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Energy minimization method for scattered data Hermite interpolation

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Abstract

Given a set of scattered data with derivatives values, we use a minimal energy method to find Hermite interpolation based on bivariate spline spaces over a triangulation of the scattered data locations. We show that the minimal energy method produces a unique Hermite spline interpolation of the given scattered data with derivative values. Also we show that the Hermite spline interpolation converges to a given sufficiently smooth function f if the data values are obtained from this f. That is, the surface of the Hermite spline interpolation resembles the given set of derivative values. Some numerical examples are presented to demonstrate our method.
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1. Introduction

Suppose $V=\{v_i=(x_i,y_i)\}_{i=1}^n$ is a set of data locations lying in a domain $\Omega\subset\mathbf{R}^2$. Let $\triangle$ be a triangulation of the data locations. Let $\{z_i^{\nu,\mu},\ 0\leqslant\nu+\mu\leqslant r,\ i=1,\ldots,n\}$ be given real values. We would like to construct a spline function $s\in C^{r+1}(\Omega)$ over triangulation $\triangle$ such that

$$D_x^\nu D_y^\mu s(v_i)=z_i^{\nu,\mu},\quad 0\leqslant\nu+\mu\leqslant r,\ i=1,\ldots,n. \tag{1.1}$$

Certainly we want s to resemble the given data values. More precisely, if $z_i^{\nu,\mu}=D_x^\nu D_y^\mu f(x_i,y_i)$, $\nu+\mu\leqslant r$, for $f\in C^m(\Omega)$ with $m\geqslant r+2$, we would like s satisfying (1.1) and

$$\|f-s\|_{L_\infty(\Omega)}\leqslant C|\triangle|^{r+2}, \tag{1.2}$$

where C is a constant which is dependent on f and $C^m(\Omega)$ is the standard space of all continuously mth differentiable functions. Here $|\triangle|$ is the size of the triangulation $\triangle$, i.e., the largest of the length of edges of $\triangle$. For $r=0$, this is

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a typical scattered data interpolation problem (e.g., [2,8,9,12,22]). For $r \geqslant 1$, this is a classical Hermite scattered data interpolate problem. It has been studied in the literature (e.g., [1,13,15,16,18,17]) by constructing C^{r+1} macro-elements. However, all the constructions require higher order derivative information than the given data values. Also, normal derivatives at edges are needed in order to make these macro-elements smooth across common edges. Still there are some other methods for solving Hermite scattered data interpolate problem (e.g., [20] and [21]). Since such higher order and normal derivative information are not available in practice, we have to use other techniques to estimate the needed information. As in the case $r = 0$, one can use a minimal energy method to construct an interpolatory spline. For the Hermite interpolation problem, we can use the minimal energy technique too. This is what we shall do in this paper. We shall first show such a minimal energy method will give a unique Hermite interpolatory spline. In addition, we will also study their approximation property. That is, we shall establish the approximation order in (1.2) for the minimal energy Hermite spline interpolation. This shows that the minimal energy Hermite spline interpolation resembles the given set of derivative values.

Next let us explain the spline spaces we shall use in this paper. Recall that $\triangle$ is a triangulation of the given data locations in a polygonal domain Ω in $\mathbf{R}^2$. That is, all the data locations (x_i, y_i), $i = 1, \dots, n$, are vertices of $\triangle$. We use the space of polynomial splines

$$S_d^{r+1}(\triangle) := \left\{ s \in C^{r+1}(\Omega) \colon s|_T \in P_d \ \forall T \in \triangle \right\},$$

where $d \geqslant 3r + 5$ and $r \geqslant 0$ are given integers and P_d is the space of bivariate polynomials of degree d. Such spline spaces have been studied by many researchers in the literature, e.g., in [19] and the references therein.

According to [4] and [5], there are spline functions $s \in S_d^{r+1}(\triangle)$ with $d \geqslant 3r + 5$ satisfying the interpolation conditions (1.1). Thus, the existence of a Hermite interpolatory spline can be easily understood. The proof of the uniqueness is a simple generalization of the counterpart when $r = 0$ (cf. [8]). Next we are interested in how well the interpolatory splines resemble the given data. When $r = 0$, the approximation of spline interpolation was studied in [10]. The researchers in [10] showed that minimal energy interpolatory splines converge to the given data values when the number of data values increases and the size of triangulation decreases. We shall generalize this result to the Hermite interpolation setting. It is not a straightforward generalization. The main difficulties are Theorem 4.3 and Lemma 4.1 which are quite different from the setting when $r = 0$. In addition, our proof of Lemma 4.2 is much simpler than the corresponding one in [10]. Our main theorem in this paper is Theorem 4.5 which gives the convergence rate of the minimal energy spline interpolation. In addition, we shall present a computational method to find the minimal energy spline Hermite interpolation which is a straightforward generalization of the computational algorithm for the minimal energy Lagrange interpolation in [2]. We have implemented our computational method in MATLAB and numerically experimented the method with various functions to verify the convergence rate.

The paper is organized as follows. In Section 2 we review some well-known Bernstein–Bézier notation. An energy minimization method is explained in Section 3 and the existence and uniqueness of spline Hermite interpolation are discussed there. In Section 4 we derive error bounds for spline Hermite interpolation using the energy minimization method. A computational method is explained in Section 5 together with some numerical examples. A numerical example for wind potential reconstruction is presented to demonstrate the usefulness of our method.

2. Preliminaries

Given a triangulation $\triangle$ and integers $0 \leqslant m < d$, we write

$$S_d^m(\triangle) := \left\{ s \in C^m(\Omega) \colon s|_T \in P_d, \ \text{for all } T \in \triangle \right\}$$

for the usual space of splines of degree d and smoothness m, where P_d is the $\binom{d+2}{2}$ dimensional space of bivariate polynomials of degree d. Throughout the paper we shall make extensive use of the well-known Bernstein–Bézier representation of splines. For each triangle $T = \langle v_1, v_2, v_3 \rangle$ in $\triangle$ with vertices v_1, v_2, v_3 the corresponding polynomial piece $s|_T$ is written in the form

$$s|_T = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^d,$$

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where B^d_{ijk} are the Bernstein–Bézier polynomials of degree d associated with T. In particular, if $(\lambda_1,\lambda_2,\lambda_3)$ are the barycentric coordinates of any point $u\in\mathbf{R}^2$ in term of the triangle T, then

$$B^d_{ijk}(u):=\frac{d!}{i!j!k!}\lambda_1^i\lambda_2^j\lambda_3^k,\quad i+j+k=d.$$

Usually, we associate the *Bernstein–Bézier coefficients* $\{c^T_{ijk}\}_{i+j+k=d}$ with the domain points $\{\xi^T_{ijk}:=(iv_1+jv_2+kv_3)/d\}_{i+j+k=d}$.

