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Algorithms for G^1 connection of multiple parametric bicubic NURBS surfaces [*]

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The objective of this paper is to introduce an innovative approach for constructing effective algorithms for removing gaps between parametric NURBS surfaces in three-space, while maintaining geometrical smoothness for the combined (or compound) surface. Similar to the degenerate case of tensor-product B-spline surfaces, if the underlying knot sequences along the connecting boundaries of two NURBS surfaces are proportional, then the parametric surfaces can be connected in a G^1 fashion. This approach can be easily extended to connecting three or four parametric NURBS surfaces. We will demonstrate the feasibility of our approach by focusing on the C^1 bicubic setting with knot sequences being equally-spaced and having double interior knots.

1. Introduction

A NURBS (or Non-Uniform Rational B-Spline) surface in the 3-dimensional space $\mathbb{R}^3$ is a biparametric surface, represented by a rational function of B-spline series. One of the major advantages in using rational functions over polynomial representations is that certain important curves and surfaces such as conic sections can be expressed by rational functions but not by polynomials. Moreover, NURBS can be used to represent a wide variety of geometric objects including not only conic-sections and free-form curves and surfaces, but also more conventional shapes such as polygonal surfaces. Hence, NURBS curves and surfaces are gaining more popularity for use in 3-dimensional computer aided design (CAD) and computer aided geometric design (CAGD), and are included in various industry standards such as IGES, STEP, and PHIGS.

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To demonstrate the main idea of our approach and to develop concrete effective algorithms, we will only focus on bicubic NURBS surfaces with knot sequences having double interior knots, i.e., the NURBS surfaces are in C^1. The objective is to connect such independently designed NURBS surfaces without gaps and in a smooth fashion. The main algorithm for connecting two NURBS surfaces without gaps in a G^1 fashion is given in section 3, after some preliminary preparations to be discussed in section 2. In section 4, we demonstrate the application of this algorithm to connecting multiple (three and four) NURBS surfaces in a G^1 fashion. Our approach depends on Bézier representations. For this reason, algorithms for converting bicubic NURBS to Bézier representations and *vice versa* will be given in the appendix.

2. Preliminaries

This section is fairly extensive. Therefore, it is divided into three subsections, devoted to the consideration of uniqueness, and of joining two curves and two surfaces, all in rational Bézier forms. Some of the results discussed in this section are known to the experts in this subject. The best references for the other readers are [5,7,10]. See also [1,2,11] for the spline literature.

2.1. *Uniqueness of local control points and weights*

To motivate the need of NURBS representation, we start with an elementary example. It is well known that the unit circle $x^2 + y^2 = 1$ in 2-space cannot be represented by using parametric polynomials with one parameter. However, it can be expressed by a rational function of quartic polynomials, for instance, of the form $\mathbf{c}(u) = (x(u), y(u))$, $u \in [0, 1]$, where

$$\begin{aligned} x(u) &= \frac{4u(1-u)(1-2u)}{(1-2u+2u^2)^2}, \\ y(u) &= \frac{(2u^2-1)(1-4u+2u^2)}{(1-2u+2u^2)^2}, \quad u \in [0,1]. \end{aligned} \tag{2.1.1}$$

It is also easy to see that $\tilde{\mathbf{c}}(\tilde{u}) = (\tilde{x}(\tilde{u}), \tilde{y}(\tilde{u}))$, where

$$\begin{aligned} \tilde{x}(\tilde{u}) &= \frac{2\sqrt{3}\tilde{u}(1-\tilde{u})(1-2\tilde{u})}{(1-\tilde{u}+\tilde{u}^2)(1-3\tilde{u}+3\tilde{u}^2)}, \\ \tilde{y}(\tilde{u}) &= \frac{-1+4\tilde{u}-\tilde{u}^2-6\tilde{u}^3+3\tilde{u}^4}{(1-\tilde{u}+\tilde{u}^2)(1-3\tilde{u}+3\tilde{u}^2)}, \quad \tilde{u} \in [0,1], \end{aligned} \tag{2.1.2}$$

is another rational Bézier representation of the same unit circle. However, there is no clear relation between (2.1.1) and (2.1.2). This raises an interesting question: When the problem of uniqueness is considered, in what sense is the representation of a rational Bézier curve unique?

To answer this question, let us first examine the structure of rational Bézier curves. Let $\mathbf{c}(t)$, $t \in [0,1]$, be an nth degree rational Bézier curve in its Bernstein form

$$\mathbf{c}(t) = \frac{\sum_{j=0}^{n} w_j \mathbf{c}_j B_{n,j}(t)}{\sum_{\ell=0}^{n} w_\ell B_{n,\ell}(t)}, \quad t \in [0,1], \tag{2.1.3}$$

where, as usual, for $k = 0, \ldots, n$,

$$B_{n,k}(x) := \binom{n}{k}(1-x)^{n-k}x^k,$$

and for $j = 0, \ldots, n$, $\mathbf{c}_j$ and w_j are *Bézier coefficients* (or local control points) and *local weights*, respectively. The unit circle has a "5th degree" rational Bézier representation with the following planar Bézier coefficients:

$$\begin{aligned}
\mathbf{c}_0 &= (0,-1),\\
\mathbf{c}_1 &= \left(\frac{2}{w_0 w_3}\sqrt{2w_0\left(5w_2^3 + w_0 w_3^2\right)}, -1\right),\\
\mathbf{c}_2 &= \left(\frac{w_3}{5w_2^3}\sqrt{2w_0\left(5w_2^3 + w_0 w_3^2\right)}, \frac{1}{5w_2^3}\left(5w_2^2 + 2w_0 w_3^2\right)\right),\\
\mathbf{c}_3 &= \left(-\frac{w_3}{5w_2^3}\sqrt{2w_0\left(5w_2^3 + w_0 w_3^2\right)}, \frac{1}{5w_2^3}\left(5w_2^2 + 2w_0 w_3^2\right)\right),\\
\mathbf{c}_4 &= \left(-\frac{2}{w_0 w_3}\sqrt{2w_0\left(5w_2^3 + w_0 w_3^2\right)}, -1\right),\\
\mathbf{c}_5 &= (0,-1),
\end{aligned}$$

which are chosen, for simplicity, symmetrically around the y-axis, and local weights given by

$$(w_0, \ldots, w_5) = \left(w_0, \frac{w_0 w_3}{5w_2}, w_2, w_3, \frac{w_0 w_3^4}{5w_2^4}, \frac{w_0 w_3^5}{w_2^5}\right),$$

where w_0 can be simply chosen as 1, but w_2 and w_3 are *free* parameters. It is interesting to note that the choice of $w_2 = w_3 = 1/5$ yields (2.1.1), while the choice of $w_2 = w_3 = 1/10$ gives (2.1.2). That is, (2.1.1) is obtained when the Bézier coefficients are

$$\begin{aligned}
&\mathbf{c}_0 = (0,-1), \qquad \mathbf{c}_1 = (4,-1), \qquad \mathbf{c}_2 = (2,3),\\
&\mathbf{c}_3 = (-2,3), \qquad \mathbf{c}_4 = (-4,-1), \qquad \mathbf{c}_5 = (0,-1),
\end{aligned}$$

and the local weights are

$$(w_0, \ldots, w_5) = \left(1, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 1\right);$$

while (2.1.2) is obtained by using the Bézier coefficients

$$\tilde{\mathbf{c}}_0 = (0, -1), \qquad \tilde{\mathbf{c}}_1 = \big(2\sqrt{3}, -1\big), \qquad \tilde{\mathbf{c}}_2 = \big(2\sqrt{3}, 5\big),$$
$$\tilde{\mathbf{c}}_3 = \big(-2\sqrt{3}, 5\big), \qquad \tilde{\mathbf{c}}_4 = \big(-2\sqrt{3}, -1\big), \qquad \tilde{\mathbf{c}}_5 = (0, -1),$$

and local weights

$$(\widetilde{w}_0, \ldots, \widetilde{w}_5) = \left(1, \frac{1}{5}, \frac{1}{10}, \frac{1}{10}, \frac{1}{5}, 1\right).$$

Note that both (2.1.1) and (2.1.2) are rational functions of quartic polynomials. These two rational Bézier representations give the same curve, namely, the unit circle.

Returning to (2.1.3), we observe that when t is replaced by

$$t = f(u) := \frac{au}{(a-1)u + 1}, \quad u \in [0, 1], \tag{2.1.4}$$

both the numerator and the denominator of $\mathbf{c}(f(u))$ are still nth degree polynomials. We will call (2.1.4) a *rational linear transformation* (RLT) [10]. To avoid discontinuity, we require $0 < a \leqslant 1$. (When $a = 1$, then $t = u$.)

In general, an nth degree rational Bézier curve with parameter $\tilde{t} \in [\tilde{u}_{2i}, \tilde{u}_{2i+2}]$ can be converted to a "new" representation by another nth degree rational Bézier curve with parameter $t \in [u_{2j}, u_{2j+2}]$, via the RLT

$$\tilde{t} = \frac{a(t - u_{2j})/(u_{2j+2} - u_{2j}) - \tilde{u}_{2i}(\tilde{u}_{2i+2} - a)/(\tilde{u}_{2i+2} - \tilde{u}_{2i})}{(t - u_{2j})/(u_{2j+2} - u_{2j}) - (\tilde{u}_{2i+2} - a)/(\tilde{u}_{2i+2} - \tilde{u}_{2i})}, \quad t \in [u_{2j}, u_{2j+2}], \tag{2.1.5}$$

where the constant $a \notin (\tilde{u}_{2i}, \tilde{u}_{2i+2})$ without changing the trace of the curve. However, it can be shown that (2.1.1) and (2.1.2) cannot be converted to each other by using RLTs. On the other hand, (2.1.2) can be converted to (2.1.1) via a function f, which is not a rational function, namely,

$$\tilde{u} = f(u) := \frac{1}{2} + \frac{\sqrt{3}}{6}\frac{4u(u-1) + \sqrt{(4u^2 - 2u + 1)(4u^2 - 6u + 3)}}{2u - 1}, \quad u \in [0, 1],$$

while (2.1.1) can be converted into (2.1.2) via f^{-1}:

$$u = f^{-1}(\tilde{u}) = \frac{1}{2} + \frac{1}{2}\frac{\sqrt{3}\,\tilde{u}(\tilde{u} - 1) + \sqrt{(\tilde{u}^2 - \tilde{u} + 1)(3\tilde{u}^2 - 3\tilde{u} + 1)}}{2\tilde{u} - 1}, \quad \tilde{u} \in [0, 1].$$

Notice that, with

$$f\left(\frac{1}{2}\right) := \lim_{u \to 1/2} f(u) = \frac{1}{2} = \lim_{\tilde{u} \to 1/2} f^{-1}(\tilde{u}) =: f^{-1}\left(\frac{1}{2}\right),$$

both f and f^{-1} are continuous and strictly increasing on $[0, 1]$.

The above discussions imply that there are many rational Bézier expressions for a given curve using different local control points and local weights. However, in the following, we show that if some 3-vectors of the differences of local control points are linearly independent (typically in 3-D curves), then the Bézier coefficients are uniquely determined and the local weights are determined up to certain affine transformations or RLTs.

Lemma 2.1. Let $\mathbf{c}(t), t \in [0,1]$, be a rational cubic Bézier curve as in (2.1.3) with $n = 3$. If the three vectors $\mathbf{c}_1 - \mathbf{c}_0$, $\mathbf{c}_2 - \mathbf{c}_0$, and $\mathbf{c}_3 - \mathbf{c}_0$ are linearly independent, then the representation of the rational cubic Bézier curve $\mathbf{c}(t)$, $t \in [0,1]$, with fixed local control points $\mathbf{c}_0, \ldots, \mathbf{c}_3$ and local weights $w_0, \ldots, w_3$, is unique up to certain affine transformations or rational linear transformations.

Proof. We wish to find the necessary relationship between t and $\tilde{t}$ such that two cubic rational Bézier curves $\mathbf{c}(t)$, $t \in [0,1]$, as in (2.1.3) with $n = 3$, and $\tilde{\mathbf{c}}(\tilde{t})$, $\tilde{t} \in [0,1]$, represent exactly the same curve, where, similar to $\mathbf{c}(t)$, $t \in [0,1]$, the curve $\tilde{\mathbf{c}}(\tilde{t})$, $\tilde{t} \in [0,1]$, is given by

$$\tilde{\mathbf{c}}(\tilde{t}) = \frac{\sum_{j=0}^{3} \widetilde{w}_j \tilde{\mathbf{c}}_j B_{3,j}(\tilde{t})}{\sum_{\ell=0}^{3} \widetilde{w}_\ell B_{3,\ell}(\tilde{t})}, \quad \tilde{t} \in [0,1]. \tag{2.1.6}$$

To this end, let $\tilde{\mathbf{c}}(\tilde{t})$, $\tilde{t} \in [0,1]$, in (2.1.6) be fixed. Observe first that $\mathbf{c}(t)$, $t \in [0,1]$, and $\tilde{\mathbf{c}}(\tilde{t})$, $\tilde{t} \in [0,1]$, can also be rewritten as

$$\mathbf{c}(t) = \frac{\sum_{j=0}^{3} w_j \mathbf{c}_j B_{3,j}(t)}{\sum_{\ell=0}^{3} w_\ell B_{3,\ell}(t)} = \mathbf{c}_0 + \frac{\sum_{j=1}^{3} w_j (\mathbf{c}_j - \mathbf{c}_0) B_{3,j}(t)}{\sum_{\ell=0}^{3} w_\ell B_{3,\ell}(t)}, \quad t \in [0,1], \tag{2.1.7}$$

$$\tilde{\mathbf{c}}(\tilde{t}) = \frac{\sum_{j=0}^{3} \widetilde{w}_j \tilde{\mathbf{c}}_j B_{3,j}(\tilde{t})}{\sum_{\ell=0}^{3} \widetilde{w}_\ell B_{3,\ell}(\tilde{t})} = \tilde{\mathbf{c}}_0 + \frac{\sum_{j=1}^{3} \widetilde{w}_j (\tilde{\mathbf{c}}_j - \tilde{\mathbf{c}}_0) B_{3,j}(\tilde{t})}{\sum_{\ell=0}^{3} \widetilde{w}_\ell B_{3,\ell}(\tilde{t})}, \quad \tilde{t} \in [0,1]. \tag{2.1.8}$$

For $\mathbf{c}(t)$, $t \in [0,1]$, and $\tilde{\mathbf{c}}(\tilde{t})$, $\tilde{t} \in [0,1]$, in (2.1.7) and (2.1.8) to represent exactly the same curve, it is necessary, by using the interpolating and tangent properties of rational Bézier curves at the endpoints, that

$$\mathbf{c}_0 = \tilde{\mathbf{c}}_0, \qquad \mathbf{c}_3 = \tilde{\mathbf{c}}_3, \tag{2.1.9}$$

$$\mathbf{c}_1 = \tilde{\mathbf{c}}_0 + \alpha\big(\tilde{\mathbf{c}}_1 - \tilde{\mathbf{c}}_0\big), \tag{2.1.10}$$

$$\mathbf{c}_2 = \tilde{\mathbf{c}}_3 + \beta\big(\tilde{\mathbf{c}}_2 - \tilde{\mathbf{c}}_3\big), \tag{2.1.11}$$

where α and β are positive real numbers. Next, let f be any differentiable function that satisfies

$$\tilde{t} = f(t), \qquad f(0) = 0, \qquad f(1) = 1. \tag{2.1.12}$$

Then it follows from

$$\mathbf{c}'(0) = \frac{3w_1}{w_0}(\mathbf{c}_1 - \mathbf{c}_0), \qquad \mathbf{c}'(1) = \frac{3w_2}{w_3}(\mathbf{c}_3 - \mathbf{c}_2),$$

$$\tilde{\mathbf{c}}'(0) = \frac{3\widetilde{w}_1}{\widetilde{w}_0}\big(\tilde{\mathbf{c}}_1 - \tilde{\mathbf{c}}_0\big), \qquad \tilde{\mathbf{c}}'(1) = \frac{3\widetilde{w}_2}{\widetilde{w}_3}\big(\tilde{\mathbf{c}}_3 - \tilde{\mathbf{c}}_2\big),$$

$$\mathbf{c}'(0) = \tilde{\mathbf{c}}'(0) f'(0), \qquad \mathbf{c}'(1) = \tilde{\mathbf{c}}'(1) f'(1),$$

that

$$\alpha = \frac{w_0\widetilde{w}_1}{\widetilde{w}_0 w_1} f'(0), \qquad \beta = \frac{\widetilde{w}_2 w_3}{w_2\widetilde{w}_3} f'(1). \tag{2.1.13}$$

