Construction of Orthonormal Wavelets of Dilation Factor 3 with Application in Image Compression and A New Construction of Multivariate Compactly Supported Tight Frame

by

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(Under the direction of Ming-Jun Lai)

Abstract

This dissertation consists of two parts. In the first part, we construct univariate compactly supported wavelets with dilation factor $q = 3$, expressing the results in terms of parameters, and apply them to image compression. Through these parameters, we found some wavelets can compress images well. At very high compression ratios, it is better than the well-known 9/7 biorthogonal wavelet with the same computing complexity. In the second part, we give a construction of tight frames from any refinable space. Some examples are shown by using B-splines. This method also works in the multivariate setting.

Index words: Scaling function, Dilation Factor, Wavelet, Orthonormal, Nonseparable Compact support, Symmetric, Vanishing moment, Tensor product, Tight Frame, Bspline, Boxspline

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Dedication

Dedicated to my mother and my father

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INTRODUCTION

The wavelet concept started to appear around two decades ago. This new concept has interdiciplinary origins from mathematics(Calderón-Zygmund operators), engineering(subband coding in signal processing, pyramidal algorithm in image processing) and physics(coherent states formalism in quantum mechanics).

The oldest example of wavelet basis was constructed by Haar in 1910 [12]. It's the simplest possible wavelet but not continuous. More complicated wavelet bases were constructed in the 1980s, such as the Battle-Lemarié spline wavelets ([1],[20],[29]) and Meyer's $\mathbb{C}^{\infty}$ wavelet [24]. Also in late 1986, Meyer and Mallat introduced the concept of multiresolution analysis(MRA), which was later used as a formal approach to construct orthogonal wavelet bases using a definite set of rules and procedures. Mallat's algorithm for the decomposition and reconstruction of an imgae is based on MRA. MRA provides the existance of scaling functions, which are used to construct the wavelets. However, the sophisticated wavelet bases constructed at that time are not compactly supported, which leads to difficulties in most applications.

In 1988, Daubechies made a breakthough. She constructed the first orthonormal basis of continuous, compactly supported wavelets for $L_2(\mathbb{R})$ with pre-assigned regularity(smoothness) in ([5],[6]). Her remarkable work had a positive impact on the study of wavelets, and her wavelets became very popular in applications.

Most efforts have been made on the scaling function for dilation $q = 2$, which is used to construct wavelets using MRA. It is interesting to investigate scaling functions for dilation $q > 2$, and hence the corresponding wavelets. Also it is shown by Daubechies that there does not exist compactly supported orthogonal symmetric scaling functions for the dilation $q = 2$, except the Haar function [5]. Chui and Lian constructed compactly supported symmetric and antisymmetric wavelets with dilation $q = 3$. Some interesting compactly supported symmetric scaling functions with dilation $q > 2$ were presented by Belogay and Wang in [2]. There are some efforts made by Welland and Lundberg to construct compactly supported wavelets with dilation $q > 2$ [32].

One major application of wavelets is digital image processing. The most common wavelets used are the tensor product of one-dimensional compactly supported orthonormal or biorthogonal wavelets. A lot of experiments were carried out for wavelets with dilation $q = 2$, but the effectiveness of compactly supported wavelets with dilation $q = 3$ in image compression is still unknown. In this dissertation, we will construct compactly supported orthonormal wavelets with dilation $q > 2$, expressing them in terms of parameters. Then we apply the wavelets with certain parameters to image compression, and finally compare the result with the well-known 9/7 biorthogonal wavelets. Our wavelets with dilation $q = 3$ have an edge over the 9/7 biorthogonal wavelets at very high compression ratio, and the computing complexities are identical.

Frames were introduced as early as 1952 by Duffin and Schaeffer in [8]; they can be viewed as overcomplete bases. The importance of the concept was not realized at that time, and the topic had been untouched for almost 30 years since then. When the wavelet era began, it was observed that frames can be used to find series expansions of functions in $L_2(\mathbb{R})$ which are similar to the expansions using orthonormal bases [7]. People started to see the potential of this topic; papers concerning frames have been booming ever since. Frames are very useful in signal transmission, since the overcompleteness of frames will reduce the influence of noise, compared to the use of orthonormal bases [23]. From the computational aspect of real world application, compactly supported tight frames are very promising.

The organization of this dissertation is as follows: The basic concepts and topics of wavelets are covered in Chapter 1. In Chapter 2, we construct scaling fucntions supported compactly supported with dilation $q = 3$ in terms of parameters. Then we use these scaling functions to construct compactly supported orthonormal wavelets in Chapter 3. The application to image compression of wavelets with $q = 3$ is shown in Chapter 4. A scaling function supported on $[0, 5/2] \times [0, 5/2]$ and the corresponding nonseperable bivariate compactly supported orthonormal wavelets are constructed in Chapter 5. The last chapter focuses on a new construction of compactly supported tight frames from any refinable space, which works in the multivariate setting. A few examples are given in both univariate and bivariate settings.

Chapter 1

PRELIMINARIES

This preliminary chapter states some basic theorems which are well-established in wavelet theory. If the original proof is to be generalized or modified, then we will mention it.

1.1 Univariate wavelets

Definition 1.1.1. *If* $F = \{f_i\}_{i\in\mathbb{Z}}$ *is a basis for* $L^2(\mathbb{R}^n)$ *such that the inner product* $\int_{\mathbb{R}^n} f_i(x)\overline{f_j(x)}\,dx$ *is 0 if* $i \neq j$ *and 1 if* $i = j$*, then* F *is an* orthonormal basis *of* $L^2(\mathbb{R}^n)$*.*

Throughout this chapter, we will fix an integer $q \geq 2$.

Definition 1.1.2. *An* orthonormal q-wavelet *consists of* $q-1$ *functions* $\psi_l \in L^2(\mathbb{R})$, $l = 1, 2, \ldots, q-1$ *such that the family of functions* $F = \{\psi_{l,j,k}, j, k \in \mathbb{Z}, l = 1, 2, \ldots, q-1\}$ *with*

$$\psi_{l,j,k}(x) = q^{-j/2}\psi_l(q^{-j}x - k), j, k \in \mathbb{Z}, l = 1, 2, \ldots, q-1$$

is an orthonormal basis in $L^2(\mathbb{R})$*.*

Definition 1.1.3. *An* orthonormal multiresolution analysis (MRA) *is a sequence of closed subspaces ...* $V_2, V_1, V_0, V_{-1}, V_{-2}$*, ... in* $L^2(\mathbb{R})$ *such that the following holds:*

1. $V_j \subset V_{j-1}, \forall j \in Z$;
2. $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$ *and* $\bigcap_j V_j = \{0\}$;
3. $f(\cdot) \in V_j \Longleftrightarrow f(q\cdot) \in V_{j-1}$;
4. $f(\cdot) \in V_0 \Longleftrightarrow f(q^{-j} \cdot -k) \in V_j$ *for all* $j,k \in \mathbb{Z}$

5. *There is a function $\phi \in L^2(\mathbb{R})$ called scaling function such that $\{\phi(x-k),\ k \in \mathbb{Z}\}$ forms an orthonormal basis of V_0.*

It implies that for all $j \in \mathbb{Z}$ $\{\phi_{j,k}, k \in \mathbb{Z}\}$ is an orthonormal basis of V_j, where $\phi_{j,k}(x) = q^{j/2}\phi(q^j x - k)$. From now on, we will use ϕ as a scaling function.