Definition 2.1. Let $\beta<\infty$. A triangulation $\triangle$ is said to be *β-quasi-uniform* provided that $|\triangle|\leqslant\beta\rho_\triangle$, where $|\triangle|$ is the maximum of the diameters of the triangles in $\triangle$, and $\rho_\triangle$ is the minimum of the radii of the incircles of triangles of $\triangle$.

It is easy to see that if $\triangle$ is β-quasi-uniform, then the smallest angle in $\triangle$ is bounded below by $2/\beta$.

Given $T:=\langle v_1,v_2,v_3\rangle$ and an integer $0\leqslant m<d$, we set $R^T_m(v_1):=\{\xi^T_{ijk}\colon i=d-m\}$. We recall that the *ring of radius m around* v_1 is the set $R_m(v_1):=\bigcup\{R^T_m(v_1)\colon T \text{ has a vertex at } v_1\}$. The rings around v_2 and v_3 are defined similarly.

Recall that a *determining set* for a spline space $S\subseteq S^0_d(\triangle)$ is a subset $\mathcal{M}$ of the set of domain points such that if $s\in S$ and $c_\xi=0$ for all $\xi\in\mathcal{M}$, then $c_\xi=0$ for all domain points. The set $\mathcal{M}$ is called a *minimal determining set (MDS)* for S if there is no smaller determining set. It is known that $\mathcal{M}$ is a *MDS* for S if and only if every spline $s\in S$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$.

Suppose that $T:=\langle v_1,v_2,v_3\rangle$ and $\widehat{T}:=\langle v_4,v_3,v_2\rangle$ are two adjoining triangles from $\triangle$ which share the edge $e:=\langle v_2,v_3\rangle$, and let

$$s|_T=\sum_{i+j+k=d}c_{ijk}B^d_{ijk},\qquad s|_{\widehat{T}}=\sum_{i+j+k=d}\widehat{c}_{ijk}\widehat{B}^d_{ijk},$$

where B^d_{ijk} and $\widehat{B}^d_{ijk}$ are the Bernstein polynomials of degree d on the triangles T and $\widehat{T}$, respectively. Given integers $0\leqslant n\leqslant j\leqslant d$, let $\tau^n_{j,e}$ be the linear functional defined on $S^0_d(\triangle)$ by

$$\tau^n_{j,e}s:=c_{n,d-j,j-n}-\sum_{\nu+\mu+\kappa=n}\widehat{c}_{\nu,\mu+j-n,\kappa+d-j}\widehat{B}^n_{\nu\mu\kappa}(v_1).$$

It is called *smoothness functional of order n*. Clearly a spline $s\in S^0_d(\triangle)$ belongs to $C^r(\Omega)$ for some $r>0$ if and only if

$$\tau^n_{m,e}s=0,\quad n\leqslant m\leqslant d,\ 0\leqslant n\leqslant r.$$

So we shall often make use of smoothness conditions to calculate one coefficient of a spline in terms of others.

Recall from [4] and [5] that for any given sufficiently smooth function $f\in\Omega$, there exists a quasi-interpolatory operator Q mapping f to $S^r_d(\triangle)$ with $d\geqslant 3r+2$ which achieves the optimal approximation order of $S^r_d(\triangle)$. That is

Theorem 2.2. *(Cf. [5].) Let $r\geqslant 1$ and $d\geqslant 3r+2$. Suppose $f\in C^m(\Omega)$ with $m\geqslant 2r$. Then there exists a spline function $Q_f\in S^r_d(\triangle)$ satisfying* (1.1) *and*

$$\left\|D^\alpha_x D^\beta_y(f-Qf)\right\|_{L_\infty(\Omega)}\leqslant K|\triangle|^{m-\alpha-\beta}|f|_{m,\infty,\Omega}$$

for $0\leqslant\alpha+\beta\leqslant m$, where $|\triangle|$ is the mesh size of $\triangle$ (i.e., the diameter of the largest triangle), and $|f|_{m,\infty}$ is the usual maximum norm of the derivatives of order m of f over Ω.

When $d<3r+2$, similar approximation results are available for some special spline spaces, see [13,15–18].

3. Existence and uniqueness of minimal energy Hermite interpolatory splines

We let $M=\dim S^{r+1}_d(\triangle)$ for a fixed integer $d\geqslant 3r+5$. Clearly, we can see that $M\gg N=\binom{r+2}{2}V$, where N denote the number of given data values. Using the locally supported basis functions $\{\phi_i,\ i=1,\dots,M\}$ in [4] or in

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[5], any spline function s in $S_d^{r+1}(\triangle)$ can be represented by $s=\sum_{i=1}^M c_i\phi_i$, for some coefficients $\{c_i\}_{i=1}^M$ with $\{c_i\}_{i=1}^N$ being the fixed coefficients given by the function values $\{z_i^{\nu,\mu}\}_{i=1}^n$, $0\leqslant \nu+\mu\leqslant r$, in the following sense: if $c_j=z_k^{\nu,\mu}$ for some k,ν,μ, then

$$D_x^\alpha D_y^\beta \phi_j(x_i,y_i)=\begin{cases}1, & \text{if } i=k,\ \alpha=\nu \text{ and } \beta=\mu,\\ 0, & \text{otherwise}\end{cases}$$

for $i=1,\dots,n$, $j=1,\dots,N$, and $\alpha+\beta\leqslant r$ as well as $D_x^\alpha D_y^\beta\phi_j(x_i,y_i)=0$ for $i=1,\dots,n$, $j=N+1,\dots,M$ and $\alpha+\beta\leqslant r$. So to determine a spline function s uniquely, we need to determine the particular set of coefficients $\{c_i\}_{i=N+1}^M$. We shall use an energy minimization method to do so.

