity, let t be rewritten as $\langle \mathbf{y}_1, \cdots, \mathbf{y}_4 \rangle$. Let i, j, k, l denote distinct elements of the set $\{1, 2, 3, 4\}$.

Part I is the union of the collections $B_1(i) = A^{6r+3}_i J_1$, $i = 1, 2, 3, 4$, where $J_1 = \{(l, m, n) : l + m + n \leq 3r + 1\}$. That is, part I is the portion of the B-net that are labled on layer l attached to each vertex of t, $0 \leq l \leq 3r + 1$.

Part II is the union of the collections $B_2(i, j) = \{\alpha : \alpha_k + \alpha_l \leq r + [(r + 1)/2]\} \backslash (B_1(i) \cup B_1(j)), i, j = 1, 2, 3, 4$ and $i < j$. Each $B_2(i, j)$ is a portion of the B-net on layer l around edge $\langle \mathbf{y}_i, \mathbf{y}_j \rangle, 0 \leq l \leq r + [(r + 1)/2]$. Associated with this portion, let $J_2 = \{C_1\alpha : \alpha \in B_2(1, 2)\}$, recalling that C_i is defined in §3.2.

Part III is the union of the collections $B_3(i, j) = \{\alpha : \alpha_k \leq r, \alpha_l \leq r, \alpha_k + \alpha_l \geq r + 1 + [(r + 1)/2]\} \backslash (B_1(i) \cup B_1(j)), i, j = 1, 2, 3, 4$ and $i < j$. Associated with this part, let $J_3 = \{C_1\alpha : \alpha \in B_3(1, 2)\}$.

Part IV is the union of collections $B_4(i, j) = B_{4,1}(i, j) \cup B_{4,2}(i, j) = \{\alpha : 2r + 2 \leq \alpha_i \leq 3r + 1, \alpha_l = r - 2m, \alpha_k = r + m + 1, m = 0, \cdots, [r/2]\} \cup \{\alpha : 2r + 2 \leq \alpha_j \leq 3r + 1, \alpha_l = r - 2m, \alpha_k = r + m + 1, m = 0, \cdots, [r/2]\}, i, j = 1, 2, 3, 4$ and $i < j$. Associated with this part of B-net, we let $J_4 = \{C_1\alpha : \alpha \in B_{4,1}(1, 2)\}$.

Part V consists of some portion of the B-net on layer l near each facet of t, $0 \le l \le r$; i.e., it is the union of the collections $B_5(i,j,k) = \{\alpha : \alpha_l \le r\} \setminus (B_1(i) \cup B_1(j) \cup B_1(k) \cup B_2(i,j) \cup B_2(i,k) \cup B_2(j,k) \cup B_3(i,j) \cup B_3(i,k) \cup B_3(j,k) \cup B_4(i,j) \cup B_4(j,k) \cup B_4(i,k), i < j < k$ and $i,j,k = 1,2,3,4$. Associated with this part, we let $J_{5,1} = \{C_1\alpha : \alpha \in B_5(1,2,3)$ and $\alpha_1 \ge 2r{+}2\}$, $J_{5,2} = \{C_2\alpha : \alpha \in B_5(1,2,3)$ and $\alpha_1 \le 2r+1, \alpha_2 \ge 2r+2\}$, and $J_{5,3} = \{C_3\alpha : \alpha \in B_5(1,2,3)$ and $\alpha_1 \le 2r+1, \alpha_2 \le 2r+1\}$.

The last part, part VI, is the collection $B_6(t) = \{\alpha : \alpha_i \ge r+1, i = 1,2,3,4\}$. Associated with the last part, let $J_6 = \{C_4\alpha : \alpha \in B_6(t)\}$.

Let us use the following two examples to illustrate how to divide these six parts of the B-net on t.

Example 3.3. Let $r = 1$. Consider the underlying index set of the B-net of polynomial of total degree ≤ 9. The six parts are described as follows. Part I is the union of collection $B_1(i) = \{\beta : \beta_i \ge 5\}$, $i = 1,2,3,4$, and let $J_1 = \{ (l,m,n) : l+m+n \le 4\}$. Part II is the union of collections $B_2(i,j) = \{ \beta : \beta_i + \beta_j \le 2\} \setminus (B_1(i) \cup B_1(j)), i,j = 1,2,3,4$ and $i < j$ and let $J_2 = \{(4,1,0)$, $(4,0,1)$, $(3,2,0)$, $(3,1,1)$, $(3,0,2)$, $(4,2,0)$, $(4,1,1)$, $(4,0,2)\}$. In this case Part III and IV are empty. Part V is the union of collections $B_5(i,j,k) = \{\beta : \beta_l \le 1\} \setminus (B_1(i) \cup B_1(j) \cup B_1(k) \cup B_2(ij) \cup B_2(ik) \cup B_2(jk)), i < j < k$ and $J_{5,1} = J_{5,2} = \{(2,2,1)\}$, $J_{5,3} = J_{5,1} \cup \{(3,3,0),(3,2,1),(2,3,1),(3,3,1)\}$. The last part is the remaining portion of the B-net, namely: $B_6 = \{\beta : \beta_i \ge 2, i = 1,2,3,4\}$ and $J_6 = \{(2,2,2),(3,2,2),(2,3,2),(2,2,3)\}$. The cardinalities of these parts are listed below:

$$\begin{aligned}
&\#\text{part I} &&= 4 \times 35\\
&\#\text{part II} &&= 6 \times 8\\
&\#\text{part V} &&= 4 \times 7\\
&\#\text{part VI} &&= 4\\
&\ \text{Total} &&= 220
\end{aligned}$$

which is the dimension of $\mathbb{P}_9$.

Example 3.4. Let $r = 2$. Consider the underlying index set of the B-net of a polynomial of total degree ≤ 15. The six parts are described as follows. Part I is the union of the collections $B_1(i) = \{\alpha : \alpha_i \ge 8\}$, $i = 1,2,3,4$ and let $J_1 = \{(l,m,n) : l+m+n \le 7\}$. Part II is the union of the collections $B_2(i,j) = \{\alpha : \alpha_i + \alpha_j \le 3\} \setminus (B_1(i) \cup B_1(j))$, $i,j = 1,2,3,4$ and $i < j$. Let $J_2 = \{(7,1,0),(7,0,1)$, $(7,2,0),(7,0,2)$, $(7,1,1),(7,3,0),(7,2,1)$, $(7,1,2)$, $(7,0,3)$, $((6,2,0)$, $(6,1,1),(6,0,2)$, $(6,3,0),(6,2,1)$, $(6,1,2)$, $(6,0,3)$, $(5,3,0)$, $(5,2,1),(5,1,2)$, $(5,0,3)\}$. Part III is the union of the collections $B_3(i,j) = \{\alpha : \alpha_k = 2, \alpha_l = 2\} \setminus (B_2(i,j) \cup B_1(i) \cup B_1(j))$, and let $J_3 = \{(4,2,2),(5,2,2)\}$. Part IV is the union of collections $B_4(i,j) = \{\alpha :$

$\alpha_i = 7, \alpha_j = 4, \alpha_k = 3, \alpha_l = 1\} \cup \{a_\alpha : \alpha_i = 6,\ \alpha_j = 5, \alpha_k = 3,\ \alpha_l = 1\}$ $\cup\{a_\alpha : \alpha_i = 4,\ \alpha_j = 7,\ \alpha_k = 3, \alpha_l = 1\} \cup \{a_\alpha : \alpha_i = 5, \alpha_j = 6, \alpha_k = 3, \alpha_l = 1\}$. Part V is the union of collections $B_5(i,j,k) = \{\alpha : \alpha_l \leq 2\} \setminus (B_1(i) \cup B_1(j) \cup B_1(k) \cup B_2(i,j) \cup B_2(i,k) \cup B_2(j,k) \cup B_3(i,j) \cup B_3(i,k) \cup B_3(j,k) \cup B_4(i,j) \cup B_4(i,k) \cup B_4(j,k))$. Associated the fifth part, $J_{5,1} = J_{5,2} = \{(4,4,0), (5,4,0), (4,5,0), (4,4,1), (4,3,2), (3,3,2), (3,4,2)\}$ and $J_{5,3} = J_{5,1} \cup \{(5,5,0), (5,4,1), (4,5,1), (5,5,1), (5,3,2), (5,4,2), (4,4,2), (4,5,2), (3,5,2), (5,5,2)\}$. Part VI is the collection $B_6 = \{\alpha : \alpha_i \geq 3, i = 1,2,3,4\}$ and $J_6 = \{(3,3,3), (3,4,3), (3,3,4), (4,3,3), (3,5,3), (3,4,4), (3,3,5), (4,3,4), (4,4,3), (5,3,3), (3,6,3), (3,5,4), (3,4,5), (3,3,6), (4,5,3), (4,4,4), (4,3,5), (5,4,3), (5,3,4), (6,3,3)\}$. The cardinalities of these six parts are as follows:

$$\begin{array}{ll}
\#\text{Part I} = & 4 \times 120 \\
\#\text{Part II} = & 6 \times 20 \\
\#\ \text{Part III \& IV} = & 6 \times 12 \\
\#\ \text{Part V} = & 4 \times 31 \\
\#\text{Part VI} = & 20
\end{array}$$

The sum of these cardinalities is 816 which is equal to the dimension of $\mathbb{P}_{15}$.

Again for t, let $e = \langle \mathbf{v}_{t,i}, \mathbf{v}_{t,j} \rangle$, $f_1 = \langle \mathbf{v}_{t,i}, \mathbf{v}_{t,j}, \mathbf{v}_{t,k} \rangle$, and $f_2 = \langle \mathbf{v}_{t,i}, \mathbf{v}_{t,j}, \mathbf{v}_{t,l} \rangle$ of t, $i < j$. Then we denote

$$J_{i,j}(t) = \begin{cases}
J_3 & \text{if } e \text{ is a nonsingular interior edge or if } e \text{ is a boundary} \\
 & \text{ edge such that } f_1 \text{ and } f_2 \text{ are interior facets;} \\
J_3 \cup J_4 \cup \bar{J}_4 & \text{if } e \text{ is a singular edge and } t = T_{e,1}\ ; \\
J_3 \cup J_4 & \text{if } e \text{ is a boundary edge and one of } f_1, f_2 \text{ is a boundary} \\
 & \text{ facet and the other is interior facet, or if } f_1 \text{ or } f_2 \text{ is a} \\
 & \text{ singular facet at } e \text{ and } t = T_{f,1}; \\
J_4 \cup \bar{J}_4 & \text{if } e \text{ is a singular edge and } t = T_{e,3}; \\
\emptyset & \text{if } e \text{ is a singular edge and } t = T_{e,2} \text{ or } T_{e,4}\ \text{, or if } f_1 \text{ or} \\
 & \text{ } f_2 \text{ is a singular facet and } t = T_{f,2}.
\end{cases}$$

In the following, we outline the procedure for constructing the fundamental vertex splines in $S^r_{6r+3}(\triangle)$. In general, we will consider five types of vertex splines of interest. They are required to satisfy the following specifications of interpolatory parameters and smoothness conditions.

(I) For each vertex $\mathbf{v} \in \mathcal{V}$ of $\triangle$ and $\gamma \in J_1$, let $V^\gamma_\mathbf{v}$ be a piecewise polynomial function of degree $6r+3$ satisfying:

$$D^\alpha V^\gamma_\mathbf{v}(\mathbf{u}) = \delta_{\alpha,\gamma}\delta_{\mathbf{v},\mathbf{u}}, \qquad \alpha \in J_1, \mathbf{u} \in \mathcal{V}; \tag{I.1}$$

$$V^\gamma_\mathbf{v} \text{ is } C^{3r+1} \text{ at each vertex } \mathbf{v} \in \mathcal{V}; \tag{I.2}$$

(I.3) $$D_e^\alpha V_{\mathbf{v}}^\gamma\Big|_{t_{e,1}}(\mathbf{v}_{e,1}) = 0, \qquad \alpha \in J_2, e \in \mathcal{E};$$

(I.4) $$V_{\mathbf{v}}^\gamma \text{ is } C^{r+[(r+1)/2]} \text{ around each edge } e \in \mathcal{E};$$

(I.5) $$D_{t,ij}^\alpha V_{\mathbf{v}}^\gamma\Big|_t(\mathbf{v}_{t,i}) = 0, \qquad \alpha \in J_{i,j}(t), i < j, i, j = 1, 2, 3, 4,\ t \in \mathcal{T};$$

(I.6) $$D_{f,l}^\alpha V_{\mathbf{v}}^\gamma\Big|_{T_{f,1}}(\mathbf{v}_{f,l}) = 0, \qquad \alpha \in J_{5,l}, l = 1, 2, 3,\ f \in \mathcal{F};$$

(I.7) $$V_{\mathbf{v}}^\gamma \text{ satisfies the } C^r \text{ smoothness conditions;}$$

(I.8) $$D_t^\alpha V_{\mathbf{v}}^\gamma\Big|_t(\mathbf{v}_{t,4}) = 0, \qquad \alpha \in I_6, t \in \mathcal{T}.$$

As usual, the symbol $\delta_{\alpha,\gamma}$ or $\delta_{\mathbf{v},\mathbf{u}}$ is the Kronecker delta.

(II) For each edge $e \in \mathcal{E}$ and $\gamma \in J_2$, let V_e^γ be a piecewise polynomial function of degree $6r + 3$ satisfying:

(II.1) $$D^\alpha V_e^\gamma(\mathbf{u}) = 0, \qquad \alpha \in J_1, \mathbf{u} \in \mathcal{V};$$

(II.2) $$V_e^\gamma \text{is } C^{3r+1} \text{ at each vertex } \mathbf{v} \in \mathcal{V};$$

(II.3) $$D_d^\alpha V_e^\gamma\Big|_{t_{d,1}}(\mathbf{v}_{d,1}) = \delta_{\alpha,\gamma}\delta_{e,d}, \qquad \alpha \in J_2, d \in \mathcal{E};$$

(II.4) $$V_e^\gamma \text{ is } C^{r+[(r+1)/2]} \text{ around each edge } d \in \mathcal{E};$$

(II.5) $$D_{t,ij}^\alpha V_e^\gamma\Big|_t(\mathbf{v}_{t,i}) = 0, \qquad \alpha \in J_{i,j}(t), t \in \mathcal{T};$$

(II.6) $$D_{f,l}^\alpha V_e^\gamma\Big|_{t_{f,1}}(\mathbf{v}_{f,l}) = 0, \qquad \alpha \in J_{5,l}, l = 1, 2, 3, f \in \mathcal{F};$$

(II.7) $$V_e^\gamma \text{ satisfies the } C^r \text{ smoothness conditions;}$$

(II.8) $$D_t^\alpha V_e^\gamma\Big|_t(\mathbf{v}_{t,4}) = 0, \qquad \alpha \in J_6, t \in \mathcal{T}.$$

(III) For each edge $e = \langle \mathbf{v}_{t,i}, \mathbf{v}_{t,j} \rangle$ of $t \in \mathcal{T}$ and $\gamma \in J_{ij}(t)$, let $V_{t,ij}^\gamma$ be a piecewise polynomial function of degree $6r + 3$ defined on $\triangle$ and satisfying the following:

(III.1) $$D^\alpha V_{t,ij}^\gamma(\mathbf{u}) = 0, \qquad \alpha \in J_1, \mathbf{u} \in \mathcal{V};$$

$$V_{t,ij}^{\gamma} \text{ is } C^{3r+1} \text{ at each vertex } \mathbf{v} \in \mathcal{V}; \tag{III.2}$$

$$D_d^{\alpha} V_{t,ij}^{\gamma}\Big|_{t_{d,1}}(\mathbf{v}_{d,1}) = 0, \qquad \alpha \in J_2, d \in \mathcal{E}; \tag{III.3}$$

$$V_{t,ij}^{\gamma} \text{ is } C^{r+[(r+1)/2]} \text{ around each edge } d \in \mathcal{E}; \tag{III.4}$$

$$D_{s,kl}^{\alpha} V_{t,ij}^{\gamma}\Big|_{s}(\mathbf{v}_{s,i}) = \delta_{\alpha,\gamma}\delta_{t,s}, \qquad \alpha \in J_{k,l}(s), s \in \mathcal{T}; \tag{III.5}$$

$$D_{f,l}^{\alpha} V_{t,ij}^{\gamma}\Big|_{T_{f,1}}(\mathbf{v}_{f,l}) = 0, \qquad \alpha \in J_{5,l}, l = 1,2,3, f \in \mathcal{F}; \tag{III.6}$$

$$V_{t,ij}^{\gamma} \text{ satisfies } C^{r} \text{ smoothness conditions;} \tag{III.7}$$

$$D_s^{\alpha} V_{t,ij}^{\gamma}\Big|_{t}(\mathbf{v}_{t,4}) = 0, \qquad \alpha \in J_6, t \in \mathcal{T}. \tag{III.8}$$