Very importantly, this function f can be computed from the implicit formulation

$$\mathbf{c}(t) = \tilde{\mathbf{c}}(\tilde{t}) = \tilde{\mathbf{c}}(f(t)), \quad t \in [0,1],$$

which, by (2.1.7)–(2.1.9), is equivalent to

$$\frac{\sum_{j=1}^{3} w_j(\mathbf{c}_j - \mathbf{c}_0)B_{3,j}(t)}{\sum_{\ell=0}^{3} w_\ell B_{3,\ell}(t)} = \frac{\sum_{j=1}^{3} \widetilde{w}_j(\tilde{\mathbf{c}}_j - \mathbf{c}_0)B_{3,j}(\tilde{t})}{\sum_{\ell=0}^{3} \widetilde{w}_\ell B_{3,\ell}(\tilde{t})}, \quad t, \tilde{t} \in [0,1]. \tag{2.1.14}$$

By introducing

$$\begin{aligned} D(t) &:= \sum_{\ell=0}^{3} w_\ell B_{3,\ell}(t), \qquad \widetilde{D}(\tilde{t}) := \sum_{\ell=0}^{3} \widetilde{w}_\ell B_{3,\ell}(\tilde{t}), \\ \mathbf{b}_j &:= \tilde{\mathbf{c}}_j - \tilde{\mathbf{c}}_0, \quad j = 1,2,3, \end{aligned} \tag{2.1.15}$$

and applying (2.1.10)–(2.1.11), we see that the identity (2.1.14) leads to

$$\begin{aligned} &\big[w_1\alpha\mathbf{b}_1 B_{3,1}(t) + w_2\big(\beta\mathbf{b}_2 + (1-\beta)\mathbf{b}_3\big)B_{3,2}(t) + w_3\mathbf{b}_3 B_{3,3}(t)\big]\widetilde{D}(t) \\ &\quad = \big[\widetilde{w}_1\mathbf{b}_1 B_{3,1}(\tilde{t}) + \widetilde{w}_2\mathbf{b}_2 B_{3,2}(\tilde{t}) + \widetilde{w}_3\mathbf{b}_3 B_{3,3}(\tilde{t})\big]D(t), \end{aligned}$$

i.e.,

$$g_1(t,\tilde{t})\mathbf{b}_1 + g_2(t,\tilde{t})\mathbf{b}_2 + g_3(t,\tilde{t})\mathbf{b}_3 = 0, \quad t,\tilde{t} \in [0,1], \tag{2.1.16}$$

where

$$g_1(t,\tilde{t}) := \alpha w_1 B_{3,1}(t)\widetilde{D}(\tilde{t}) - \widetilde{w}_1 B_{3,1}(\tilde{t})D(t), \tag{2.1.17}$$

$$g_2(t,\tilde{t}) := \beta w_2 B_{3,2}(t)\widetilde{D}(\tilde{t}) - \widetilde{w}_2 B_{3,2}(\tilde{t})D(t), \tag{2.1.18}$$

$$g_3(t,\tilde{t}) := \big[(1-\beta)w_2 B_{3,2}(t) + w_3 B_{3,3}(t)\big]\widetilde{D}(\tilde{t}) - \widetilde{w}_3 B_{3,3}(\tilde{t})D(t). \tag{2.1.19}$$

Suppose that the three vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are linearly independent. Then it follows from (2.1.16)–(2.1.18) that

$$\alpha w_1\widetilde{w}_2 B_{3,1}(t)B_{3,2}(\tilde{t}) = \beta\widetilde{w}_1 w_2 B_{3,1}(\tilde{t})B_{3,2}(t),$$

which leads to

$$\tilde{t} = \frac{at}{(a-1)t+1}, \quad \text{where} \quad a := \frac{\beta\widetilde{w}_1 w_2}{\alpha w_1\widetilde{w}_2}, \tag{2.1.20}$$

and (2.1.19) is then automatically satisfied. Hence, substituting (2.1.20) into $g_1(t,\tilde{t}) = 0$, $t,\tilde{t} \in [0,1]$, we have

$$\begin{aligned} &\alpha = \beta = 1, \qquad && a = \frac{\widetilde{w}_0}{w_0}\frac{w_1}{\widetilde{w}_1}, \\ &w_2 = \frac{\widetilde{w}_0}{w_0}\frac{w_1^2}{\widetilde{w}_1^2}\widetilde{w}_2, \qquad && w_3 = \frac{\widetilde{w}_0^2}{w_0^2}\frac{w_1^3}{\widetilde{w}_1^3}\widetilde{w}_3, \end{aligned} \tag{2.1.21}$$

where w_1 is a free parameter. Therefore, from (2.1.9)–(2.1.11), we have $\mathbf{c}_j = \tilde{\mathbf{c}}_j,\ j = 0,\dots,3$. For $a = 1$, the local weight w_1 can be chosen as $w_1 := (\widetilde{w}_1/\widetilde{w}_0)w_0$, so that from (2.1.21), it follows that $w_2 = (\widetilde{w}_2/\widetilde{w}_0)w_0$ and $w_3 = (\widetilde{w}_3/\widetilde{w}_0)w_0$. In other words, the linear independence of $\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3$ uniquely determines the weights, and hence, the rational cubic Bézier curves. This completes the proof of the lemma. □

Remark 1. To consider the rare situation when the matrix with column vectors $\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3$ has rank 2, we assume, without loss of generality, that $\mathbf{b}_1$ and $\mathbf{b}_2$ are linearly independent, and

$$\mathbf{b}_3 = \xi\mathbf{b}_1 + \eta\mathbf{b}_2, \tag{2.1.22}$$

for some constants ξ and η. Then, from (2.1.16)–(2.1.19), we have $g_1 = -\xi g_3$ and $g_2 = -\eta g_3$, which implies both

$$\begin{aligned}&\bigl[\alpha w_1 B_{3,1}(t) + \xi\bigl((1-\beta)w_2 B_{3,2}(t) + w_3 B_{3,3}(t)\bigr)\bigr]\bigl[\eta\widetilde{w}_1 B_{3,1}(\tilde{t}) - \xi\widetilde{w}_2 B_{3,2}(\tilde{t})\bigr]\\&\quad = \bigl[\eta\alpha w_1 B_{3,1}(t) - \xi\beta w_2 B_{3,2}(t)\bigr]\bigl[\widetilde{w}_1 B_{3,1}(\tilde{t}) + \xi\widetilde{w}_3 B_{3,3}(\tilde{t})\bigr],\end{aligned} \tag{2.1.23}$$

and

$$\frac{D(t)}{\widetilde{D}(\tilde{t})} = \frac{\eta\alpha w_1 B_{3,1}(t) - \xi\beta w_2 B_{3,2}(t)}{\eta\widetilde{w}_1 B_{3,1}(\tilde{t}) - \xi\widetilde{w}_2 B_{3,2}(\tilde{t})}, \quad t,\tilde{t} \in [0,1]. \tag{2.1.24}$$

It follows from (2.1.23) that

$$h_2(t)\tilde{t}^2 + h_1(t)\tilde{t} + h_0(t) = 0, \quad t,\tilde{t} \in [0,1], \tag{2.1.25}$$

where $h_0(t),\ h_1(t)$, and $h_2(t)$ are quadratic polynomials in t. Similarly, the identity (2.1.24) leads to

$$k_3(t)\tilde{t}^3 + k_2(t)\tilde{t}^2 + k_1(t)\tilde{t} + k_0(t) = 0, \quad t,\tilde{t} \in [0,1], \tag{2.1.26}$$

where $k_0(t),\dots,k_3(t)$ are cubic polynomials in t. Solving (2.1.25) for $\tilde{t}$ and choosing the solution $\tilde{t} = f(t)$ that satisfies (2.1.12), we see that η must satisfy

$$\eta = \frac{\beta - (w_1/\widetilde{w}_1)^2(\widetilde{w}_0/w_0)(\widetilde{w}_2/w_2)\alpha^2}{\beta - 1}.$$

Under certain conditions, (2.1.26) is automatically satisfied by $f(t)$. We omit further non-essential details here. In summary, similar to (2.1.1)–(2.1.2), the rational cubic functions representing the curve $\mathbf{c}(t),\ t \in [0,1]$, as in (2.1.7), are not unique, even up to RLTs. Finally, when the matrix with column vectors $\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3$ has rank 1, the rational curve reduces to a straight line segment, but we omit the discussion of this rare situation.

2.2. C^1 smoothness conditions for rational cubic Bézier curves

Consider two rational cubic Bézier curves

$$\tilde{\mathbf{c}}(u) = \frac{\sum_{j=0}^{3} \widetilde{w}_j \tilde{\mathbf{c}}_j B_{3,j,a,b}(u)}{\sum_{\ell=0}^{3} \widetilde{w}_\ell B_{3,\ell,a,b}(u)}, \quad u \in [a,b], \tag{2.2.1}$$

$$\mathbf{c}(u) = \frac{\sum_{j=0}^{3} w_j \mathbf{c}_j B_{3,j,b,c}(u)}{\sum_{\ell=0}^{3} w_\ell B_{3,\ell,b,c}(u)}, \quad u \in [b,c], \tag{2.2.2}$$

where $a < b < c$, and where $B_{3,k,a,b}$, $k = 0, \ldots, 3$, denote the cubic Bernstein polynomials relative to the interval $[a,b]$, i.e.,

$$B_{3,k,a,b}(x) := \binom{3}{k} \left(\frac{b-x}{b-a}\right)^{3-k} \left(\frac{x-a}{b-a}\right)^{k}, \quad k = 0, \ldots, 3. \tag{2.2.3}$$

We have the following.

Lemma 2.2. The two rational cubic Bézier curves $\tilde{\mathbf{c}}(u)$, $u \in [a,b]$, and $\mathbf{c}(u)$, $u \in [b,c]$, in (2.2.1), (2.2.2) are joined continuously (or in C^0) at $u = b$ if and only if

$$\mathbf{c}_0 = \tilde{\mathbf{c}}_3; \tag{2.2.4}$$

and they are joined in a C^1 fashion at $u = b$ if and only if both (2.2.4) and

$$\mathbf{c}_1 = \tilde{\mathbf{c}}_3 + \frac{c-b}{b-a} \frac{w_0}{w_1} \frac{\widetilde{w}_2}{\widetilde{w}_3} (\tilde{\mathbf{c}}_3 - \tilde{\mathbf{c}}_2) \tag{2.2.5}$$

are satisfied.

Proof. Consider first a rational cubic Bézier curve with parameter $t \in [0,1]$ as in (2.1.7). Then introduce

$$\begin{aligned} \mathbf{A}(t) &:= \sum_{j=0}^{3} w_j \mathbf{c}_j B_{3,j}(t), \\ W(t) &:= \sum_{\ell=0}^{3} w_\ell B_{3,\ell}(t), \end{aligned} \tag{2.2.6}$$

so that $\mathbf{c}(t)$ in (2.1.7) can be written as

$$\mathbf{c}(t) = \frac{\mathbf{A}(t)}{W(t)}, \quad t \in [0,1]. \tag{2.2.7}$$

From (2.2.6), we have

$$\begin{aligned} \mathbf{A}'(t) &= 3 \sum_{j=0}^{2} (w_{j+1} \mathbf{c}_{j+1} - w_j \mathbf{c}_j) B_{2,j}(t), \\ W'(t) &= 3 \sum_{\ell=0}^{2} (w_{\ell+1} - w_\ell) B_{2,\ell}(t), \end{aligned} \tag{2.2.8}$$

so that

$$\mathbf{A}'(0) = 3(w_1\mathbf{c}_1 - w_0\mathbf{c}_0), \qquad \mathbf{A}'(1) = 3(w_3\mathbf{c}_3 - w_2\mathbf{c}_2),$$
$$W'(0) = 3(w_1 - w_0), \qquad W'(1) = 3(w_3 - w_2). \tag{2.2.9}$$

It follows from (2.2.7)–(2.2.9) that

$$\mathbf{c}'(t) = \mathbf{A}'(t)\frac{1}{W(t)} - \mathbf{A}(t)W'(t)\frac{1}{(W(t))^2}, \tag{2.2.10}$$

and

$$\mathbf{c}'(0) = 3(w_1\mathbf{c}_1 - w_0\mathbf{c}_0)\frac{1}{w_0} - w_0\mathbf{c}_0 \cdot 3(w_1 - w_0)\frac{1}{w_0^2} = \frac{3w_1}{w_0}(\mathbf{c}_1 - \mathbf{c}_0),$$
$$\mathbf{c}'(1) = 3(w_3\mathbf{c}_3 - w_2\mathbf{c}_2)\frac{1}{w_3} - w_3\mathbf{c}_3 \cdot 3(w_3 - w_2)\frac{1}{w_3^2} = \frac{3w_2}{w_3}(\mathbf{c}_3 - \mathbf{c}_2). \tag{2.2.11}$$

Clearly, $\tilde{\mathbf{c}}(u)$, $u \in [a,b]$, and $\mathbf{c}(u)$, $u \in [b,c]$, are joined continuously (or in C^0) at $u = b$ if and only if (2.2.4) holds. We now show (2.2.5). First, C^1 conditions are equivalent to both (2.2.4) and

$$\mathbf{c}'(b) = \tilde{\mathbf{c}}'(b). \tag{2.2.12}$$

This, in turn, is equivalent to

$$\frac{1}{c-b}\frac{3w_1}{w_0}(\mathbf{c}_1 - \mathbf{c}_0) = \frac{1}{b-a}\frac{3\widetilde{w}_2}{\widetilde{w}_3}\left(\tilde{\mathbf{c}}_3 - \tilde{\mathbf{c}}_2\right),$$

which, together with (2.2.4), leads to (2.2.5). □

2.3. *G^1 connection of rational bicubic Bézier surface patches*

Since any NURBS surface can be rewritten in its rational Bernstein Bézier form, let us first focus on two generic bicubic Bézier surface patches:

$$\mathbf{s}(u,v) = \frac{\sum_{k=0}^{3}\sum_{\ell=0}^{3} w_{k,\ell}\mathbf{c}_{k,\ell}B_{3,k,u_{2i},u_{2i+2}}(u)B_{3,\ell,v_{2j},v_{2j+2}}(v)}{\sum_{k=0}^{3}\sum_{\ell=0}^{3} w_{k,\ell}B_{3,k,u_{2i},u_{2i+2}}(u)B_{3,\ell,v_{2j},v_{2j+2}}(v)},$$
$$(u,v) \in [u_{2i}, u_{2i+2}] \times [v_{2j}, v_{2j+2}], \tag{2.3.1}$$

$$\tilde{\mathbf{s}}(\tilde{u},\tilde{v}) = \frac{\sum_{k=0}^{3}\sum_{\ell=0}^{3} \widetilde{w}_{k,\ell}\tilde{\mathbf{c}}_{k,\ell}B_{3,k,\tilde{u}_{2p},\tilde{u}_{2p+2}}(\tilde{u})B_{3,\ell,\tilde{v}_{2q},\tilde{v}_{2q+2}}(\tilde{v})}{\sum_{k=0}^{3}\sum_{\ell=0}^{3} \widetilde{w}_{k,\ell}B_{3,k,\tilde{u}_{2p},\tilde{u}_{2p+2}}(\tilde{u})B_{3,\ell,\tilde{v}_{2q},\tilde{v}_{2q+2}}(\tilde{v})},$$
$$\left(\tilde{u},\tilde{v}\right) \in \left[\tilde{u}_{2p}, \tilde{u}_{2p+2}\right] \times \left[\tilde{v}_{2q}, \tilde{v}_{2q+2}\right], \tag{2.3.2}$$

where, similar to rational Bézier curves (2.1.3), the 3-vectors $\mathbf{c}_{k,\ell}$ and $\tilde{\mathbf{c}}_{k,\ell}$, $k,\ell = 0,\dots,3$, are *Bézier coefficients*, while the scalars $w_{k,\ell}$ and $\widetilde{w}_{k,\ell}$, $k,\ell = 0,\dots,3$, are called *local weights*. The two rational Bézier surface patches $\mathbf{s}(u,v)$ and $\tilde{\mathbf{s}}(\tilde{u},\tilde{v})$ in (2.3.1) and (2.3.2) are said to be *joined continuously* (or in C^0), if there exists a reparameterization $\tilde{u} = g_i(u)$, which is necessarily a strictly increasing function, such that

$$g_i(u) \in \left[\tilde{u}_{2p}, \tilde{u}_{2p+2}\right], \qquad \text{when } u \in [u_{2i}, u_{2i+2}], \tag{2.3.3}$$

$$\mathbf{s}(u, v_{2j}) = \tilde{\mathbf{s}}\big(g_i(u), \tilde{v}_{2q}\big), \qquad \text{for } u \in [u_{2i}, u_{2i+2}]. \tag{2.3.4}$$