Since $\phi \in V_0$ and $V_0 \subset V_{-1}$, we have $\phi \in V_{-1}$. If follows that we can expand $\phi = \sum_n h_n \phi_{-1,n}$, where

$$h_n = \sqrt{q}\int_{\mathbb{R}} \phi(x)\overline{\phi(qx-n)}dx \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |h_n|^2 = 1, \tag{1.1.1}$$

i.e. we have the following dilation equation

$$\phi(x) = \sqrt{q}\sum_n h_n \phi(qx-n), \quad x \in \mathbb{R}. \tag{1.1.2}$$

The one-dimensional Fourier transform is defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ix\xi}dx. \tag{1.1.3}$$

By taking the Fourier transform both sides of (1.1.2), we get that

$$\hat{\phi}(\xi) = m_0(\xi/q)\hat{\phi}(\xi/q), \quad \xi \in \mathbb{R}, \tag{1.1.4}$$

where

$$m_0(\xi) = \frac{1}{\sqrt{q}}\sum_n h_n e^{-in\xi}. \tag{1.1.5}$$

m_0 is called the symbol(mask) of ϕ.

Since $\{\phi_{0,n}\}_{n\in\mathbb{Z}}$ is orthonormal, we know

$$\begin{aligned}\delta_{n0} &= \int_{\mathbb{R}} \phi(x)\overline{\phi(x-n)}dx = \int_{\mathbb{R}} |\hat{\phi}(\xi)|^2 e^{in\xi}d\xi \\ &= \int_0^{2\pi} (\sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+2k\pi)|^2)e^{in\xi}d\xi,\end{aligned}$$

We conclude that

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+2k\pi)|^2 = \frac{1}{2\pi} \text{ a.e.} \tag{1.1.6}$$

Since m_0 is 2π-periodic, by (1.1.4) and (1.1.6) we find that

$$\frac{1}{2\pi} = \sum_{k\in\mathbb{Z}} |\hat{\phi}(q\xi + 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} |m_0(\xi + 2k\pi/q)|^2 |\hat{\phi}(\xi + 2k\pi/q)|^2$$

$$= |m_0(\xi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + 2k\pi)|^2 + |m_0(\xi + 2\pi/q)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi/q + 2k\pi)|^2$$

$$+ \cdots + |m_0(\xi + 2(q-1)\pi/q)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + 2(q-1)\pi/q + 2k\pi)|^2.$$

Hence

$$|m_0(\xi)|^2 + |m_0(\xi + 2\pi/q)|^2 + \cdots + |m_0(\xi + 2(q-1)\pi/q)|^2 = 1 \text{ a.e.} \tag{1.1.7}$$

If scaling function ϕ is compactly supported, then m_0 is naturally a trigonometric polynomial, hence we can drop almost everywhere from the above equation.

For $j = 0$, there is a orthogonal complement W_0 of V_0 in V_{-1}. Since V_0 is generated by integer translates of $\phi(x)$ and V_{-1} is generated by integer translates of q functions, $\phi(qx)$, $\phi(qx-1)$, ..., $\phi(qx-q+1)$, we expect to need $q-1$ functions to generate W_0 under integer translates.More precisely, we seek functions ψ_1, ψ_2, ..., ψ_{q-1} such that

$$\{\psi_i(x-k) : i = 1, 2, \ldots, q-1, k \in \mathbb{Z}\} \text{ forms an orthonormal basis for } W_0. \tag{1.1.8}$$

Decompose W_0 as the direct sum of $q-1$ subspaces, say W_0^1, W_0^2, ..., W_0^{q-1}. Each $W_0^i \subset V_{-1}$ is spanned by ψ_j in the sense that

$$W_0^i \;=\; \text{span}\{\psi_i(x-k), k \in Z\} \text{ for } i = 1, 2, \cdots, q-1 \tag{1.1.9}$$

In general, we set W_j^i which inherit the scaling property from V_j, i.e.

$$f \in W_j^i \Longleftrightarrow f(q^j \cdot) \in W_0^i. \tag{1.1.10}$$

as

$$W_0^i \;=\; \text{span}\{\psi_i(q^j x-k), k \in Z\} \text{ for } i = 1, 2, \cdots, q-1$$

We have V_j as

$$V_{j-1} = V_j \oplus W_j^1 \oplus W_j^2 \oplus \cdots \oplus W_j^{q-1} \tag{1.1.11}$$

and

$$W_j^i \perp W_j^{i'} \text{ if } i \neq i'. \tag{1.1.12}$$

It implies

$$L^2(R) = \bigoplus_{j\in Z} \bigoplus_{i=1}^{q-1} W_j^i \tag{1.1.13}$$

By (1.1.13) and (1.1.10), $\psi_i, i = 1, 2, \cdots, q-1$ form an orthonormal q-wavelet.

Next, we discuss a necessary and sufficient condition such that 1.1.8 holds, if scaling function ϕ is given.

Since for each $j = 1, 2, \ldots, q-1$, $\psi_j \in W_0$ implies that $\psi_j \in V_{-1}$ and $\psi_j \perp V_0$. We have

$$\psi_j(x) = \sqrt{q} \sum_n g_n^j \phi(qx - n), \tag{1.1.14}$$

with $g_n^j = \langle \psi_j, \phi_{-1,n} \rangle$, $\{g_n^j\} \in l^2(\mathbb{Z})$. This implies

$$\hat{\psi}_j(\xi) = m_j(\xi/q)\hat{\phi}(\xi/q), \tag{1.1.15}$$

where

$$m_j(\xi) = \frac{1}{\sqrt{q}} \sum_n g_n^j e^{-in\xi} \tag{1.1.16}$$

is a 2π-periodic function in $L^2([0, 2\pi])$, called the symbol of ψ_j.

Lemma 1.1.4. *Let ϕ be a scaling function with m_0 the corresponding symbol. Let ψ_1, $\psi_2, \ldots \psi_{q-1} \in W_0$, take m_1, m_2, $\ldots$, m_{q-1} as the corresponding symbols by* (1.1.14)-(1.1.16). *Then the family $\{\psi_{j,0,n}, j = 1, 2, \cdots, q-1\}_{n\in\mathbb{Z}}$ is an orthonormal basis of W_0, the orthogonal complement of V_0 in V_{-1} if and only if the matrix*

$$\begin{bmatrix} m_0(\xi) & m_0(\xi + 2\pi/q) & \cdots & m_0(\xi + 2(q-1)\pi/q) \\ m_1(\xi) & m_1(\xi + 2\pi/q) & \cdots & m_1(\xi + 2(q-1)\pi/q) \\ \vdots & \vdots & \ddots & \vdots \\ m_{q-1}(\xi) & m_{q-1}(\xi + 2\pi/q) & \cdots & m_{q-1}(\xi + 2(q-1)\pi/q) \end{bmatrix} \tag{1.1.17}$$

is unitary almost everywhere, where m_j is the symbol of $\psi_{j,0,n}$ for $j = 1, 2, \cdots, q-1$.