(IV) For each facet $f \in \mathcal{F}$ any for each index $\gamma \in J_{5,l}, l = 1, 2, 3$, let $V_{f,l}^{\gamma}$ be a piecewise polynomial function of degree $6r+3$ satisfying the following:

$$D^{\alpha} V_{f,l}^{\gamma}(\mathbf{u}) = 0, \qquad \alpha \in J_1, \mathbf{u} \in \mathcal{V}; \tag{IV.1}$$

$$V_{f,l}^{\gamma} \text{ is } C^{3r+1} \text{ at each vertex } \mathbf{v} \in \mathcal{V}; \tag{IV.2}$$

$$D_d^{\alpha} V_{f,l}^{\gamma}\Big|_{t_{d,1}}(\mathbf{v}_{d,1}) = 0, \qquad \alpha \in J_2, d \in \mathcal{E}; \tag{IV.3}$$

$$V_{f,l}^{\gamma} \text{ is } C^{r+[(r+1)/2]} \text{ around each edge } d \in \mathcal{E}; \tag{IV.4}$$

$$D_{t,ij}^{\alpha} V_{f,l}^{\gamma}\Big|_{t}(\mathbf{v}_{t,i}) = 0, \qquad \alpha \in J_{i,j}(t), t \in \mathcal{T}; \tag{IV.5}$$

$$D_{f,m}^{\alpha} V_{f,l}^{\gamma}\Big|_{t_{f,m}}(\mathbf{v}_{f,m}) = \delta_{\alpha,\gamma}\delta_{l,m}, \quad \alpha \in J_{5,m},\ m = 1,2,3, f \in \mathcal{F}; \tag{IV.6}$$

$$V_{f,l}^{\gamma} \text{ satisfies the } C^{r} \text{ smoothness conditions;} \tag{IV.7}$$

$$D_t^{\alpha} V_{f,l}^{\gamma}\Big|_{t}(\mathbf{v}_{t,4}) = 0, \qquad \alpha \in J_6, t \in \mathcal{T}. \tag{IV.8}$$

(V) For each tetrahedron $t \in \mathcal{T}$ and $\gamma \in J_6$, let V_t^γ be a piecewise polynomial function of degree $6r+3$ satisfying the following:

(V.1) $$D^\alpha V_t^\gamma(\mathbf{u}) = 0, \qquad \alpha \in J_1, \mathbf{u} \in \mathcal{V};$$

(V.2) $$V_t^\gamma \text{ is } C^{3r+1} \text{ at each vertex } \mathbf{v} \in \mathcal{V};$$

(V.3) $$D_d^\alpha V_t^\gamma\Big|_{t_{d,1}}(\mathbf{v}_{d,1}) = 0, \qquad \alpha \in J_2, d \in \mathcal{E};$$

(V.4) $$V_t^\gamma \text{ is } C^{r+[(r+1)/2]} \text{ around each edge } d \in \mathcal{E};$$

(V.5) $$D_{s,ij}^\alpha V_t^\gamma\Big|_s(\mathbf{v}_{s,i}) = 0, \qquad \alpha \in J_{i,j}(s), s \in \mathcal{T};$$

(V.6) $$D_{f,m}^\alpha V_t^\gamma\Big|_{T_{f,m}}(\mathbf{v}_{f,m}) = 0, \qquad \alpha \in J_{5,m}, m = 1,2,3, f \in \mathcal{F};$$

(V.7) $$V_t^\gamma \text{ satisfies the } C^r \text{ smoothness conditions;}$$

(V.8) $$D_s^\alpha V_t^\gamma\Big|_s(\mathbf{v}_{s,4}) = \delta_{\alpha,\gamma}\delta_{t,s}, \qquad \alpha \in J_6, s \in \mathcal{T}.$$

The outline for constructing these vertex splines can be described in following five steps. Let V stand for one of the above vertex splines and $t = \langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4 \rangle$ be a tetrahedron in $\triangle$.

Step 1. Determination of B-nets with indices in part I.

The B-coefficients of $V|_\delta$ indexed in $A_i^{6r+3}J_1$ are simply zero when V is required to satisfy $D^\alpha V(\mathbf{y}_i) = 0$. When V is required to satisfy the interpolation conditions $D^\alpha V(\mathbf{y}_i) = \delta_{\alpha,\gamma}$, we first convert the partial derivatives D^α at $\mathbf{y}_i$ into derivatives relative to t at $\mathbf{y}_i$, and then use the values of $D_{t,i}^\beta V|_t(\mathbf{y}_i)$ to determine the B-coefficients of $V|_t$ with underlying indices in $A_i^{6r+3}J_1$.

Step 2. Determination of B-nets with indices in part II.

Let $e \subset t$ be an edge. The B-coefficients of $V|_t$ with indices on layer l around e located in part II can be directly obtained from the requirements in one of (I.3)–(V.3) if $t = t_{e,1}$, $0 \le l \le r + [(r+1)/2]$. Otherwise, they can be determined by using Lemma 3.3.3 and the corresponding portions of the B-coefficients of $V|_{t_{e,1}}$. In the other words, first compute $D_e^\alpha V|_t$ from $V|_{t_{e,1}}$ and then use the resulting directional derivatives to determine the remaining portion of B-coefficients of $V|_t$ on layer l

around e, $0 \le l \le r+[(r+1)/2]$. Alternatively, we can apply the smoothness conditions on two adjacent simplices sharing a common edge in [39] to find the portion of the B-coefficients of $V|_t$ from the corresponding B-coefficients of $V|_{t_{e,1}}$.

Step 3. Determination of B-nets with indices in part III.

Case 1: Suppose that both $\langle \mathbf{y}_i, \mathbf{y}_j, \mathbf{y}_k \rangle$ and $\langle \mathbf{y}_i, \mathbf{y}_j, \mathbf{y}_l \rangle$ are not singular facets at $e = \langle \mathbf{y}_i, \mathbf{y}_j \rangle$ or suppose that e is a singular edge and $t = t_{e,1}$. Then we directly apply one of (I.5)–(V.5) to obtain the portion of the B-coefficients of $V|_t$ indexed in $B_3(i, j)$.

Case 2: Suppose that $e = [\mathbf{y}_i, \mathbf{y}_j]$ is a singular edge and $t \ne t_{e,1}$ or suppose that $f = \langle \mathbf{y}_i, \mathbf{y}_j, \mathbf{y}_k \rangle$ is a singular facet at e and $t \ne T_{f,1}$. We will obtain the portion of the B-coefficients of $V|_t$ with indices in $B_3(i, j)$ by using the smoothness conditions in Lemma 3.1 or Lemma 3.5 from the corresponding part of the B-coefficients of $V|_{t'}$, where t' is the neighboring tetrahedron in $\mathcal{T}$ sharing a common facet f with t.

Step 4. Determination of B-nets with indices in Part IV & V.

Case 1: Suppose that $f = \langle \mathbf{y}_i, \mathbf{y}_j, \mathbf{y}_k \rangle$ is a bound ary facet or suppose that f is a singular facet at $\langle \mathbf{y}_i, \mathbf{y}_j \rangle$ and $t = T_{f,1}$. Then the B-coefficients of $V|_t$ with indices in the one-fourth portion of parts IV and V closest to $\langle \mathbf{y}_i, \mathbf{y}_j, \mathbf{y}_k \rangle$ are obtained by applying the requirement in one of (I.6)–(V.6).

Case 2: Suppose that a facet $f = \langle \mathbf{y}_i, \mathbf{y}_j, \mathbf{y}_k \rangle$ is singular at an edge $\langle \mathbf{y}_i, \mathbf{y}_j \rangle$ and $t = T_{f,2}$. Then the B-coefficients of $V|_t$ with indices in $B_4(i, j)$ will be obtained by using Lemma 3.5 and the corresponding part of the B-coefficients of $V|_{T_{f,1}}$.

Case 3: This is the remaining case. To determine the remaining B-coefficients of $V|_t$ with indices in parts IV and V, we need to use all of (I.6) and (I.7), etc. and apply Lemma 3.3.4, 3.3.5, or 3.3.6 accordingly. Let t' be a tetrahedron sharing $f = \langle \mathbf{y}_i, \mathbf{y}_j, \mathbf{y}_k \rangle$ with t. Consider the B-coefficients of $V|_{t \cup t'}$ on layer $3r + 2$ attached to $\mathbf{y}_i$. To find the undetermined B-coefficients on this layer located in parts IV and V, we may use the same technique as mentioned in Step 4 in §3.4. The undetermined B-coefficients of $V|_{t \cup t'}$ on the other layer l where $3r + 3 \le l \le 4r + 1$ attached to y_i located in parts IV and V can be obtained in a similar manner. Similarly, the undetermined B-coefficients of $V|_{t \cup t'}$ on layer l attached at other vertices $\mathbf{y}_j$ and $\mathbf{y}_k$ located in parts IV and V can be found too.

Step 5. Determination of B-nets in part VI.

After the all B-coefficients of $V|_t$ in parts I–V are determined, we can determine the B-coefficients of $V|_t$ with indices in part VI by using the requirements in one of (I.8)–(V.8).

In the construction, we know that the support of the vertex splines $V_{\mathbf{v}}^{\gamma}$ is the union of all tetrahedra in $\mathcal{T}$ with $\mathbf{v}$ as their common vertex; the support of V_e^{γ} is the union of all tetrahedra of $\triangle$ with e as their common edge; the support of $V^{\gamma}_{t,ij}$ is some part of the union of all tetrahedra sharing edge $\langle \mathbf{v}_{t,i}, \mathbf{v}_{t,j}\rangle$; the support of $V^{\gamma}_{f,l}$ is the union of all the tetrahedra in $\triangle$ with f as their common facet and the support of V^{γ}_{t} is the tetrahedron t.

Consider the space

$$\begin{aligned}\widehat{S}^r_{6r+3}(\triangle) = \text{span} \quad & \{V^{\gamma}_{\mathbf{v}}, \gamma \in J_1, \mathbf{v} \in \mathcal{V}\} \cup \{V^{\gamma}_{e}, \gamma \in J_2, e \in \mathcal{E}\} \\ & \cup \{V^{\gamma}_{f,i}, \gamma \in J_{5,i}, i = 1,2,3, f \in \mathcal{F}\} \\ & \cup \{V^{\gamma}_{t,ij} : \gamma \in J_{ij}(t), t \in \mathcal{T}\} \\ & \cup \{V^{\gamma}_{t} : \gamma \in J_6, t \in \mathcal{T}\}\,.\end{aligned}$$

Clearly, $\widehat{S}^r_{6r+3}(\triangle)$ is a subspace of $S^r_{6r+3}(\triangle)$. For each sufficiently smooth function g, we define

$$\begin{aligned} Lg(\mathbf{x}) = & \sum_{\mathbf{v}\in\mathcal{V}} \sum_{\gamma\in J_1} D^{\gamma} g(\mathbf{v}) V^{\gamma}_{\mathbf{v}}(\mathbf{x}) \\ & + \sum_{e\in\mathcal{E}} \sum_{\gamma\in J_2} D^{\gamma}_{e} g(\mathbf{v}_{e,1}) V^{\gamma}_{e}(\mathbf{x}) \\ & + \sum_{t\in\mathcal{T}} \sum_{1\le i<j\le 4} \sum_{\gamma\in J_{ij}(t)} D^{\gamma}_{t,ij} g(\mathbf{v}_{t,i}) V^{\gamma}_{t,ij}(\mathbf{x}) \\ & + \sum_{f\in\mathcal{F}} \sum_{i=1}^{3} \sum_{\gamma\in J_{5,i}} D^{\gamma}_{f,i} g(\mathbf{v}_{f,i}) V^{i}_{f}(\mathbf{x}) \\ & + \sum_{t\in\mathcal{T}} \sum_{\gamma\in J_6} D^{\gamma}_{t} g(\mathbf{v}_{t,4}) V^{\gamma}_{t}(\mathbf{x}). \end{aligned} \tag{3.5.1}$$

We are now ready to derive some properties of the super spline space $\widehat{S}^r_{6r+3}(\triangle)$.

LEMMA 3.5.1 *$Lp = p$ for any polynomial p of total degree $6r+3$.*

Proof. We use mathematical induction on the number of tetrahedra in $\triangle$ to prove this lemma. For $n = 1$, L is an interpolatory operator based on t, the only tetrahedron of $\triangle$. Since the sets of interpolation conditions associated with each vertex of t are lower sets and induce a partition of Λ_{6r+3}, we see that $Lp = p$ for all p of total degree $6r+3$ by Proposition 3.1. Suppose now that the result holds for $m = \#\{t : t \in \triangle\}$. Let $\#\{t : t \in \triangle\} = m+1$ and set $\triangle = \{t_i : i = 1, \cdots, m+1\}$. By relabeling if necessary, assume that $t_{m+1} = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\rangle$ has at least one boundary facet, and for the time being, assume that it has only one interior facet $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\rangle$, say. Let $\triangle' = \{t_i : i = 1, \cdots, m\} = \triangle \backslash \{t_{m+1}\}$. Observing the uniqueness in Lemma 3.3.5 and applying Theorem 4.1.3 in [39], we can see that the smoothness of Lp across $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\rangle$ may be rewritten as appropriate interpolation conditions (directional derivatives relative to the edges and relative to the facet) such that $L_{\triangle}p\Big|_{\triangle'} = L_{\triangle'}p$ and $L_{\triangle}p\Big|_{t_{m+1}} = L_{t_{m+1}}p$, where linear operators $L_{\triangle}, L_{\triangle'}, L_{t_{m+1}}$ are restrictions of L on $\triangle, \triangle', t_{m+1}$, respectively. By the induction hypothesis, we have $L_{\triangle}p\Big|_{\triangle'} = p$ and $L_{\triangle}p\Big|_{t_{m+1}} = p$. Hence, $Lp = p$ on $\triangle$. The proof is similar if t_{m+1} contains two or three interior facets. This completes the proof.

If Lg is interpreted as

$$\begin{aligned} Lg(\mathbf{x}) \ &= \sum_{\mathbf{v}\in\mathcal{V}}\sum_{\gamma\in J_1} D^{\gamma}g(\mathbf{v})V_{\mathbf{v}}^{\gamma}(\mathbf{x}) \\ &+\sum_{e\in\mathcal{E}}\sum_{\gamma\in J_2} D_e^{\gamma}g\Big|_{t_{e,1}}(\mathbf{v}_{e,1})V_e^{\gamma}(\mathbf{x}) \\ &+\sum_{t\in\mathcal{T}}\sum_{1\le i<j\le 4}\sum_{\gamma\in J_{ij}(t)} D_{t,ij}^{\gamma}g\Big|_{t}(\mathbf{v}_{t,i})V_{t,ij}^{\gamma}(\mathbf{x}) \\ &+\sum_{f\in\mathcal{F}}\sum_{i=1}^{3}\sum_{\gamma\in J_{5,i}} D_{f,i}^{\gamma}g\Big|_{t^{f,1}}(\mathbf{v}_{f,i})V_f^{i}(\mathbf{x}) \\ &+\sum_{t\in\mathcal{T}}\sum_{\gamma\in J_6} D_t^{\gamma}g\Big|_{t}(\mathbf{v}_{t,4})V_t^{\gamma}(\mathbf{x}), \end{aligned}$$

we can improve the above lemma as follows.

LEMMA 3.5.2 *$Lg = g$ for any $g \in \widehat{S}^r_{6r+3}(\triangle)$.*

Let S denote a spline space defined by

$$\begin{aligned} S = \{s \in S^r_{6r+3} : \ & s \in C^{3r+1} \text{at each vertex of } \triangle \\ & \text{and } s \in C^{r+[(r+1)/2]} \text{around each edge.} \end{aligned}$$

Any function s in S is called a super spline since s satisfies extra smoothness conditions at each vertex and around each edge of $\triangle$. Actually, this spline space S is spanned by the fundamental vertex splines constructed above. This result is the consequence of Lemma 3.5.2.