Furthermore, $\mathbf{s}(u, v)$ and $\tilde{\mathbf{s}}(\tilde{u}, \tilde{v})$ are said to be *joined* G^1 *continuously* (or in G^1, see [9]), if $\mathbf{s}(u, v)$ and $\tilde{\mathbf{s}}(\tilde{u}, \tilde{v})$ are joined continuously, and if there exist three polynomials Θ, Φ, and Ψ such that, with the same function g_i,

$$\Theta(u)\frac{\partial}{\partial v}\mathbf{s}(u, v_{2j}) = \Phi(u)\frac{\partial}{\partial \tilde{v}}\tilde{\mathbf{s}}\big(g_i(u), \tilde{v}_{2q}\big) + \Psi(u)\frac{\partial}{\partial \tilde{u}}\tilde{\mathbf{s}}\big(g_i(u), \tilde{v}_{2q}\big), \quad u \in [u_{2i}, u_{2i+2}]. \tag{2.3.5}$$

We are now in a position to establish a sufficient condition for connecting two rational Bézier surface patches in a G^1 fashion along the boundaries, say $u = u_{2i}$ and $\tilde{u} = \tilde{u}_{2p}$.

Lemma 2.3. Two rational bicubic Bézier surface patches $\mathbf{s}(u, v)$ and $\tilde{\mathbf{s}}(\tilde{u}, \tilde{v})$ in (2.3.1)–(2.3.2) are joined G^1 continuously, if

$$\tilde{\mathbf{c}}_{k,0} = \mathbf{c}_{k,0}, \qquad \widetilde{w}_{k,0} = w_{k,0}, \quad k = 0, \ldots, 3, \tag{2.3.6}$$

and

$$\tilde{\mathbf{c}}_{k,1} = \mathbf{c}_{k,0} - \frac{\tilde{v}_{2q+2} - \tilde{v}_{2q}}{v_{2j+2} - v_{2j}}\frac{w_{k,1}}{\widetilde{w}_{k,1}}(\mathbf{c}_{k,1} - \mathbf{c}_{k,0}), \tag{2.3.7}$$

$$\widetilde{w}_{k,1} = w_{k,0} - \frac{\tilde{v}_{2q+2} - \tilde{v}_{2q}}{v_{2j+2} - v_{2j}}(w_{k,1} - w_{k,0}), \quad k = 0, \ldots, 3, \tag{2.3.8}$$

where, without loss of generality, the normalization $w_{0,0} = \widetilde{w}_{0,0}$ is assumed.

Proof. By lemma 2.1, since the three 3-vectors $\tilde{\mathbf{c}}_{1,0} - \tilde{\mathbf{c}}_{0,0}$, $\tilde{\mathbf{c}}_{2,0} - \tilde{\mathbf{c}}_{0,0}$, and $\tilde{\mathbf{c}}_{3,0} - \tilde{\mathbf{c}}_{0,0}$ are, in general, linearly independent, the C^0 condition (2.3.6) between $\tilde{\mathbf{s}}(\tilde{u}, \tilde{v})$ and $\mathbf{s}(u, v)$ is clear. To study the other G^1 condition in (2.3.7)–(2.3.8), let us first take the partial derivatives of $\mathbf{s}$. To this end, again we may focus on Bézier surface patches with parameters $(u, v) \in [0, 1]^2$, namely,

$$\mathbf{s}_0(u, v) = \frac{\sum_{k=0}^{3}\sum_{\ell=0}^{3} w_{k,\ell}\mathbf{c}_{k,\ell}B_{3,k}(u)B_{3,\ell}(v)}{\sum_{k=0}^{3}\sum_{\ell=0}^{3} w_{k,\ell}B_{3,k}(u)B_{3,\ell}(v)}, \quad (u, v) \in [0, 1]^2, \tag{2.3.9}$$

and introduce

$$\mathbf{A}(u, v) := \sum_{k=0}^{3}\sum_{\ell=0}^{3} w_{k,\ell}\mathbf{c}_{k,\ell}B_{3,k}(u)B_{3,\ell}(v), \tag{2.3.10}$$

$$W(u, v) := \sum_{k=0}^{3}\sum_{\ell=0}^{3} w_{k,\ell}B_{3,k}(u)B_{3,\ell}(v), \tag{2.3.11}$$

so that

$$\mathbf{s}_0(u,v) = \frac{\mathbf{A}(u,v)}{W(u,v)}, \quad (u,v) \in [0,1]^2. \tag{2.3.12}$$

It then follows from (2.3.11)–(2.3.12) that

$$\begin{aligned}
\frac{\partial \mathbf{s}_0(u,v)}{\partial u} &= \frac{1}{W(u,v)}\frac{\partial \mathbf{A}(u,v)}{\partial u} - \frac{\mathbf{A}(u,v)}{W(u,v)^2}\frac{\partial W(u,v)}{\partial u} \\
&= \frac{3}{W(u,v)}\sum_{k=0}^{2}\sum_{\ell=0}^{3}(w_{k+1,\ell}\mathbf{c}_{k+1,\ell} - w_{k,\ell}\mathbf{c}_{k,\ell})B_{3,\ell}(v)B_{2,k}(u) \\
&\quad - 3\frac{\mathbf{A}(u,v)}{W(u,v)^2}\sum_{k=0}^{2}\sum_{\ell=0}^{3}(w_{k+1,\ell} - w_{k,\ell})B_{3,\ell}(v)B_{2,k}(u), \qquad (2.3.13)\\
\frac{\partial \mathbf{s}_0(u,v)}{\partial v} &= \frac{1}{W(u,v)}\frac{\partial \mathbf{A}(u,v)}{\partial v} - \frac{\mathbf{A}(u,v)}{W(u,v)^2}\frac{\partial W(u,v)}{\partial v} \\
&= \frac{3}{W(u,v)}\sum_{\ell=0}^{2}\sum_{k=0}^{3}(w_{k,\ell+1}\mathbf{c}_{k,\ell+1} - w_{k,\ell}\mathbf{c}_{k,\ell})B_{3,k}(u)B_{2,\ell}(v) \\
&\quad - 3\frac{\mathbf{A}(u,v)}{W(u,v)^2}\sum_{\ell=0}^{2}\sum_{k=0}^{3}(w_{k,\ell+1} - w_{k,\ell})B_{3,k}(u)B_{2,\ell}(v). \qquad (2.3.14)
\end{aligned}$$

Hence, for $u \in [0,1]$, we have

$$\begin{aligned}
\left.\frac{\partial \mathbf{s}_0(u,v)}{\partial u}\right|_{v=0} &= \frac{3}{\sum_{k=0}^{3} w_{k,0}B_{3,k}(u)}\sum_{k=0}^{2}(w_{k+1,0}\mathbf{c}_{k+1,0} - w_{k,0}\mathbf{c}_{k,0})B_{2,k}(u) \\
&\quad - \frac{3\sum_{k=0}^{3} w_{k,0}\mathbf{c}_{k,0}B_{3,k}(u)}{(\sum_{k=0}^{3} w_{k,0}B_{3,k}(u))^2}\sum_{k=0}^{2}(w_{k+1,0} - w_{k,0})B_{2,k}(u), \qquad (2.3.15)\\
\left.\frac{\partial \mathbf{s}_0(u,v)}{\partial v}\right|_{v=0} &= \frac{3}{\sum_{k=0}^{3} w_{k,0}B_{3,k}(u)}\sum_{k=0}^{3}(w_{k,1}\mathbf{c}_{k,1} - w_{k,0}\mathbf{c}_{k,0})B_{3,k}(u) \\
&\quad - \frac{\sum_{k=0}^{3} w_{k,0}\mathbf{c}_{k,0}B_{3,k}(u)}{(\sum_{k=0}^{3} w_{k,0}B_{3,k}(u))^2}3\sum_{k=0}^{3}(w_{k,1} - w_{k,0})B_{3,k}(u). \qquad (2.3.16)
\end{aligned}$$

By applying the identity

$$B_{m,p}(x)B_{n,q}(x) = \frac{\binom{m}{p}\binom{n}{q}}{\binom{m+n}{p+q}}B_{m+n,p+q}(x), \tag{2.3.17}$$

we see that both (2.3.15) and (2.3.16) can be written as 6th degree rational Bézier formulation as follows:

$$\frac{1}{3}\frac{\partial \mathbf{s}_0(u,v)}{\partial u}\bigg|_{v=0} = \frac{\sum_{k=0}^{6} w_k^{\mathrm{u}} \mathbf{c}_k^{\mathrm{u}} B_{6,k}(u)}{\sum_{k=0}^{6} w_k^{\mathrm{u}} B_{6,k}(u)}, \tag{2.3.18}$$

$$\frac{1}{3}\frac{\partial \mathbf{s}_0(u,v)}{\partial v}\bigg|_{v=0} = \frac{\sum_{k=0}^{6} w_k^{\mathrm{u}} \mathbf{c}_k^{\mathrm{v}} B_{6,k}(u)}{\sum_{k=0}^{6} w_k^{\mathrm{u}} B_{6,k}(u)}, \tag{2.3.19}$$

where the local weights $w_k^{\mathrm{u}},\ 0 \leqslant k \leqslant 6$, are given by

$$\begin{aligned}
w_0^{\mathrm{u}} &= w_{0,0}^2,\\
w_1^{\mathrm{u}} &= w_{0,0}w_{1,0},\\
w_2^{\mathrm{u}} &= \frac{2}{5}w_{0,0}w_{2,0} + \frac{3}{5}w_{1,0}^2,\\
w_3^{\mathrm{u}} &= \frac{1}{10}w_{0,0}w_{3,0} + \frac{9}{10}w_{1,0}w_{2,0},\\
w_4^{\mathrm{u}} &= \frac{2}{5}w_{1,0}w_{3,0} + \frac{3}{5}w_{2,0}^2,\\
w_5^{\mathrm{u}} &= w_{2,0}w_{3,0},\\
w_6^{\mathrm{u}} &= w_{3,0}^2,
\end{aligned} \tag{2.3.20}$$

while the local control points $\mathbf{c}_k^{\mathrm{u}}$ and $\mathbf{c}_k^{\mathrm{v}},\ k = 0, \ldots, 6$, are defined in terms of the local weights, namely,

$$\begin{aligned}
w_0^{\mathrm{u}}\mathbf{c}_0^{\mathrm{u}} &= w_{1,0}w_{0,0}(\mathbf{c}_{1,0} - \mathbf{c}_{0,0}),\\
w_1^{\mathrm{u}}\mathbf{c}_1^{\mathrm{u}} &= \frac{1}{3}w_{1,0}w_{0,0}(\mathbf{c}_{1,0} - \mathbf{c}_{0,0}) + \frac{1}{3}w_{2,0}w_{0,0}(\mathbf{c}_{2,0} - \mathbf{c}_{0,0}),\\
w_2^{\mathrm{u}}\mathbf{c}_2^{\mathrm{u}} &= \frac{1}{15}w_{1,0}w_{0,0}(\mathbf{c}_{1,0} - \mathbf{c}_{0,0}) + \frac{4}{15}w_{2,0}w_{0,0}(\mathbf{c}_{2,0} - \mathbf{c}_{0,0})\\
&\quad + \frac{1}{5}w_{2,0}w_{1,0}(\mathbf{c}_{2,0} - \mathbf{c}_{1,0}) + \frac{1}{15}w_{3,0}w_{0,0}(\mathbf{c}_{3,0} - \mathbf{c}_{0,0}),\\
w_3^{\mathrm{u}}\mathbf{c}_3^{\mathrm{u}} &= \frac{1}{10}w_{3,0}w_{0,0}(\mathbf{c}_{3,0} - \mathbf{c}_{0,0}) + \frac{3}{10}w_{2,0}w_{1,0}(\mathbf{c}_{2,0} - \mathbf{c}_{1,0})\\
&\quad + \frac{1}{10}w_{3,0}w_{1,0}(\mathbf{c}_{3,0} - \mathbf{c}_{1,0}) + \frac{1}{10}w_{2,0}w_{0,0}(\mathbf{c}_{2,0} - \mathbf{c}_{0,0}),\\
w_4^{\mathrm{u}}\mathbf{c}_4^{\mathrm{u}} &= \frac{1}{15}w_{3,0}w_{2,0}(\mathbf{c}_{3,0} - \mathbf{c}_{2,0}) + \frac{4}{15}w_{3,0}w_{1,0}(\mathbf{c}_{3,0} - \mathbf{c}_{1,0})\\
&\quad + \frac{1}{5}w_{2,0}w_{1,0}(\mathbf{c}_{2,0} - \mathbf{c}_{1,0}) + \frac{1}{15}w_{3,0}w_{0,0}(\mathbf{c}_{3,0} - \mathbf{c}_{0,0}),\\
w_5^{\mathrm{u}}\mathbf{c}_5^{\mathrm{u}} &= \frac{1}{3}w_{3,0}w_{2,0}(\mathbf{c}_{3,0} - \mathbf{c}_{2,0}) + \frac{1}{3}w_{3,0}w_{1,0}(\mathbf{c}_{3,0} - \mathbf{c}_{1,0}),\\
w_6^{\mathrm{u}}\mathbf{c}_6^{\mathrm{u}} &= w_{3,0}w_{2,0}(\mathbf{c}_{3,0} - \mathbf{c}_{2,0}),
\end{aligned} \tag{2.3.21}$$

and

$$
\begin{aligned}
w_0^{\mathrm{u}}\mathbf{c}_0^{\mathrm{v}} &= w_{0,1}w_{0,0}(\mathbf{c}_{0,1}-\mathbf{c}_{0,0}),\\
w_1^{\mathrm{u}}\mathbf{c}_1^{\mathrm{v}} &= \frac{1}{2}w_{1,1}w_{0,0}(\mathbf{c}_{1,1}-\mathbf{c}_{0,0})+\frac{1}{2}w_{0,1}w_{1,0}(\mathbf{c}_{0,1}-\mathbf{c}_{1,0}),\\
w_2^{\mathrm{u}}\mathbf{c}_2^{\mathrm{v}} &= \frac{1}{5}w_{2,1}w_{0,0}(\mathbf{c}_{2,1}-\mathbf{c}_{0,0})+\frac{3}{5}w_{1,1}w_{1,0}(\mathbf{c}_{1,1}-\mathbf{c}_{1,0})+\frac{1}{5}w_{0,1}w_{2,0}(\mathbf{c}_{0,1}-\mathbf{c}_{2,0}),\\
w_3^{\mathrm{u}}\mathbf{c}_3^{\mathrm{v}} &= \frac{1}{20}w_{3,1}w_{0,0}(\mathbf{c}_{3,1}-\mathbf{c}_{0,0})+\frac{9}{20}w_{2,1}w_{1,0}(\mathbf{c}_{2,1}-\mathbf{c}_{1,0})\\
&\quad+\frac{9}{20}w_{1,1}w_{2,0}(\mathbf{c}_{1,1}-\mathbf{c}_{2,0})+\frac{1}{20}w_{0,1}w_{3,0}(\mathbf{c}_{0,1}-\mathbf{c}_{3,0}),\\
w_4^{\mathrm{u}}\mathbf{c}_4^{\mathrm{v}} &= \frac{1}{5}w_{3,1}w_{1,0}(\mathbf{c}_{3,1}-\mathbf{c}_{1,0})+\frac{3}{5}w_{2,1}w_{2,0}(\mathbf{c}_{2,1}-\mathbf{c}_{2,0})+\frac{1}{5}w_{1,1}w_{3,0}(\mathbf{c}_{1,1}-\mathbf{c}_{3,0}),\\
w_5^{\mathrm{u}}\mathbf{c}_5^{\mathrm{v}} &= \frac{1}{2}w_{3,1}w_{2,0}(\mathbf{c}_{3,1}-\mathbf{c}_{2,0})+\frac{1}{2}w_{2,1}w_{3,0}(\mathbf{c}_{2,1}-\mathbf{c}_{3,0}),\\
w_6^{\mathrm{u}}\mathbf{c}_6^{\mathrm{v}} &= w_{3,1}w_{3,0}(\mathbf{c}_{3,1}-\mathbf{c}_{3,0}).
\end{aligned}
\tag{2.3.22}
$$