Proof: We will first prove the necessary condition. Orthonormality of $\{\psi_{j,0,k}\}$ implies

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}_j(\xi+2k\pi)|^2 = \frac{1}{2\pi} \text{ a.e.} \tag{1.1.18}$$

Since m_j is 2π-periodic, by (1.1.15) and (1.1.18)

$$\begin{aligned}\frac{1}{2\pi} &= \sum_{k\in\mathbb{Z}} |\hat{\psi}_j(q\xi+2k\pi)|^2 = \sum_{k\in\mathbb{Z}} |m_j(\xi+2k\pi/q)|^2|\hat{\phi}(\xi+2k\pi/q)|^2 \\ &= |m_j(\xi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+2k\pi)|^2 + |m_j(\xi+2\pi/q)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+2\pi/q+2k\pi)|^2 \\ &+ \cdots + |m_j(\xi+2(q-1)\pi/q)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+2(q-1)\pi/q+2k\pi)|^2\end{aligned}$$

With (1.1.6), it implies

$$|m_j(\xi)|^2 + |m_j(\xi+2\pi/q)|^2 + \cdots + |m_j(\xi+2(q-1)\pi/q)|^2 = 1 \text{ a.e.} \tag{1.1.19}$$

where $1 \le j \le q-1$.

The constraint $\psi_j \perp V_0$ implies $\psi_j \perp \phi_{0,k}$ for all k, where $j = 1, 2, \cdots, q-1$ i.e.,

$$\int_{\mathbb{R}} e^{-ik\xi}\hat{\phi}(\xi)\overline{\hat{\psi}_j(\xi)}\, d\xi = 0.$$

Substituting (1.1.4) and (1.1.15) into the above equation, we get

$$\int_{\mathbb{R}} e^{-ik\xi} m_0(\xi/q)\overline{m_j(\xi/q)}|\hat{\phi}(\xi/q)|^2\, d\xi = 0,$$

and

$$\begin{aligned}\int_0^{2\pi} e^{-ik\xi}\big[m_0(\xi/q)\overline{m_j(\xi/q)} &\sum_l |\hat{\phi}(\xi/q+2\pi l)|^2 \\ + m_0(\xi/2+2\pi/q)\overline{m_j(\xi/q+2\pi/q)} &\sum_l |\hat{\phi}(\xi/2+2\pi/q+2\pi l)|^2 + \cdots \\ + m_0(\xi/2+2(q-1)\pi/q)\overline{m_j(\xi/q+2(q-1)\pi/q)} &\sum_l |\hat{\phi}(\xi/2+2(q-1)\pi/q+2\pi l)|^2\big]\, d\xi = 0.\end{aligned}$$

Using (1.1.6), we have

$$\begin{aligned}\int_0^{2\pi} e^{-ik\xi}\big[\, m_0(\xi/q)\overline{m_j(\xi/q)} + m_0(\xi/q+2\pi/q)\overline{m_j(\xi/q+2\pi/q)} + \cdots + \\ m_0(\xi/q+2(q-1)\pi/q)\overline{m_j(\xi/q+2(q-1)\pi/q)}\,\big]\, d\xi = 0.\end{aligned}$$

It implies that

$$m_0(\xi)\overline{m_j(\xi)} + m_0(\xi + 2\pi/q)\overline{m_j(\xi + 2\pi/q)} + \cdots + \\ m_0(\xi + 2(q-1)\pi/q)\overline{m_j(\xi + 2(q-1)\pi/q)} = 0 \text{ for } j = 1, 2, \cdots, q-1, \text{a.e.} \quad (1.1.20)$$

Similarily, another contraint $\psi_{j_1} \perp W_0^{j_2}$ where $1 \leq j_1, j_2 \leq q-1$ and $j_1 \neq j_2$ leads to the following equation

$$m_{j_1}(\xi)\overline{m_{j_2}(\xi)} + m_{j_1}(\xi + 2\pi/q)\overline{m_{j_2}(\xi + 2\pi/q)} + \cdots + \\ m_{j_1}(\xi + 2(q-1)\pi/q)\overline{m_{j2}(\xi + 2(q-1)\pi/q)} = 0 \text{ a.e.} \quad (1.1.21)$$

The matrix version of equations (1.1.7), (1.1.19), (1.1.20), and (1.1.21) is that (1.1.17) is unitary a.e..

Conversely, the above arguments is reversible, which gives the sufficient condition.

Again, we can drop almost everywhere if ϕ is compactly supported.

If a trigonometric polynomial m_0 is associated with a multiresolution analysis, then we have

$$\phi(x) = \sqrt{q} \sum_n h_n \phi(qx - n).$$

If the corresponding scaling function $\phi \in L^1(\mathbb{R})$, then $\hat{\phi}$ is continuous, and

$$\hat{\phi}(\xi) = m_0(\xi/q)\hat{\phi}(\xi/q).$$

The condition $\hat{\phi}(0) \neq 0$ is necessary in this case [6], hence

$$m_0(0) = 1. \quad (1.1.22)$$

Because of (1.1.7), it follows that $m_0(2j\pi/q) = 0$, where $j = 1, 2, \cdots, q-1$, and $|\hat{\phi}(0)| = \frac{1}{\sqrt{2\pi}}$. For convenience, we may normalize ϕ so that

$$\int_{\mathbb{R}} \phi(x)dx = 1. \quad (1.1.23)$$

To find compactly supported wavelets $\psi_j, j = 1, 2, \cdots, q-1$, an easy approach is to choose an orthonormal compactly supported scaling function ϕ. From the definition of h_n, only finitely many h_n are nonzero, and each ψ_j becomes a finite linear combination of compactly supported functions, so that it is compactly supported as well.

For compactly supported ϕ, the 2π-periodic function

$$m_0(\xi) = \frac{1}{\sqrt{q}} \sum_n h_n e^{-in\xi}$$

is a trigonometric polynomial. Orthonormality of the $\phi_{0,n}$ implies (1.1.7), that is,

$$|m_0(\xi)|^2 + |m_0(\xi + 2\pi/q)|^2 + \cdots + |m_0(\xi + 2(q-1)\pi/q)|^2 = 1.$$

Vanishing moment is a key factor for the regularity of wavelets. Daubechies found the link between vanishing moment of the wavelet and regularity(smoothness) of the wavelet and scaling function.

DEFINITION 1.1.5. *A wavelet $\psi(x)$ has K* vanishing moments *if*

$$\int x^k \psi(x)\, dx = 0 \text{ for } 0 \leq k \leq K$$

.

A necessary and sufficient condition for this to hold is that, for each k, $0 \leq k \leq K$, there exist constants c_l such that $x^k = \sum c_l \phi(x - l)$.

In order to make ϕ and $\psi_j, j = 1, 2, \cdots, q-1$ reasonably regular, m_0 should be of the form

$$m_0(\xi) = \left(\frac{1 + e^{-i\xi} + e^{-i2\xi} + \cdots + e^{-i(q-1)\xi}}{q} \right)^N \mathcal{L}(\xi) \tag{1.1.24}$$

with $N \geq 1$, and $\mathcal{L}(\xi) = \sum_{k=0}^{m} h_k e^{-ik\xi}$.

The following is a generalization of a result by S. Mallat [22] from $q = 2$ to general $q \geq 2$.