Recall that an energy functional $E(f)$ is an expression for the amount of potential energy in a thin elastic plate f that passes through the data points V. The potential energy of the thin plate is given by

$$E=\int_\Omega \left[aH^2+bK\right]dx\,dy, \tag{3.1}$$

where H and K are mean curvature and Gaussian curvature of the surface S and a and b are constants which depend on the material of the plate (cf. [23]). In particular,

$$H=\frac{\kappa_1+\kappa_2}{2}=\frac{(1+f_x^2)f_{yy}-2f_xf_yf_{xy}+(1+f_y^2)f_{xx}}{(1+f_x^2+f_y^2)^{3/2}}$$

and

$$K=\kappa_1\kappa_2=\frac{f_{xx}f_{yy}-f_{xy}^2}{(1+f_x^2+f_y^2)^2},$$

where κ_1 and κ_2 are the principle curvatures of the surface of the plate. Assume that $f_x\approx 0$ and $f_y\approx 0$ when the plate has small deflections. The potential energy E can be simplified in the following form:

$$E(f)=\int_\Omega \left[a(f_{xx}+f_{yy})^2-2(1-\omega)\left(f_{xx}f_{yy}-f_{xy}^2\right)\right]dx\,dy,$$

where the parameter ω is a constant depending on the material at hand (e.g., [9]). For simplicity, we choose $a=1$ and $\omega=0$. That is,

$$E(f)=\int_\Omega \left[f_{xx}^2+2f_{xy}^2+f_{yy}^2\right]dx\,dy=\int_\Omega\left[\sum_{k=0}^2\binom{2}{k}\left[\left(\frac{\partial}{\partial x}\right)^k\left(\frac{\partial}{\partial y}\right)^{2-k}f\right]^2\right]dx\,dy$$

which is commonly used in the literature, e.g., [8].

In this paper, we use a generalized version of the energy functional $E(f)$ which can be represented as

$$E(f)=\int_\Omega\left[\sum_{k=0}^{r+2}\binom{r+2}{k}\left[\left(\frac{\partial}{\partial x}\right)^k\left(\frac{\partial}{\partial y}\right)^{r+2-k}f\right]^2\right]dx\,dy. \tag{3.2}$$

This energy functional is not brand new and it was used in [7] for D^m splines.

For $s=\sum_{i=1}^M c_i\phi_i\in S_d^{r+1}(\triangle)$, we can see

$$E(s)=\int_\Omega\left[\sum_{k=0}^{r+2}\binom{r+2}{k}\left[\sum_{i=1}^M c_i\left(\frac{\partial}{\partial x}\right)^k\left(\frac{\partial}{\partial y}\right)^{r+2-k}\phi_i\right]^2\right]dx\,dy. \tag{3.3}$$

Since the coefficients $\{c_i\}_{i=1}^N$ are already determined, $E(s)$ is a function of the $M-N$ coefficients $\{c_i\}_{i=N+1}^M$. That is, we can write $E(s)=E(c_{N+1},\dots,c_M)$. Thus the minimal energy Hermite interpolation problem can be formulated as follows: find a spline $s_*\in S_d^{r+1}(\triangle)$ such that

$$D_x^\nu D_y^\mu s_*(x_i,y_i)=z_i^{\nu,\mu},\quad 0\leqslant\nu+\mu\leqslant r,\ i=1,\dots,n,$$

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and

$$E(s_*) = \min\left\{E(s)\colon D_x^\nu D_y^\mu s(x_i, y_i) = z_i^{\nu,\mu},\ 0 \leqslant \nu + \mu \leqslant r,\ i = 1, \ldots, n,\ s \in S_d^{r+1}(\triangle)\right\}. \tag{3.4}$$

This is a straightforward generalization of the minimal energy Lagrange interpolation using bivariate splines (cf., e.g., [10]). The existence and uniqueness of our problem (3.3) and (3.4) follow from the similar analysis discussed in [10].

For the self-containedness, we include a short and explicit proof. Define

$$\langle \phi_i, \phi_j \rangle = \int_\Omega \left[\sum_{k=0}^{r+2} \binom{r+2}{k} \left[\left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y}\right)^{r+2-k} \phi_i \left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y}\right)^{r+2-k} \phi_j \right]\right] dx\, dy$$

to be a bi-linear form for any $\phi_i, \phi_j \in S_d^{r+1}(\triangle)$.

In order to minimize $E(s)$, we need to have $\frac{\partial}{\partial c_j} E(s_*) = 0$ for each $c_j \in \{c_{N+1}, \ldots, c_M\}$. A direct computation yields

$$\frac{\partial}{\partial c_j} E(s) = 2 \sum_{i=N+1}^{M} c_i \langle \phi_i, \phi_j \rangle + 2 \sum_{i=1}^{N} c_i \langle \phi_i, \phi_j \rangle$$

or

$$\sum_{i=N+1}^{M} c_i \langle \phi_i, \phi_j \rangle = -\sum_{i=1}^{N} c_i \langle \phi_i, \phi_j \rangle$$

for each $j \in \{N+1, \ldots, M\}$. Thus we have a linear system of $M-N$ equations in $M-N$ unknown coefficients $\{c_{N+1}, \ldots, c_M\}$, i.e., $\mathbf{Ac} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} \langle \phi_{N+1}, \phi_{N+1} \rangle & \langle \phi_{N+2}, \phi_{N+1} \rangle & \cdots & \langle \phi_M, \phi_{N+1} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \phi_{N+1}, \phi_M \rangle & \langle \phi_{N+2}, \phi_M \rangle & \cdots & \langle \phi_M, \phi_M \rangle \end{pmatrix},$$

$$\mathbf{b} = -\begin{pmatrix} \sum_{i=1}^N c_i \langle \phi_i, \phi_{N+1} \rangle \\ \vdots \\ \sum_{i=1}^N c_i \langle \phi_i, \phi_M \rangle \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} c_{N+1} \\ \vdots \\ c_M \end{pmatrix}.$$

Now the existence and uniqueness follow from the fact that $\mathbf{A}$ is invertible. Indeed, if there exists a nonzero vector $\mathbf{c}$ such that $\mathbf{Ac} = 0$, then $\mathbf{c}^T\mathbf{Ac} = 0$ which is exactly $E(s_0) = 0$, where $s_0 = \sum_{i=N+1}^M c_i \phi_i$. But $E(s_0) = 0$ implies that s_0 is a polynomial of degree $r+1$ over each triangle T. For each triangle $T = \langle v_1, v_2, v_3 \rangle$, we have

$$\left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^k s_0(v_i) = 0, \quad j + k \leqslant r,$$

for $i = 1, 2, 3$ since the interpolation conditions (3.3). We claim that $s_0 \equiv 0$. First we use

$$\left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^k f(v_1) = 0, \quad j + k \leqslant r,$$

to determine all the domain points except those on $R_{r+1}^T(v_1)$. Next we use

$$\left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^k s(v_2) = 0, \quad j + k \leqslant r,$$

to determine the remain points except one at v_3. Finally the coefficient at v_3 can be determined by

$$\left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^k s(v_3) = 0, \quad j + k \leqslant r.$$