To connect $\mathbf{s}$ and $\tilde{\mathbf{s}}$ in (2.3.1) and (2.3.2) in a G^1 fashion, we simply choose $\Theta(u) = 1$, $\Phi(u) = -1$, and $\Psi(u) = 0$ in (2.3.5), and $g_i(u)$ as a linear transformation, i.e.,

$$\tilde{u} = g_i(u) := \tilde{u}_{2p}\frac{u_{2i+2}-u}{u_{2i+2}-u_{2i}} + \tilde{u}_{2p+2}\frac{u-u_{2i}}{u_{2i+2}-u_{2i}}, \quad u \in [u_{2i}, u_{2i+2}]. \tag{2.3.23}$$

With g_i in (2.3.23), we see that $\tilde{\mathbf{s}}$ in (2.3.2) can be rewritten as

$$
\begin{aligned}
\tilde{\mathbf{s}}\big(g_i(u),\tilde{v}\big) &= \frac{\sum_{k=0}^{3}\sum_{\ell=0}^{3}\widetilde{w}_{k,\ell}\tilde{\mathbf{c}}_{k,\ell}B_{3,k,u_{2i},u_{2i+2}}(u)B_{3,\ell,\tilde{v}_{2q},\tilde{v}_{2q+2}}(\tilde{v})}{\sum_{k=0}^{3}\sum_{\ell=0}^{3}\widetilde{w}_{k,\ell}B_{3,k,u_{2i},u_{2i+2}}(u)B_{3,\ell,\tilde{v}_{2q},\tilde{v}_{2q+2}}(\tilde{v})},\\
&\big(u,\tilde{v}\big) \in [u_{2i}, u_{2i+2}] \times \big[\tilde{v}_{2q}, \tilde{v}_{2q+2}\big].
\end{aligned}
\tag{2.3.24}
$$

Then, by applying (2.3.19), the first order derivative of $\mathbf{s}$ with respect to v, along $v = v_{2j}$, is given by

$$\frac{1}{3}\frac{\partial}{\partial v}\mathbf{s}(u, v_{2j}) = \frac{1}{v_{2j+2}-v_{2j}}\frac{\sum_{k=0}^{6}w_k^{\mathrm{u}}\mathbf{c}_k^{\mathrm{v}}B_{6,k,u_{2i},u_{2i+2}}(u)}{\sum_{k=0}^{6}w_k^{\mathrm{u}}B_{6,k,u_{2i},u_{2i+2}}(u)}, \quad u \in [u_{2i}, u_{2i+2}], \tag{2.3.25}$$

where w_k^{u} and $w_k^{\mathrm{u}}\mathbf{c}_k^{\mathrm{v}}$ are given by using (2.3.20) and (2.3.22). Analogously, we also have

$$\frac{1}{3}\frac{\partial}{\partial \tilde{v}}\tilde{\mathbf{s}}\big(g_i(u), \tilde{v}_{2q}\big) = \frac{1}{\tilde{v}_{2q+2}-\tilde{v}_{2q}}\frac{\sum_{k=0}^{6}\widetilde{w}_k^{\mathrm{u}}\tilde{\mathbf{c}}_k^{\mathrm{v}}B_{6,k,u_{2i},u_{2i+2}}(u)}{\sum_{k=0}^{6}\widetilde{w}_k^{\mathrm{u}}B_{6,k,u_{2i},u_{2i+2}}(u)}, \quad u \in [u_{2i}, u_{2i+2}], \tag{2.3.26}$$

where $\widetilde{w}_k^{\mathrm{u}}$ are defined exactly the same as w_k^{u} in (2.3.20), with w replaced by $\widetilde{w}$, while $\widetilde{w}_k^{\mathrm{u}}\tilde{\mathbf{c}}_k^{\mathrm{v}}$ are determined by

$$\begin{aligned}
\widetilde{w}_0^{\mathrm{u}}\tilde{\mathbf{c}}_0^{\mathrm{v}} &= \widetilde{w}_{0,0}\widetilde{w}_{0,1}\big(\tilde{\mathbf{c}}_{0,1}-\tilde{\mathbf{c}}_{0,0}\big),\\
\widetilde{w}_1^{\mathrm{u}}\tilde{\mathbf{c}}_1^{\mathrm{v}} &= \frac{1}{2}\widetilde{w}_{0,0}\widetilde{w}_{1,1}\big(\tilde{\mathbf{c}}_{1,1}-\tilde{\mathbf{c}}_{0,0}\big)+\frac{1}{2}\widetilde{w}_{1,0}\widetilde{w}_{0,1}\big(\tilde{\mathbf{c}}_{0,1}-\tilde{\mathbf{c}}_{1,0}\big),\\
\widetilde{w}_2^{\mathrm{u}}\tilde{\mathbf{c}}_2^{\mathrm{v}} &= \frac{1}{5}\widetilde{w}_{0,0}\widetilde{w}_{2,1}\big(\tilde{\mathbf{c}}_{2,1}-\tilde{\mathbf{c}}_{0,0}\big)+\frac{3}{5}\widetilde{w}_{1,0}\widetilde{w}_{1,1}\big(\tilde{\mathbf{c}}_{1,1}-\tilde{\mathbf{c}}_{1,0}\big)+\frac{1}{5}\widetilde{w}_{2,0}\widetilde{w}_{0,1}\big(\tilde{\mathbf{c}}_{0,1}-\tilde{\mathbf{c}}_{2,0}\big),\\
\widetilde{w}_3^{\mathrm{u}}\tilde{\mathbf{c}}_3^{\mathrm{v}} &= \frac{1}{20}\widetilde{w}_{0,0}\widetilde{w}_{3,1}\big(\tilde{\mathbf{c}}_{3,1}-\tilde{\mathbf{c}}_{0,0}\big)+\frac{9}{20}\widetilde{w}_{1,0}\widetilde{w}_{2,1}\big(\tilde{\mathbf{c}}_{2,1}-\tilde{\mathbf{c}}_{1,0}\big)\\
&\quad+\frac{9}{20}\widetilde{w}_{2,0}\widetilde{w}_{1,1}\big(\tilde{\mathbf{c}}_{1,1}-\tilde{\mathbf{c}}_{2,0}\big)+\frac{1}{20}\widetilde{w}_{3,0}\widetilde{w}_{0,1}\big(\tilde{\mathbf{c}}_{0,1}-\tilde{\mathbf{c}}_{3,0}\big),\\
\widetilde{w}_4^{\mathrm{u}}\tilde{\mathbf{c}}_4^{\mathrm{v}} &= \frac{1}{5}\widetilde{w}_{1,0}\widetilde{w}_{3,1}\big(\tilde{\mathbf{c}}_{3,1}-\tilde{\mathbf{c}}_{1,0}\big)+\frac{3}{5}\widetilde{w}_{2,0}\widetilde{w}_{2,1}\big(\tilde{\mathbf{c}}_{2,1}-\tilde{\mathbf{c}}_{2,0}\big)+\frac{1}{5}\widetilde{w}_{3,0}\widetilde{w}_{1,1}\big(\tilde{\mathbf{c}}_{1,1}-\tilde{\mathbf{c}}_{3,0}\big),\\
\widetilde{w}_5^{\mathrm{u}}\tilde{\mathbf{c}}_5^{\mathrm{v}} &= \frac{1}{2}\widetilde{w}_{2,0}\widetilde{w}_{3,1}\big(\tilde{\mathbf{c}}_{3,1}-\tilde{\mathbf{c}}_{2,0}\big)+\frac{1}{2}\widetilde{w}_{3,0}\widetilde{w}_{2,1}\big(\tilde{\mathbf{c}}_{2,1}-\tilde{\mathbf{c}}_{3,0}\big),\\
\widetilde{w}_6^{\mathrm{u}}\tilde{\mathbf{c}}_6^{\mathrm{v}} &= \widetilde{w}_{3,0}\widetilde{w}_{3,1}\big(\tilde{\mathbf{c}}_{3,1}-\tilde{\mathbf{c}}_{3,0}\big).
\end{aligned} \tag{2.3.27}$$

Hence, a sufficient condition for

$$\frac{\partial}{\partial v}s(u,v_{2j}) = -\frac{\partial}{\partial\tilde{v}}\tilde{s}\big(g_i(u),\tilde{v}_{2q}\big),\quad u\in[u_{2i},u_{2i+2}],$$

which is (2.3.5) with $\Theta(u)=1$, $\Phi(u)=-1$, and $\Psi(u)=0$, is given by

$$\widetilde{w}_k^{\mathrm{u}} = w_k^{\mathrm{u}}, \tag{2.3.28}$$

and

$$\frac{1}{\tilde{v}_{2q+2}-\tilde{v}_{2q}}\tilde{\mathbf{c}}_k^{\mathrm{v}} = -\frac{1}{v_{2j+2}-v_{2j}}\mathbf{c}_k^{\mathrm{v}},\quad k=0,\ldots,6. \tag{2.3.29}$$

Note that for $\widetilde{w}_{0,0}=w_{0,0}$, the conditions in (2.3.6) already imply that (2.3.28) holds. Finally, from (2.3.21) and (2.3.27), it only takes some straightforward calculation to see that the equalities in (2.3.29) are satisfied by the choice of $\widetilde{w}_{k,1}$ and $\tilde{\mathbf{c}}_{k,1}$ in (2.3.7)–(2.3.8), $k=0,\ldots,6$. This completes the proof of lemma 2.3. □

3. Algorithm for G^1 connection of two NURBS surfaces

We are now ready to study parametric NURBS surfaces in $\mathbb{R}^3$. From the practical point of view, it is important to be able to join multiple individually *predesigned* NURBS surfaces without gaps and to satisfy certain requirements, such as interpolating a given common set of points on the boundaries and smoothness across the boundaries (see [4] for joining spline surfaces). A classical method for joining multiple predesigned NURBS surfaces is to carve away portions of the surfaces along the boundaries and *fill* the gaps by introducing additional surface patches.

The main objective of this paper is to study the feasibility of removing gaps simply by manipulating the control points and weights, but without disturbing the interpolation data, while minimizing the modification of the NURBS surfaces patches. The important constraints are that no additional surfaces could be used to fill the gaps, that the modification (if needed) is supposed to be very minimal, and that the combined surface, without gaps, should be smooth.

Let us first discuss the notion of NURBS surfaces studied in [5,6,8,10]. For the parametric domain $[0,1]^2$, let

$$\mathbf{u} = \{0 = u_0 = \cdots = u_3 < u_4 = u_5 < u_6 = u_7 \\ < \cdots < u_{2m} = u_{2m+1} < u_{2m+2} = \cdots = u_{2m+5} = 1\}, \tag{3.1}$$

and

$$\mathbf{v} = \{0 = v_0 = \cdots = v_3 < v_4 = v_5 < v_6 = v_7 \\ < \cdots < v_{2n} = u_{2n+1} < v_{2n+2} = \cdots = v_{2n+5} = 1\} \tag{3.2}$$

be parametric *knot sequences*; $M_{4,\mathbf{u},i}$ and $M_{4,\mathbf{v},j}$ be the corresponding 4th order normalized B-splines [1,2,10] with knots $u_i, \ldots, u_{i+4}$ and $v_j, \ldots, v_{j+4}$, respectively, $i = 0, \ldots, 2m+1,\ j = 0, \ldots, 2n+1$. Let

$$\mathbf{w} := \{w_{i,j}\colon\ i = 0, \ldots, 2m+1,\ j = 0, \ldots, 2n+1\} \tag{3.3}$$

be a (global) *weight sequence*, where $w_{i,j} > 0$ is the (global) *weight* corresponding to the knot position (u_i, v_j), $i = 0, \ldots, 2m+1,\ j = 0, \ldots, 2n+1$, and set

$$N_{\mathbf{u},\mathbf{v},\mathbf{w},i,j}(u,v) := \frac{w_{i,j} M_{4,\mathbf{u},i}(u) M_{4,\mathbf{v},j}(v)}{\sum_{k=0}^{2m+1} \sum_{\ell=0}^{2n+1} w_{k,\ell} M_{4,\mathbf{u},k}(u) M_{4,\mathbf{v},\ell}(v)}, \\ i = 0, \ldots, 2m+1,\ j = 0, \ldots, 2n+1. \tag{3.4}$$

We will study *bicubic NURBS surfaces* with two parameters u and v of the form

$$S_1\colon\quad \mathbf{f}(u,v) = \sum_{i=0}^{2m+1} \sum_{j=0}^{2n+1} \mathbf{d}_{i,j} N_{\mathbf{u},\mathbf{v},\mathbf{w},i,j}(u,v) \\ = \sum_{i=0}^{2m+1} \sum_{j=0}^{2n+1} \mathbf{d}_{i,j} \frac{w_{i,j} M_{4,\mathbf{u},i}(u) M_{4,\mathbf{v},j}(v)}{\sum_{k=0}^{2m+1} \sum_{\ell=0}^{2n+1} w_{k,\ell} M_{4,\mathbf{u},k}(u) M_{4,\mathbf{v},\ell}(v)}, \quad (u,v) \in [0,1]^2, \tag{3.5}$$

where $\mathbf{d}_{i,j}$ is the (global) *control point* relative to the parametric position (u_i, v_j), $i = 0, \ldots, 2m+1,\ j = 0, \ldots, 2n+1$.