LEMMA 1.1.6. *If m_0 is a 2π-periodic continuous function satisfying* (1.1.7)*, and if*

$$\frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(q^{-j}\xi)$$

converges pointwise almost everywhere, then its limit $\hat{\phi}(\xi)$ is in $L^2(\mathbb{R})$.

Proof: Here is the simple modification of the original Mallat proof. First, we denote $M(\xi) = |m_0(\xi)|^2$ and denote by $M_k(\xi)$ $(k \geq 1)$ the continuous fucntion defined by

$$M_k(\xi) = \begin{cases} 0 & \text{if } |\xi| > 2q^k\pi, \\ M(\xi/q)M(\xi/q^2)\cdots M(\xi/q^k) & \text{if } |\xi| \leq 2q^k\pi. \end{cases}$$

For all $k \in \mathbb{N}, k \neq 0$, we denote I_k^n as the following integral ,

$$I_k^n = \int_{-\infty}^{\infty} M_k(\xi)e^{i2n\pi\xi}\, d\xi.$$

The integral will then be divided into two parts:

$$I_k^n = \int_{-2q^k\pi}^{0} M_k(\xi)e^{i2n\pi\xi}\, d\xi \; + \; \int_{0}^{2q^k\pi} M_k(\xi)e^{i2n\pi\xi}\, d\xi \qquad (1.1.25)$$

Since $M_k(\xi)e^{i2n\pi\xi}$ is 2π periodic, we only need to compute the first part. The fisrt part can be further divided into q integrals, and written as

$$\int_{-2q^k\pi}^{-2q^k\pi+2q^{k-1}\pi} L(\xi)\, d\xi + \int_{-2q^k\pi+2q^{k-1}\pi}^{-2q^k\pi+4q^{k-1}\pi} L(\xi)\, d\xi + \cdots + \int_{-2q^k\pi+2(q-1)q^{k-1}\pi}^{0} L(\xi)\, d\xi \quad (1.1.26)$$

where $L(\xi) = M_k(\xi)e^{i2n\pi\xi}$.

Since $M(q^{-j}\xi + 2q^{k-j}\pi) = M(q^{-j}\xi)$ for $0 \neq j < k$ and

$$M(q^{-k}\xi) + M(q^{-k}\xi + 2\pi/q) + \cdots + M(q^{-k}\xi + 2(q-1)\pi/q) = 1,$$

by changing variables in each integral of (1.1.26) as follows: $\xi' = \xi + 2q^k\pi$ in the first integral, $\xi' = \xi + 2q^k\pi - 2q^{k-1}\pi$ in the second integral, $\ldots$, $\xi' = \xi + 2q^k\pi - 2(q-1)q^{k-1}\pi$ in the last integral, we obtain the first part as

$$\int_0^{2q^{k-1}\pi} M(\xi/q)M(\xi/q^2)\cdots M(\xi/q^{k-1})e^{i2n\pi\xi}\, d\xi.$$

Since $M(\xi)$ is 2π-periodic,

$$\begin{aligned} I_k^n &= 2\int_0^{2q^{k-1}\pi} M(\xi/q)M(\xi/q^2)\cdots M(\xi/q^{k-1})e^{i2n\pi\xi}\, d\xi \\ &= \int_{-2q^{k-1}\pi}^{2q^{k-1}\pi} M(\xi/q)M(\xi/q^2)\cdots M(\xi/q^{k-1})e^{i2n\pi\xi}\, d\xi \\ &= I_{k-1}^n \end{aligned}$$

Hence, we derive that

$$I_k^n = I_{k-1}^n = \cdots = I_1^n = \begin{cases} 4\pi & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Since $0 \leq M(\xi) \leq 1$, $\lim_{k\to\infty} M_k(\xi) = \prod_{j=1}^{\infty} M(q^{-j}\xi) = 2\pi|\hat{\phi}(\xi)|^2$ converges. From Fatou's lemma, we derive that

$$\int_{-\infty}^{\infty} \lim_{k\to\infty} M_k(\xi)d\xi \leq \lim_{k\to\infty} \int_{-\infty}^{\infty} M_k(\xi)d\xi = 4\pi.$$

Thus, $\hat{\phi}$ is in $L^2(\mathbb{R})$.

1.2 Multivariable wavelets

Throughout this section, we fix $d \geq 2$.

Definition 1.2.1. *An s by s integer matrix D is called a* dilation matrix *if* $\lim_{k\to\infty} D^{-k} = 0$.

Definition 1.2.2. *An* orthonormal wavelet set *associated with a dilation matrix D is a finite set of functions $\psi_r \in L^2(\mathbb{R}^d), r = 1, 2, ..., s$ such that the system*

$$\{|detD|^{-j/2}\psi_r(D^{-j}x - k)\}$$

with $r = 1, 2, ..., s,\ j \in \mathbb{Z}, k \in \mathbb{Z}^d$, is an orthonormal basis of $L^2(\mathbb{R}^d)$.

Definition 1.2.3. *An* orthonormal multiresolution analysis *associated with a dilation matrix D is a sequence of closed subspaces ... $V_2, V_1, V_0, V_{-1}, V_{-2}$, ... in $L^2(\mathbb{R}^d)$ satisfying*

1. $V_j \subset V_{j-1}, \forall j \in \mathbb{Z}$;
2. $\overline{\bigcup_j V_j} = L^2(\mathbb{R}^d)$ *and* $\bigcap_j V_j = \{0\}$;
3. $f(\cdot) \in V_j \Longleftrightarrow f(D\cdot) \in V_{j-1}$;
4. $f(\cdot) \in V_0 \Longleftrightarrow f(D^{-j} \cdot -k) \in V_j$ *for all* $j \in \mathbb{Z}, k \in \mathbb{Z}^d$

5. *there exists a scaling function* ϕ *such that the distributions* $|detD|^{j/2}\phi(D^jx-k)$, $k \in \mathbb{Z}^d$, *form an orthonormal basis of* V_j.

The eigenvalues of D must have absolute value strictly greater than 1, thus the matrix D dilates in every direction. The entries of D have to be integers, i.e. $D\mathbb{Z}^d \subset \mathbb{Z}^d$ [30].

Each compactly supported ϕ takes the form

$$\phi(x) = \sum_{k\in\Lambda} c_k \phi(Dx-k), \tag{1.2.1}$$

where Λ is a finite subset of the lattice $\mathbb{Z}^d$.

And Y. Meyer proved the following theorem[25].

THEOREM 1.2.4. *Let* $\{V_j\}_{j\in\mathbb{Z}}$ *be an orthonormal multiresolution analysis with dilation matrix* D*. Then there exist* $|detD|-1$ *wavelets* $\psi_1, \psi_2, \ldots, \psi_{|detD|-1} \in V_{-1}$*, that generate an orthonormal basis of the orthogonal complement of* V_0 *in* V_{-1}*, i.e.*

$$\{\psi^l_{j;k}(\cdot) = |detD|^{-j/2}\psi_l(D^{-j}\ \cdot -k):\ l = 1, \ldots, |detD|-1, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$.

However, it does not guarantee that we can get compactly supported orthonormal wavelets in the high dimension automatically even if the scaling function is compactly supported. Thus in this dissertation, we try to construct compactly supported tight-frame instead of orthonormal wavelet.