THEOREM 3.5.1 *The collection*

$$\begin{aligned} \mathcal{B} := \ & \{V_{\mathbf{v}}^{\gamma} : \mathbf{v} \in \mathcal{V}, \gamma \in J_1\} \cup \{V_e^{\gamma} : \gamma \in J_2, e \in \mathcal{E}\} \\ & \cup\{V_{f,i}^{\gamma} : i = 1,2,3, \gamma \in J_{5,i}, f \in \mathcal{F}\} \\ & \cup\{V_{t,ij}^{\gamma} : \gamma \in J_{ij}(t), t \in \mathcal{T}, i<j, i,j = 1,2,3,4\} \\ & \cup\{V_t^{\gamma} : \gamma \in J_6, t \in \mathcal{T}\} \end{aligned}$$

is a basis of S. Therefore $S = \widehat{S}^r_{6r+3}(\triangle)$.

Let $G \subset \sup\{t : t \in \triangle\}$ and for $g \in C^k(G)$, denote

$$\|D^k g\| = \max_{|\alpha|=k} \|D^{\alpha}g\|_{C(G)}$$

and

$$\operatorname{dist}(f,S)=\inf_{s\in S}\|f-s\|.$$

For the given $\triangle$, let $|\triangle|$ denote the maximum of the diameter of all $t\in\triangle$. Clearly, vertex splines in $S_d^r, d\geq 6r+3$, can be constructed by using the same idea as above and similar results may be obtained. Therefore, we can state a more general theorem.

THEOREM 3.5.2. *Let $d\geq 6r+3$. There exists a linear operator L with range $\widehat{S}_d^r$ such that*

$$\|Lg-g\|\leq C\|D^{d+1}g\||\triangle|^{d+1}$$

for all sufficiently smooth functions g, where C is a constant independent of g and $|\triangle|$ but is dependent on the geometry of $\triangle$. In particular, for $d=6r+3$, L can be chosen to be (3.5.1). Consequently

$$\operatorname{dist}\,(g,S_d^r)\leq C\|D^{d+1}g\||\triangle|^{d+1}.$$

Proof. For $d\geq 8r+1$, this result was proved in [88] and [39]. Here, we only prove the special and most important case where $d=6r+3$, since a similar argument yields the desired result for $6r+3<d<8r+1, r>1$. Fix a point $\mathbf{x}\in G$ and consider the linear functional

$$F(g)=Lg(\mathbf{x})-g(\mathbf{x}).$$

It is easy to see that F satisfies the following:

(i) $|F(g)|\leq K_1\sum_{j=0}^{6r+3}\|D^jg\||\triangle|^j$

(ii) $F(p)=0$ for all $p\in\mathbb{P}_{6r+3}$,

where the constant K_1 may be dependent on the geometry of $\triangle$.

Indeed, (ii) follows from Lemma 3.5.1. Since each vertex spline is a bounded function whose bound may depends on the geometry of $\triangle$, (i) holds if $|\triangle|=1$. If $|\triangle|<1$, by letting $\tilde{g}(\mathbf{y})=g(|\triangle|\mathbf{y})$, we see that

$$\begin{aligned} |F(g)| &= |F(\tilde{g})| \\ &\leq K_1 \sum_{j=0}^{6r+3} \|D^j \tilde{g}\|_{\tilde{G}} \\ &= K_1 \sum_{j=0}^{6r+3} \|D^j g\| |\triangle|^j. \end{aligned}$$

By a result in Bramble and Hilbert [24], there exist a constant K independent of g, $\mathbf{x}$, and $|\triangle|$ such that

$$|Lg(\mathbf{x}) - g(\mathbf{x})| \leq K \|D^{6r+4} g\| |\triangle|^{6r+4}.$$

Therefore, we have established the theorem.

As we know $\widehat{S}^r_{6r+3}$ is a proper subspace of S^r_{6r+3}. In fact, as a consequence of Theorem 3.5.1, the exact dimensions of $\widehat{S}^r_9$ and $\widehat{S}^2_{15}$ can be written down as follows.

THEOREM 3.5.3.

$$\dim \widehat{S}^1_9 = 35N_v + 8N_e + 4N_t + 7N_f,$$

and

$$\dim \widehat{S}^2_{15} = 120N_v + 20N_e + 40N_{be} + 20N_s + 20N_t + 31N_f$$

where $N_v, N_b, N_e, N_s, N_f, N_t$ *denote the number of vertices, boundary edges, edges, singular edges, facets, and tetrahedra of* $\triangle$*, respectively.*

3.6. Vertex Splines on Mixed Partition Regions

We are now going to study how vertex splines on a region $R \subset \mathbb{R}^3$ of interest are constructed. In particular, we consider that R has been partitioned into patches (tetrahedra, prisms, and parallelepipeds). This kind of partitioned region is called mixed partition region. The precise definition will be given soon. For a given mixed partition consisting of all these three types of geometric configurations and for $r \geq 0$, the degree d of polynomials which will be used in the construction will be assumed to satisfy $d \geq 8r+1$, where r denotes as usual the order of smoothness.

Let us begin with two definitions.

DEFINITION 3.1. A region $R \subset \mathbb{R}^3$ which is the union of a finite number of tetrahedra, prisms and parallelepipeds denoted by $t_1, \cdots, t_N$ is called a mixed partitioned region if it satisfies:

(1) $\operatorname{int}(t_i) \cap \operatorname{int}(t_j) = \emptyset, i \neq j$;

(2) either $t_i \cap t_j = \emptyset$ or $t_i \cap t_j$ is a common vertex, or common edge, or a common facet of t_i and t_j.

We denote by $\triangle = \{t_i : i = 1, \cdots, N\}$ a mixed partition of R.

For integers $r, d \in \mathbb{Z}_+$ with $0 \leq r < d$, let

$$S_d^r(\triangle) = \{g \in C^r(R) : g\Big|_{t_i} \in \pi_d(t_i), i = 1, \cdots, N\}$$

be the multivariate splines space of degree d and smoothness r on D, where $\pi(t_i)$ is the polynomial spaces of "degree" $\leq d$ as defined in §3.1, $\forall i$.

Splines which will be considered and constructed in this section are piecewise polynomial functions in a subspace of $S_d^r(\triangle)$ as defined below:

DEFINITION 3.2. Let

$$\begin{aligned}\widehat{S}_d^r(\triangle) = \{f \in S_d^r(\triangle) : \ & f \in C^{4r} \text{ at each vertex of } \triangle \\ & f \in C^{2r} \text{ around each edge of } \triangle\}.\end{aligned}$$

As before, $\widehat{S}_d^r$ will be called a super spline space and any spline $f \in \widehat{S}_d^r$ is called a super spline.

For convenience, we only consider the special and most important case where $d = 8r+1$. The other cases where $d > 8r+1$ can be treated essentially the same as the special case. In the following, we will subdivide the indices of the B-net of a polynomial on a tetrahedron (or prism or parallelepiped) into several parts. The

B-net of a spline on a patch (tetrahedron, or prism, or parallelepiped) with indices in different parts will be determined by different methods.

First, consider a tetrahedron $T_1 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle$. Denote by i, j, k, l distinct elements of $\{1, 2, 3, 4\}$. We divide the underlying indices of the B-net $\{a_\alpha : |\alpha| = 8r + 1\}$ of a polynomial on T_1 with B-coefficients a'_αs into four parts.

Part I is the union of collections $B_1(i) = A_i^{8r+1} I_1$, $i = 1, \cdots, 4$, where $I_1 = \{(l, m, n) : l + m + n \leq 4r\}$.

Part II is the union of the collections $B_2(i, j) = \{\alpha : \alpha_k + \alpha_l \leq 2r\} \backslash (B_1(i) \cup B_2(j)), i < j$ and $i, j = 1, 2, 3, 4$. Associated with this part, let $I_2 = \{C_1 \alpha : \alpha \in B_2(12)\}$ where C_1 is a map as defined in §3.2.

Part III is the union of collections $B_3(i, j, k) = \{\alpha : \alpha_l \leq r\} \backslash (B_1(i) \cup B_1(j) \cup B_1(k) \cup B_2(i, j) \cup B_2(i, k) \cup B_2(j, k))$, $i < j < k$, and $i, j, k = 1, 2, 3, 4$. Associated with this part, let $I_3 = \{C_1 \alpha : \alpha \in B_3(123)\}$.

Part IV is the collection of the remaining indices $B_4 = \{\alpha : \alpha_i \geq r + 1, i = 1, 2, 3, 4\}$. Associated with the last part, let $I_4 = \{C_1 \alpha : \alpha \in B_4\}$.

Next let $T_2 = \langle \mathbf{y}_1, \cdots, \mathbf{y}_6 \rangle$ be a prism. We divide the underlying indices of the B-net $\{\bar{a}_\beta : \beta_1 + \beta_2 + \beta_3 = n, 0 \leq \beta_4 \leq n\}$ of a polynomial on T_2 into four parts.

Part I is the union of collections $\bar{B}_1(i) = \{\bar{A}_i^{8r+1} I_1\}$, $i = 1, \cdots, 6$, where $I_1 = \{(l, m, n) : l + m + n \leq 4r\}$.

Part II is the union of collections $\bar{B}_2(i, i+3) = \{\beta : \beta_i \leq 2r\} \backslash (\bar{B}_1(i) \cup \bar{B}_1(i+3))$, $i = 1, 2, 3$, $\bar{B}_2(i, j) = \{\beta : \beta_k + \beta_4 \leq 2r\} \backslash \bar{B}_1(i) \cup \bar{B}_1(j))$, $i, j \in \{1, 2, 3\}$ and $\bar{B}_2(i, j) = \{\beta : \beta_k + 8r + 1 - \beta_4 \leq 2r\} \backslash \bar{B}_1(i) \cup \bar{B}_1(j))$, $i, j \in \{4, 5, 6\}$. Associated with this part, let $\bar{I}_{2,1} = \{C_1 \beta : \beta \in \bar{B}_2(1, 2)\}$ and $\bar{I}_{2,2} = \{C_1 \beta : \beta \in \bar{B}_2(1, 4)\}$.

Part III is the union of collections $\bar{B}_3(1, 2, 3) = \{\beta : \beta_4 \geq 6r + 1\} \backslash \bar{B}_1(1) \cup \bar{B}_1(2) \cup \bar{B}_1(3) \cup \bar{B}_2(1, 2) \cup \bar{B}_2(1, 3) \cup \bar{B}_2(2, 3))$, $\bar{B}_3(4, 5, 6) = \{\beta : \beta_4 \leq 2r\} \backslash \bar{B}_1(4) \cup \bar{B}_1(5) \cup \bar{B}_1(6) \cup \bar{B}_2(4, 5) \cup \bar{B}_2(4, 6) \cup \bar{B}_2(5, 6))$, $\bar{B}_3(i, j, i + 3, j + 3) = \{\beta : \beta_k \leq 2r\} \backslash (\bar{B}_1(i) \cup \bar{B}_1(j) \cup \bar{B}_1(i+3) \cup \bar{B}_1(j+3) \cup \bar{B}_2(ij) \cup \bar{B}_2(i+3, j+3) \cup \bar{B}_2(i, i+3) \cup \bar{B}_2(j, j+3)), i, j \in \{1, 2, 3\}$. Associated with this part, let $\bar{I}_{3,1} = \{C_1 \beta : \beta \in \bar{B}_3(1, 2, 3)\}$ and $\bar{I}_{3,2} = \{C_1 \beta : \beta \in \bar{B}_3(1, 2, 4, 5)\}$.

Part IV is the collection of the remaining indices $\bar{B}_4 = \{\beta : \beta_i \geq r + 1, i = 1, 2, 3; r + 1 \leq \beta_4 \leq 7r\}$. Associated with the last part, let $\bar{I}_4 = \{C_1 \beta : \beta \in \bar{B}_4\}$.

Finally, let $T_3 = \langle \mathbf{z}_1, \cdots, \mathbf{z}_8 \rangle$ be a parallelepiped. We again divide the underlying indices of the B-net $\{\tilde{a}_\beta : \beta \leq (8r + 1, 8r + 1, 8r + 1)\}$ of a polynomial on T_3 with B-coefficients $\tilde{a}$'s into four parts.

Part I is the union of collections $\tilde{B}_1(i) = \tilde{A}_i^{8r+1} I_1$, where $I_1 = \{(l, m, n) : l + m + n \leq 4r\}$, $i = 1, \cdots, 8$.

Part II is the union of the collections $\tilde{B}_2(i, i+4) = \{\tilde{A}_i^{8r+1}\eta : \eta = (i, j, k), i + j \leq 2r\}\backslash(\tilde{B}_1(i) \cup \tilde{B}_1(i+4))$, $\tilde{B}_2(i, i+1) = \{\tilde{A}_i^{8r+1}\eta : \eta = (i, j, k), j+k \leq 2r\}\backslash(\tilde{B}_1(i) \cup \tilde{B}_1(i+1)), i = 1, 2, 3, 5, 6, 7$, $\tilde{B}_2(4, 1) = \{\tilde{A}_4^{8r+1}\eta : \eta = (i, j, k), j + k \leq 2r\}\backslash(\tilde{B}_1(4) \cup \tilde{B}_1(1))$, and $\tilde{B}_2(8, 4) = \{\tilde{A}_8^{8r+1}\eta : \eta = (i, j, k), j + k \leq 2r\}\backslash(\tilde{B}_1(8) \cup \tilde{B}_1(1))$. Associated with this part, let $\tilde{I}_{2,1} = B_2(1, 4)$, and $\tilde{I}_{2,2} = \tilde{B}_2(1, 2)$.

Part III is the union of the collections $\tilde{B}_3(1, 2, 3, 4) = \{\beta : \beta_3 \leq r\}\backslash(\cup_{i=1}^4 \tilde{B}_1(i) \cup \tilde{B}_2(1, 2) \cup \tilde{B}_2(2, 3) \cup \tilde{B}_2(3, 4) \cup \tilde{B}_2(4, 1))$, $\tilde{B}_3(5, 6, 7, 8) = \{\beta : \beta_3 \geq 7r+1\}\backslash(\cup_{i=5}^8 B_1(i) \cup \tilde{B}_2(5, 6) \cup \tilde{B}_3(6, 7) \cup \tilde{B}_2(7, 8) \cup \tilde{B}_2(8, 1))$ and etc.. Associated with this part, let $\tilde{I}_3 = \tilde{B}_3(1, 2, 3, 4)$.

Part IV is the collection of the remaining indices $\tilde{B}_4 = \{\beta : r + 1 \leq \beta_i \leq 7r\}$ and $\tilde{I}_4 = \{\beta : \beta \in B_4\}$.

In the following, we give an example to illustrate the partition of the underlying indices of B-net on a patch (tetrahedron or prism or parallelepiped).

Example 3.5. Consider a tetrahedron. Let $r = 1$ and $d = 9$. Then part I is the union of indices in layer $l, 0 \leq l \leq 4$, attached to each vertex of the tetrahedron. Part II is the union of the remaining indices in layer $l, 0 \leq l \leq 2$, around each edge. Part III is the union of the remaining indices in layer $l, 0 \leq l \leq 1$, near each facet of the tetrahedron. Part IV is the indices in the l^{th} core, $r + 1 \leq l$. (Every part of indices of B-net on a prism or a parallelepiped is similar to the corresponding part on the tetrahedron as considered above. We omit the details here.) Next the index set associated with each part of the underlying indices of the B-net on a tetrahedron (or prism or parallelepiped) is listed below.

$I_1 = \{(l, m, n) : l + m + n \leq 4\}$; $I_2 = \{(4, 1, 0), (4, 0, 1), (3, 2, 0), (3, 1, 1), (3, 0, 2), (4, 2, 0), (4, 1, 1), (4, 0, 2)\}$; $I_3 = \{(2, 2, 1), (3, 3, 0), (3, 2, 1), (2, 3, 1), (3, 3, 1)\}$; and $I_4 = \{(2, 2, 2), (3, 2, 2), (2, 2, 3), (2, 3, 2)\}$.

$\bar{I}_{2,1} = \{(1, 4, 0), (2, 4, 0), (2, 3, 0), (0, 4, 1), (0, 5, 1), (1, 5, 1), (1, 4, 1), (1, 3, 1), (0, 3, 2), (0, 4, 2), (0, 5, 2), (0, 6, 2)\}$; $\bar{I}_{2,2} = \{(1, 0, 5), (1, 0, 4), (0, 1, 4), (0, 1, 5), (1, 1, 6), (1, 1, 5), (1, 1, 4), (1, 1, 3), (2, 0, 6), (2, 0, 5), (2, 0, 4), (2, 0, 3), (0, 2, 6), (0, 2, 5), (0, 2, 4), (0, 2, 3)\}$; $\bar{I}_{3,1} = \{(3, 3, 0), (2, 2, 1), (3, 2, 1), (2, 3, 1), (3, 3, 1), (2, 4, 1), (4, 2, 1), (2, 5, 1), (3, 4, 1), (4, 3, 1), (5, 2, 1)\}$; $\bar{I}_{3,2} = \{(1, i, j) : 2 \leq i, j \leq 6\} \cup \{(0, i, j) : 3 \leq i, j \leq 6\}$ and $\bar{I}_4 = \{(i, j, k) : 2 \leq k \leq 7, i, j \geq 2, i + j \leq 7\}$.