We begin with connecting only two NURBS surfaces. To this end, we need another NURBS surface S_2 with knot sequences

$$\tilde{\mathbf{u}} := \bigl\{0 = \tilde{u}_0 = \cdots = \tilde{u}_3 < \tilde{u}_4 = \tilde{u}_5 < \tilde{u}_6 = \tilde{u}_7 \\ < \cdots < \tilde{u}_{2\tilde{m}} = \tilde{u}_{2\tilde{m}+1} < \tilde{u}_{2\tilde{m}+2} = \cdots = \tilde{u}_{2\tilde{m}+5} = 1\bigr\}, \tag{3.6}$$

$$\tilde{\mathbf{v}} := \bigl\{0 = \tilde{v}_0 = \cdots = \tilde{v}_3 < \tilde{v}_4 = \tilde{v}_5 < \tilde{v}_6 = \tilde{v}_7 \\ < \cdots < \tilde{v}_{2\tilde{n}} = \tilde{v}_{2\tilde{n}+1} < \tilde{v}_{2\tilde{n}+2} = \cdots = \tilde{v}_{2\tilde{n}+5} = 1\bigr\}, \tag{3.7}$$

weight sequence $\widetilde{\mathbf{w}} = \{\widetilde{w}_{i,j}\}_{i=0,\ldots,2\widetilde{m}+1;\ j=0,\ldots,2\tilde{n}+1}$, and (global) control points $\tilde{\mathbf{d}}_{i,j}$, $i = 0,\ldots,2\widetilde{m}+1, j = 0,\ldots,2\tilde{n}+1$. That is,

$$S_2: \quad \tilde{\mathbf{f}}(\tilde{u},\tilde{v}) = \sum_{i=0}^{2\widetilde{m}+1}\sum_{j=0}^{2\tilde{n}+1} \tilde{\mathbf{d}}_{i,j} N_{\tilde{\mathbf{u}},\tilde{\mathbf{v}},\widetilde{\mathbf{w}},i,j}(\tilde{u},\tilde{v})$$
$$= \sum_{i=0}^{2\widetilde{m}+1}\sum_{j=0}^{2\tilde{n}+1} \tilde{\mathbf{d}}_{i,j} \frac{\widetilde{w}_{i,j} M_{4,\tilde{\mathbf{u}},i}(\tilde{u}) M_{4,\tilde{\mathbf{v}},j}(\tilde{v})}{\sum_{k=0}^{2\widetilde{m}+1}\sum_{\ell=0}^{2\tilde{n}+1} \widetilde{w}_{k,\ell} M_{4,\tilde{\mathbf{u}},k}(\tilde{u}) M_{4,\tilde{\mathbf{v}},\ell}(\tilde{v})},$$
$$(\tilde{u},\tilde{v}) \in [0,1]^2. \tag{3.8}$$

As illustrated in figure 1, we assume that the first Bézier surface patch of S_1 along the u-direction corresponds to the rth Bézier surface patch of S_2 along the $\tilde{u}$-direction, and that $m + r - 1 \leqslant \widetilde{m}$.

In order to *keep* the NURBS representations after appropriate adjustments of the Bézier coefficients and local weights of one or both NURBS surfaces, the underlying knot sequences of S_1 and S_2 along the "common" boundaries are required to be *proportional* [3]. For this reason, we only consider *equally-spaced* knot sequences along the common boundaries, i.e.,

$$\alpha_i := \frac{v_{2i+2} - v_{2i}}{v_{2i+2} - v_{2i-2}} = \frac{1}{2}, \quad i = 1,\ldots,m-1, \quad \text{and}$$
$$\tilde{\alpha}_i := \frac{\tilde{v}_{2i+2} - \tilde{v}_{2i}}{\tilde{v}_{2i+2} - \tilde{v}_{2i-2}} = \frac{1}{2}, \quad i = 1,\ldots,\widetilde{m}-1.$$

In other words, for $\mathbf{u}$ and $\mathbf{v}$ in (3.1)–(3.2),

$$u_{2i+2} - u_{2i} = \textit{constant} := L_u = \frac{1}{m}, \quad i = 1,\ldots,m,$$
$$v_{2j+2} - v_{2j} = \textit{constant} := L_v = \frac{1}{n}, \quad j = 1,\ldots,n; \tag{3.9}$$

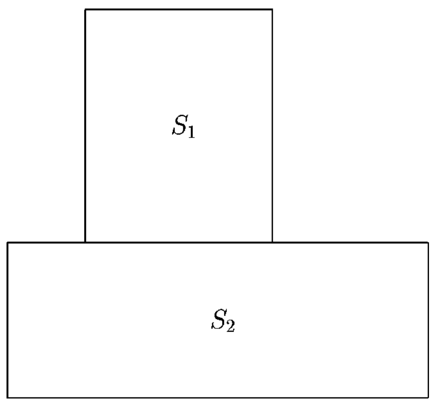

Figure 1. Schematic diagram of parametric domains of two parametric NURBS surfaces S_1 and S_2.

and for $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ in (3.6)–(3.7),

$$\begin{aligned}\tilde{u}_{2i+2} - \tilde{u}_{2i} &= \textit{constant} := L_{\tilde{u}} = \frac{1}{\widetilde{m}}, \quad i = 1, \dots, \widetilde{m},\\ \tilde{v}_{2j+2} - \tilde{v}_{2j} &= \textit{constant} := L_{\tilde{v}} = \frac{1}{\tilde{n}}, \quad j = 1, \dots, \tilde{n}.\end{aligned} \tag{3.10}$$

Due to (3.9)–(3.10), the two NURBS surfaces S_1 and S_2 in (3.5) and (3.8) can be rewritten as

$$\begin{aligned} S_1\colon \quad \mathbf{f}(u,v) &= \sum_{i=0}^{2m+1} \sum_{j=0}^{2n+1} \mathbf{d}_{i,j} N_{\mathbf{u},\mathbf{v},\mathbf{w},i,j}(u,v)\\ &= \sum_{i=0}^{2m+1} \sum_{j=0}^{2n+1} \mathbf{d}_{i,j} \frac{w_{i,j} M_{4,\mathbf{u},i}(u) M_{4,\mathbf{v},j}(v)}{\sum_{k=0}^{2m+1} \sum_{\ell=0}^{2n+1} w_{k,\ell} M_{4,\mathbf{u},k}(u) M_{4,\mathbf{v},\ell}(v)},\\ &(u,v) \in [0,m] \times [0,n], \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} S_2\colon \quad \tilde{\mathbf{f}}(\tilde{u},\tilde{v}) &= \sum_{i=0}^{2\tilde{m}+1} \sum_{j=0}^{2\tilde{n}+1} \tilde{\mathbf{d}}_{i,j} N_{\tilde{\mathbf{u}},\tilde{\mathbf{v}},\tilde{\mathbf{w}},i,j}(\tilde{u},\tilde{v})\\ &= \sum_{i=0}^{2\tilde{m}+1} \sum_{j=0}^{2\tilde{n}+1} \tilde{\mathbf{d}}_{i,j} \frac{\widetilde{w}_{i,j} M_{4,\tilde{\mathbf{u}},i}(\tilde{u}) M_{4,\tilde{\mathbf{v}},j}(\tilde{v})}{\sum_{k=0}^{2\tilde{m}+1} \sum_{\ell=0}^{2\tilde{n}+1} \widetilde{w}_{k,\ell} M_{4,\tilde{\mathbf{u}},k}(\tilde{u}) M_{4,\tilde{\mathbf{v}},\ell}(\tilde{v})},\\ &(\tilde{u},\tilde{v}) \in [0,\widetilde{m}] \times [0,\tilde{n}], \end{aligned} \tag{3.12}$$

where the knots are

$$\begin{aligned} &u_0 = u_1 = 0, \quad && u_{2i} = u_{2i+1} = i-1, \quad && i = 1, \dots, m+1,\\ &v_0 = v_1 = 0, && v_{2j} = v_{2j+1} = j-1, && j = 1, \dots, n+1;\\ &\tilde{u}_0 = \tilde{u}_1 = 0, && \tilde{u}_{2i} = \tilde{u}_{2i+1} = i-1, && i = 1, \dots, \widetilde{m}+1,\\ &\tilde{v}_0 = \tilde{v}_1 = 0, && \tilde{v}_{2j} = \tilde{v}_{2j+1} = j-1, && j = 1, \dots, \tilde{n}+1. \end{aligned}$$

A NURBS surface S_1 is said to be *connectible* along, say, the lower boundary strip, if the global weights along this boundary strip satisfy

$$\begin{aligned} &2w_{i,0} - w_{i,1} > 0, \quad i = 0, \dots, 2m+1, \quad \text{and}\\ &2w_{0,0} - w_{1,0} > 0, \quad 2w_{2m+1,0} - w_{2m,0} > 0. \end{aligned} \tag{3.13}$$

Similarly, it is *connectible* along the upper, left, and right boundary strips if

$$\begin{aligned} &2w_{i,2n+1} - w_{i,2n} > 0, && i = 0, \dots, 2m+1,\\ &2w_{0,2n+1} - w_{1,2n+1} > 0, && 2w_{2m+1,2n+1} - w_{2m,2n+1} > 0; \end{aligned} \tag{3.14}$$

$$\begin{aligned} &2w_{0,j} - w_{1,j} > 0, && j = 0, \dots, 2n+1,\\ &2w_{0,0} - w_{0,1} > 0, && 2w_{0,2n+1} - w_{0,2n} > 0; \quad \text{and} \end{aligned} \tag{3.15}$$

$$\begin{aligned} &2w_{2m+1,j} - w_{2m,j} > 0, && j = 0, \dots, 2n+1,\\ &2w_{2m+1,0} - w_{2m+1,1} > 0, && 2w_{2m+1,2n+1} - w_{2m+1,2n} > 0, \end{aligned} \tag{3.16}$$

respectively. Now assume that S_1 is connectible along its lower boundary, i.e., (3.13) has been satisfied. By applying algorithm A.1 in the appendix and lemma 2.3, we can connect S_1 and S_2 by adjusting the (global) control points and global weights of S_2 along its upper boundary strip by applying the following.

Algorithm 1 (G^1 connection of two NURBS surfaces S_1 and S_2, see figure 1).

Keep S_1 intact and modify the first and second lines of control points along the upper boundary strip of S_2 as follows, while keeping the other control points of S_2 unchanged. Then S_1 and S_2 are connected in a G^1 fashion.

1°. At the lower-left corner position of the parametric domain of S_1, set

$$\begin{aligned}
\tilde{\mathbf{d}}_{2r-2,0} &= \frac{2w_{0,0}\mathbf{d}_{0,0} - w_{1,0}\mathbf{d}_{1,0}}{2w_{0,0} - w_{1,0}}, \\
\widetilde{w}_{2r-2,0} &= 2w_{0,0} - w_{1,0}; \\
\tilde{\mathbf{d}}_{2r-2,1} &= \frac{2(2w_{0,0}\mathbf{d}_{0,0} - w_{0,1}\mathbf{d}_{0,1}) - (2w_{1,0}\mathbf{d}_{1,0} - w_{1,1}\mathbf{d}_{1,1})}{2(2w_{0,0} - w_{0,1}) - (2w_{1,0} - w_{1,1})}, \\
\widetilde{w}_{2r-2,1} &= 2(2w_{0,0} - w_{0,1}) - (2w_{1,0} - w_{1,1}).
\end{aligned} \tag{3.17}$$

2°. Along the corresponding (interior) m pairs of boundary patches of the parametric domains of S_1 and S_2, set

$$\begin{aligned}
\tilde{\mathbf{d}}_{2r-2+k,0} &= \mathbf{d}_{k,0}, \\
\widetilde{w}_{2r-2+k,0} &= w_{k,0}; \\
\tilde{\mathbf{d}}_{2r-2+k,1} &= \frac{2w_{k,0}\mathbf{d}_{k,0} - w_{k,1}\mathbf{d}_{k,1}}{2w_{k,0} - w_{k,1}}, \\
\widetilde{w}_{2r-2+k,1} &= 2w_{k,0} - w_{k,1}, \quad k = 1, \ldots, 2m.
\end{aligned} \tag{3.18}$$

3°. At the lower-right corner position of the parametric domain of S_1, set

$$\begin{aligned}
\tilde{\mathbf{d}}_{2(r+m-1)+1,0} &= \frac{2w_{2m+1,0}\mathbf{d}_{2m+1,0} - w_{2m,0}\mathbf{d}_{2m,0}}{2w_{2m+1,0} - w_{2m,0}}, \\
\widetilde{w}_{2(r+m-1)+1,0} &= 2w_{2m+1,0} - w_{2m,0}; \\
\tilde{\mathbf{d}}_{2(r+m-1)+1,1} &= \frac{2(2w_{2m+1,0}\mathbf{d}_{2m+1,0} - w_{2m+1,1}\mathbf{d}_{2m+1,1})}{2(2w_{2m+1,0} - w_{2m+1,1}) - (2w_{2m,0} - w_{2m,1})} \\
&\quad - \frac{2w_{2m,0}\mathbf{d}_{2m,0} - w_{2m,1}\mathbf{d}_{2m,1}}{2(2w_{2m+1,0} - w_{2m+1,1}) - (2w_{2m,0} - w_{2m,1})}, \\
\widetilde{w}_{2(r+m-1)+1,1} &= 2(2w_{2m+1,0} - w_{2m+1,1}) - (2w_{2m,0} - w_{2m,1}).
\end{aligned} \tag{3.19}$$

4. Algorithms for G^1 connection of multiple NURBS surfaces

The techniques in the previous section can be extended to treat the connection of multiple NURBS surfaces. We only consider connecting three and four NURBS surfaces in this section.

In addition to S_1 and S_2 in (3.17)–(3.18), let S_3 be a NURBS surface with knot sequences $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, i.e.,

$$\begin{aligned}\hat{\mathbf{u}} := \{0 = \hat{u}_0 = \cdots = \hat{u}_3 < \hat{u}_4 = \hat{u}_5 < \hat{u}_6 = \hat{u}_7 \\ < \cdots < \hat{u}_{2\hat{m}} = \hat{u}_{2\hat{m}+1} < \hat{u}_{2\hat{m}+2} = \cdots = \hat{u}_{2\hat{m}+5} = \widehat{m}\},\end{aligned} \tag{4.1}$$

$$\begin{aligned}\hat{\mathbf{v}} := \{0 = \hat{v}_0 = \cdots = \hat{v}_3 < \hat{v}_4 = \hat{v}_5 < \hat{v}_6 = \hat{v}_7 \\ < \cdots < \hat{v}_{2\hat{n}} = \hat{v}_{2\hat{n}+1} < \hat{v}_{2\hat{n}+2} = \cdots = \hat{v}_{2\hat{n}+5} = \hat{n}\},\end{aligned} \tag{4.2}$$

weight sequence $\widehat{\mathbf{w}} = \{\widehat{w}_{i,j}\}_{i=0,\ldots,2\hat{m}+1;j=0,\ldots,2\hat{n}+1}$, and control points $\hat{\mathbf{d}}_{i,j}$, $i = 0,\ldots,2\widehat{m}+1$, $j = 0,\ldots,2\hat{n}+1$, so that

$$\begin{aligned} S_3\text{:}\quad \hat{\mathbf{f}}(\hat{u},\hat{v}) &= \sum_{i=0}^{2\hat{m}+1}\sum_{j=0}^{2\hat{n}+1} \hat{\mathbf{d}}_{i,j} N_{\hat{\mathbf{u}},\hat{\mathbf{v}},\hat{\mathbf{w}},i,j}(\hat{u},\hat{v}) \\ &= \sum_{i=0}^{2\hat{m}+1}\sum_{j=0}^{2\hat{n}+1} \hat{\mathbf{d}}_{i,j} \frac{\widehat{w}_{i,j} M_{4,\hat{\mathbf{u}},i}(\hat{u}) M_{4,\hat{\mathbf{v}},j}(\hat{v})}{\sum_{k=0}^{2\hat{m}+1}\sum_{\ell=0}^{2\hat{n}+1} \widehat{w}_{k,\ell} M_{4,\hat{\mathbf{u}},k}(\hat{u}) M_{4,\hat{\mathbf{v}},\ell}(\hat{v})}, \\ &(\hat{u},\hat{v}) \in [0,\widehat{m}] \times [0,\hat{n}]. \end{aligned} \tag{4.3}$$

Again, the knots are

$$\hat{u}_0 = \hat{u}_1 = 0, \qquad \hat{u}_{2i} = \hat{u}_{2i+1} = i-1, \quad i = 1,\ldots,\widehat{m}+1,$$

$$\hat{v}_0 = \hat{v}_1 = 0, \qquad \hat{v}_{2j} = \hat{v}_{2j+1} = j-1, \quad j = 1,\ldots,\hat{n}+1.$$

In addition to the assumption that S_1 is connectible along its lower boundary strip, we also require both S_1 and S_2 to be connectible along their right boundary strips. The second algorithm is to connect S_1, S_2 and S_3, where one of the boundary knot of S_1 must correspond to a boundary knot of S_2, namely, $m + r - 1 = \widetilde{m}$ in algorithm 1. In other words, after applying algorithm 1 to S_1 and S_2, the third NURBS surface S_3 is connected to the compound surface of S_1 and S_2 by algorithm 1 again, as illustrated in figure 2.