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So it follows that $s_0 \equiv 0$. But $\{\phi_i\}_{N+1}^M$ are basis functions and linearly independent. Therefore this implies $c_i = 0$ for all $i \in \{N+1, \dots, M\}$. That is, $\mathbf{c} = 0$ which is a contradiction. These prove that $\mathbf{A}$ is nonsingular. We have thus established the following

Theorem 3.1. *There exists a unique solution $s_* \in S_d^{r+1}(\triangle)$ which solves the minimal energy Hermite interpolation problem* (3.4). *That is, s_* minimizes the energy* (3.2) *and satisfies the interpolation conditions* (1.1) *in the bivariate spline space $S_d^{r+1}(\triangle)$.*

4. Error bounds for the minimal energy interpolatory splines

In this section we derive error bounds for bivariate Hermite interpolatory splines which are calculated by minimizing the energy functional in the previous section. The error bounds for Lagrange interpolation problem can be found in [10]. In this paper, we generalize the error bounds from Lagrange interpolation to Hermite interpolation setting. All the results are similar with some modifications. Some of the modifications are sufficiently different, e.g., the proof of Lemma 4.2. It will be interesting to present a complete theory here.

First we convert the minimal energy interpolation problem (3.4) into a standard approximation problem in Hilbert space. Let

$$X := \left\{ f \in B(\Omega):\ f|_T \in W_\infty^{r+2}(T),\ \text{all triangles } T \text{ in } \triangle \right\},$$

where $B(\Omega)$ is the set of for all bounded real-valued functions on Ω. It is clear that $S_d^{r+1}(\triangle)$ is a subspace of X. Let us extend the inner product $\langle \phi_i, \phi_j \rangle$ on $S_d^{r+1}(\triangle)$ to X. More precisely, let

$$\langle f, g\rangle_X := \sum_{T\in\triangle} \int_T \left[\sum_{k=0}^{r+2} \binom{r+2}{k} \left[\left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y}\right)^{r+2-k} f \left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y}\right)^{r+2-k} g \right] \right] dx\, dy.$$

Then $\langle f, g\rangle_X$ defines a semi-definite inner-product on X. Let $\|f\|_X$ be the associated semi-norms.

Next let

$$W := \left\{ s \in S_d^{r+1}(\triangle):\ \left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^k s(v) = 0,\ j+k \leqslant r,\ v \in V \right\}. \tag{4.1}$$

It is easy to see that $\langle f, f\rangle_X$ for $f \in W$ is a norm. Indeed, if $\langle w, w\rangle_X = 0$ for some $w \in W$, then w is a function of degree $r+1$ over each triangle of $\triangle$. For a triangle $T = \langle v_1, v_2, v_3\rangle$, we have

$$\left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^k w(v_i) = 0, \quad j+k \leqslant r,$$

for $i = 1, 2, 3$ by the interpolation conditions (3.3). Similar to the proof of Theorem 3.1, we can prove that $w \equiv 0$. Hence W equipped with the inner-product $\langle \cdot, \cdot\rangle_X$ is a Hilbert space. Let

$$U_f := \left\{ s \in S_d^{r+1}(\triangle):\ \left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^k s(v) = \left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^k f(v),\ j+k \leqslant r,\ v \in V \right\} \tag{4.2}$$

be the set of all splines in S that interpolate f at the points of V. Then our minimal energy Hermite interpolation problem can be rewritten as follows: find a spline $S_f \in U_f$ such that

$$E(S_f) = \min_{s\in U_f} E(s). \tag{4.3}$$

Given f, suppose the set U_f defined in (4.2) is not empty. There is an $s_f \in U_f$. Then it is easy to see that the solution S_f to the minimal energy problem (4.3) is equal to $s_f - Ps_f$, where P is the linear projector $P: X \to W$ defined by

$$E(g - Pg) = \min_{w\in W} E(g - w)$$

for all $g \in X$. By Theorem 3.1, Pg is uniquely defined, and is characterized by

$$\langle g - Pg, w\rangle_X = 0, \quad \text{for all } w \in W.$$

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Moreover, using the Cauchy–Schwarz inequality, it is easy to see that

$$\|Pg\|_X \leqslant \|g\|_X \tag{4.4}$$

for all $g \in X$.

Given a triangle T, let $\operatorname{star}^0(T) = T$, and

$$\operatorname{star}^q(T) := \bigcup\{T \in \triangle\colon\ T \cap \operatorname{star}^{q-1}(T) \neq \emptyset\}, \quad q \geqslant 1.$$

Lemma 4.1. *Let $T = \langle v_1, v_2, v_3\rangle$ be a triangle. Suppose that $f \in C^{r+2}(T)$ satisfies*

$$\left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^k f(v_i) = 0$$

for $i = 1, 2, 3$ and $j + k \leqslant r$. Then for all $v \in T$,

$$\left|f(v)\right| \leqslant C_1 |T|^{r+2} |f|_{r+2,\infty,T}. \tag{4.5}$$

Proof. Given $v \in T$, we can write $v = v_1 + t(v_2 - v_1) + u(v_3 - v_1)$ with (t, u) in a standard triangle $S := \{(t, u), t, u \geqslant 0, t + u \leqslant 1\}$. Let $g(t, u) = f(v_1 + t(v_2 - v_1) + u(v_3 - v_1))$ for $(t, u) \in S$. Since

$$\left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^k f(v_i) = 0$$

for $i = 1, 2, 3$ and $j + k \leqslant r$, we can get

$$\left(\frac{\partial}{\partial u}\right)^j \left(\frac{\partial}{\partial t}\right)^k g(0,0) = 0, \qquad \left(\frac{\partial}{\partial u}\right)^j \left(\frac{\partial}{\partial t}\right)^k g(1,0) = 0, \qquad \left(\frac{\partial}{\partial u}\right)^j \left(\frac{\partial}{\partial t}\right)^k g(0,1) = 0,$$

for any $j + k \leqslant r$. By Taylor's expansion, we have

$$0 = \left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial u}\right)^k g(1,0) = \left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial u}\right)^k g(0,0) + \left(\frac{\partial}{\partial t}\right)^{j+1} \left(\frac{\partial}{\partial u}\right)^k g(0,0) + \frac{1}{2}\left(\frac{\partial}{\partial t}\right)^{j+2} \left(\frac{\partial}{\partial u}\right)^k g(\xi,0)$$

for any $j + k = r$ and $\xi \in (0, 1)$. So

$$\left|\left(\frac{\partial}{\partial t}\right)^{j+1} \left(\frac{\partial}{\partial u}\right)^k g(0,0)\right| \leqslant \frac{1}{2}|g|_{r+2,\infty,S}.$$