1.2.1 Two-dimensional wavelets with dilation factor 3

In this dissertation, we are interested in constructing a bivariate nonseperable compactly supported wavelet with dilation factor 3, which is associated with the standard dilation matrix $D = \left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$. Here, we start with a refinable function $\phi \in L^2(\mathbb{R}^2)$ which generates a two-dimensional MRA(MultiResolution Analysis) with respect to the dilation matrix D.

Explicitly, it satisfies the dilation equation

$$\phi(x,y) = 3 \sum_{(k_1,k_2)\in\mathbb{Z}^2} h_{k_1,k_2}\phi(3x-k_1, 3y-k_2), \tag{1.2.2}$$

for some sequence h_{k_1,k_2}, $(k_1,k_1) \in \mathbb{Z}^2$. ϕ can be normalized by

$$\int_{\mathbb{R}^2} \phi(x)dx = 1. \tag{1.2.3}$$

Taking the Fourier transform of (1.2.2) yields

$$\hat{\phi}(\xi,\eta) = m_0(\xi/3,\eta/3)\hat{\phi}(\xi/3,\eta/3), \tag{1.2.4}$$

with

$$m_0(\xi,\eta) = \frac{1}{3}\sum_{k_1,k_2} h_{k_1,k_2}e^{-i(k_1\xi+k_2\eta)}. \tag{1.2.5}$$

And

$$m_0(0,0) = 1. \tag{1.2.6}$$

It follows that

$$\hat{\phi}(\xi,\eta) = \frac{1}{2\pi}\prod_{j=1}^{\infty} m_0(3^{-j}\xi, 3^{-j}\eta). \tag{1.2.7}$$

As we explained before in one-dimensional case, the orthonormality of ϕ implies that the trigonometric polynomial function m_0 has to satisfy

$$\begin{aligned} &|m_0(\xi,\eta)|^2 + |m_0(\xi+2\pi/3,\eta)|^2 + |m_0(\xi,\eta+2\pi/3)|^2 + |m_0(\xi+4\pi/3,\eta)|^2 \\ &+ |m_0(\xi,\eta+4\pi/3)|^2 + |m_0(\xi+2\pi/3,\eta+2\pi/3)|^2 + |m_0(\xi+4\pi/3,\eta+2\pi/3)|^2 \\ &+ |m_0(\xi+2\pi/3,\eta+4\pi/3)|^2 + |m_0(\xi+4\pi/3,\eta+4\pi/3)|^2 = 1. \end{aligned} \tag{1.2.8}$$

Therefore (1.2.6) implies that

$$\begin{aligned} &m_0(2\pi/3,0) = 0, m_0(0,2\pi/3) = 0, m_0(4\pi/3,0) = 0, m_0(0,4\pi/3) = 0, \\ &m_0(2\pi/3,2\pi/3) = 0, m_0(2\pi/3,4\pi/3) = 0, m_0(4\pi/3,2\pi/3) = 0, m_0(4\pi/3,4\pi/3) = 0. \end{aligned} \tag{1.2.9}$$

Lemma 1.2.5. *If m_0 is a 2π-periodic function of the form*

$$m_0(\xi,\eta) = \frac{1}{q}\sum_{k_1,k_2} h_{k_1,k_2}e^{-i(k_1\xi+k_2\eta)}$$

for finitely many k_1, k_2 and if

$$\frac{1}{2\pi}\prod_{j=1}^{\infty} m_0(q^{-j}\xi, q^{-j}\eta)$$

converges pointwise almost everywhere, then its limit $\hat{\phi}(\xi,\eta)$ is in $L^2(\mathbb{R}^2)$.

It implies that the function ϕ defined as in (1.2.7) is in $L^2(\mathbb{R}^2)$, since m_0 satisfies (1.2.6) and (1.2.8). According to Theorem 1.2.4, there are eight wavelets spanning the orthogonal complements of V_0 in V_{-1}. They are defined by

$$\psi_l(x,y) = 3\sum_{k_1,k_2} g^l_{k_1,k_2}\phi(3x-k_1, 3y-k_2), \quad l = 1,2,\cdots,8 \tag{1.2.10}$$

for some sequence $g^l_k, k = (k_1,k_2), \in \mathbb{Z}^2$.

we have

$$V_{j-1} = V_j \oplus W_j = V_j \overset{8}{\underset{l=1}{\oplus}} W^l_j. \tag{1.2.11}$$

Each W^l_j is generated by its orthonormal basis $\{\psi_{l,j;k_1,k_2} : \ \psi_{l,j;k_1,k_2}(x,y) = 3^{-j}\psi_l(3^{-j}x - k_1, 3^{-j}y - k_2), k_1, k_2 \in \mathbb{Z}\}$.

Taking Fourier transform of (1.2.10), it yields

$$\hat{\psi}_l(\xi,\eta) = m_l(\xi/3,\eta/3)\hat{\phi}(\xi/3,\eta/3), \tag{1.2.12}$$

with

$$m_l(\xi,\eta) = \frac{1}{3}\sum_{k_1,k_2} g^l_{k_1,k_2} e^{-i(k_1\xi+k_2\eta)}, \quad \text{for each } l = 1,2,\cdots,8 \tag{1.2.13}$$

The orthonormality of ϕ and ψ_l, $l = 1,2,\cdots,8$ implies the following matrix is unitary.

$$\begin{aligned}&[m_j(\xi,\eta), m_j(\xi+\sigma_1,\eta), m_j(\xi,\eta+\sigma_1), m_j(\xi+\sigma_1,\eta+\sigma_1), m_j(\xi+\sigma_2,\eta),\\ &m_j(\xi,\eta+\sigma_2), m_j(\xi+\sigma_1,\eta+\sigma_2), m_j(\xi+\sigma_2,\eta+\sigma_1), m_j(\xi+\sigma_2,\eta+\sigma_2)]_{j=0,1,\cdots,8},\end{aligned} \tag{1.2.14}$$

where $\sigma_1 = 2\pi/3,$ and $\sigma_2 = 4\pi/3$.

1.2.2 Two-dimensional separable wavelets

If a bivariate wavelet $\psi(x,y)$ can be written as a product of two one-dimensional functions $f(x)g(y)$, then $\psi(x,y)$ is seperable. The set of seperable wavelet functions is just a subset of all bivariate wavelets. The simplest separable bivariate wavelet is the tensor product of univariate wavelets [6].

If $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA of $L^2(\mathbb{R})$, ϕ and ψ^1, ψ^2 are the scaling function and orthonormal wavelets associated with the MRA respectively, then the tensor product of the subspace V_j with itself is given by

$$\mathbf{V}_j = V_j \otimes V_j = \overline{\operatorname{span}\{F(x,y) = f(x)g(y) : f, g \in V_j\}}.$$

It can be shown that

$$\Phi_{j;k_1,k_2}(x,y) = 3^{-j}\phi(3^{-j}x - k_1)\phi(3^{-j}y - k_2),\ \ k_1, k_2 \in \mathbb{Z},$$

constitutes an orthonormal basis for $\mathbf{V}_j$, and $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ is an MRA associated with the dilation matrix $D = \left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$ of $L^2(\mathbb{R}^2)$.