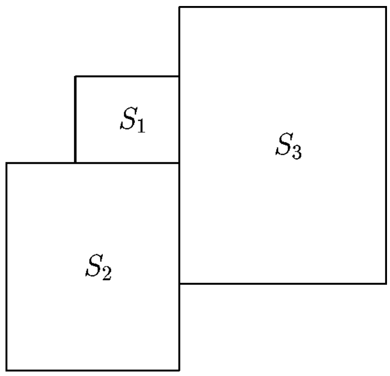

Figure 2. Schematic diagram of parametric domains of three parametric NURBS surfaces S_1, S_2, and S_3.

Algorithm 2 (G^1 connection of three NURBS surfaces $S_1,\ S_2$, and S_3, see figure 2).

1°. Apply algorithm 1 to S_1 and S_2. Denote the compound surface by S_{12}.

2°. Keep S_{12} intact, and connect the third NURBS surface S_3 to S_{12} by applying another variation of algorithm 1 along the right boundary strip of S_{12} and/or the left boundary strip of S_3.

Next, let us consider connecting four NURBS surfaces in a G^1 fashion. In addition to $S_1,\ S_2$ and S_3 in (3.17), (3.18) and (4.3), let S_4 be a NURBS surface with knot sequences $\check{\mathbf{u}}$ and $\check{\mathbf{v}}$, i.e.,

$$\begin{aligned}\check{\mathbf{u}} := \big\{0 = \check{u}_0 = \cdots = \check{u}_3 < \check{u}_4 = \check{u}_5 < \check{u}_6 = \check{u}_7 \\ < \cdots < \check{u}_{2\check{m}} = \check{u}_{2\check{m}+1} < \check{u}_{2\check{m}+2} = \cdots = \check{u}_{2\check{m}+5} = \check{m}\big\},\end{aligned} \tag{4.4}$$

$$\begin{aligned}\check{\mathbf{v}} := \big\{0 = \check{v}_0 = \cdots = \check{v}_3 < \check{v}_4 = \check{v}_5 < \check{v}_6 = \check{v}_7 \\ < \cdots < \check{v}_{2\check{n}} = \check{v}_{2\check{n}+1} < \check{v}_{2\check{n}+2} = \cdots = \check{v}_{2\check{n}+5} = \check{n}\big\},\end{aligned} \tag{4.5}$$

weight sequence $\check{\mathbf{w}} = \{\check{w}_{i,j}\}_{i=0,\ldots,2\check{m}+1;j=0,\ldots,2\check{n}+1}$, and control points $\check{\mathbf{d}}_{i,j},\ i = 0,\ldots,2\check{m}+1,\ j = 0,\ldots,2\check{n}+1$, so that

$$\begin{aligned} S_4:\quad \check{\mathbf{f}}(\check{u},\check{v}) &= \sum_{i=0}^{2\check{m}+1}\sum_{j=0}^{2\check{n}+1} \check{\mathbf{d}}_{i,j} N_{\check{\mathbf{u}},\check{\mathbf{v}},\check{\mathbf{w}},i,j}(\check{u},\check{v}) \\ &= \sum_{i=0}^{2\check{m}+1}\sum_{j=0}^{2\check{n}+1} \check{\mathbf{d}}_{i,j} \frac{\check{w}_{i,j} M_{4,\check{\mathbf{u}},i}(\check{u}) M_{4,\check{\mathbf{v}},j}(\check{v})}{\sum_{k=0}^{2\check{m}+1}\sum_{\ell=0}^{2\check{n}+1} \check{w}_{k,\ell} M_{4,\check{\mathbf{u}},k}(\check{u}) M_{4,\check{\mathbf{v}},\ell}(\check{v})}, \\ &(\check{u},\check{v}) \in [0,\check{m}] \times [0,\check{n}]. \end{aligned} \tag{4.6}$$

Again, the knots are

$$\begin{aligned} &\check{u}_0 = \check{u}_1 = 0, \qquad \check{u}_{2i} = \check{u}_{2i+1} = i-1, \quad i = 1,\ldots,\check{m}+1, \\ &\check{v}_0 = \check{v}_1 = 0, \qquad \check{v}_{2j} = \check{v}_{2j+1} = j-1, \quad j = 1,\ldots,\check{n}+1. \end{aligned}$$

The third algorithm is to connect the four NURBS surface $S_1,\ldots,S_4$ in a G^1 fashion. To do so, by algorithm 1, we see that S_1 and S_2, as well as S_3 and S_4, can be connected in a G^1 fashion. If we denote the two compound NURBS surfaces by S_{12} and S_{34}, then, by applying algorithm 1 again, the two new NURBS surfaces S_{12} and S_{34} can be connected in a G^1 fashion, as illustrated in figure 3.

Algorithm 3 (G^1 connection of four NURBS surfaces $S_1,\ldots,S_4$, see figure 3).

1°. Apply algorithm 1 to S_1 and S_2. Denote the compound surface by S_{12}.

2°. Apply algorithm 1 to S_3 and S_4. Denote the compound surface by S_{34}.

3°. Keep S_{12} intact, and connect S_{34} to S_{12} by applying the same variation of algorithm 1, along the right boundary stripe of S_{12} and/or the left boundary strip of S_{34}.

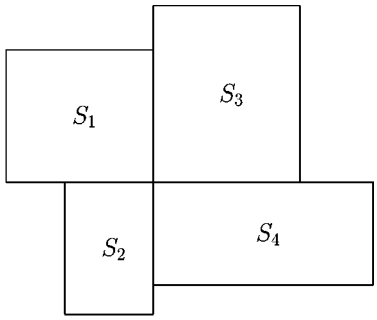

Figure 3. Schematic diagram of parametric domains of four parametric NURBS surfaces S_1, S_2, S_3 and S_4.

Appendix. Algorithms for converting NURBS to Bézier representations and vice versa

Various continuous joining conditions between *two* NURBS surfaces can be established *through* smoothing conditions between rational Bézier surface patches. To this end, we will give, in this section, algorithms for converting a bicubic NURBS to Bézier representations, and *vice versa*.

First, the representation (3.5) can be rewritten as

$$S_1:\quad \mathbf{f}(u,v) = \frac{\sum_{i=1}^m \sum_{j=1}^n \sum_{k=0}^3 \sum_{\ell=0}^3 w_{i,j,k,\ell}\mathbf{d}_{i,j,k,\ell} B_{3,k,u_{2i},u_{2i+2}}(u) B_{3,\ell,v_{2j},v_{2j+2}}(v)}{\sum_{i=1}^m \sum_{j=1}^n \sum_{k=0}^3 \sum_{\ell=0}^3 w_{i,j,k,\ell} B_{3,k,u_{2i},u_{2i+2}}(u) B_{3,\ell,v_{2j},v_{2j+2}}(v)}, \tag{A.1}$$

where

$$(u,v) \in [0,1]^2 = \bigcup_{i=1}^m \bigcup_{j=1}^n [u_{2i}, u_{2i+2}] \times [u_{2j}, u_{2j+2}],$$

$\mathbf{d}_{i,j,k,\ell}$, $k,\ell = 0,\ldots,3$, are *Bézier coefficients* (or local control points) and $w_{i,j,k,\ell}$, $k,\ell = 0,\ldots,3$, are *local weights*, corresponding to the parametric domains $[u_{2i}, u_{2i+2}] \times [u_{2j}, u_{2j+2}]$, $i = 1,\ldots,m$, $j = 1,\ldots,n$; and $B_{3,j,a,b}$ denotes the truncated cubic Bernstein polynomials relative to the interval $[a,b]$, i.e.,

$$B_{3,k,a,b}(x) := \binom{3}{k}\left(\frac{b-x}{b-a}\right)^{3-k}\left(\frac{x-a}{b-a}\right)^k \chi_{[a,b]}(x), \quad k = 0,\ldots,3.$$

To find the Bézier coefficients and local weights in (A.1) from (3.1), define

$$\mathbf{s}_{i,j}(u,v) := \mathbf{f}(u,v)\big|_{[u_{2i},u_{2i+2}]\times[u_{2j},u_{2j+2}]}, \quad i = 1,\ldots,m,\ j = 1,\ldots,n. \tag{A.2}$$

Then by the compact supportedness property of B-splines, we have

$$\mathbf{s}_{i,j}(u,v) = \frac{\sum_{k=0}^3 \sum_{\ell=0}^3 w_{2i-2+k,2j-2+\ell}\mathbf{d}_{2i-2+k,2j-2+\ell} M_{4,\mathbf{u},i}(u) M_{4,\mathbf{v},j}(v)}{\sum_{k=0}^3 \sum_{\ell=0}^3 w_{2i-2+k,2j-2+\ell} M_{4,\mathbf{u},i}(u) M_{4,\mathbf{v},j}(v)},$$
$$(u,v) \in [u_{2i}, u_{2i+2}] \times [u_{2j}, u_{2j+2}]. \tag{A.3}$$

Observe that

$$M_{4,\mathbf{u},2i}(u) = B_{3,2,u_{2i},u_{2i+2}}(u) + \alpha_i B_{3,3,u_{2i},u_{2i+2}}(u) + \alpha_i B_{3,0,u_{2i+2},u_{2i+4}}, \quad \text{and}$$
$$M_{4,\mathbf{u},2i+1}(u) = \beta_i B_{3,3,u_{2i},u_{2i+2}}(u) + \beta_i B_{3,0,u_{2i+2},u_{2i+4}}(u) + B_{3,1,u_{2i+2},u_{2i+4}}(u), \quad 0 \leqslant i \leqslant m, \tag{A.4}$$

where

$$\alpha_i := \frac{u_{2i+4} - u_{2i+2}}{u_{2i+4} - u_{2i}}, \qquad \beta_i := \frac{u_{2i+2} - u_{2i}}{u_{2i+4} - u_{2i}} = 1 - \alpha_i, \quad 0 \leqslant i \leqslant m; \tag{A.5}$$

and

$$M_{4,\mathbf{v},2j}(v) = B_{3,2,v_{2j},v_{2j+2}}(v) + \xi_j B_{3,3,v_{2j},v_{2j+2}}(v) + \xi_j B_{3,0,v_{2j+2},v_{2j+4}}, \quad \text{and}$$
$$M_{4,\mathbf{v},2j+1}(v) = \eta_j B_{3,3,v_{2j},v_{2j+2}}(v) + \eta_j B_{3,0,v_{2j+2},v_{2j+4}}(v) + B_{3,1,v_{2j+2},v_{2j+4}}(v), \quad 0 \leqslant j \leqslant n, \tag{A.6}$$

where

$$\xi_j := \frac{v_{2j+4} - v_{2j+2}}{v_{2j+4} - v_{2j}}, \qquad \eta_j := \frac{v_{2j+2} - v_{2j}}{v_{2j+4} - v_{2j}} = 1 - \xi_j, \quad 0 \leqslant j \leqslant n. \tag{A.7}$$

We next introduce the following pairs of arrays:

(i) along the horizontal parametric grid lines, $\{w^{\mathrm{h}}_{k,\ell}\}$ and $\{\mathbf{d}^{\mathrm{h}}_{k,\ell}\}$ of size $(2m+2) \times (n+1)$;

(ii) along the vertical parametric grid lines, $\{w^{\mathrm{v}}_{k,\ell}\}$ and $\{\mathbf{d}^{\mathrm{v}}_{k,\ell}\}$ of size $(m+1)\times(2n+2)$; and

(iii) at the corner positions of the parametric domain, $\{w^{\mathrm{c}}_{k,\ell}\}$ and $\{\mathbf{d}^{\mathrm{c}}_{k,\ell}\}$ of size $(m+1) \times (n+1)$, defined by

$$\begin{aligned}
w^{\mathrm{h}}_{2i,j} &:= \xi_{j-1} w_{2i,2j-2} + \eta_{j-1} w_{2i,2j-1},\\
w^{\mathrm{h}}_{2i+1,j} &:= \xi_{j-1} w_{2i+1,2j-2} + \eta_{j-1} w_{2i+1,2j-1},\\
w^{\mathrm{h}}_{2i,j}\mathbf{d}^{\mathrm{h}}_{2i,j} &= \xi_{j-1} w_{2i,2j-2}\mathbf{d}_{2i,2j-2} + \eta_{j-1} w_{2i,2j-1}\mathbf{d}_{2i,2j-1},\\
w^{\mathrm{h}}_{2i+1,j}\mathbf{d}^{\mathrm{h}}_{2i+1,j} &= \xi_{j-1} w_{2i+1,2j-2}\mathbf{d}_{2i+1,2j-2} + \eta_{j-1} w_{2i+1,2j-1}\mathbf{d}_{2i+1,2j-1},\\
& i = 0,\dots,m,\ j = 1,\dots,n+1;
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
w^{\mathrm{v}}_{i,2j} &:= \alpha_{i-1} w_{2i-2,2j} + \beta_{i-1} w_{2i-1,2j},\\
w^{\mathrm{v}}_{i,2j+1} &:= \alpha_{i-1} w_{2i-2,2j+1} + \beta_{i-1} w_{2i-1,2j+1},\\
w^{\mathrm{v}}_{i,2j}\mathbf{d}^{\mathrm{v}}_{i,2j} &= \alpha_{i-1} w_{2i-2,2j}\mathbf{d}_{2i-2,2j} + \beta_{i-1} w_{2i-1,2j}\mathbf{d}_{2i-1,2j},\\
w^{\mathrm{v}}_{i,2j+1}\mathbf{d}^{\mathrm{v}}_{i,2j+1} &= \alpha_{i-1} w_{2i-2,2j+1}\mathbf{d}_{2i-2,2j+1} + \beta_{i-1} w_{2i-1,2j+1}\mathbf{d}_{2i-1,2j+1},\\
& i = 1,\dots,m+1,\ j = 0,\dots,n;
\end{aligned} \tag{A.9}$$

and

$$\begin{aligned} w^{\mathrm{c}}_{i,j} &:= \xi_{j-1} w^{\mathrm{v}}_{i,2j-2} + \eta_{j-1} w^{\mathrm{v}}_{i,2j-1} \\ &= \alpha_{i-1} w^{\mathrm{h}}_{2i-2,j} + \beta_{i-1} w^{\mathrm{h}}_{2i-1,j}, \\ w^{\mathrm{c}}_{i,j}\mathbf{d}^{\mathrm{c}}_{i,j} &= \xi_{j-1} w^{\mathrm{v}}_{i,2j-2}\mathbf{d}^{\mathrm{v}}_{i,2j-2} + \eta_{j-1} w^{\mathrm{v}}_{i,2j-1}\mathbf{d}^{\mathrm{v}}_{i,2j-1} \\ &= \alpha_{i-1} w^{\mathrm{h}}_{2i-2,j}\mathbf{d}^{\mathrm{h}}_{2i-2,j} + \beta_{i-1} w^{\mathrm{h}}_{2i-1,j}\mathbf{d}^{\mathrm{h}}_{2i-1,j}, \\ & \quad i = 1,\ldots,m+1,\ j = 1,\ldots,n+1. \end{aligned} \tag{A.10}$$

Then we have the following.

Algorithm A.1 (Conversion from NURBS to Bézier representations).