Similarly,

$$\left|\left(\frac{\partial}{\partial t}\right)^{j} \left(\frac{\partial}{\partial u}\right)^{k+1} g(0,0)\right| \leqslant \frac{1}{2}|g|_{r+2,\infty,S}.$$

Thus

$$\begin{aligned}\left|f(v)\right| = \left|g(t,u)\right| &\leqslant \left|g(0,0)\right| + \cdots \\ &\quad + \frac{1}{(r+1)!}\sum_{j+k=r+1}\binom{r+1}{j}\left|\left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial u}\right)^k g(0,0)\right| + K_3|g|_{r+2,\infty,S} \\ &\leqslant K_4|g|_{r+2,\infty,S}.\end{aligned}$$

Since $|g|_{r+2,\infty,S} \leqslant K_5 |f|_{r+2,\infty,S} |T|^{r+2}$, we conclude that (4.5) holds. $\square$

Lemma 4.2. *There exist positive constants C_2 and C_3 such that for any $u \in W$,*

$$C_2 \int_\Omega u^2 \leqslant |\triangle|^{2r+4} \|u\|_X^2 \leqslant C_3 \int_\Omega u^2. \tag{4.6}$$

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Proof. We first show the left side of the inequality. By applying Lemma 4.1, we have

$$\int_\Omega u^2 = \sum_{T\in\triangle}\int_T u^2 \leqslant \sum_{T\in\triangle} A_T\|u\|^2_{\infty,T} \leqslant \sum_{T\in\triangle} A_T\big(C_1|\triangle|^{r+2}|u|_{r+2,\infty,T}\big)^2$$
$$\leqslant C_1^2K_6|\triangle|^{2r+4}\sum_{T\in\triangle}|u|^2_{r+2,2,T} = C_1^2K_6|\triangle|^{2r+4}\|u\|^2_X,$$

where we have used the fact that the restriction of u to each triangle is a polynomial. Next by applying Markov's inequality, we have

$$\|u\|^2_X = \sum_{T\in\triangle}|u|^2_{r+2,2,T} \leqslant \left(\frac{K_7}{|\triangle|^{r+2}}\right)^2\sum_{T\in\triangle}|u|^2_{0,2,T} = \left(\frac{K_7}{|\triangle|^{r+2}}\right)^2\int_\Omega u^2.$$

Let $C_2 = \frac{1}{C_1^2K_6}$ and $C_3 = K_7^2$, we conclude the inequalities. □

Recall from [14] that when $d \geqslant 3r+5$, $S_d^{r+1}(\triangle)$ possesses a stable local basis $\{B_\xi\}_{\xi\in\mathcal{M}}$ corresponding to a minimal determining set $\mathcal{M}$. Thus, for any $\xi\in\mathcal{M}$, there is a vertex v_ξ of $\triangle$ with $\operatorname{supp}(B_\xi)\subseteq \operatorname{star}^l(v_\xi)$ and for any $\{c_\xi\}_{\xi\in\mathcal{M}}$

$$C_4|\triangle|^2\sum_{\xi\in\mathcal{M}}c_\xi^2 \leqslant \left\|\sum_{\xi\in\mathcal{M}}c_\xi B_\xi\right\|_2^2 \leqslant C_5|\triangle|^2\sum_{\xi\in\mathcal{M}}c_\xi^2. \tag{4.7}$$

Theorem 4.3. *Suppose W is defined in* (4.1). *Let g be a function in X with support in a triangle T in $\triangle$, and let τ be another triangle which lies outside of* $\operatorname{star}^q(T)$ *for some $q\geqslant 1$. Then*

$$\|Pg\|_{X_\tau} \leqslant C_6\sigma^q\|g\|_X, \tag{4.8}$$

for some constants $0<\sigma<1$ and C_6 dependent only on l,d and β as in Definition 2.1.

Proof. The proof uses a similar argument as in the proof of Theorem 3.1 of [11]. Letting

$$\mathcal{M}_0^T := \{\xi\in\mathcal{M}:\ \operatorname{supp}(B_\xi)\cap T\neq\emptyset\},$$
$$\mathcal{M}_q^T := \{\xi\in\mathcal{M}:\ \operatorname{supp}(B_\xi)\cap \operatorname{star}^{2ql}T\neq\emptyset\},$$
$$\mathcal{N}_0^T := \mathcal{M}_0^T,$$
$$\mathcal{N}_q^T := \mathcal{M}_q^T\setminus\mathcal{M}_{q-1}^T.$$

Suppose Pg is given by

$$Pg = \sum_{\xi\in\mathcal{M}}c_\xi B_\xi,$$

and let

$$u_q := \sum_{\xi\in\mathcal{M}_q^T}c_\xi B_\xi,\qquad w_q := Pg-u_q,\qquad a_q = \sum_{\xi\in\mathcal{N}_q^T}c_\xi^2,$$

for any $q\geqslant 0$. We start with

$$\|w_q\|_2^2 \leqslant \frac{1}{C_2}|\triangle|^{2r+4}\|w_q\|_X^2 = \frac{1}{C_2}|\triangle|^{2r+4}\langle Pg-u_q, w_q\rangle_X$$
$$= \frac{1}{C_2}|\triangle|^{2r+4}\langle g-u_q, w_q\rangle_X = -\frac{1}{C_2}|\triangle|^{2r+4}\langle u_q, w_q\rangle_X$$
$$\leqslant \frac{1}{C_2}|\triangle|^{2r+4}\left\|\sum_{\xi\in\mathcal{N}_q^T}c_\xi B_\xi\right\|_X\|w_q\|_X$$

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$$\leqslant \frac{1}{C_2}|\triangle|^{2r+4}\left(\frac{C_3^{1/2}}{|\triangle|^{r+2}}\left\|\sum_{\xi\in N_q^T} c_\xi B_\xi\right\|_2\right)\left(\frac{C_3^{1/2}}{|\triangle|^{r+2}}\|w_q\|_2\right)$$
$$= \frac{C_3}{C_2}\left\|\sum_{\xi\in\mathcal{N}_q^T} c_\xi B_\xi\right\|_2\|w_q\|_2.$$

Here we have used Lemma 4.1. Thus by (4.7):

$$\|w_q\|_2^2 \leqslant \frac{C_3^2}{C_2^2}\left\|\sum_{\xi\in\mathcal{N}_q^T} c_\xi B_\xi\right\|_2^2 \leqslant \frac{C_3^2 C_5}{C_2^2}|\triangle|^2 a_q$$

and

$$\sum_{j\geqslant q+1} a_j = \sum_{\xi\notin\mathcal{M}_q^T} c_\xi^2 \leqslant \frac{\|w_q\|_2^2}{C_4|\triangle|^2} \leqslant \frac{C_3^2 C_5}{C_2^2 C_4} a_q, \quad q \geqslant 0.$$