$\tilde{I}_2 = \bar{I}_{2,2}$; $\tilde{I}_3 = \{(i, j, 0) : 3 \leq i, j \leq 6\} \cup \{(i, j, 1) : 2 \leq i, j \leq 7\}$ and $\tilde{I}_4 = \{(i, j, k) : 2 \leq i, j, k \leq 7\}$.

For the partition $\triangle$, we denote all its vertices by $\{\mathbf{v}_1, \cdots, \mathbf{v}_L\}$. Let $\mathcal{V}, \mathcal{E}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ denote collections of all vertices, edges, triangular facets, parallelepiped facets, tetrahedra, prisms, and parallelepipeds, respectively. For each $e \in \mathcal{E}$ with two endpoints $\mathbf{v}_i$ and $\mathbf{v}_j$, where $i < j$, we rewrite it as $e = \langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2} \rangle$, where $\mathbf{v}_{e,1} = \mathbf{v}_i$ and $\mathbf{v}_{e,2} = \mathbf{v}_j$. For each triangular facet $f \in \mathcal{F}_1$ with vertices $\mathbf{v}_i, \mathbf{v}_j$, and $\mathbf{v}_k$, where $i < j < k$, we rewrite it as $f = \langle \mathbf{v}_{f,1}, \mathbf{v}_{f,2}, \mathbf{v}_{f,3} \rangle$ where $\mathbf{v}_{f,1} = \mathbf{v}_i$, $\mathbf{v}_{f,2} = \mathbf{v}_j$ and $\mathbf{v}_{f,3} = \mathbf{v}_k$. For each parallelepiped facet $f \in \mathcal{F}_2$ with four vertices $\mathbf{v}_i, \mathbf{v}_i, \mathbf{v}_k, \mathbf{v}_l$ where $i < j < k < l$, we rewrite it as $\langle \mathbf{v}_{f,1}, \mathbf{v}_{f,2}, \mathbf{v}_{f,3}, \mathbf{v}_{f,4} \rangle$, etc.. For a tetrahedron $t \in \mathcal{T}_1$ with vertices $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_l$ where $i < j < k < l$, we similarly rewrite it as $t = \langle \mathbf{v}_{t,1}, \mathbf{v}_{t,2}, \mathbf{v}_{t,3}, \mathbf{v}_{t,4} \rangle$. For a prism $t \in \mathcal{T}_2$, we write it as $t = \langle \mathbf{v}_{t,1}, \cdots, \mathbf{v}_{t,6} \rangle$ and for a parallelepiped $t \in \mathcal{T}_3$, $t = \langle \mathbf{v}_{t,1}, \cdots, \mathbf{v}_{t,8} \rangle$.

For each edge $e \in \mathcal{E}$, let $t_{e,i}, i = 1, \cdots, l(e)$ be the elements in $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ sharing e as their common edge. If at least one of them is a tetrahedron, we may assume without loss of generality that $t_{e,1}$ is a tetrahedron and call e a t-edge. We will also rewrite it as $t_{e,1} = \langle \mathbf{v}_{e,1}, \mathbf{v}_{e,2}, \mathbf{v}_{e,3}, \mathbf{v}_{e,4} \rangle$. If $\{t_{e,i} : i = 1, \cdots, l(e)\}$ does not contain a tetrahedron but contain at least one prism, we will call e an m-edge and we will assume that $t_{e,1}$ is a prism; i.e., $t_{e,1} = \langle \mathbf{v}_{e,1}, \cdots, \mathbf{v}_{e,6} \rangle$ and assume that $\langle \mathbf{v}_{e,2}, \mathbf{v}_{e,1} \rangle$, $\langle \mathbf{v}_{e,3}, \mathbf{v}_{e,1} \rangle$, and $\langle \mathbf{v}_{e,3}, \mathbf{v}_{e,1} \rangle$ are three of its edges. If $\{t_{e,i} : i = 1, \cdots, l(e)\}$ is a set of parallelepipeds, we call e a p-edge and $t_{e,1}$ is a parallelepiped denoted by $t_{e,1} = \langle \mathbf{v}_{e,1}, \cdots, \mathbf{v}_{e,8} \rangle$ with $\langle \mathbf{v}_{e,2}, \mathbf{v}_{e,1} \rangle$, $\langle \mathbf{v}_{e,3}, \mathbf{v}_{e,1} \rangle$, and $\langle \mathbf{v}_{e,4}, \mathbf{v}_{e,1} \rangle$ as three of its edges. We may define the derivatives relative to the edge e as same as that in the previous section. We let

$$I_e = \begin{cases} I_2 & \text{if } t_{e,1} \text{ is a tetrahedron} \\ \bar{I}_{2,1} & \text{if } t_{e,1} \text{ is prism} \\ \tilde{I}_2 & \text{ if } t_{e,1} \text{ is a parallelepiped} \end{cases}$$

be the index set associated with e.

For each triangular facet f, let $t_{f,1}$ and $t_{f,2}$ be two elements in $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ sharing f, if f is an interior facet. If one of $t_{f,1}$ and $t_{f,2}$ is a tetrahedron, we may assume that $t_{f,1}$ is this one and call f a t-facet. Otherwise, $t_{f,1}$ is a prism and f is called an m-facet. If f is a boundary facet, let $t_{f,1}$ be the element containing f. We may write it as $t_{f,1} = \langle \mathbf{v}_{f,1}, \mathbf{v}_{f,2}, \mathbf{v}_{f,3}, \mathbf{v}_{f,4} \rangle$ if $t_{e,1}$ is a tetrahedron, we call it a t-facet, or $t_{f,1} = \langle \mathbf{v}, \cdots, \mathbf{v}_{f,6} \rangle$ if $t_{f,1}$ is a prism, we will call it an m-facet.

For each parallelepiped facet f, let $t_{f,1}$ and $t_{f,2}$ be two elements in $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ sharing f, if f is an interior facet. If one of $t_{f,1}$ and $t_{f,2}$ is a prism, we may assume that $t_{f,1}$ is this one and call it a m-facet. Otherwise, $t_{f,1}$ is a parallelepiped, and we call f a p-facet. If f is a boundary facet, let $t_{f,1}$ be the element containing f. We may write it as $t_{f,1} = \langle \mathbf{v}_{f,1}, \cdots, \mathbf{v}_{f,6} \rangle$ if $t_{e,1}$ is a prism, and call f a m-facet, or $t_{f,1} = \langle \mathbf{v}_{f,1}, \cdots, \mathbf{v}_{f,8} \rangle$ if $t_{f,1}$ is a parallelepiped, and we call f a p-facet. In the later case, we denote it by $t_{f,1}$ and $\langle \mathbf{v}_{e,2}, \mathbf{v}_{e,1} \rangle$, $\langle \mathbf{v}_{e,3}, \mathbf{v}_{e,1} \rangle$ and $\langle \mathbf{v}_{e,4}, \mathbf{v}_{e,1} \rangle$ are assumed to be three of its edges. We will use the notation $D_f^\alpha := D_{f,1}^\alpha$ for the derivatives relative to f as defined in the previous section. We also let

$$I_f = \begin{cases} I_3 & \text{if } t_{f,1} \text{ is a tetrahedron} \\ \bar{I}_{3,1} & \text{if } t_{f,1} \text{ is a prism} \\ \tilde{I}_3 & \text{if } t_{f,1} \text{ is a parallelepiped} \end{cases}$$

be the index set associated with each facet f.

Let t be an element of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. If $t = \langle \mathbf{v}_{t,1}, \cdots, \mathbf{v}_{t,4} \rangle$, we rewrite it as $\langle \mathbf{y}_{t,1}, \cdots, \mathbf{y}_{t,4} \rangle$ with $\mathbf{v}_{t,4} = \mathbf{y}_{t,1}$; if $t = \langle \mathbf{v}_{t,1}, \cdots, \mathbf{v}_{t,6} \rangle$, we rewrite it as $\langle \mathbf{y}_{t,1}, \cdots, \mathbf{y}_{t,6} \rangle$ with $\mathbf{v}_{t,6} = \mathbf{y}_{t,1}$ and $\langle \mathbf{y}_{t,2}, \mathbf{y}_{t,1} \rangle$, $\langle \mathbf{y}_{t,3}, \mathbf{y}_{t,1} \rangle$ and $\langle \mathbf{y}_{t,4}, \mathbf{y}_{t,1} \rangle$ are three of the edges of t; and if $t = \langle \mathbf{v}_{t,1}, \cdots, \mathbf{v}_{t,8} \rangle$, we rewrite it as $\langle \mathbf{y}_{t,1}, \cdots, \mathbf{y}_{t,8} \rangle$ with $\mathbf{y}_{t,1} = \mathbf{v}_{t,8}$ and $\langle \mathbf{y}_{t,2}, \mathbf{y}_{t,1} \rangle$, $\langle \mathbf{y}_{t,3}, \mathbf{y}_{t,1} \rangle$ and $\langle \mathbf{y}_{t,4}, \mathbf{y}_{t,1} \rangle$ are three of the edges of t. As before, the derivatives relative to t are denoted by

$$D_t^\alpha = (D_{\mathbf{y}_{t,2}-\mathbf{y}_{t,1}})^{\alpha_1} (D_{\mathbf{y}_{t,3}-\mathbf{y}_{t,1}})^{\alpha_2} (D_{\mathbf{y}_{t,4}-\mathbf{y}_{t,1}})^{\alpha_3}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_+^3$ and we let

$$I_t = \begin{cases} I_4 & \text{if } t \text{ is a tetrahedron} \\ \bar{I}_4 & \text{if } t \text{ is a prism} \\ \tilde{I}_4 & \text{if } t \text{ is a parallelepiped} \end{cases}$$

be the index set associated with each t.

We are now in a position to construct vertex splines on this mixed partitioned region R. In general, we will consider four types of vertex splines of interest. They are required to satisfy the following specifications of interpolatory parameters and smoothness conditions.

(I) For each vertex $\mathbf{v} \in \mathcal{V}$ and $\gamma \in I_1$, let $V_{\mathbf{v}}^\gamma$ be a piecewise polynomial function of "degree" $\leq 8r + 1$ satisfying the following:

$$D_{\mathbf{u}}^\alpha V_{\mathbf{v}}^\gamma(\mathbf{u}) = \delta_{\mathbf{v},\mathbf{u}} \delta_{\gamma,\alpha}, \alpha \in I_1, \mathbf{u} \in \mathcal{V}; \tag{I.1}$$

$$V_{\mathbf{v}}^\gamma \in C^{4r} \text{ at each vertex of } \triangle; \tag{I.2}$$

$$D_e^\alpha V_{\mathbf{v}}^\gamma \Big|_{t_{e,1}} (\mathbf{v}_{e,1}) = 0, \alpha \in I_e, e \in \mathcal{E}; \tag{I.3}$$

$$V_{\mathbf{v}}^\gamma \in C^{2r} \text{ around each edge of } \triangle; \tag{I.4}$$

$$D_f^\alpha V_{\mathbf{v}}^\gamma \Big|_{t_{f,1}} (\mathbf{y}_{f,1}) = 0, \gamma \in I_f, f \in \mathcal{F}_1 \cup \mathcal{F}_2; \tag{I.5}$$

(I.6) $$V_{\mathbf{v}}^{\gamma} \in C^r \text{ across each facet of } \triangle;$$

(I.7) $$D_t^{\alpha} V_{\mathbf{v}}^{\gamma}\Big|_t(\mathbf{y}_{t,1}) = 0, \alpha \in I_t, t \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3,$$

where $\delta_{\mathbf{v},\mathbf{u}}, \delta_{\gamma,\alpha}$, as usual, denote the Kronecker delta.

(II) For each edge $e \in \mathcal{E}$ and $\gamma \in I_e$, let V_e^{γ} be a piecewise polynomial function of "degree" $\leq 8r+1$ satisfying the following:

(II.1) $$D_{\mathbf{v}}^{\alpha} V_e^{\gamma}(\mathbf{v}) = 0, \alpha \in I_1, \mathbf{v} \in \mathcal{V};$$

(II.2) $$V_e^{\gamma} \in C^{4r} \text{ at each vertex of } \triangle;$$

(II.3) $$D_d^{\alpha} V_e^{\gamma}\Big|_{t_{d,1}}(\mathbf{v}_{d,1}) = \delta_{e,d}\delta_{\gamma,\alpha}, \alpha \in I_d, d \in \mathcal{E};$$

(II.4) $$V_e^{\gamma} \in C^{2r} \text{ around each edge of } \triangle;$$

(II.5) $$D_f^{\alpha} V_e^{\gamma}\Big|_{t_{f,1}}(\mathbf{y}_{f,1}) = 0, \alpha \in I_f, f \in \mathcal{F}_1 \cup \mathcal{F}_2;$$

(II.6) $$V_e^{\gamma} \in C^r \text{ across each facet of } \triangle;$$

(II.7) $$D_t^{\alpha} V_e^{\gamma}\Big|_t(\mathbf{y}_{t,1}) = 0, \alpha \in I_t, t \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3.$$

(III) For each facet $f \in \mathcal{F}_1 \cup \mathcal{F}_2$ and $\gamma \in I_f$, let V_f^{γ} be a piecewise polynomial function of "degree" $\leq 8r+1$ satisfying the following:

(III.1) $$D_{\mathbf{v}}^{\alpha} V_f^{\gamma}(\mathbf{v}) = 0, \alpha \in I_1, \mathbf{v} \in \mathcal{V};$$

(III.2) $$V_f^{\gamma} \in C^{4r} \text{ at each vertex of } \triangle;$$

(III.3) $$D_d^{\alpha} V_f^{\gamma}\Big|_{t_{d,1}}(\mathbf{v}_{d,1}) = 0, \alpha \in I_d, d \in \mathcal{E};$$

(III.4) $$V_f^{\gamma} \in C^{2r} \text{ around each edge of } \triangle;$$

(III.5) $$D_g^{\alpha} V_f^{\gamma}\Big|_{t_{g,1}}(\mathbf{y}_{g,1}) = \delta_{f,g}\delta_{\gamma,\alpha}, \alpha \in I_g, g \in \mathcal{F}_1 \cup \mathcal{F}_2;$$

(III.6) $$V_e^\gamma \in C^r \text{ across each facet of } \triangle;$$

(III.7) $$D_t^\alpha V_f^\gamma\Big|_t(\mathbf{y}_{t,1}) = 0, \alpha \in I_t, t \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3.$$

(IV) For each element $t \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ and $\gamma \in I_t$, let V_t^γ be a piecewise polynomial function of "degree" $\leq 8r+1$ satisfying the following:

(IV.1) $$D_\mathbf{v}^\alpha V_t^\gamma(\mathbf{v}) = 0, \alpha \in I_1, \mathbf{v} \in \mathcal{V};$$

(IV.2) $$V_t^\gamma \in C^{4r} \text{ at each vertex of } \triangle;$$

(IV.3) $$D_d^\alpha V_t^\gamma\Big|_{t_{d,1}}(\mathbf{v}_{d,1}) = 0, \alpha \in I_d, d \in \mathcal{E};$$

(IV.4) $$V_t^\gamma \in C^{2r} \text{ around each edge of } \triangle;$$

(IV.5) $$D_f^\alpha V_t^\gamma\Big|_{t_{f,1}}(\mathbf{y}_{f,1}) = 0, \gamma \in I_f, f \in \mathcal{F}_1 \cup \mathcal{F}_2;$$

(IV.6) $$V_t^\gamma \in C^r \text{ across each facet of } \triangle;$$

(IV.7) $$D_s^\alpha V_t^\gamma\Big|_s(\mathbf{y}_{s,1}) = \delta_{t,s}\delta_{\gamma,\alpha}, \alpha \in I_s, s \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3.$$

The outline for constructing these vertex splines can be described as follows. Let us first consider the construction procedure of $V_\mathbf{v}^\gamma$. Suppose t is an element in $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. The requirements (I.1) and (I.2) specify the portion of the B-net of $V_\mathbf{v}^\gamma$ with indices on layer l at each vertex of t, $0 \leq l \leq 4r$. Around each edge e of t, the requirement (I.3) determines the B-net of $V_\mathbf{v}^\gamma\Big|_{t_{e,1}}$ with indices in layer l around e, $0 \leq l \leq 2r$ if $t = t_{e,1}$. Otherwise, we apply Lemmas 3.3.5 and Lemma 3.3.1 to determine

$$D_t^\alpha V_\mathbf{v}^\gamma\Big|_t$$

from $V_\mathbf{v}^\gamma\Big|_{t_{e,1}}$ and use the resulting directional derivatives to determine the portion of the B-net of $V_\mathbf{v}^\gamma\Big|_t$ in layer l around e located in part II, $0 \leq l \leq 2r$. Suppose that a facet $f \subset t$ is an each interior facet. If $t = t_{f,1}$, then we may directly obtain the portion of the B-net of $V_\mathbf{v}^\gamma$ with indices on layer l near f located in part III, $0 \leq l \leq r$, by using the requirements in (I.5). Otherwise, after requiring that $V_\mathbf{v}^\gamma|_{t_{f,1}}$ satisfies (I.5), we use Lemmas 3.3.6-3.3.12 whichever applies, to determine the portion of the B-net of $V_{\mathbf{v}}^{\gamma}|_t$ on the layer l near f located in part III, $0 \le l \le r$. Finally, we directly apply the requirement in (I.7) to determine the rest of the B-net of $V_{\mathbf{v}}^{\gamma}|_t$. From the construction procedure, we know that the support of $V_{\mathbf{v}}^{\gamma}$ is the union of all elements $t \in T_1 \cup T_2 \cup T_3$ sharing $\mathbf{v}$ as their common vertex.