1°. The mn 4×4 coefficient matrices of the 16 local weights are given by, for $i = 1,\ldots,m,\ j = 1,\ldots,n,$

$$[w_{i,j,\ell,3-k}]_{0\leqslant k,\ell\leqslant 3} = t_{i,j} \begin{bmatrix} w^{\mathrm{c}}_{i,j+1} & w^{\mathrm{h}}_{2i-1,j+1} & w^{\mathrm{h}}_{2i,j+1} & w^{\mathrm{c}}_{i+1,j+1} \\ w^{\mathrm{v}}_{i,2j} & w_{2i-1,2j} & w_{2i,2j} & w^{\mathrm{v}}_{i+1,2j} \\ w^{\mathrm{v}}_{i,2j-1} & w_{2i-1,2j-1} & w_{2i,2j-1} & w^{\mathrm{v}}_{i+1,2j-1} \\ w^{\mathrm{c}}_{i,j} & w^{\mathrm{h}}_{2i-1,j} & w^{\mathrm{h}}_{2i,j} & w^{\mathrm{c}}_{i+1,j} \end{bmatrix}, \tag{A.11}$$

where $t_{i,j}$ are arbitrary positive constants.

2°. The corresponding mn 4×4 coefficient matrices of the 16 Bézier coefficients, with 3-vector entries, are given by, for $i = 1,\ldots,m,\ j = 1,\ldots,n,$

$$[\mathbf{d}_{i,j,\ell,3-k}]_{0\leqslant k,\ell\leqslant 3} = \begin{bmatrix} \mathbf{d}^{\mathrm{c}}_{i,j+1} & \mathbf{d}^{\mathrm{h}}_{2i-1,j+1} & \mathbf{d}^{\mathrm{h}}_{2i,j+1} & \mathbf{d}^{\mathrm{c}}_{i+1,j+1} \\ \mathbf{d}^{\mathrm{v}}_{i,2j} & \mathbf{d}_{2i-1,2j} & \mathbf{d}_{2i,2j} & \mathbf{d}^{\mathrm{v}}_{i+1,2j} \\ \mathbf{d}^{\mathrm{v}}_{i,2j-1} & \mathbf{d}_{2i-1,2j-1} & \mathbf{d}_{2i,2j-1} & \mathbf{d}^{\mathrm{v}}_{i+1,2j-1} \\ \mathbf{d}^{\mathrm{c}}_{i,j} & \mathbf{d}^{\mathrm{h}}_{2i-1,j} & \mathbf{d}^{\mathrm{h}}_{2i,j} & \mathbf{d}^{\mathrm{c}}_{i+1,j} \end{bmatrix}. \tag{A.12}$$

For connecting multiple NURBS surfaces, we need the Bézier coefficients and local weights along the "common" or connecting boundary strips of all NURBS surfaces. So let us write down (A.11)–(A.12) more explicitly, say, along the "lower" boundary strip. It follows, from (A.5) and (A.7)–(A.10), that when $i = j = 1$,

$$\begin{aligned} &[w_{1,1,\ell,3-k}]_{0\leqslant k,\ell\leqslant 3} \\ &= t_{1,1} \begin{bmatrix} \xi_1 w_{0,2} + \eta_1 w_{0,3} & \xi_1 w_{1,2} + \eta_1 w_{1,3} & \xi_1 w_{2,2} + \eta_1 w_{2,3} & w^{\mathrm{c}}_{2,2} \\ w_{0,2} & w_{1,2} & w_{2,2} & \alpha_1 w_{2,2} + \beta_1 w_{3,2} \\ w_{0,1} & w_{1,1} & w_{2,1} & \alpha_1 w_{2,1} + \beta_1 w_{3,1} \\ w_{0,0} & w_{1,0} & w_{2,0} & \alpha_1 w_{2,0} + \beta_1 w_{3,0} \end{bmatrix}, \end{aligned} \tag{A.13}$$

with $t_{1,1}$ an arbitrary positive constant and

$$\begin{aligned} w^{\mathrm{c}}_{2,2} &= \xi_1(\alpha_1 w_{2,2} + \beta_1 w_{3,2}) + \eta_1(\alpha_1 w_{2,3} + \beta_1 w_{3,3}) \\ &= \alpha_1(\xi_1 w_{2,2} + \eta_1 w_{2,3}) + \beta_1(\xi_1 w_{3,2} + \eta_1 w_{3,3}); \end{aligned} \tag{A.14}$$

and

$$[\mathbf{d}_{1,1,\ell,3-k}]_{0\leqslant k,\ell\leqslant 3} = \begin{bmatrix} \mathbf{d}^{c}_{1,2} & \mathbf{d}^{h}_{1,2} & \mathbf{d}^{h}_{2,2} & \mathbf{d}^{c}_{2,2} \\ \mathbf{d}_{0,2} & \mathbf{d}_{1,2} & \mathbf{d}_{2,2} & \mathbf{d}^{v}_{2,2} \\ \mathbf{d}_{0,1} & \mathbf{d}_{1,1} & \mathbf{d}_{2,1} & \mathbf{d}^{v}_{2,1} \\ \mathbf{d}_{0,0} & \mathbf{d}_{1,0} & \mathbf{d}_{2,0} & \mathbf{d}^{c}_{2,1} \end{bmatrix}, \tag{A.15}$$

with

$$\begin{aligned}
\mathbf{d}^{c}_{1,2} &= \frac{\xi_1 w_{0,2}\mathbf{d}_{0,2} + \eta_1 w_{0,3}\mathbf{d}_{0,3}}{\xi_1 w_{0,2} + \eta_1 w_{0,3}},\\
\mathbf{d}^{h}_{1,2} &= \frac{\xi_1 w_{1,2}\mathbf{d}_{1,2} + \eta_1 w_{1,3}\mathbf{d}_{1,3}}{\xi_1 w_{1,2} + \eta_1 w_{1,3}},\\
\mathbf{d}^{h}_{2,2} &= \frac{\xi_1 w_{2,2}\mathbf{d}_{2,2} + \eta_1 w_{2,3}\mathbf{d}_{2,3}}{\xi_1 w_{2,2} + \eta_1 w_{2,3}},\\
\mathbf{d}^{c}_{2,2} &= \frac{\xi_1(\alpha_1 w_{2,2}\mathbf{d}_{2,2} + \beta_1 w_{3,2}\mathbf{d}_{3,2}) + \eta_1(\alpha_1 w_{2,3}\mathbf{d}_{2,3} + \beta_1 w_{3,3}\mathbf{d}_{3,3})}{\xi_1(\alpha_1 w_{2,2} + \beta_1 w_{3,2}) + \eta_1(\alpha_1 w_{2,3} + \beta_1 w_{3,3})},\\
\mathbf{d}^{v}_{2,2} &= \frac{\alpha_1 w_{2,2}\mathbf{d}_{2,2} + \beta_1 w_{3,2}\mathbf{d}_{3,2}}{\alpha_1 w_{2,2} + \beta_1 w_{3,2}},\\
\mathbf{d}^{v}_{2,1} &= \frac{\alpha_1 w_{2,1}\mathbf{d}_{2,1} + \beta_1 w_{3,1}\mathbf{d}_{3,1}}{\alpha_1 w_{2,1} + \beta_1 w_{3,1}},\\
\mathbf{d}^{c}_{2,1} &= \mathbf{d}^{v}_{2,0} = \frac{\alpha_1 w_{2,0}\mathbf{d}_{2,0} + \beta_1 w_{3,0}\mathbf{d}_{3,0}}{\alpha_1 w_{2,0} + \beta_1 w_{3,0}}.
\end{aligned} \tag{A.16}$$

When $i = 2, \ldots, m-1,\ j = 1,$

$$[w_{i,1,\ell,3-k}]_{0\leqslant k,\ell\leqslant 3} = t_{i,1} \begin{bmatrix} w^{c}_{i,2} & w^{h}_{2i-1,2} & w^{h}_{2i,2} & w^{c}_{i+1,2} \\ w^{v}_{i,2} & w_{2i-1,2} & w_{2i,2} & w^{v}_{i+1,2} \\ w^{v}_{i,1} & w_{2i-1,1} & w_{2i,1} & w^{v}_{i+1,1} \\ w^{c}_{i,1} & w_{2i-1,0} & w_{2i,0} & w^{c}_{i+1,1} \end{bmatrix}, \tag{A.17}$$

with $t_{i,1}$ arbitrary positive constants and

$$\begin{aligned}
w^{c}_{i,2} &= \xi_1(\alpha_{i-1} w_{2i-2,2} + \beta_1 w_{2i-1,2}) + \eta_1(\alpha_{i-1} w_{2i-2,3} + \beta_1 w_{2i-1,3}),\\
w^{h}_{2i-1,2} &= \xi_1 w_{2i-1,2} + \eta_1 w_{2i-1,3},\\
w^{h}_{2i,2} &= \xi_1 w_{2i,2} + \eta_1 w_{2i,3},\\
w^{c}_{i+1,2} &= \xi_1(\alpha_i w_{2i,2} + \beta_1 w_{2i+1,2}) + \eta_1(\alpha_i w_{2i,3} + \beta_1 w_{2i+1,3}),\\
w^{v}_{i,2} &= \alpha_{i-1} w_{2i-2,2} + \beta_1 w_{2i-1,2},
\end{aligned} \tag{A.18}$$

$$
\begin{aligned}
w^{\mathrm{v}}_{i+1,2} &= \alpha_i w_{2i,2} + \beta_i w_{2i+1,2},\\
w^{\mathrm{v}}_{i,1} &= \alpha_{i-1} w_{2i-2,1} + \beta_1 w_{2i-1,1},\\
w^{\mathrm{v}}_{i+1,1} &= \alpha_i w_{2i,1} + \beta_i w_{2i+1,1},\\
w^{\mathrm{c}}_{i,1} &= w^{\mathrm{v}}_{i,0} = \alpha_{i-1} w_{2i-2,0} + \beta_1 w_{2i-1,0},\\
w^{\mathrm{c}}_{i+1,1} &= w^{\mathrm{v}}_{i+1,0} = \alpha_i w_{2i,0} + \beta_i w_{2i+1,0};
\end{aligned}
$$

and

$$
[\mathbf{d}_{i,1,\ell,3-k}]_{0\leqslant k,\ell\leqslant 3} = \begin{bmatrix} \mathbf{d}^{\mathrm{c}}_{i,2} & \mathbf{d}^{\mathrm{h}}_{2i-1,2} & \mathbf{d}^{\mathrm{h}}_{2i,2} & \mathbf{d}^{\mathrm{c}}_{i+1,2} \\ \mathbf{d}^{\mathrm{v}}_{i,2} & \mathbf{d}_{2i-1,2} & \mathbf{d}_{2i,2} & \mathbf{d}^{\mathrm{v}}_{i+1,2} \\ \mathbf{d}^{\mathrm{v}}_{i,1} & \mathbf{d}_{2i-1,1} & \mathbf{d}_{2i,1} & \mathbf{d}^{\mathrm{v}}_{i+1,1} \\ \mathbf{d}^{\mathrm{c}}_{i,1} & \mathbf{d}_{2i-1,0} & \mathbf{d}_{2i,0} & \mathbf{d}^{\mathrm{c}}_{i+1,1} \end{bmatrix}, \tag{A.19}
$$

with

$$
\begin{aligned}
\mathbf{d}^{\mathrm{c}}_{i,2} &= \frac{\xi_1(\alpha_{i-1} w_{2i-2,2}\mathbf{d}_{2i-2,2} + \beta_{i-1} w_{2i-1,2}\mathbf{d}_{2i-1,2})}{\xi_1(\alpha_{i-1} w_{2i-2,2} + \beta_{i-1} w_{2i-1,2}) + \eta_1(\alpha_{i-1} w_{2i-2,3} + \beta_{i-1} w_{2i-1,3})}\\
&\quad + \frac{\eta_1(\alpha_{i-1} w_{2i-2,3}\mathbf{d}_{2i-2,3} + \beta_{i-1} w_{2i-1,3}\mathbf{d}_{2i-1,3})}{\xi_1(\alpha_{i-1} w_{2i-2,2} + \beta_{i-1} w_{2i-1,2}) + \eta_1(\alpha_{i-1} w_{2i-2,3} + \beta_{i-1} w_{2i-1,3})},\\
\mathbf{d}^{\mathrm{h}}_{2i-1,2} &= \frac{\xi_1 w_{2i-1,2}\mathbf{d}_{2i-1,2} + \eta_1 w_{2i-1,3}\mathbf{d}_{2i-1,3}}{\xi_1 w_{2i-1,2} + \eta_1 w_{2i-1,3}},\\
\mathbf{d}^{\mathrm{h}}_{2i,2} &= \frac{\xi_1 w_{2i,2}\mathbf{d}_{2i,2} + \eta_1 w_{2i,3}\mathbf{d}_{2i,3}}{\xi_1 w_{2i,2} + \eta_1 w_{2i,3}},\\
\mathbf{d}^{\mathrm{c}}_{i+1,2} &= \frac{\xi_1(\alpha_i w_{2i,2}\mathbf{d}_{2i,2} + \beta_i w_{2i+1,2}\mathbf{d}_{2i+1,2})}{\xi_1(\alpha_i w_{2i,2} + \beta_i w_{2i+1,2}) + \eta_1(\alpha_i w_{2i,3} + \beta_i w_{2i+1,3})}\\
&\quad + \frac{\eta_1(\alpha_i w_{2i,3}\mathbf{d}_{2i,3} + \beta_i w_{2i+1,3}\mathbf{d}_{2i+1,3})}{\xi_1(\alpha_i w_{2i,2} + \beta_i w_{2i+1,2}) + \eta_1(\alpha_i w_{2i,3} + \beta_i w_{2i+1,3})},\\
\mathbf{d}^{\mathrm{v}}_{i,2} &= \frac{\alpha_{i-1} w_{2i-2,2}\mathbf{d}_{2i-2,2} + \beta_1 w_{2i-1,2}\mathbf{d}_{2i-1,2}}{\alpha_{i-1} w_{2i-2,2} + \beta_1 w_{2i-1,2}},\\
\mathbf{d}^{\mathrm{v}}_{i+1,2} &= \frac{\alpha_i w_{2i,2}\mathbf{d}_{2i,2} + \beta_i w_{2i+1,2}\mathbf{d}_{2i+1,2}}{\alpha_i w_{2i,2} + \beta_i w_{2i+1,2}},\\
\mathbf{d}^{\mathrm{v}}_{i,1} &= \frac{\alpha_{i-1} w_{2i-2,1}\mathbf{d}_{2i-2,1} + \beta_1 w_{2i-1,1}\mathbf{d}_{2i-1,1}}{\alpha_{i-1} w_{2i-2,1} + \beta_1 w_{2i-1,1}},\\
\mathbf{d}^{\mathrm{v}}_{i+1,1} &= \frac{\alpha_i w_{2i,1}\mathbf{d}_{2i,1} + \beta_i w_{2i+1,1}\mathbf{d}_{2i+1,1}}{\alpha_i w_{2i,1} + \beta_i w_{2i+1,1}},\\
\mathbf{d}^{\mathrm{c}}_{i,1} &= \mathbf{d}^{\mathrm{v}}_{i,0} = \frac{\alpha_{i-1} w_{2i-2,0}\mathbf{d}_{2i-2,0} + w_{2i-1,0}\beta_1\mathbf{d}_{2i-1,0}}{\alpha_{i-1} w_{2i-2,0} + \beta_1 w_{2i-1,0}},\\
\mathbf{d}^{\mathrm{c}}_{i+1,1} &= \mathbf{d}^{\mathrm{v}}_{i+1,0} = \frac{\alpha_i w_{2i,0}\mathbf{d}_{2i,0} + \beta_i w_{2i+1,0}\mathbf{d}_{2i+1,0}}{\alpha_i w_{2i,0} + \beta_i w_{2i+1,0}}.
\end{aligned} \tag{A.20}
$$