Let $\mathbf{W}_j$ be the orthogonal complement space of $\mathbf{V}_j$ in $\mathbf{V}_{j-1}$. Since $V_{j-1} = V_j \bigoplus_{l=1}^{2} W_j^l$, we have

$$\begin{aligned}
\mathbf{V}_{j-1} &= V_{j-1} \otimes V_{j-1} \\
&= V_j \otimes V_j \oplus [(V_j \otimes W_j^1) \oplus (W_j^1 \otimes V_j) \oplus (V_j \otimes W_j^2) \oplus (W_j^2 \otimes V_j) \\
&\quad \oplus (W_j^1 \otimes W_j^2) \oplus (W_j^2 \otimes W_j^1) \oplus (W_j^1 \otimes W_j^1) \oplus (W_j^2 \otimes W_j^2)] \\
&= \mathbf{V}_j \oplus \mathbf{W}_j.
\end{aligned}$$

Since $\{\phi_{j,n}\}_{n\in\mathbb{Z}}$ and $\{\psi^l_{j,n}, l = 1, 2\}_{n\in\mathbb{Z}}$ are orthonormal bases of V_j and W_j, we can derive that $\{\phi_{j,k_1}(x)\psi^1_{j,k_2}(y)$, $\psi^1_{j,k_1}(x)\phi_{j,k_2}(y)$, $\phi_{j,k_1}(x)\psi^2_{j,k_2}(y)$, $\psi^2_{j,k_1}(x)\phi_{j,k_2}(y)$,$\psi^1_{j,k_1}(x)\psi^1_{j,k_2}(y)$, $\psi^1_{j,k_1}(x)\psi^2_{j,k_2}(y)$, $\psi^2_{j,k_1}(x)\psi^1_{j,k_2}(y)$, $\psi^2_{j,k_1}(x)\psi^2_{j,k_2}(y)\}_{(k_1,k_2)\in\mathbb{Z}^2}$ is an orthonormal basis of $\mathbf{W}_j$.

Chapter 2

CONSTRUCTION OF UNIVARIATE SCALING FUNCTION

In this chapter, we will construct two compactly supported univariate scaling functions of dilation 3 supported on $[0, 5/2]$ and $[0, 4]$ respectively, with the coefficients parameterized.

Definition 2.0.1. *A finitely supported sequence* $\mathbf{c}=\{c_k,\ k\in\mathbb{Z}\}$ *is q-orthonormal if* $\sum_{k\in\mathbb{Z}} c_k = 1$ *and*

$$\sum_{k\in\mathbb{Z}} c_k c_{k+qj} = \begin{cases} \frac{1}{q} & \text{if } j=0, \\ 0 & \text{if } j\neq 0. \end{cases} \tag{2.0.1}$$

There are several properties of the q-orthonormal sequence.

Proposition 2.0.2. *1. A finitely supported sequence* $\mathbf{p}$ *is q-orthonormal if and only if*

$$\sum_{k=0}^{q-1} \left| M_p\left(\omega + \frac{2\pi k}{q}\right)\right|^2 = 1, \quad for\ all\ \omega\in\mathbb{R}$$

where $M_p(z) = \sum_k p_k e^{ik\omega}$.

2. If a sequence $\mathbf{p}$ *is q-orthonormal, then* $\sum_{j\in\mathbb{Z}} p_{k+qj} = \frac{1}{q}$ *for all* $k\in\mathbb{Z}$.

The proof of (1) can be seen in [11], proof of (2) in [4].

2.1 General case

Let $H(z) = \sum_k p_k z^k$ with $z = e^{i\xi}$ be the symbol of a scaling function. We need to find real valued p_k such that $H(x)$ will satisfy

$$H(1) = 1 \tag{2.1.1}$$

$$|H(z)|^2 + |H(ze^{i2\pi/3})|^2 + |H(ze^{i4\pi/3})|^2 = 1 \tag{2.1.2}$$

Let $H_6(z) = p_0 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + p_5 z^5$ be the symbol of the scaling function supported on $[0, 5/2]$, where $z = e^{i\xi}$.

LEMMA 2.1.1. *$H_6(z)$ satisfies $H_6(1) = 1$ and*

$$|H_6(z)|^2 + |H_6(z\tau)|^2 + \left|H_6(z\tau^2)\right|^2 = 1, \forall z = e^{i\xi}, \xi \in \mathbb{R},\ \tau = e^{i\frac{2\pi}{3}}$$

if and only if

$$p_0 = \frac{1}{6} + \frac{\sqrt{3}}{6}\cos\theta \tag{2.1.3a}$$
$$p_1 = \frac{1}{6} + \frac{\sqrt{3}}{6}\sin\theta\cos\alpha \tag{2.1.3b}$$
$$p_2 = \frac{1}{6} + \frac{\sqrt{3}}{6}\sin\theta\sin\alpha \tag{2.1.3c}$$
$$p_3 = \frac{1}{6} - \frac{\sqrt{3}}{6}\cos\theta \tag{2.1.3d}$$
$$p_4 = \frac{1}{6} - \frac{\sqrt{3}}{6}\sin\theta\cos\alpha \tag{2.1.3e}$$
$$p_5 = \frac{1}{6} - \frac{\sqrt{3}}{6}\sin\theta\sin\alpha \tag{2.1.3f}$$

for some θ, α in $[0, 2\pi]$.

Proof: Since $|H_6(z)|^2 + |H_6(z\tau)|^2 + |H_6(z\tau^2)|^2 = 1, \forall z = e^{i\omega}, \omega \in R,\ \tau = e^{i\frac{2\pi}{3}}$, by Proposition 2.0.2, $\mathbf{p} = \{p_k, k = 0, 1, 2, 3, 4, 5\}$ is a 3-orthonormal sequence. Hence

$$\begin{aligned} p_0^2 + p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 &= \frac{1}{3} \\ p_0p_3 + p_1p_4 + p_2p_5 &= 0 \\ p_0 + p_3 = p_1 + p_4 = p_2 + p_5 &= \frac{1}{3} \end{aligned} \tag{2.1.4}$$

From first two euqations of (2.1.4), we have

$$(p_0 - p_3)^2 + (p_1 - p_4)^2 + (p_2 - p_5)^2 = \frac{1}{3}.$$

Then,

$$p_0 - p_3 = \frac{\sqrt{3}}{3}\cos\theta, \quad p_1 - p_4 = \frac{\sqrt{3}}{3}\sin\theta\cos\alpha \text{ and } p_2 - p_5 = \frac{\sqrt{3}}{3}\sin\theta\sin\alpha$$

With the third equation in (2.1.4), we get the result by solving this linear equation system. Here we finished proving the necessary condition. The above computation is conversible; so it gives sufficient condition.

Let $H_9(z) = p_0 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + p_5 z^5 + p_1 z + p_6 z^6 + p_7 z^7 + p_8 z^8$ be the symbol of a scaling function supported on $[0, 4]$, where $z = e^{i\xi}$.