Then applying Lemma 2 in [3] with $\gamma := \frac{C_3^2 C_5}{C_2^2 C_4}$, we see that

$$a_q \leqslant (\gamma+1)\sigma^{2q} a_0,$$

where $\sigma := [\gamma/(\gamma+1)]^{1/2}$. Since $\|Pg\|_X^2 \leqslant \|g\|_X^2$ and by (4.7), we have

$$\|Pg\|_2^2 \leqslant \frac{C_3}{C_2}\|g\|_2^2$$

and

$$a_0 \leqslant \sum_{j\geqslant 0} a_j = \sum_{\xi\in\mathcal{M}} c_\xi^2 \leqslant \frac{1}{C_4|\triangle|^2}\|Pg\|_2^2 \leqslant \frac{C_3}{C_2 C_4|\triangle|^2}\|g\|_2^2.$$

Finally by (4.6) and (4.7), we have

$$\|Pg\|_{X_\tau}^2 \leqslant \frac{C_3}{|\triangle|^{2r+4}}\left\|\sum_{\xi\notin\mathcal{M}_q^T} c_\xi B_\xi\right\|_2^2 \leqslant \frac{C_3 C_5|\triangle|^2}{|\triangle|^{2r+4}}\sum_{\xi\notin\mathcal{M}_q^T} c_\xi^2$$
$$= \frac{C_3 C_5|\triangle|^2}{|\triangle|^{2r+4}}\sum_{j\geqslant q+1} a_j \leqslant \frac{C_3^4 C_5^2}{C_2^3 C_4^2|\triangle|^{2r+4}}(\gamma+1)\sigma^{2q}\|g\|_2^2$$
$$\leqslant \frac{C_3^4 C_5^2}{C_2^4 C_4^2}(\gamma+1)\sigma^{2q}\|g\|_X^2$$

which gives (4.8) with $C_6 = \frac{C_3^2 C_5}{C_2^2 C_4}(\gamma+1)^{1/2}$. □

Theorem 4.4. *Suppose $\triangle$ is a β-quasi-uniform triangulation $\triangle$. Suppose that $\{B_\xi\}_{\xi\in\mathcal{M}}$ is a stable local basis for $S_d^{r+1}(\triangle)$ with $d \geqslant 3r+5$ corresponding to a minimal determining set $\mathcal{M}$ containing the set V of vertices of $\triangle$. Then*

$$|Pg|_{r+2,\infty,\Omega} \leqslant C_7|g|_{r+2,\infty,\Omega} \quad \text{for all } g \in X, \tag{4.9}$$

where C_7 depends only on d,l, r, and β.

Proof. The proof is the same as that of Theorem 5.5 in [10]. For simplicity, we omit the detail here. □

Theorem 4.5. *Suppose $\triangle$ is a β-quasi-uniform triangulation. Suppose that $f \in C^m(\Omega)$ with $m \geqslant 2r$. Then there exists a constant C depending only on d, β and f such that the minimum energy interpolant S_f defined in* (4.3) *satisfies*

$$\|f - S_f\|_{L_\infty(\Omega)} \leqslant C|\triangle|^{r+2}. \tag{4.10}$$

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Proof. Given a function $f \in C^m(\Omega)$, let $s_f \in U_f$ be the quasi-interpolant spline of f as in Theorem 2.2. We know that

$$\|f - s_f\|_{L_\infty(\Omega)} \leqslant K|\triangle|^m |f|_{m,\infty,\Omega}$$

and

$$|s_f|_{r+2,\infty,\Omega} \leqslant |f|_{r+2,\infty,\Omega} + K|\triangle|^{m-r-2}|f|_{m,\infty,\Omega} =: C_0.$$

That is, C_0 is a constant dependent only on d, β, and f with the conventional assumption of $|\triangle| \leqslant 1$. We recall that $Ps_f = s_f - S_f$. By Theorem 4.4,

$$|s_f - S_f|_{r+2,\infty,\Omega} = |Ps_f|_{r+2,\infty,\Omega} \leqslant C_7|s_f|_{r+2,\infty,\Omega} \leqslant C_7C_0.$$

Since

$$\left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^k \bigl(s_f(v) - S_f(v)\bigr) = 0, \quad j + k \leqslant r,$$

for all vertices v of $\triangle$, by Lemma 4.1,

$$\|s_f - S_f\|_{L_\infty(\Omega)} \leqslant C_1|\triangle|^{r+2}|s_f - S_f|_{r+2,\infty,\Omega}$$

and hence,

$$\|s_f - S_f\|_{L_\infty(\Omega)} \leqslant C_1C_7C_0|\triangle|^{r+2}|f|_{r+2,\infty,\Omega}.$$

Then the error bound (4.10) follows from

$$\|f - S_f\|_{L_\infty(\Omega)} \leqslant \|f - s_f\|_{L_\infty(\Omega)} + \|s_f - S_f\|_{L_\infty(\Omega)}.$$

This completes the proof. □

5. A computational method for spline Hermite interpolation

In this section we describe a computational algorithm to solve the minimal energy Hermite interpolation problem using a spline space $S_d^{r+1}(\triangle)$ with $d > r$, e.g., $d \geqslant 3r + 5$. It is a straightforward generalization of the computational method given in [2]. Let $\triangle$ be a triangulation of the given data locations. For each spline function $s \in S_d^{r+1}(\triangle)$, let $\mathbf{c} = (c_{ijk}^t, i + j + k = d, t \in \triangle)$ be the coefficient vector associated with s. Since $s \in S_d^{r+1}(\triangle)$, s satisfies the smoothness conditions which can be expressed by a linear system $H\mathbf{c} = 0$ (cf. [2]). Also the energy functional $E(s)$ can be written in terms of $\mathbf{c}$ as

$$E(s) = \mathbf{c}^T K\mathbf{c},$$

where $K = \operatorname{diag}(K_T, T \in \triangle)$ is a block diagonal matrix with

$$K_T = \left[\int_T \sum_{k=0}^{r+2} \binom{r+2}{k} \left[\left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y}\right)^{r+2-k} B_{ijk}^d \left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y}\right)^{r+2-k} B_{pqr}^d\right] dx\,dy\right]_{p+q+r=d}^{i+j+k=d}.$$

Let $\mathbf{f} = (z_i^{\nu,\mu},\ \nu + \mu \leqslant r,\ i = 1, \ldots, n)$ be the data value vector. Then the Hermite interpolation conditions can be expressed by another linear system $I\mathbf{c} = \mathbf{f}$.