Next, the outline for the construction procedures for other vertex splines are the same as that of $V_{\mathbf{v}}^{\gamma}$. We omit their details.

From the construction procedures, we know that the support of V_e^{γ} is the union of all elements $t \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ sharing edge e, the support of V_f^{γ} is the union of all elements $t \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ sharing facet f; the support of V_t^{γ} is the element t only.

Consider the space

$$\begin{aligned} \widehat{S}_{8r+1}^{r}(\triangle) = \text{span} \quad & \{V_{\mathbf{v}}^{\gamma} : \gamma \in I_1, \mathbf{v} \in \mathcal{V}\} \cup \{V_e^{\gamma} : \gamma \in I_e, e \in \mathcal{E}\} \\ & \cup \{V_f^{\gamma} : \gamma \in I_f, f \in \mathcal{F}_1 \cup \mathcal{F}_2\} \cup \{V_t^{\gamma} : \gamma \in I_t, t \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3\}. \end{aligned}$$

Clearly, $\widehat{S}_{8r+1}^{r}(\triangle)$ is a subspace of $S_{8r+1}^{r}(\triangle)$. For each sufficiently smooth function g, we define

$$\begin{aligned} \text{(3.6.1)} \qquad Lg(\mathbf{x}) \;=\; & \sum_{\mathbf{v} \in \mathcal{V}} \sum_{|\gamma| \le 4r} D^{\gamma} g(\mathbf{v}) V_{\mathbf{v}}^{\gamma}(\mathbf{x}) + \sum_{e \in \mathcal{E}} \sum_{\gamma \in I_e} D_e^{\gamma} g(\mathbf{v}_{e,1}) V_e^{\gamma}(\mathbf{x}) \\ + \; & \sum_{f \in \mathcal{F}_1 \cup \mathcal{F}_2} \sum_{\gamma \in I_f} D_f^{\gamma} g(\mathbf{y}_{f,1}) V_f^{\gamma}(\mathbf{x}) + \sum_{t \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3} \sum_{\gamma \in I_t} D_t^{\gamma} g(\mathbf{y}_{t,1}) V_t^{\gamma}(\mathbf{x}). \end{aligned}$$

We are now ready to derive some properties of the super spline space $\widehat{S}_{8r+1}^{r}(\triangle)$.

LEMMA 3.6.1. *$Lp = p$ for any polynomial p of total degree* $8r+1$.

Proof. We use mathematical induction on the number of tetrahedra in $\triangle$ to establish this lemma. For $n = 1$, L is an interpolatory operator based on t, the only tetrahedron of $\triangle$. Since the sets of interpolation conditions associated with each vertex of t are lower sets and induce a partition of Λ_{8r+1}, we see that $Lp = p$ for all p of total degree $8r+1$ by Proposition 3.1. Suppose now that the result holds for $m = \#\{t : t \in \triangle\}$. Let $\#\{t : t \in \triangle\} = m+1$ and set $\triangle = \{t_i : i = 1, \cdots, m+1\}$. By relabeling if necessary, assume that $t_{m+1} = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle$ has at least one boundary facet, and for the time being, assume that it has only one interior facet $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$, say. Let $\triangle' = \{t_i : i = 1, \cdots, m\} = \triangle \backslash \{t_{m+1}\}$. Observing the uniqueness in Lemma 3.3.5 and applying Theorem 4.1.3 in [39], we can see that the smoothness of Lp across $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ may be expressed in terms of certain appropriate interpolation conditions (i.e., derivatives relative to the edges and related to the facet) such that $L_{\triangle} p\Big|_{\triangle'} = L_{\triangle'} p$ and $L_{\triangle} p\Big|_{t_{m+1}} = L_{t_{m+1}} p$, where $L_{\triangle}, L_{\triangle'}, L_{t_{m+1}}$ are restrictions of L on $\triangle, \triangle', t_{m+1}$, respectively. By the induction hypothesis, we have $L_{\triangle} p\Big|_{\triangle'} = p$ and $L_{\triangle} p\Big|_{t_{m+1}} = p$. Hence,

$Lp = p$ on $\triangle$. The proof is similar if t_{m+1} contains two or three interior facets. This completes the proof.

If Lg is interpreted as

$$\begin{aligned} Lg(\mathbf{x}) &= \sum_{\mathbf{v}\in\mathcal{V}} \sum_{|\gamma|\le 4r} D^\gamma g(\mathbf{v}) V_{\mathbf{v}}^\gamma(\mathbf{x}) \\ &+ \sum_{e\in\mathcal{E}} \sum_{\gamma\in I_e} D_e^\gamma g\Big|_{t_{e,1}}(\mathbf{v}_{e,1}) V_e^\gamma(\mathbf{x}) \\ &+ \sum_{f\in\mathcal{F}_1\cup\mathcal{F}_2} \sum_{\gamma\in I_f} D_f^\gamma g\Big|_{t_{f,1}}(\mathbf{y}_{f,1}) V_f^\gamma(\mathbf{x}) \\ &+ \sum_{t\in\mathcal{T}_1\cup\mathcal{T}_2\cup\mathcal{T}_3} \sum_{\gamma\in I_t} D_t^\gamma g\Big|_t(\mathbf{y}_{t,1}) V_t^\gamma(\mathbf{x}), \end{aligned}$$

then the following result is also derived from the above argument.

LEMMA 3.6.2. *$Lg = g$ for any $g \in \widehat{S}^r_{8r+1}(\triangle)$.*

We have the following consequence of Lemma 3.5.2.

THEOREM 3.6.1. *The collection*

$$\begin{aligned} \mathcal{B} := \ & \{V_{\mathbf{v}}^\gamma : \mathbf{v}\in\mathcal{V}, |\gamma|\le 4r\} \cup \{V_e^\gamma : \gamma\in I_e, e\in\mathcal{E}\} \cup \{V_f^\gamma : \gamma\in I_f, f\in\mathcal{F}_1\cup\mathcal{F}_2\} \\ & \cup\{V_t^\gamma : \gamma\in I_t, t\in\mathcal{T}_1\cup\mathcal{T}_2\cup\mathcal{T}_3\} \end{aligned}$$

is a basis of S and consequently $S = \widehat{S}^r_{8r+1}(\triangle)$.

Let $G \subset \sup\{t : t\in\triangle\}$ and for $g\in C^k(G)$, denote

$$\|D^k g\| = \max_{|\alpha|=k} \|D^\alpha g\|_{C(G)}$$

and

$$\operatorname{dist}(f, S) = \inf_{s\in S} \|f - s\|.$$

Clearly, the vertex splines in $S^r_d, d \ge 8r+1$, can also be constructed by using same idea as above and similar results may be derived. Therefore, we can state the following more general theorem.

THEOREM 3.6.2. *Let $d \ge 8r+1$. There exists a linear operator L with range $\widehat{S}^r_d$ such that*

$$\|Lg - g\| \le K\|D^{d+1}g\||\triangle|^{d+1}$$

for all sufficiently smooth function g, where K is a constant independent of g and $\triangle$. In particular, for $d = 8r+1$, L can be chosen to be (3.6.1). Consequently,

$$\operatorname{dist}(g, \widehat{S}^r_d) \le K\|D^{d+1}g\||\triangle|^{d+1}.$$

Proof. Here, we only prove the case that $d = 8r+1$, since a similar argument yields the desired result for $d > 8r+1$. Fix a point $\mathbf{x} \in G$ and consider a linear functional

$$F(g) = Lg(\mathbf{x}) - g(\mathbf{x}).$$

It is easy to see that F satisfies the following:

(i) $|F(g)| \leq K_1 \sum_{j=0}^{8r+1} \|D^j g\| |\triangle|^j$

(ii) $F(p) = 0$ for all $p \in \mathbb{P}_{8r+1}$,

By an argument similar to that in Bramble and Hilbert [24], there exists a constant K independent of g, $\mathbf{x}$, and $|\triangle|$ such that

$$|Lg(\mathbf{x}) - g(\mathbf{x})| \leq K\|D^{8r+2}g\||\triangle|^{8r+2}.$$

Therefore, we have established the theorem.

Note that $\widehat{S}^r_{8r+1}$ is a proper subspace of S^r_{8r+1}. In fact, the exact dimension of $\widehat{S}^r_9$ is given in the following

COROLLARY 3.6.3.

$$\begin{aligned} \dim \widehat{S}^1_9 &= 35N_v + 8N_{e,t} + 12N_{e,s} + 16N_{e,p} + 7N_{f,t} \\ &\quad +11N_{f,s} + 46N_{f,p} + 4N_{t,t} + 60N_{t,s} + 216N_{t,p} \end{aligned}$$

where $N_v, N_{e,t}, N_{e,s}, N_{e,p}, N_{f,t}, N_{f,m}, N_{f,p}, N_{t,t}, N_{t,m}, N_{t,p}$ *denote the numbers of vertices, t-edges, m-edges, p-edge, t-facets, m-facets, p-facets, tetrahedra, prisms, parallelepipeds of* $\triangle$*, respectively.*

4. FINAL CONCLUSIONS AND REMARKS

In sections 2.1–2.6 and 3.1–3.5 we mainly dealt with the construction of vertex splines in various spline spaces $S_d^r(\triangle)$ in the bivariate and trivariate settings and proved that approximation formulas based on the vertex splines may be used to realize the full approximation order. Our results show that these vertex splines in $S_d^r(\triangle)$ span a super spline subspace and the full approximation order can be achieved by using its subspace. We should comment on the application aspects of these vertex splines. Due to the facts that they have local supports and that they can be easily constructed on an arbitrary grid partition, vertex splines should find important applications in engineering and other applied areas. The following are three of these applications.

(1) Assume that we are given some partial information of a unknown function f on scattered data set of an interested region R and we are looking for a surface (such as a piecewise polynomial function) with certain preassigned smoothness to approximate the unknown function f. The mathematical model can be described as follows. Let $\mathcal{V} = \{\mathbf{v}_1, \cdots, \mathbf{v}_N\}$ be the set of scattered data and r be the preassigned smoothness requirement. Assume that the region R of interest has been partitioned into a simplicial region $\triangle$ (or in general, a mixed partition) with $\mathbf{v}_i, i = 1, \cdots, N$, as its vertices. For convenience, we assume that $R \subset \mathbb{R}^2$. Denote $I := \{\alpha \in \mathbb{Z}_+^2 : |\alpha| \leq r + [(r+1)/2]\}$ and denote $I_c \subset \mathcal{V} \times I$ the index set of interpolatory constraints. In other words, $D^\alpha f(\mathbf{v}_i), (\mathbf{v}, \alpha) \in I_c$ are the known information. In addition, we know the certain moments of f on R. Then the problem is

Find a spline $S_f \in \widehat{S}_d^r(\triangle)$ such that

$$\begin{aligned} \|f - S_f\|_2 \ &= \inf\{\|f - s\|_2 : \ s \in \widehat{S}_d^r(\triangle) \\ &\qquad\qquad\qquad\qquad D^\alpha s(\mathbf{v}) = D^\alpha f(\mathbf{v}), (\mathbf{v}, \alpha) \in I_c\} \end{aligned}$$

Here, $\|g\|_2 = (\int_R |g|^2)^{1/2}$ and $d = 3r + 2$.

Note that when $I_c = \emptyset$, the problem is reduced to the usual L^2 approximation problem.

By using vertex splines in $\widehat{S}_d^r(\triangle)$, this problem can be easily solved. The resulting surface has the full approximation order to the unknown function f if f is assumed to be sufficiently smooth. We may solve the discrete version of the above problem as well. The reader is referred to [28] and [39] for reference.

(2) Assume that we are given the same problem as (1) except that we do not have the information on the certain moment of f mentioned above. Then we may consider

the following:

Find a spline $s_f \in \widehat{S}_d^r(\triangle)$ *such that*

$$\begin{aligned} \|D^2 s_f\|_2 &= \inf\{\|D^2 s\|_2 : \ s \in \widehat{S}_d^r(\triangle), \ \textit{and} \\ &\qquad\qquad D^\alpha s(\mathbf{v}) = D^\alpha f(\mathbf{v}), (\mathbf{v},\alpha) \in I_c\}. \end{aligned}$$

Here, $\|D^2 g\| := \sum_{|\alpha|=2} \|D^\alpha g\|_2$. *That is, the spline we look for is of minimum "energy".*

By using fundamental vertex splines of $\widehat{S}_d^r(\triangle)$, this problem is turned into one that requires solving a linear system. Thus, we can easily solve this problem and obtain a desired surface. (cf. [27, 28].)

(3)Another application of vertex splines is to solve partial differential equations (PDE's). We may use them as trial functions instead of finite elements. Let us consider a wave equation of optical waveguide:

$$\begin{cases} (\nabla^2 + (\frac{k_0}{\beta})^2)u &= 0 \\ \frac{\partial}{\partial n} u &= 0 \quad \text{on the magnetic wall} \\ u &= 0 \quad \text{on the electrical wall.} \end{cases}$$

where β is the cutoff wavelength to be determined and u represents the axial field components of E_z or H_z for the electromagnetic field.

Now let $u = \sum_{i=1}^{L} f_i V_i$, where V_i's are the fundamental vertex splines in $\widehat{S}_d^r(\triangle)$. Multiplying from the left by u and integrating over the region R, we then apply Green identities to obtain

$$\int\int_R \nabla u \cdot \nabla u - (\frac{k_0}{\beta})^2 \int\int_R u^2 = 0$$

or

$$\mathbf{f}^t S \mathbf{f} - (\frac{k_0}{\beta})^2 \mathbf{f}^t T \mathbf{f} = 0$$

where $\mathbf{f} := (f_1, \cdots, f_L)^t$ and S and T are resulting matrices. After variational calculus, we will obtain an eigenvalue problem with $(k_0/\beta)^2$ as its eigenvalue. Then solving for the eigenvalues and corresponding eigenfunctions, we obtain the desired results on the electromagnetic field of a waveguide.

The advantages of using vertex splines instead of finite elements are the following: in the bivariate setting, the degree of the polynomial pieces is $3r+2$ instead of $4r+1$ for $r \geq 1$; in the trivariate setting, the degree of the polynomial pieces is $6r+3$ instead of $8r+1$ for $r \geq 1$; no normal derivatives inside of an edge of each triangle (or tetrahedron) are needed; no mapping to the standard triangle or rectangle is necessary; finding inner products of vertex splines is easier (their B-nets and exact formulas are used) instead of numerical integration. We may use this technique to solve other PDE's. See [27, 28] for other applications.

Besides the above remarks, we also comment on the other aspects of vertex splines as well as the theory of MSA.

It should be noted that the construction of vertex splines is based on the assumption that a partition of the region R of interest is given beforehand. Many methods for generating simplicial partitions can be found in the literature (cf. [6, 110] for references) and some study based on mixed partitions can be found in [79, 104].