Lemma 2.1.2. *$H_9(z)$ satisfies $H_9(1) = 1$ and*

$$|H_9(z)|^2 + |H_9(z\tau)|^2 + |H_9(z\tau^2)|^2 = 1, \forall z = e^{i\xi}, \xi \in \mathbb{R},\ \tau = e^{i\frac{2\pi}{3}} \tag{2.1.5}$$

if and only if

$$p_0 = \frac{1}{12} + \frac{\sqrt{3}}{12}\cos\alpha\cos\beta + \frac{1}{6}r\cos\theta\cos\gamma \tag{2.1.6a}$$

$$p_1 = \frac{1}{12} + \frac{\sqrt{3}}{12}\sin\alpha + \frac{1}{6}r\sin\theta \tag{2.1.6b}$$

$$p_2 = \frac{1}{12} + \frac{\sqrt{3}}{12}\cos\alpha\sin\beta + \frac{1}{6}r\cos\theta\sin\gamma \tag{2.1.6c}$$

$$p_3 = \frac{1}{6}(1 - \sqrt{3}\cos\alpha\cos\beta) \tag{2.1.6d}$$

$$p_4 = \frac{1}{6}(1 - \sqrt{3}\sin\alpha) \tag{2.1.6e}$$

$$p_5 = \frac{1}{6}(1 - \sqrt{3}\cos\alpha\sin\beta) \tag{2.1.6f}$$

$$p_6 = \frac{1}{12} + \frac{\sqrt{3}}{12}\cos\alpha\cos\beta - \frac{1}{6}r\cos\theta\cos\gamma \tag{2.1.6g}$$

$$p_7 = \frac{1}{12} + \frac{\sqrt{3}}{12}\sin\alpha - \frac{1}{6}r\sin\theta \tag{2.1.6h}$$

$$p_8 = \frac{1}{12} + \frac{\sqrt{3}}{12}\cos\alpha\sin\beta - \frac{1}{6}r\cos\theta\sin\gamma \tag{2.1.6i}$$

where

$$r = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{6}(\cos\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha)}$$

for some α, β, γ in $[0, 2\pi]$.

Proof: Since $|H_9(z)|^2 + |H_9(z\tau)|^2 + |H_9(z\tau^2)|^2 = 1, \forall z = e^{i\omega}, \omega \in R,\ \ \tau = e^{i\frac{2\pi}{3}}$, by Proposition 2.0.2, $\mathbf{p} = \{p_k, k = 0, 1, \cdots, 8\}$ is a 3-orthonormal sequence. Hence

$$p_0^2 + p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + p_6^2 + p_7^2 + p_8^2 = \frac{1}{3} \tag{2.1.7a}$$

$$p_0p_3 + p_1p_4 + p_2p_5 + p_3p_6 + p_4p_7 + p_5p_8 = 0 \tag{2.1.7b}$$

$$p_0p_6 + p_1p_7 + p_2p_8 = 0 \tag{2.1.7c}$$

$$p_0 + p_3 + p_6 = p_1 + p_4 + p_7 = p_2 + p_5 + p_8 = \frac{1}{3} \tag{2.1.7d}$$

From (2.1.7a), (2.1.7b) and (2.1.7c),we get

$$(p_0 - p_3 + p_6)^2 + (p_1 + p_4 + p_7)^2 + (p_2 + p_5 + p_8)^2 = \frac{1}{3}.$$

Thus,

$$p_0 - p_3 + p_6 = \frac{\sqrt{3}}{3}\cos\alpha\cos\beta, \tag{2.1.8a}$$

$$p_1 - p_4 + p_7 = \frac{\sqrt{3}}{3}\sin\alpha \tag{2.1.8b}$$

$$p_2 - p_5 + p_8 = \frac{\sqrt{3}}{3}\cos\alpha\sin\beta. \tag{2.1.8c}$$

With (2.1.7d), we can solve for $p_3,\ p_4$ and p_5 first,

$$p_3 = \frac{1}{6}(1 - \sqrt{3}\cos\alpha\cos\beta),$$
$$p_4 = \frac{1}{6}(1 - \sqrt{3}\sin\alpha)$$
$$p_5 = \frac{1}{6}(1 - \sqrt{3}\cos\alpha\sin\beta).$$

Also,

$$p_0 + p_6 = \frac{1}{6}(1 + \sqrt{3}\cos\alpha\cos\beta) \tag{2.1.9a}$$

$$p_1 + p_7 = \frac{1}{6}(1 + \sqrt{3}\sin\alpha) \tag{2.1.9b}$$

$$p_2 + p_8 = \frac{1}{6}(1 + \sqrt{3}\cos\alpha\sin\beta) \tag{2.1.9c}$$

Then, we rewrite equation (2.1.7a) as

$$p_0^2 + p_1^2 + p_2^2 + p_6^2 + p_7^2 + p_8^2 = r^2, \text{where } r^2 = \frac{1}{3} - (p_3^2 + p_4^2 + p_5^2).$$

and combine (2.1.7c) to get

$$(p_0 - p_6)^2 + (p_1 - p_7)^2 + (p_2 - p_8)^2 = r^2$$

where r is defined above. Hence

$$p_0 - p_6 = r\cos\theta\cos\gamma \tag{2.1.10a}$$

$$p_1 + p_7 = r\cos\theta\sin\gamma \tag{2.1.10b}$$

$$p_2 + p_8 = r\sin\theta \tag{2.1.10c}$$

With (2.1.9) and (2.1.10), we are able to solve for the rest coefficients. Again, the argument is reversible to get sufficiency.

2.2 Symmetric scaling functions

We are trying to build symmetric scaling function after finding the general expression of scaling functions. First, we will explore symmetric scaling funcitons supported on $[0, 5/2]$. In order to find the relationship between the symmetry of scaling function and the symmetry of the coeffient sequence, we need the following Lemma which is proved by Belogay and Wang [2].

Lemma 2.2.1. *A scaling function is symmetric if and only if the coefficient sequence of its associated trigonometric polynomial is symmetric.*

Lemma 2.2.2. *A scaling function supported on* $[0, 5/2]$ *is symmetric if and only if the coefficients of its associated trigonometric polynomial* H_6 *satisfies* (2.1.3)*(a-f) and* $\tan\theta = -1,\ \sin\alpha = 1$ *or* $\tan\theta = 1,\ \sin\alpha = -1$

Proof: From Lemma 2.1.1, we know that the associated trigonometric polynomial of scaling function supported on $[0, 5/2]$ is $H_6(z)$ whose coefficients satisfy equations (2.1.3)(a-f) If scaling function is symmetric, then

$$p_0 = p_5, p_1 = p_4 \text{ and } p_2 = p_3.$$

$p_2 = p_3$ implies

$$\sin\theta\cos\alpha = 0, \tag{2.2.1}$$

and the other two equations imply

$$\sin\theta\sin\alpha = -\cos\theta. \tag{2.2.2}$$

From (2.2.1), we will have either $\sin\theta = 0$ or $\cos\alpha = 0$. But if $\sin\theta = 0$, then from (2.2.2), $\cos\theta$ has to be 0, which is impossible. Hence $\cos\alpha$ has to be 0, $\sin\alpha = \pm 1$. From (2.2.2), we conclude $\tan\theta = \mp 1$.

Now we start to investigate symmetric scaling functions supported on $[0, 4]$.