Note that the minimal energy Hermite interpolation problem is equivalent to the following constrained minimization problem:

$$\min\bigl\{\mathbf{c}^T K\mathbf{c}\colon\ H\mathbf{c} = 0,\ I\mathbf{c} = \mathbf{f}\bigr\}.$$

By Lagrange multiplier method, let

$$L(\mathbf{c}, \alpha, \beta) := \mathbf{c}^T K\mathbf{c} + \alpha^T H\mathbf{c} + \beta^T D\mathbf{c}$$

be a Lagrangian function. We need to find a local minimizer of $L(\mathbf{c}, \alpha, \beta)$: that is

$$\frac{\partial}{\partial \mathbf{c}} L(\mathbf{c}, \alpha, \beta) = 0, \qquad \frac{\partial}{\partial \alpha} L(\mathbf{c}, \alpha, \beta) = 0, \qquad \frac{\partial}{\partial \beta} L(\mathbf{c}, \alpha, \beta) = 0.$$

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Then we have

$$\begin{bmatrix} H^T & I^T & 2K \\ 0 & 0 & H \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{f} \end{bmatrix}. \tag{5.1}$$

To solve the above (singular) linear system, we rewrite it in the following matrix form:

$$\begin{bmatrix} B^T & A \\ 0 & B \end{bmatrix} \begin{bmatrix} \lambda \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix} \tag{5.2}$$

with the singular matrix A. Next we use the following algorithm to solve the system.

Algorithm. (Cf. [2].) Fix $\varepsilon > 0$. Given an initial guess $\lambda^{(0)} \in \mathrm{Im}(B)$, e.g., $\lambda^{(0)} = 0$, we first compute

$$\mathbf{c}^{(1)} = \left(A + \frac{1}{\varepsilon} B^T B\right)^{-1} \left(F + \frac{1}{\varepsilon} B^T G - B^T \lambda^{(0)}\right)$$

and iteratively compute

$$\mathbf{c}^{(k+1)} = \left(A + \frac{1}{\varepsilon} B^T B\right)^{-1} \left(A\mathbf{c}^{(k)} + \frac{1}{\varepsilon} B^T G\right)$$

for $k = 1, 2, \ldots,$ where $\mathrm{Im}(B)$ is the range of B.

The existence and uniqueness of the interpolatory spline in $S_d^{r+1}(\triangle)$ imply that the above algorithm is well-defined and convergent (cf. [2]). Once we obtain the spline coefficients $\mathbf{c}$, we can use the well-known de Casteljau Algorithm to evaluate the Hermite interpolatory spline with coefficient vector $\mathbf{c}$.

In the following we present some numerical experiments.

Example 5.1. Suppose $\diamond$ is a uniform partition of the unit square domain $\Omega := [0, 1] \times [0, 1]$ into N^2 subsquares. Let $S_8^2(\triangle_\diamond)$ be a C^2 spline space, where $\triangle_\diamond$ is the triangulation obtained by inserting diagonals of each subsquares in $\diamond$. We use the following test functions:

$$\begin{aligned} f_1(x, y) &= -2x^3 + y^3, \\ f_2(x, y) &= \sin\bigl(2(x - y)\bigr), \\ f_3(x, y) &= 0.75\exp\bigl(-0.25(9x-2)^2 - 0.25(9y-2)^2\bigr) \\ &\quad + 0.75\exp\bigl(-(9x+1)^2/49 - (9y+1)/10\bigr) \\ &\quad + 0.5\exp\bigl(-0.25(9x-7)^2 - 0.25(9y-3)^2\bigr) \\ &\quad - 0.2\exp\bigl(-(9x-4)^2 - (9y-7)^2\bigr), \end{aligned}$$

where f_3 is the well-known Franke function. We use these function values and derivative values at the grid points $(i/N, j/N)$, $i, j = 0, \ldots, N$, to have a set of scattered Hermite data. We approximated these functions for choices $N = 2, 4, 8, 16, 32$ which corresponds to repeatedly halving the mash size. Table 1 gives the maximum error computed based on 100×100 equally-spaced points over Ω. Table 2 presents the corresponding ratios of errors for successive values of N. From Table 2 we can see the approximation order is close to 8 which is what we expect for order 3 convergence.

Table 1
Maximum errors for various test functions

N	2	4	8	16	32
f_1	0.01314	0.00188	2.38×10^{-4}	2.97×10^{-5}	3.83×10^{-6}
f_2	0.01259	0.00189	2.27×10^{-4}	2.78×10^{-5}	3.41×10^{-6}
f_3	0.15858	0.05576	6.20×10^{-3}	1.05×10^{-3}	1.48×10^{-4}

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Table 2
Rates of convergence for various test functions

f_1	6.989	7.899	8.013	7.755
f_2	6.661	8.326	8.165	8.152
f_3	2.844	8.994	5.905	7.095

Table 3
Rates of convergence for various test functions

f_1	7.508	7.964	7.991	7.851
f_2	7.778	7.928	7.974	8.036
f_3	3.497	8.720	5.908	7.149

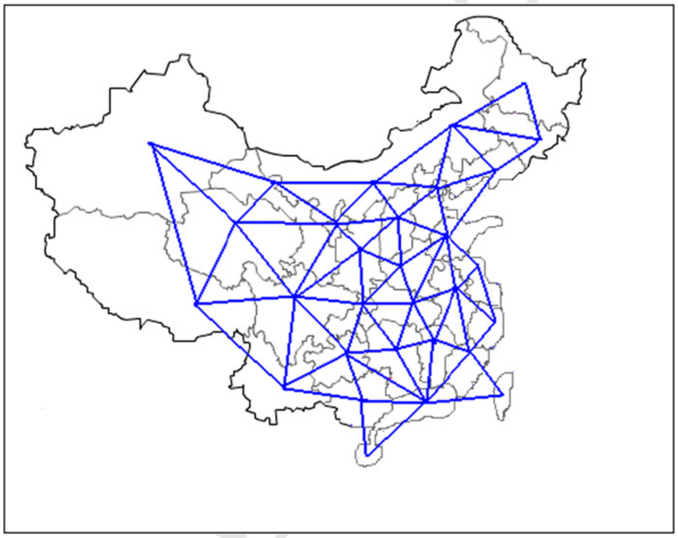

Fig. 1. A triangulation $\triangle$ of the data locations over China.

Example 5.2. We repeat Example 5.1 using spline space $S_9^2(\triangle_\diamond)$. Our numerical results show that the convergence rates are almost the same as that in Example 5.1 (see Table 3). This confirms Theorem 4.5 that the convergence rates are independent of the degree of spline space.