In addition to the study on simplicial B-splines, box splines, and vertex splines, the discussion on dimensions of various bivariate spline spaces and thin-plate splines which are of global support has been carried out simultaneously. Readers are referred to [1–3, 5, 33, 43–48, 74, 75, 78, 108, 109, 111, 115, 116] for the information on the dimensions of bivariate spline and super spline spaces and to [113, 114] and etc. for the discussion on thin-plate splines. Also, see [67–69, 80, 93, 96] for the study of using radial functions in multivariate interpolation and [29–31, 34, 73, 91, 93, 95, 97, 100, 102] for multivariate spline interpolation and surface fitting.

In [19], the authors conjectured that in the trivariate setting, the full approximation order of S_d^r is obtained as soon as $d \geq 4r + 3$. The conjecture is still an open problem.

References

[1] Alfeld, P., *A case study of multivariate piecewise polynomials,* in Geometric Modeling, G. Farin, ed., Society for Industrial and Applied Mathematics, Philadelphia, 1987, pp. 149–160.

[2] ——, *On the dimension of multivariate piecewise polynomials*, in Proceedings of the Biennial Dundee Conference on Numerical Analysis, Pitman, Harlow, Essex, 1985.

[3] Alfeld, P., B. Piper and L. L. Schumaker, *Minimally supported bases for spaces of bivariate piecewise polynomials of smoothness* r *and degree* $d \geq 4r+1$, Comput. Aided Geom. Design, 3 (1987), pp. 189–198.

[4] ——, *An explicit basis for* C^1 *quatic bivariate splines,* SIAM J. Numer. Anal., 24 (1987), pp. 891–911.

[5] ALFELD, P. AND L. L. SCHUMAKER, *The dimension of bivariate spline spaces of smoothness r for degree $d \geq 4r+1$*, Constr. Approx., 3 (1987), pp. 189–197.

[6] BAKER, B. S., E. GROSSE, AND C. S. RAFFERTY, *Non-obtuse triangulation of a polygon,* Bell Lab. Rpt., Murray Hill, 1985.

[7] BAMBERGER, L., *Zweidimensionale Splines auf regulären Triangulationen*, Ph. D. Thesis, University of Munich, 1985.

[8] BARNHILL, R. E. AND G. FARIN, *C^1 quintic interpolation over triangles: two explicit representations*, J. Numer. Meth. Engr., 17 (1981), pp. 1763–1778.

[9] BÖHM, W., *Subdividing multivariate splines,* Computer-aided Design, 15 (1983), pp. 345-352.

[10] BÖHM, W., G. FARIN, AND J. KAHMANN, *A survey of curve and surface methods in* CAGD, Comput. Aided Geom. Design, 1 (1984), pp. 1–60.

[11] DE BOOR, C., *B-form basics*, in Geometric Modeling, G. Farin, ed., Society for Industrial and Applied Mathematics, Philadelphia, 1987, pp. 131–148.

[12] ———, *The polynomials in the linear span of integer translates of a compactly supported function,* Constr. Approx., 3 (1987), pp. 199–208.

[13] DE BOOR, C. AND R. DEVORE, *Approximation by smooth multivariate splines,* Trans. Amer. Math. Soc., 276 (1983), pp. 775–788.

[14] DE BOOR, C., R. DEVORE, AND K. HÖLLIG, *Approximation order from smooth bivariate pp functions*, in Approximation Theory IV, C. K. Chui, L. L. Schumaker, and J. D. Ward, eds., Academic Press, New York, 1983, pp. 353–357.

[15] DE BOOR, C. AND K. HÖLLIG, *Recurrence relations for multivariate B-splines,* Proc. Amer. Math. Soc., 85 (1982), pp. 397–400.

[16] ———, *B-splines from parallelepipeds,* J. Analyse Math., 42 (1982), pp. 99–115.

[17] ———, *Approximation order from bivariate C^1-cubics: a counterexample*, Proc. Amer. Math. Soc., 87 (1983), pp. 649–655.

[18] ———, *Bivariate box splines and smooth pp functions on a three-direction mesh,* J. Comput. Appl. Math., 9 (1983), pp. 13–28.

[19] ———, *Approximation power of smooth bivariate pp functions,* Math. Z., 197(1988), pp. 343-363.

[20] de Boor, C., K. Höllig, and S. Riemenschneider, *Bivariate cardinal interpolation by splines on a three-direction mesh,* Illinois J. Math., 29 (1985), pp. 533–566.

[21] ———, *The limits of multivariate cardinal splines,* in Multivariate Approximation Theory III, W. Schempp & K. Zeller, eds., Birkhäuser, Basel, 1985, pp. 47–50.

[22] ———, *Convergence of bivariate cardinal interpolation,* Constr. Approx., 1 (1985), pp. 183–193.

[23] de Boor, C. and R. Q. Jia, *Controlled approximation and a characterization of the local approximation order,* Proc. Amer. Math. Soc., 95 (1985), pp. 547–553.

[24] Bramble, J. and Hilbert, *Bounds for a class of linear functionals with applications to Hermite interpolation,* Num. Math., 16(1971), pp. 362-369.

[25] Bramble, J. and M. Zlamal, *Triangular elements in the finite element method,* Math. Comp., 24 (1970), pp. 809–820.

[26] Chen, G., C.K. Chui and M. J. Lai, *Construction of real-time spline quasi-interpolation schemes*, Approx. Theory Applic., 4(1988), pp. 61-75.

[27] Chui, C. K., *Multi-dimensional spline techniques for fitting of surface to wind fields over complex terrain,* Final Report, Battelle, Aug. 1986.

[28] ———, Multivariate Splines, CBMS vol. 54, SIAM Publication, Philadelphia, 1988.

[29] Chui, C. K. and H. Diamond, *A natural formulation of quasi-interpolation by multivariate splines,* Proc. Amer. Math. Soc., 99 (1987), pp. 643–646.

[30] Chui, C. K., H. Diamond, and L. A. Raphael, *Interpolation by multivariate splines,* Math. Comp., 51 (1988), pp. 203-218.

[31] ———, *Shape-preserving quasi-interpolation and interpolation by box spline surfaces,* J. Comp. Appl. Math., 25 (1989), pp. 1-30.

[32] Chui, C. K. and T. X. He, *On minimal and quasi-minimal supported bivariate splines,* J. Approx. Theory, 52 (1988), pp. 217–238.

[33] ———, *On the dimension of bivariate super spline spaces*, Math. Comp., to appear.

[34] ———, *On bivariate C^1 quadratic finite elements and vertex splines*, Math. Comp., to appear.

[35] Chui, C. K., K. Jetter, and J. D. Ward, *Cardinal interpolation by multivariate splines,* Math. Comp., 48 (1987), pp. 711–724.

[36] Chui, C. K. and M. J. Lai, *On bivariate vertex splines,* in Multivariate Approximation Theory III, W. Schempp & K. Zeller, eds., Birkhäuser, Basel, 1985, pp. 84–115.

[37] ———, *On bivariate super vertex splines,* Constr. Approx., to appear.

[38] ———, *A multivariate analog of Marsden's identity and a quasi-interpolation scheme,* Constr. Approx., 3 (1987), pp. 111-122.

[39] ———, *Multivariate vertex splines and finite elements,* J. Approx. Theory, to appear.

[40] ———, *On multivariate vertex splines and applications*, in Topics in Multivariate Approximation, C. K. Chui, L. L. Schumaker, and F. Utreras, eds., Academic Press, New York, 1987, pp. 19–36.

[41] ———, *Computation of box splines and B-splines on triangulations of nonuniform rectangular partitions*, Approx. Theory Appl., 3 (1987), pp. 37–62.

[42] Chui, C. K., M. J. Lai and S. R. Bowers, *An algorithm for generating B-nets and graphically displaying box surfaces,* CAT Report # 181, Texas A&M University, 1989.

[43] Chui, C. K. and L. L. Schumaker, *On spaces of piecewise polynomials with boundary conditions,* I. *Rectangles*, in Multivariate Approximation Theory II, W. Schempp & K. Zeller, eds., Birkhäuser, Basel, 1982, pp. 69–80.

[44] Chui, C. K., L. L. Schumaker, and R. H. Wang, *On spaces of piecewise polynomials with boundary conditions,* II. *Type*-1 *triangulations*, in Second Edmonton

Conference on Approximation Theory, Z. Ditzian, A. Meir, S. Riemenschneider, and A. Sharma, eds., American Mathematical Society, Providence, 1983, pp. 51–66.

[45] ——, *On spaces of piecewise polynomials with boundary conditions*, III. *Type*-2 *triangulations*, in Second Edmonton Conference on Approximation Theory, Z. Ditzian, A. Meir, S. Riemenschneider, and A. Sharma, eds., American Mathematical Society, Providence, 1983, pp. 67–80.

[46] Chui, C. K. and R. H. Wang, *Spaces of bivariate cubic and quartic splines on type*-1 *triangulations,* J. Math. Anal. Appl., 101 (1984), pp. 540–554.

[47] ——, *On a bivariate B-spline basis,* Scientia Sinica , 27 (1984), pp. 1129–1142.

[48] ——, *Concerning* C^1 *B-splines on triangulations of non-uniform rectangular partitions,* Approx. Theory Appl., 1 (1984), pp. 11–18.

[49] Cohen, E., T. Lyche, and R. Riesenfeld, *Discrete box splines and refinement algorithms,* Comput. Aided Geom. Design, 1 (1984), pp. 131–141.

[50] ——, *Cones and recurrence relations for simplex splines,* Constr. Approx., 3 (1987), pp. 131–142.

[51] Dahmen, W., *Multivariate B-splines—Recurrence relations and linear combinations of truncated powers*, in Multivariate Approximation Theory, W. Schempp & K. Zeller, eds., Birkhäuser, Basel, 1979, pp. 64–82.

[52] ——, *On multivariate B-splines*, SIAM J. Numer. Anal., 17 (1980), pp. 179–190.

[53] ——, *Bernstein-Bézier representation of polynomial surfaces*, in Extension of B-spline Curve Algorithms to Surfaces, Siggraph 86, organized by C. de Boor, Dallas, 1986.

[54] ——, *Subdivision algorithms converge quadratically,* J. Comput. Appl. Math., 16 (1986), pp. 145–158.

[55] Dahmen, W., N. Dyn, and D. Levin, *On the convergence rates of subdivision algorithms for box spline surfaces*, Constr. Approx., 1 (1985), pp. 305–322.

[56] Dahmen, W. and C. A. Micchelli, *On the linear independence of multivariate B-splines* I. *Triangulations of simploids*, SIAM J. Numer. Anal., 19 (1982), pp. 992–1012.

[57] _____, *On the linear independence of multivariate B-splines* II. *Complete configurations,* Math. Comp., 41 (1983), pp. 141–164.

[58] _____, *Translates of multivariate splines,* Linear Algebra Appl., 52 (1983), pp. 217–234.

[59] _____, *Recent progress in multivariate splines,* in Approximation Theory IV, C. K. Chui, L. L. Schumaker, and J. D. Ward, eds., Academic Press, New York, 1983, pp. 27–121.

[60] _____, *Subdivision algorithms for the generation of box-spline surfaces,* Comput. Aided Geom. Design, 1 (1984), pp. 115–129.

[61] _____, *On the approximation order from certain multivariate spline spaces*, J. Austral. Math. Soc. Ser. B, 26 (1984), pp. 233–246.

[62] _____, *On the optimal approximation rates for criss-cross finite element spaces,* J. Comput. Appl. Math., 10 (1984), pp. 255–273.

[63] _____, *On the solution of certain systems of partial difference equations and linear dependence of translates of box splines,* Trans. Amer. Math. Soc., 292 (1985), pp. 305–320.

[64] _____, *On the local linear independence of translates of a box spline,* Studia Math., 82 (1985), pp. 243–262.

[65] _____, *Algebraic properties of discrete box splines,* Constr. Approx., 3 (1987), pp. 209–221.

[66] _____, *Convexity of multivariate Bernstein polynomials and box spline surfaces,* Studia Sci. Math. Hungar., to appear.

[67] Dyn, N., *Interpolation of scattered data by radial functions,* in Topics in Multivariate Approximation, C. K. Chui, L. L. Schumaker, and F. I. Utreras, eds., Academic Press, New York, 1987, pp. 47-62.

[68] Dyn, N., T. Goodman, and C. Micchelli, *Positive powers of certain conditionally negative definite matrices,* Nederl. Akad. Wetensch. Proc. Ser. A, 89 (1986), pp. 163–178.

[69] Dyn, N., D. Levin, and S. Rippa, *Numerical procedures for global surface fitting of scattered data by radial functions,* SIAM J. Sci. Stat. Comp., 7 (1986), pp. 639–659.

[70] Farin, G., *Subsplines über Dreiecken,* Ph. D. thesis, University of Braunschweig, 1979.

[71] ——, *Triangular Bernstein-Bézier patches,* Comp. Aided Geom. Design, 3(1986), 87-127.

[72] ——, ed., *Geometric Modeling*, Society for Industrial and Applied Mathematics, Philadelphia, 1987.

[73] Franke, R., *Recent advances in the approximation of surfaces from scattered data,* in Topics in Multivariate Approximation, C. K. Chui, L. L. Schumaker, and F. I. Utreras, eds., Academic Press, New York, 1987, pp. 79-98.

[74] Gmelig Meyling, R. H. J., *Approximation by cubic C^1-splines on arbitrary triangulations,* Numer. Math., 51 (1987), pp. 65–85.

[75] Ibrahim, A. KH., and L. L. Schumaker, *Super spline spaces of smoothness r and degree $d \geq 3r + 2$,* in Approximation Theory VI, C. K. Chui, L. L. Schumaker, and J. D. Ward, eds., Academic Press, New York, 1989, to appear.

[76] Höllig, K., *Multivariate splines,* SIAM J. Numer. Anal., 19 (1982), pp. 1013–1031.

[77] ——, *Box Splines,* in Approximation Theory V, C. K. Chui, L. L. Schumaker, and J. D. Ward, eds., Academic Press, New York, 1986, pp. 71–95.

[78] Hong, D., *On the dimensions of bivariate spline spaces,* Master thesis, Zhejiang University, Hangzhou, 1987.

[79] Jensen, T., *Assembling triangular and rectangular patches and multivariate splines*, *Geometric Modeling*, G. Farin ed., Society for Industrial and Applied Mathematics, Philadelphia, 1987, pp. 203–220.

[80] Jackson, I. R. H., *Convergence properties of radial basis functions,* DAMTP Report, Cambridge University, Cambridge, 1986.

[81] Jetter, K., *A short survey on cardinal interpolation by box splines,* in Topics in Multivariate Approximation, C. K. Chui, L. L. Schumaker, and F. I. Utreras, eds., Academic Press, New York, 1987, pp. 125–139.

[82] Jetter, K. and S. Riemenschneider, *Cardinal interpolation with box splines on submodules of* $\mathbf{Z}^d$*,* in Approximation Theory V, C. K. Chui, L. L. Schumaker, and J. D. Ward, eds., Academic Press, New York, 1986, pp. 403–406.

[83] _____, *Cardinal interpolation, submodules, and the 4-direction mesh,* Constr. Approx., 3 (1987), pp. 169–188.

[84] Jia, R. Q., *On the linear independence of translates of box splines,* J. Approx. Theory, 40 (1984), pp. 158–160.

[85] _____, *Local linear independence of the translates of a box splines,* Constr. Approx., 1 (1985), pp. 175–182.

[86] _____, *Approximation order from certain spaces of smooth bivariate splines on a three-direction mesh,* Trans. Amer. Math. Soc., 295 (1986), pp. 199-212.

[87] _____, *Approximation order of smooth bivariate piecewise polynomial functions on a three direction mesh,* in manuscript, 1989. (Dept. of Math. Zhejiang Univ.)

[88] Le Mehauté, A., *On Hermite elements of class* C^q *in* $\mathbb{R}^3$*,* in Approximation Theory IV, C. K. Chui, L. L. Schumaker, and J. D. Ward, eds., Academic Press, New York, 1983, pp. 581–586.

[89] _____, *Interpolation et approximation par des fonctions polynomiales par morceaux dans* $\mathbb{R}^n$*,* Thèse, Université de Rennes, 1984.

[90] _____, *Unisolvent interpolation in* $\mathbb{R}^n$ *and the simplicial polynomial finite element method,* in Topics in Multivariate Approximation, C. K. Chui, L. L. Schumaker, and F. I. Utreras, eds., Academic Press, New York 1987, pp. 141–151.

[91] Meinguet, J., *Surface spline interpolation: basic theory and computational aspects,* in Approximation Theory and Spline Functions, S. P. Singh, J. H. W. Burry, and B. Watson, eds., Reidel, Dordrecht, 1984, pp. 127–142.