Finally, when $i = m$, $j = 1$, we have

$$[w_{m,1,\ell,3-k}]_{0\leqslant k,\ell\leqslant 3} = t_{m,1}\begin{bmatrix} w^{\mathrm{c}}_{m,2} & w^{\mathrm{h}}_{2m-1,2} & w^{\mathrm{h}}_{2m,2} & \xi_1 w_{2m+1,2}+\eta_1 w_{2m+1,3} \\ w^{\mathrm{v}}_{m,2} & w_{2m-1,2} & w_{2m,2} & w_{2m+1,2} \\ w^{\mathrm{v}}_{m,1} & w_{2m-1,1} & w_{2m,1} & w_{2m+1,1} \\ w^{\mathrm{c}}_{m,1} & w_{2m-1,0} & w_{2m,0} & w_{2m+1,0} \end{bmatrix}, \tag{A.21}$$

with $t_{m,1}$ being an arbitrary positive constant,

$$\begin{aligned} w^{\mathrm{c}}_{m,2} &= \xi_1\left(\alpha_{m-1}w_{2m-2,2}+\beta_{m-1}w_{2m-1,2}\right) \\ &\quad + \eta_1\left(\alpha_{m-1}w_{2m-2,3}+\beta_{m-1}w_{2m-1,3}\right), \\ w^{\mathrm{h}}_{2m-1,2} &= \xi_1 w_{2m-1,2}+\eta_1 w_{2m-1,3}, \\ w^{\mathrm{h}}_{2m,2} &= \xi_1 w_{2m,2}+\eta_1 w_{2m,3}, \\ w^{\mathrm{v}}_{m,2} &= \alpha_{m-1}w_{2m-2,2}+\beta_{m-1}w_{2m-1,2}, \\ w^{\mathrm{v}}_{m,1} &= \alpha_{m-1}w_{2m-2,1}+\beta_{m-1}w_{2m-1,1}, \\ w^{\mathrm{c}}_{m,1} &= w^{\mathrm{v}}_{m,0} = \alpha_{m-1}w_{2m-2,0}+\beta_{m-1}w_{2m-1,0}; \end{aligned} \tag{A.22}$$

and

$$[\mathbf{d}_{m,1,\ell,3-k}]_{0\leqslant k,\ell\leqslant 3} = \begin{bmatrix} \mathbf{d}^{\mathrm{c}}_{m,2} & \mathbf{d}^{\mathrm{h}}_{2m-1,2} & \mathbf{d}^{\mathrm{h}}_{2m,2} & \mathbf{d}^{\mathrm{c}}_{m+1,2} \\ \mathbf{d}^{\mathrm{v}}_{m,2} & \mathbf{d}_{2m-1,2} & \mathbf{d}_{2m,2} & \mathbf{d}_{2m+1,2} \\ \mathbf{d}^{\mathrm{v}}_{m,1} & \mathbf{d}_{2m-1,1} & \mathbf{d}_{2m,1} & \mathbf{d}_{2m+1,1} \\ \mathbf{d}^{\mathrm{c}}_{m,1} & \mathbf{d}_{2m-1,0} & \mathbf{d}_{2m,0} & \mathbf{d}_{2m+1,0} \end{bmatrix}, \tag{A.23}$$

with

$$\begin{aligned} \mathbf{d}_{m,2} &= \left[\xi_1(\alpha_{m-1}w_{2m-2,2}\mathbf{d}_{2m-2,2}+\beta_{m-1}w_{2m-1,2}\mathbf{d}_{2m-1,2}) \right.\\ &\quad \left. + \eta_1(\alpha_{m-1}w_{2m-2,3}\mathbf{d}_{2m-2,3}+\beta_{m-1}w_{2m-1,3}\mathbf{d}_{2m-1,3})\right] \\ &\quad \Big/\left[\xi_1(\alpha_{m-1}w_{2m-2,2}+\beta_{m-1}w_{2m-1,2}) \right.\\ &\quad \left. + \eta_1(\alpha_{m-1}w_{2m-2,3}+\beta_{m-1}w_{2m-1,3})\right], \\ \mathbf{d}^{\mathrm{c}}_{m+1,2} &= \mathbf{d}^{\mathrm{h}}_{2m+1,2} = \frac{\xi_1 w_{2m+1,2}\mathbf{d}_{2m+1,2}+\eta_1 w_{2m+1,3}\mathbf{d}_{2m+1,3}}{\xi_1 w_{2m+1,2}+\eta_1 w_{2m+1,3}}, \\ \mathbf{d}^{\mathrm{h}}_{2m-1,2} &= \frac{\xi_1 w_{2m-1,2}\mathbf{d}_{2m-1,2}+\eta_1 w_{2m-1,3}\mathbf{d}_{2m-1,3}}{\xi_1 w_{2m-1,2}+\eta_1 w_{2m-1,3}}, \\ \mathbf{d}^{\mathrm{h}}_{2m,2} &= \frac{\xi_1 w_{2m,2}\mathbf{d}_{2m,2}+\eta_1 w_{2m,3}\mathbf{d}_{2m,3}}{\xi_1 w_{2m,2}+\eta_1 w_{2m,3}}, \end{aligned} \tag{A.24}$$

$$\mathbf{d}^{\mathrm{v}}_{m,2} = \frac{\alpha_{m-1}w_{2m-2,2}\mathbf{d}_{2m-2,2} + \beta_{m-1}w_{2m-1,2}\mathbf{d}_{2m-1,2}}{\alpha_{m-1}w_{2m-1,2} + \eta_{m-1}w_{2m-1,2}},$$
$$\mathbf{d}^{\mathrm{v}}_{m,1} = \frac{\alpha_{m-1}w_{2m-2,1}\mathbf{d}_{2m-2,1} + \beta_{m-1}w_{2m-1,1}\mathbf{d}_{2m-1,1}}{\alpha_{m-1}w_{2m-1,1} + \eta_{m-1}w_{2m-1,1}},$$
$$\mathbf{d}^{\mathrm{c}}_{m,1} = \mathbf{d}^{\mathrm{v}}_{m,0} = \frac{\alpha_{m-1}w_{2m-2,0}\mathbf{d}_{2m-2,0} + \beta_{m-1}w_{2m-1,0}\mathbf{d}_{2m-1,0}}{\alpha_{m-1}w_{2m-1,0} + \eta_{m-1}w_{2m-1,0}}.$$

We remark that equations (A.13)–(A.24) have been used for the G^1 joining of multiple NURBS surfaces in sections 3 and 4.

On the other hand, to convert a C^1 surface from its piecewise Bézier form (A.1) back to its NURBS representation (3.5); i.e., to convert Bézier coefficients and local weights in (A.1) back to control points and global weights in (3.5), the piecewise bicubic Bézier surface patches must satisfy certain conditions. More precisely, the Bézier coefficients and local weights need to be adjusted as follows.

A. Uniformly adjustment of the local weights.

1°. Along the lower boundary patches,

$$w_{i+1,1,k,\ell} \longleftarrow \frac{w_{i,1,3,0}}{w_{i+1,1,0,0}} w_{i+1,1,k,\ell},$$

i.e.,

$$w_{i+1,1,k,\ell} = w_{1,1,k,\ell} \prod_{q=1}^{i} \frac{w_{q,1,3,0}}{w_{q+1,1,0,0}}, \quad k,\ell = 0,\dots,3;\ i = 1,\dots,m-1. \tag{A.25}$$

2°. From bottom up and along the v-direction,

$$w_{i,j+1,k,\ell} \longleftarrow \frac{w_{i,j,0,3}}{w_{i,j+1,0,0}} w_{i,j+1,k,\ell},$$

i.e.,

$$w_{i,j+1,k,\ell} = w_{i,1,k,\ell} \prod_{q=1}^{j} \frac{w_{i,q,0,3}}{w_{i,q+1,0,0}}, \quad k,\ell = 0,\dots,3;$$
$$j = 1,\dots,n-1,\ i = 1,\dots,m. \tag{A.26}$$

B. Along the $m-1$ interior vertical grid lines of the parametric domain, the Bézier coefficients and local weights must satisfy

$$\mathbf{d}_{i,j,0,\ell} = \mathbf{d}_{i-1,j,3,\ell} = \frac{\alpha_{i-1}w_{i-1,j,2,\ell}\mathbf{d}_{i-1,j,2,\ell} + \beta_{i-1}w_{i,j,1,\ell}\mathbf{d}_{i,j,1,\ell}}{\alpha_{i-1}w_{i-1,j,2,\ell} + \beta_{i-1}w_{i,j,1,\ell}}, \tag{A.27}$$

$$\frac{w_{i,j,0,0}}{w_{i-1,j,3,0}} = \frac{w_{i,j,0,1}}{w_{i-1,j,3,1}} = \frac{w_{i,j,0,2}}{w_{i-1,j,3,2}} = \frac{w_{i,j,0,3}}{w_{i-1,j,3,3}}, \tag{A.28}$$

$$w_{i,j,1,\ell} = \frac{w_{i,j,0,\ell}}{w_{i-1,j,3,\ell}} \frac{w_{i-1,j,3,\ell} - \alpha_{i-1}w_{i-1,j,2,\ell}}{\beta_{i-1}}, \tag{A.29}$$
$$\ell = 0,\dots,3;\ i = 2,\dots,m,\ j = 1,\dots,n.$$

C. Along the $n-1$ interior horizontal grid lines of the parametric domain, the Bézier coefficients and local weights must satisfy

$$\mathbf{d}_{i,j,k,0} = \mathbf{d}_{i,j-1,k,3} = \frac{\xi_{j-1} w_{i,j-1,k,2} \mathbf{d}_{i,j-1,k,2} + \eta_{j-1} w_{i,j,k,1} \mathbf{d}_{i,j,k,1}}{\xi_{j-1} w_{i,j-1,k,2} + \eta_{j-1} w_{i,j,k,1}}, \tag{A.30}$$

$$\frac{w_{i,j,0,0}}{w_{i,j-1,0,3}} = \frac{w_{i,j,1,0}}{w_{i,j-1,1,3}} = \frac{w_{i,j,2,0}}{w_{i,j-1,2,3}} = \frac{w_{i,j,3,0}}{w_{i,j-1,3,3}}, \tag{A.31}$$

$$w_{i,j,k,1} = \frac{w_{i,j,k,0}}{w_{i,j-1,k,3}} \frac{w_{i,j-1,k,3} - \xi_{j-1} w_{i,j-1,k,2}}{\eta_{j-1}}, \tag{A.32}$$

$$k = 0, \ldots, 3; \ \ i = 1, \ldots, m, j = 2, \ldots, n.$$

Under these assumptions, the conversion algorithm is given below.

Algorithm A.2 (Conversion from Bézier to NURBS representations).

1°. At the 4 *corners* of the parametric domain $[0,1]^2$,

$$\begin{aligned} w_{0,0} &= w_{1,1,0,0}, & w_{2m+1,0} &= w_{m,1,3,0}, \\ w_{0,2n+1} &= w_{1,n,0,3}, & w_{2m+1,2n+1} &= w_{m,n,3,3}, \end{aligned} \tag{A.33}$$

and

$$\begin{aligned} \mathbf{d}_{0,0} &= \mathbf{d}_{1,1,0,0}, & \mathbf{d}_{2m+1,0} &= \mathbf{d}_{m,1,3,0}, \\ \mathbf{d}_{0,2n+1} &= \mathbf{d}_{1,n,0,3}, & \mathbf{d}_{2m+1,2n+1} &= \mathbf{d}_{m,n,3,3}. \end{aligned} \tag{A.34}$$

2°. Along the *lower* and *top* boundaries of $[0,1]^2$,

$$\begin{aligned} w_{2i-1,0} &= w_{i,1,1,0}, & w_{2i,0} &= w_{i,1,2,0}, \\ w_{2i-1,2n+1} &= w_{i,n,1,3}, & w_{2i,2n+1} &= w_{i,n,2,3}, \quad i = 1, \ldots, m, \end{aligned} \tag{A.35}$$

and

$$\begin{aligned} \mathbf{d}_{2i-1,0} &= \mathbf{d}_{i,1,1,0}, & \mathbf{d}_{2i,0} &= \mathbf{d}_{i,1,2,0}, \\ \mathbf{d}_{2i-1,2n+1} &= \mathbf{d}_{i,n,1,3}, & \mathbf{d}_{2i,2n+1} &= \mathbf{d}_{i,n,2,3}, \quad i = 1, \ldots, m. \end{aligned} \tag{A.36}$$

3°. Along the *left* and *right* boundaries of $[0,1]^2$,

$$\begin{aligned} w_{0,2j-1} &= w_{1,j,0,1}, & w_{0,2j} &= w_{1,j,0,2}, \\ w_{2m+1,2j-1} &= w_{m,j,3,1}, & w_{2m+1,2j} &= w_{m,j,3,2}, \quad j = 1, \ldots, n, \end{aligned} \tag{A.37}$$

and

$$\begin{aligned} \mathbf{d}_{0,2j-1} &= \mathbf{d}_{1,j,0,1}, & \mathbf{d}_{0,2j} &= \mathbf{d}_{1,j,0,2}, \\ \mathbf{d}_{2m+1,2j-1} &= \mathbf{d}_{m,j,3,1}, & \mathbf{d}_{2m+1,2j} &= \mathbf{d}_{m,j,3,2}, \quad j = 1, \ldots, n. \end{aligned} \tag{A.38}$$

4°. In the *interior* of $[0,1]^2$,

$$\begin{bmatrix} w_{2i-1,2j} & w_{2i,2j} \\ \\ w_{2i-1,2j-1} & w_{2i,2j-1} \end{bmatrix} = \begin{bmatrix} w_{i,j,1,2} & w_{i,j,2,2} \\ \\ w_{i,j,1,1} & w_{i,j,2,1} \end{bmatrix}, \tag{A.39}$$

and

$$\begin{bmatrix} \mathbf{d}_{2i-1,2j} & \mathbf{d}_{2i,2j} \\ \mathbf{d}_{2i-1,2j-1} & \mathbf{d}_{2i,2j-1} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_{i,j,1,2} & \mathbf{d}_{i,j,2,2} \\ \mathbf{d}_{i,j,1,1} & \mathbf{d}_{i,j,2,1} \end{bmatrix}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n. \tag{A.40}$$

References

[1] C. de Boor, *A Practical Guide to Splines* (Springer, Berlin, 1978).

[2] C.K. Chui, *Multivariate Splines* (SIAM, Philadelphia, PA, 1988).

[3] C.K. Chui, J. Hanisch, M.J. Lai, J.-A. Lian and P. Cassidy, Smoothing and gap removal of compound C^1 cubic NURBS surface patches, in: *Proc. of 1998 NSF Design and Manufacturing Grantees Conf.*, Monterey, Mexico (January 5–8, 1998) pp. 141–142.

[4] C.K. Chui, M.J. Lai, J.-A. Lian and P. Cassidy, Effective algorithms for joining NURBS surfaces in a smooth fashion, in: *Proc. of 1999 NSF Design and Manufacturing Grantees Conf.*, Queen Mary, Long Beach, CA (January 5–8, 1999).

[5] G. Farin, *Geometric Modeling: Algorithms and New Trends* (SIAM, Philadelphia, PA, 1987).

[6] G. Farin and B. Hamann, Current trends in geometric modeling and selected computational applications, J. Comput. Phys. 138 (1997) 1–15.

[7] J. Hoschek and D. Lasser, *Fundamentals of Computer Aided Geometric Design*, Transl. by L.L. Schumaker (A.K. Peters, Wellesley, MA, 1993).

[8] K. Konno and H. Chiyokura, An approach of designing and controlling free-form surfaces by using NURBS boundary Gregory patches, Comput. Aided Geom. Design 13 (1996) 825–849.

[9] D. Liu and J. Hoschek, GC^1 continuity conditions between adjacent rectangular and triangular Bézier surface patches, Comput. Aided Design 21 (1989) 194–200.

[10] L. Piegal and W. Tiller, *The NURBS Book* (Springer, Berlin, 1995).

[11] L.L. Schumaker, *Spline Functions: Basic Theory* (Wiley, New York, 1981).