Finally we present the following example to illustrate an application of our spline Hermite interpolation.

Example 5.3. We consider the reconstruction of a wind potential function. We are given a set of wind velocity measurements over 30 major cities in China in one day and required to construct the wind potential function W. Let $\{(x_i, y_i, DxW(x_i, y_i), DyW(x_i, y_i)),\ i = 1, \ldots, 30\}$ be the given wind velocity values. In order to uniquely determine the wind potential, we assume that $W(x_1, y_1) = 0$. Let $\triangle$ be a triangulation of the data locations $\{(x_i, y_i),\ i = 1, \ldots, 30\}$ as shown in Fig. 1 and we use the spline space $S_8^2(\triangle)$. We find the spline function $s_W \in S_8^2(\triangle)$ satisfying

$$s_W(x_1, y_1) = 0, \qquad \frac{\partial}{\partial x} s_W(x_i, y_i) = DxW(x_i, y_i), \qquad \frac{\partial}{\partial y} s_W(x_i, y_i) = DxW(x_i, y_i)$$

for $i = 1, \ldots, 30$ and

$$E_3(s_W) = \min\left\{ E_3(s),\ s(x_1, y_1) = 0,\ \frac{\partial}{\partial x} s(x_i, y_i) = DxW(x_i, y_i),\right.$$
$$\left. \frac{\partial}{\partial y} s(x_i, y_i) = DxW(x_i, y_i),\ i = 1, \ldots, 30 \right\},$$

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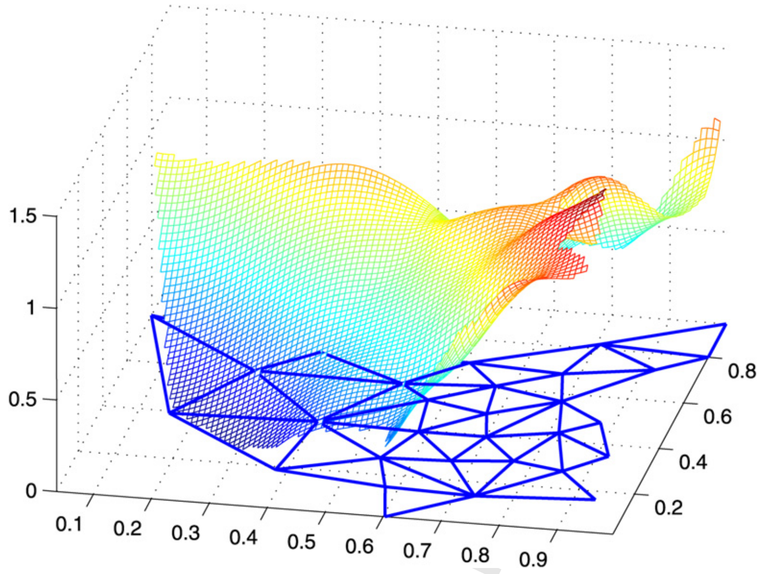

Fig. 2. The wind potential function.

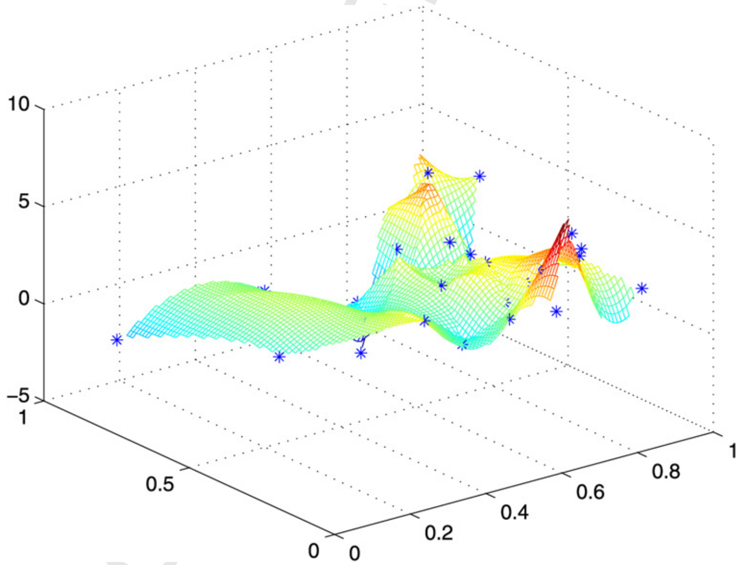

Fig. 3. The wind velocity in X direction.

where

$$E_3(s) = \int_{\Omega} \left[\sum_{k=0}^{3} \binom{3}{k} \left[\left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y}\right)^{3-k} s \right]^2 \right] dx\, dy. \tag{5.3}$$

We can show that there exists a unique solution s_W in any spline space $S^r_d(\triangle)$ of smoothness $r \geqslant 2$ and $d \geqslant 3r+2$. The proof is almost the same as Theorem 3.1. The detail is left to the interested reader. In Fig. 2, we show the spline reconstruction of the wind potential function. The wind velocity in X direction and in Y direction are shown in Fig. 3 and Fig. 4. From these two figures, we can see the derivatives are matched quite well.

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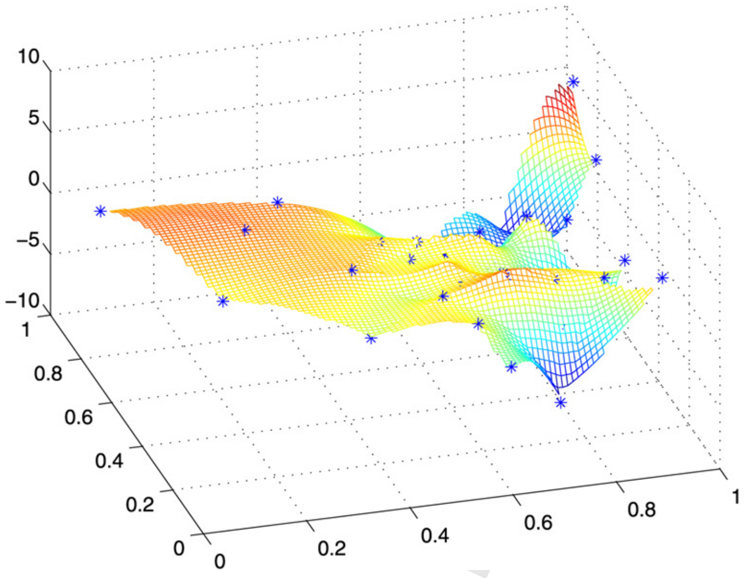

Fig. 4. The wind velocity in Y direction.

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