[92] Micchelli, C. A., *A constructive approach to Kergin interpolation in* $\mathbb{R}^k$*: multivariate* B-*splines and Lagrange interpolation,* Rocky Mountain J. Math., 10 (1982), pp. 485–497.

[93] ———, *Interpolation of scattered data: distance matrices and conditionally positive definite functions,* in Approximation Theory and Spline Functions, S. P. Singh, J. H. W. Burry, and B. Watson, eds., Reidel, Dordrecht, 1984, pp. 143–145.

[94] ———, *Subdivision algorithms for curves and surfaces,* in Extension of B-spline Curve Algorithms to Surfaces, Siggraph 86, organized by C. de Boor, Dallas, 1986.

[95] Powell, M. J. D., *Piecewise quadratic surface fitting for contour plotting,* in Software for Numerical Mathematics, D. J. Evans, ed., Academic Press, London, 1974, pp. 253–271.

[96] ———, *Radial basis functions for multivariable interpolation: a review,* in Algorithms for the Approximation of Functions and Data, M. G. Cox and J. C. Mason, eds., Oxford University Press, 1987.

[97] Powell, M. J. D. and M. A. Sabin, *Piecewise quadratic approximations on triangles,* ACM Trans. Math. Software, 3 (1977), pp. 316–325.

[98] Prautzsch, H., *Unterteilungsalgorithmen für multivariate Splines,* Ph. D. thesis, University of Braunschweig, 1984.

[99] ———, *Generalized subdivision and convergence,* Comput. Aided Geom. Design, 2 (1985), pp. 69–75.

[100] Renka, R. J., *Triangulation and bivariate interpolation for irregularly distributed data points,* Ph. D. thesis, University of Texas, Austin, 1981.

[101] Riemenschneider, S. and K. Scherer, *Cardinal Hermite interpolation with box splines,* Constr. Approx., 3 (1987), pp. 223–238.

[102] Sabin, M. A., *The use of piecewise forms for the numerical representation of shape,* Ph. D. thesis, Hungarian Academy of Sciences, Budapest, 1977.

[103] Sablonnière, P., *Bases de Bernstein et approximants splines,* Ph. D. thesis, Université de Lille I., 1982.

[104] ———, *Interpolation by quadratic splines on triangles and squares,* Computers in Industry, 3 (1982), pp. 45–52.

[105] ———, *Composite finite elements of class C^k,* J. Comp. Appl. Math., 12 (1984), pp. 541–550.

[106] ———, *Bernstein-Bézier methods for the construction of bivariate spline approximants,* Comput. Aided Geom. Design, 2 (1985), pp. 29–36.

[107] SCHUMAKER, L. L., *Fitting surfaces to scattered data,* in Approximation Theory II, C. K. Chui, G. G. Lorentz, and L. L. Schumaker, eds., Academic Press, New York, 1976, pp. 203–268.

[108] ———, *On the dimension of spaces of piecewise polynomials in two variables,* in Multivariate Approximation Theory, W. Schempp and K. Zeller, eds., Birkhäuser, Basel, 1979, pp. 396–412.

[109] ———, *Bounds on the dimension of spaces of multivariate piecewise polynomials,* Rocky Mountain J. Math., 14 (1984), pp. 251–264.

[110] ———, *Triangulation methods,* in Topics in Multivariate Approximation, C. K. Chui, L. L. Schumaker, and F. I. Utreras, eds., Academic Press, New York, 1987, pp. 219–232.

[111] ———, *On super splines and finite elements*, CAT Report # 150, 1987.

[112] SCHUMAKER, L. L. AND W. VOLK, *Efficient algorithms for evaluating multivariate polynomials,* Comput. Aided Geom. Design, 3 (1986), pp. 149–154.

[113] UTRERAS, F. I., *Positive thin plate splines,* Approx. Theory Appl., 1 (1985), pp. 77–108.

[114] ———, *Constrained surface construction,* in Topics in Multivariate Approximation, C. K. Chui, L. L. Schumaker, and F. I. Utreras, eds., Academic Press, New York, 1987, pp. 233–254.

[115] WANG, R. H., *The structural characterization and interpolation for multivariate splines,* Acta Math. Sinica, 18 (1975), pp. 91–106.

[116] ———, *The dimension and basis of spaces of multivariate splines,* J. Comp. Appl. Math., 12 (1985), pp. 163–177.

[117] WARD, J. D., *Polynomial reproducing formulas and the commutator of a locally supported spline,* in Topics in Multivariate Approximation, C. K. Chui, L. L. Schumaker, and F. I. Utreras, eds., Academic Press, New York, 1987, pp. 255-263.

[118] ŽENIŠEK, A., *Interpolation polynomials on the triangle,* Numer. Math., 15 (1970), pp. 283–296.

[119] ———, *Polynomial approximation on tetrahedrons in the finite element method,* J. Approx. Theory, 7 (1973), pp. 334–351.

[120] ———, *A general theorem on triangular C^m elements,* RAIRO Numer. Anal., 22 (1974), pp. 119–127.

APPENDIX

Examples of Vertex Splines on A Mixed Partition

In this appendix, explicit formulation of the vertex splines $V_{\mathbf{v}}^{\gamma}$, and V_e^{γ} in the super spline space $\widehat{S}_5^1$ are given in terms of their B-coefficents on each triangle of their supports as shown in Figure A.1–A.7 as well as their graphs as shown in Figure A.8–A.14 and Figure A.16–A.22 with supports as shown in Figure A.15 and A.23. In Figure A.1–A.7, we only consider formulation of each vertex spline $V_{\mathbf{v}}^{\gamma}$ on a triangle inside its support which consists of triangles only for simplicity. However, we present graphs of those vertex splines whose support contain both parallelograms and triangles in Figure A.8–A.14 with support as shown in A.15.

Let $\mathbf{x}_i = (x_i, y_i)$ be a vertex in $\triangle$ and $\langle \mathbf{x}_i, \mathbf{x}_{i,k}, \mathbf{x}_{i,k+1}\rangle, k = 1, \cdots, l(\mathbf{x}_i)$, be the triangles in $\triangle$ that share $\mathbf{x}_i$ as their common vertex. Let $e = [\mathbf{x}_{e,1}, \mathbf{x}_{e,2}]$ be an interior edge of $\triangle$ and $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3}\rangle$ and $\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,4}\rangle$ be the two triangles of $\triangle$ that share e as their common edge. For simplicity, we may assume that e is not a singular edge. Also, we have assumed that all vertices of $\triangle$ have been enumerated and all edges have been assigned a direction which is denoted by $[\mathbf{x}_{e,1}, \mathbf{x}_{e,2}]$.

Denote $< \mathbf{x}, \mathbf{y} >= \{\mathbf{x} + t(\mathbf{y} - \mathbf{x}) : 0 \le t \le 1\}$. As usual, we denote $\mathbf{x} = (x, y)$ and denote by

$$\delta\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\rangle = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

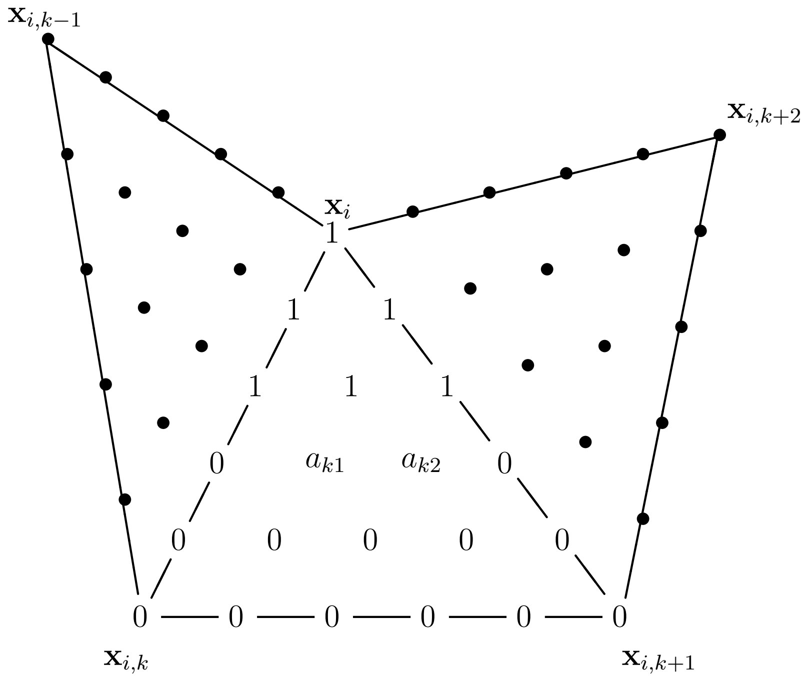
the signed area of the triangle $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\rangle$. In Figure A.1–A.6 we need the following notation.

$$\alpha_k = \frac{\delta\langle \mathbf{x}_{i,k-1}, \mathbf{x}_{i,k}, \mathbf{x}_{i,k+1}\rangle}{\delta\langle \mathbf{x}_{i,k}, \mathbf{x}_i, \mathbf{x}_{i,k-1}\rangle} \qquad \beta_k = -\frac{\delta\langle \mathbf{x}_{i,k}, \mathbf{x}_{i,k+1}, \mathbf{x}_i\rangle}{\delta\langle \mathbf{x}_{i,k}, \mathbf{x}_i, \mathbf{x}_{i,k-1}\rangle}$$

$$\gamma_k = \frac{\delta\langle \mathbf{x}_{i,k}, \mathbf{x}_{i,k+1}, \mathbf{x}_{i,k+2}\rangle}{\delta\langle \mathbf{x}_{i,k+1}, \mathbf{x}_{i,k+2}, \mathbf{x}_i\rangle}$$

and

$$\begin{aligned}
b_k &= \frac{1}{5}(x_{i,k} - x_i), \qquad c_k = \frac{1}{5}(y_{i,k} - y_i), \qquad d_k = \frac{1}{20}(x_{i,k} - x_i)^2, \\
e_k &= \frac{1}{20}(y_{i,k} - y_i)^2, \qquad f_k = \frac{1}{20}(x_{i,k} - x_i)(x_{i,k+1} - x_i), \\
h_k &= \frac{1}{20}(y_{i,k} - y_i)(y_{i,k+1} - y_i), \quad g_k = \frac{1}{10}(x_{i,k} - x_i)(y_{i,k} - y_i), \\
\tilde{g}_k &= \frac{1}{20}[(x_{i,k+1} - x_i)(y_{i,k} - y_i) + (y_{i,k+1} - y_i)(x_{i,k} - x_i)].
\end{aligned}$$



I. $<\mathbf{x}_{i,k}, \mathbf{x}_i>, <\mathbf{x}_{i,k+1}, \mathbf{x}_i>$ are interior edges

$$a_{k1} = \begin{cases} \alpha_k + \beta_k & \text{if } <\mathbf{x}_{i,k}, \mathbf{x}_i> = [\mathbf{x}_{i,k}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

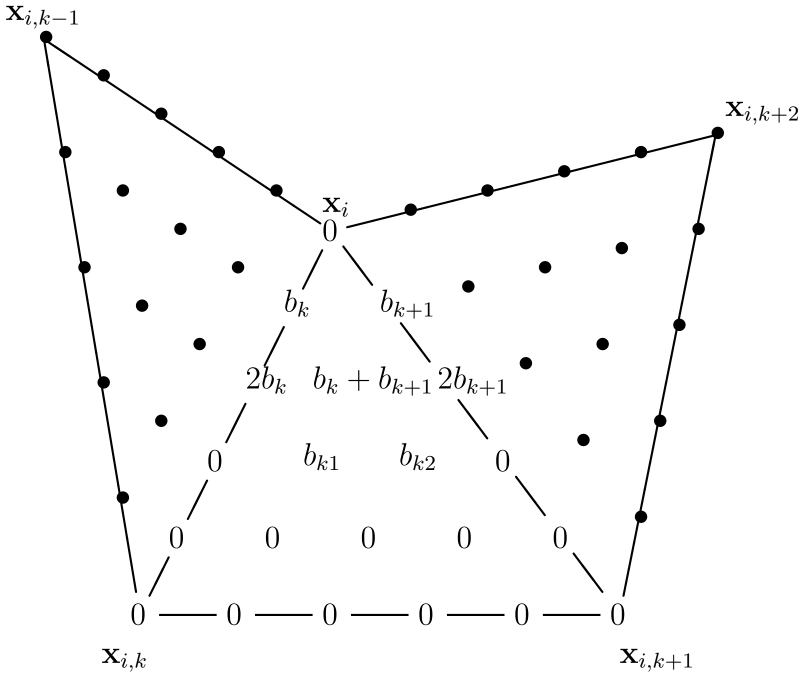
$$a_{k2} = \begin{cases} 1 & \text{if } <\mathbf{x}_{i,k+1}, \mathbf{x}_i> = [\mathbf{x}_{i,k+1}, \mathbf{x}_i] \\ \gamma_k & \text{otherwise} \end{cases}$$

II. $<\mathbf{x}_{i,k}, \mathbf{x}_i>, <\mathbf{x}_{i,k+1}, \mathbf{x}_i>$ are boundary edges

$$a_{k1} = \begin{cases} 1 & \text{if } <\mathbf{x}_{i,k}, \mathbf{x}_i> = [\mathbf{x}_{i,k}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

$$a_{k2} = \begin{cases} 1 & \text{if } <\mathbf{x}_{i,k+1}, \mathbf{x}_i> = [\mathbf{x}_{i,k+1}, \mathbf{x}_i] \\ \gamma_k & \text{otherwise} \end{cases}$$

Fig. A.1 Vertex spline $V_{\mathbf{x}_i}^{(0,0)}$



I. $<\mathbf{x}_{i,k}, \mathbf{x}_i>, <\mathbf{x}_{i,k+1}, \mathbf{x}_i>$ are interior edges

$$b_{k1} = \begin{cases} 2b_k\alpha_k + (b_{k-1} + 2b_k)\beta_k & \text{if } <\mathbf{x}_{i,k}, \mathbf{x}_i> = [\mathbf{x}_{i,k}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

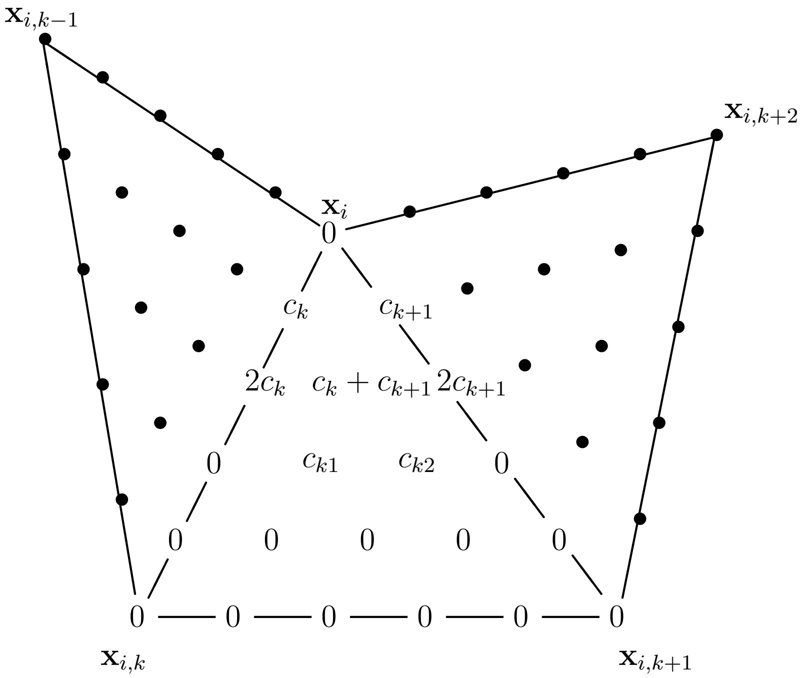
$$b_{k2} = \begin{cases} b_k + 2b_{k+1} & \text{if } <\mathbf{x}_{i,k+1}, \mathbf{x}_i> = [\mathbf{x}_{i,k+1}, \mathbf{x}_i] \\ 2b_{k+1}\gamma_k & \text{otherwise} \end{cases}$$

II. $<\mathbf{x}_{i,k}, \mathbf{x}_i>, <\mathbf{x}_{i,k+1}, \mathbf{x}_i>$ are boundary edges

$$b_{k1} = \begin{cases} 2b_k + b_{k+1} & \text{if } <\mathbf{x}_{i,k}, \mathbf{x}_i> = [\mathbf{x}_{i,k}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

$$b_{k2} = \begin{cases} b_k + 2b_{k+1} & \text{if } <\mathbf{x}_{i,k+1}, \mathbf{x}_i> = [\mathbf{x}_{i,k+1}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

Fig. A.2 Vertex spline $V_{\mathbf{x}_i}^{(1,0)}$



I. $< \mathbf{x}_{i,k}, \mathbf{x}_i >, < \mathbf{x}_{i,k+1}, \mathbf{x}_i >$ are interior edges

$$c_{k1} = \begin{cases} 2c_k\alpha_k + (c_{k-1} + 2c_k)\beta_k & \text{if } < \mathbf{x}_{i,k}, \mathbf{x}_i >= [\mathbf{x}_{i,k}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

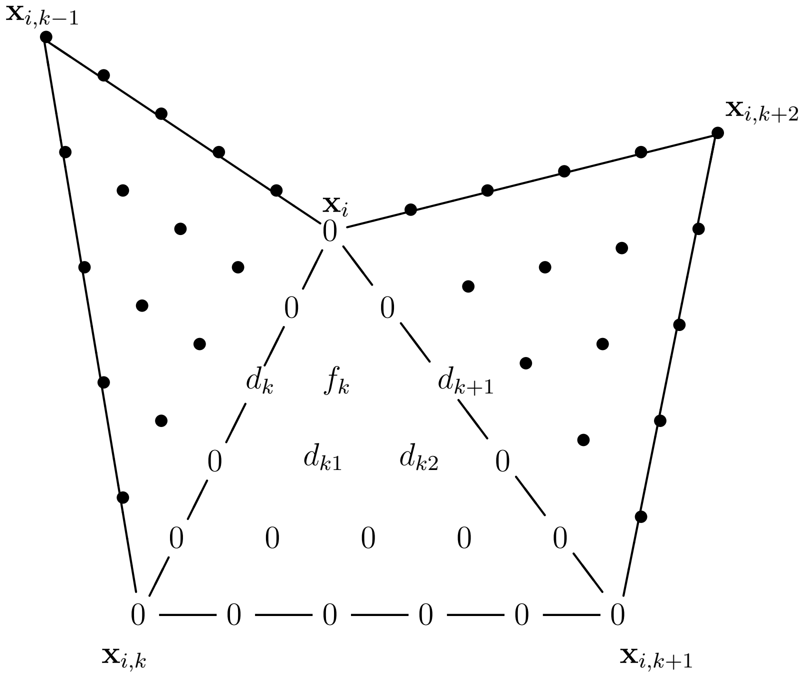
$$c_{k2} = \begin{cases} c_k + 2c_{k+1} & \text{if } < \mathbf{x}_{i,k+1}, \mathbf{x}_i >= [\mathbf{x}_{i,k+1}, \mathbf{x}_i] \\ 2c_{k+1}\gamma_k & \text{otherwise} \end{cases}$$

II. $< \mathbf{x}_{i,k}, \mathbf{x}_i >, < \mathbf{x}_{i,k+1}, \mathbf{x}_i >$ are boundary edges

$$c_{k1} = \begin{cases} 2c_k + c_{k+1} & \text{if } < \mathbf{x}_{i,k}, \mathbf{x}_i >= [\mathbf{x}_{i,k}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

$$c_{k2} = \begin{cases} c_k + 2c_{k+1} & \text{if } < \mathbf{x}_{i,k+1}, \mathbf{x}_i >= [\mathbf{x}_{i,k+1}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

Fig. A.3 Vertex spline $V_{\mathbf{x}_i}^{(0,1)}$



I. $< \mathbf{x}_{i,k}, \mathbf{x}_i >, < \mathbf{x}_{i,k+1}, \mathbf{x}_i >$ are interior edges

$$d_{k1} = \begin{cases} d_k\alpha_k + (d_k + 2f_{k-1})\beta_k & \text{if } < \mathbf{x}_{i,k}, \mathbf{x}_i > = [\mathbf{x}_{i,k}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

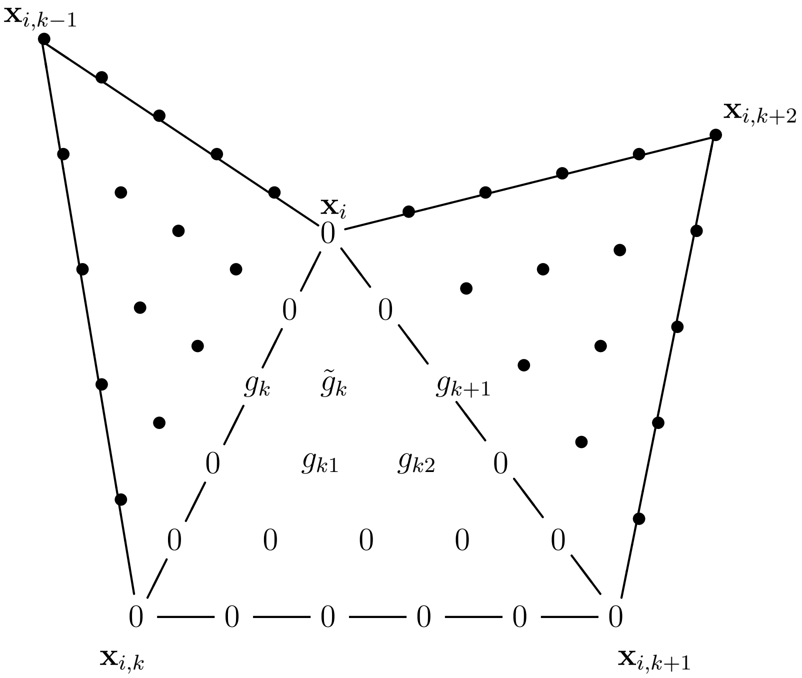
$$d_{k2} = \begin{cases} d_{k+1} + 2f_k & \text{if } < \mathbf{x}_{i,k+1}, \mathbf{x}_i > = [\mathbf{x}_{i,k+1}, \mathbf{x}_i] \\ d_{k+1}\gamma_k & \text{otherwise} \end{cases}$$

II. $< \mathbf{x}_{i,k}, \mathbf{x}_i >, < \mathbf{x}_{i,k+1}, \mathbf{x}_i >$ are boundary edges

$$d_{k1} = \begin{cases} d_k + 2f_k & \text{if } < \mathbf{x}_{i,k}, \mathbf{x}_i > = [\mathbf{x}_{i,k}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

$$d_{k2} = \begin{cases} d_{k+1} + 2f_k & \text{if } < \mathbf{x}_{i,k+1}, \mathbf{x}_i > = [\mathbf{x}_{i,k+1}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

Fig. A.4 Vertex spline $V_{\mathbf{x}_i}^{(2,0)}$



I. $<\mathbf{x}_{i,k}, \mathbf{x}_i>, <\mathbf{x}_{i,k+1}, \mathbf{x}_i>$ are interior edges

$$g_{k1} = \begin{cases} g_k\alpha_k + (g_k + 2\tilde{g}_{k-1})\beta_k & \text{if } <\mathbf{x}_{i,k}, \mathbf{x}_i> = [\mathbf{x}_{i,k}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

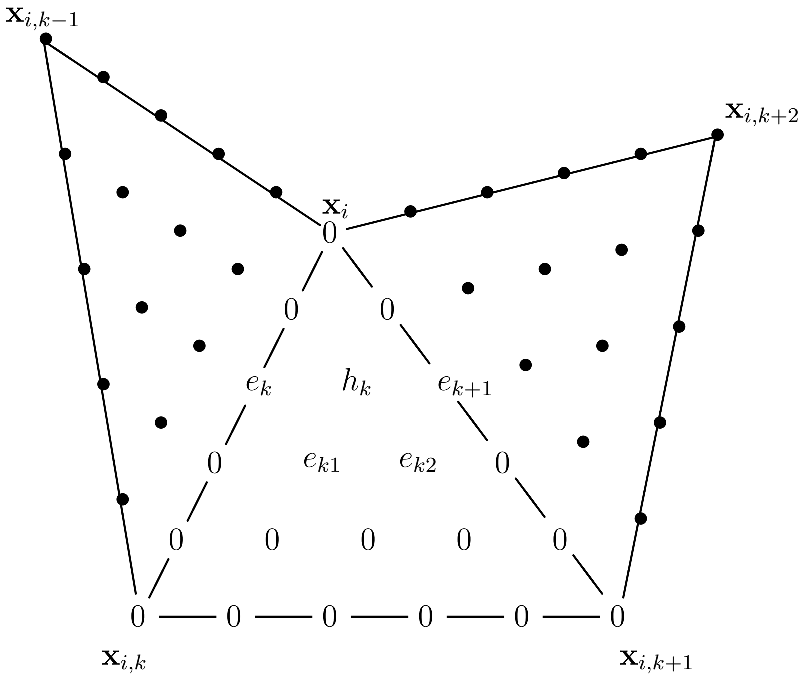
$$g_{k2} = \begin{cases} g_{k+1} + 2\tilde{g}_k & \text{if } <\mathbf{x}_{i,k+1}, \mathbf{x}_i> = [\mathbf{x}_{i,k+1}, \mathbf{x}_i] \\ g_{k+1}\gamma_k & \text{otherwise} \end{cases}$$

II. $<\mathbf{x}_{i,k}, \mathbf{x}_i>, <\mathbf{x}_{i,k+1}, \mathbf{x}_i>$ are boundary edges

$$g_{k1} = \begin{cases} g_k + 2\tilde{g}_k & \text{if } <\mathbf{x}_{i,k}, \mathbf{x}_i> = [\mathbf{x}_{i,k}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

$$g_{k2} = \begin{cases} g_{k+1} + 2\tilde{g}_k & \text{if } <\mathbf{x}_{i,k+1}, \mathbf{x}_i> = [\mathbf{x}_{i,k+1}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

Fig. A.5 Vertex spline $V_{\mathbf{x}_i}^{(1,1)}$



I. $< \mathbf{x}_{i,k}, \mathbf{x}_i >, < \mathbf{x}_{i,k+1}, \mathbf{x}_i >$ are interior edges

$$e_{k1} = \begin{cases} e_k\alpha_k + (e_k + 2h_{k-1})\beta_k & \text{if } < \mathbf{x}_{i,k}, \mathbf{x}_i >= [\mathbf{x}_{i,k}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

$$e_{k2} = \begin{cases} e_{k+1} + 2h_k & \text{if } < \mathbf{x}_{i,k+1}, \mathbf{x}_i >= [\mathbf{x}_{i,k+1}, \mathbf{x}_i] \\ e_{k+1}\gamma_k & \text{otherwise} \end{cases}$$

II. $< \mathbf{x}_{i,k}, \mathbf{x}_i >, < \mathbf{x}_{i,k+1}, \mathbf{x}_i >$ are boundary edges

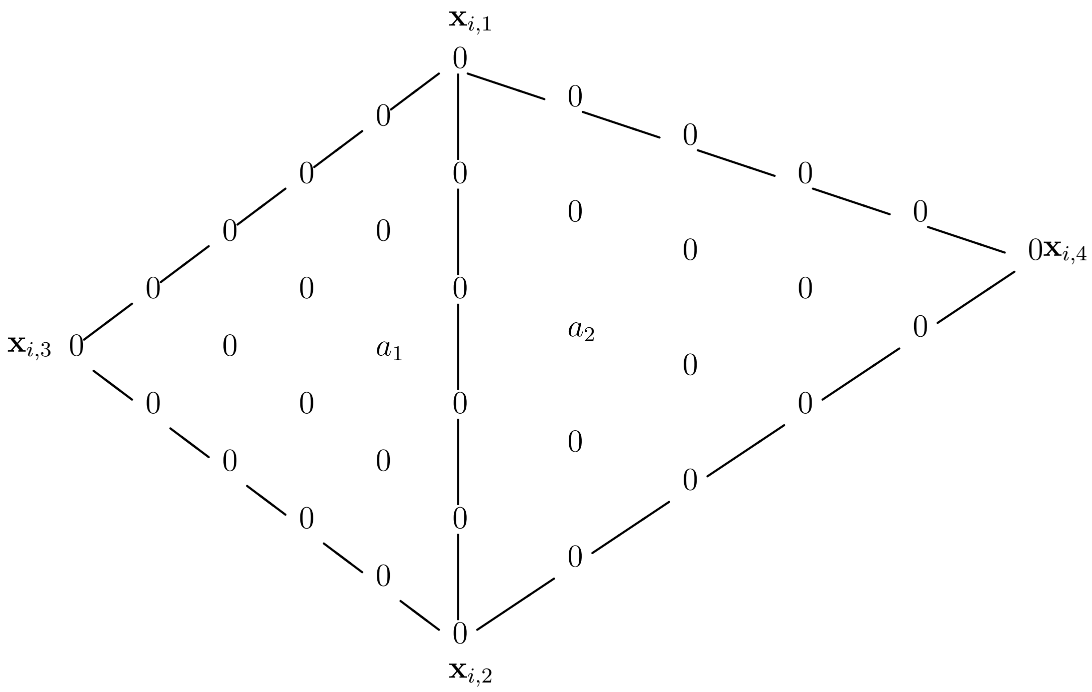
$$e_{k1} = \begin{cases} e_k + 2h_k & \text{if } < \mathbf{x}_{i,k}, \mathbf{x}_i >= [\mathbf{x}_{i,k}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

$$e_{k2} = \begin{cases} e_{k+1} + 2h_k & \text{if } < \mathbf{x}_{i,k+1}, \mathbf{x}_i >= [\mathbf{x}_{i,k+1}, \mathbf{x}_i] \\ 0 & \text{otherwise} \end{cases}$$

Fig. A.6 Vertex spline $V_{\mathbf{x}_i}^{(0,2)}$

In addition, let

$$l_e = \delta\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,3}\rangle, \quad \bar{l}_e = \delta\langle \mathbf{x}_{e,1}, \mathbf{x}_{e,2}, \mathbf{x}_{e,4}\rangle.$$



I. $< \mathbf{x}_{i,1}, \mathbf{x}_{i,2} >$ is an interior edge

$$a_1 = \begin{cases} l_e & \text{if } < \mathbf{x}_{i,1}, \mathbf{x}_{i,2} >= [\mathbf{x}_{i,1}, \mathbf{x}_{i,2}] \\ -l_e & \text{otherwise} \end{cases}$$

$$a_2 = \begin{cases} \bar{l}_e & \text{if } < \mathbf{x}_{i,1}, \mathbf{x}_{i,2} >= [\mathbf{x}_{i,1}, \mathbf{x}_{i,2}] \\ -\bar{l}_e & \text{otherwise} \end{cases}$$

II. $< \mathbf{x}_{i,1}, \mathbf{x}_{i,2} >$ is a boundary edge. Assume $< \mathbf{x}_{i,1}, \mathbf{x}_{i,2}, \mathbf{x}_{i,3} >$ is the only triangle of $\triangle$ containing $< \mathbf{x}_{i,1}, \mathbf{x}_{i,2} >$.

$a_1 = l_e$

Fig. A.7 Vertex spline V_e

VITA

I. Personal Data

- Name: Ming-Jun Lai
- Birthday: Oct. 26, 1956
- Birthplace: Hangzhou, China
- Citizenship: Chinese
- Social Security Number: 459-67-8163
- Marital Status: Married, wife's name: Ma, Lingyun
- Degree: B.S. Mathematics in January 1982, Hangzhou University, China
- Temporary Address: Dept. of Math. The University of Utah, Salt Lake City, Utah 84112, U.S.A.
- Permanent Address: c/o Mr. Ma, Wenjun Guan Jia Long #33, Yuyao, Zhejiang, China

II. Publications

[1] A multivariate analog of Marsden's identity and a quasi-interpolation scheme, (with C.K. Chui), *Constructive Approximation*, **3** (1987), pp. 111-122.

[2] On multivariate vertex splines and applications, (with C.K. Chui), in *Topics in Multivariate Approximation*, Chui, C.K., L.L. Schumaker, and F. Utreras eds. Academic Press, 1987, pp. 19-36.

[3] VanderMonde determinants and Lagrange interpolation in $\mathbf{R}^s$, (with C.K. Chui), *Nonlinear and Convex analysis*, B.L.Lin & S.Simons eds., Marcel Dekker, New York, 1987, pp. 23-35.

[4] On multivariate Newtonian interpolation, (with X.H. Wang), *Scientia Sinica*, **29** (1986), pp. 23-32.

[5] On bivariate super vertex splines, (with C.K. Chui), to appear in *Constr. Approx.*

This dissertation was typed in LaTeXby the author Ming-Jun Lai.