LEMMA 2.2.3. *A scaling function supported on* $[0, 4]$ *is symmetric if and only if the coefficients of its associated trigonometric function* $H_9(z)$ *satisfies* (2.1.6)*(a-i) and* $\sin\theta = 0$*,* $\cos\alpha = 0$*,* $\tan\beta = 1$*, and* $\tan\gamma = -1$*.*

Proof: Similar to the proof of Lemma 2.2.2, we start to look at the symmetry of the scaling function. If scaling function is symmetric, then $p_i = p_{8-i}$ for $i = 0, 1, 2, 3$. $p_1 = p_7$ implies $\sin\theta = -\sin\theta$, thus $\sin\theta = 0$. $p_3 = p_5$ implies $\cos\alpha(\cos\beta - \sin\beta) = 0$, so either $\cos\alpha = 0$ or $\tan\beta = 1$. $p_2 = p_6$ and $p_0 = p_8$ both imply $\sqrt{3}/4\cos\alpha(\cos\beta - \sin\beta) = 1/2r\cos\theta(\cos\gamma + \sin\gamma)$ where r is defined in Lemma 2.1.2, hence $\cos\gamma + \sin\gamma = 0$ since $\cos\theta \neq 0$.

Chapter 3

CONSTRUCTION OF UNIVARIATE ORTHONORMAL WAVELETS

Once a scaling function is determined, we can start to find the corresponding orthonormal wavelets. In this chapter, we will construct the wavelets of the given scaling function in previous chapter.

First, we rewrite the trigonometric polynomial $M(z)$ associated to the scaling function in the polyphase form, let $M(\omega) = \sum_{m=0}^{2} f_{0,m}(z^3)z^m$ where $f_{0,m} = \sum_{j=0}^{2} c_{j,m}z^j$ and $z = e^{-i\omega}$. Then we have the following Lemma.

Lemma 3.1. *A trigonometric polynomial $M(z)$ satisfies* (2.1.2) *if and only if*

$$|f_{0,0}|^2 + |f_{0,1}|^2 + |f_{0,2}|^2 = \frac{1}{3}.$$

It can be easily checked by straightforward computation. Now we extend the construction method of bivariate wavelets in [15] to our setting of orthonormal wavelets with dilation $q = 3$.

Write $[f_{0,0}, f_{0,1}, f_{0,2}]^T = c_0 + c_1 z + c_2 z^2$ with $c_i = [c_{i,0}, c_{i,1}, c_{i,2}]^T$. Let $L = [c_0, c_1, c_2]$. Then there exists an orthonormal matrix H (by Householder transform) such that HL is a lower triangle matrix. Then let

$$[\tilde{f}_{0,0}, \tilde{f}_{0,1}, \tilde{f}_{0,2}]^T = HL[1, z, z^2]^T.$$

Note that

$$\sum_{j=0}^{2} |\tilde{f}_{0,j}|^2 = \sum_{j=0}^{2} |f_{0,j}|^2 = \frac{1}{3}.$$

If $\tilde{f}_{0,0} = \frac{1}{\sqrt{3}}$, then $\tilde{f}_{0,j} = 0$ for $j = 1, 2$. Otherwise, let

$$v = [\tilde{f}_{0,0}, \tilde{f}_{0,1}, \tilde{f}_{0,2}]^T - \frac{1}{\sqrt{3}}[1, 0, 0]^T$$

and

$$H(v) = I_3 - \frac{2}{v^*v} vv^*$$

be a householder matrix such that

$$H(v)[\tilde{f}_{0,0}, \tilde{f}_{0,1}, \tilde{f}_{0,2}]^T = [\frac{1}{\sqrt{3}}, 0, 0]^T.$$

Thus $[f_{0,0}, f_{0,1}, f_{0,2}]^T = HH(v)[\frac{1}{\sqrt{3}}, 0, 0,]^T$. Hence $[f_{0,0}, f_{0,1}, f_{0,2}] = [\frac{1}{\sqrt{3}}, 0, 0]H(v)H$. By choosing $P(z) = \frac{1}{\sqrt{3}} H(v)H$, we have $P(z)P^*(z) = \frac{1}{3} I_3$ with $[f_{0,0}, f_{0,1}, f_{0,2}]$ in the first row of $P(z)$, now we can define the polynomial $Q_1,\ Q_2$ as follows:

$$\begin{bmatrix} M(\omega) & M(\omega + \frac{2\pi}{3}) & M(\omega + \frac{4\pi}{3}) \\ \\ Q_1(\omega) & Q_1(\omega + \frac{2\pi}{3}) & Q_1(\omega + \frac{4\pi}{3}) \\ \\ Q_2(\omega) & Q_2(\omega + \frac{2\pi}{3}) & Q_2(\omega + \frac{4\pi}{3}) \end{bmatrix} = P(e^{3i\omega}) \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2\pi}{3}} & e^{\frac{4\pi}{3}} \\ 1 & e^{\frac{4\pi}{3}} & e^{\frac{2\pi}{3}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\omega} & 0 \\ 0 & 0 & e^{2i\omega} \end{bmatrix}. \tag{3.0.1}$$

Then we obtain the following theorem.

THEOREM 3.2. *Let $M(z), Q_1(z)$ and $Q_2(z)$ be trigonometric polynomials constructed in* (3.0.1). *Let ϕ be the scaling function generated by $M(z)$ as*

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{n=1}^{\infty} M(3^{-n}\omega). \tag{3.0.2}$$

Define ψ_k as

$$\hat{\psi}_k = Q_k(\omega/3)\hat{\phi}(\omega/3), \tag{3.0.3}$$

for $k = 1, 2$. Then $\{\psi_k, k = 1, 2\}$ is an orthonormal wavelet set in $L^2(\mathbb{R})$. That is,

$$\{3^{-j/2}\psi_k(3^{-j}x - l);\ l \in \mathbb{Z}, j \in \mathbb{Z}, k = 1, 2\}$$

is an orthonormal basis of $L^2(\mathbb{R})$.

One example of symmetric length 6 scaling function is to take $\theta = 3\pi/4$ and $\alpha = \pi/2$. Using the above method, we are able to find the wavelets corresponding to the scaling function in the previous chapter. First, we apply Theorem 3.2 to get the polynomials exhibited in Table 3.1.

Table 3.1: The coefficients of $M(z), Q_1(z)$ and $Q_2(z)$ for symmetric scaling function.

$M(z)$	$Q_1(z)$	$Q_2(z)$
-0.0374574785652648	-0.1975659009915497	0.0182037942994759
0.1666666666666667	0.3569019230097092	-0.0328850735827839
0.3707908118985982	-0.3676693553514927	0.0338771887327142
0.3707908118985982	0.1423256765821654	-0.1463217975972670
0.1666666666666667	0.1014314103236241	0.3407561176270030
-0.0374574785652648	0.0062429130942108	0.2385783818624835
	-0.0308992343248832	-0.3353495962937979
	-0.0138888888888889	-0.1507362037805418
	0.0031214565471054	0.0338771887327142

Then we use the cascade algorithm [5] to graph the scaling function and its wavelets.

Next, we want to show the graphs of another scaling function supported on $[0, 5/2]$, and the corresponding wavelets. Note that, this scaling function is not symmetric. We call this wavelet as Q3L6B; it will be applied to image compression in the next Chapter.

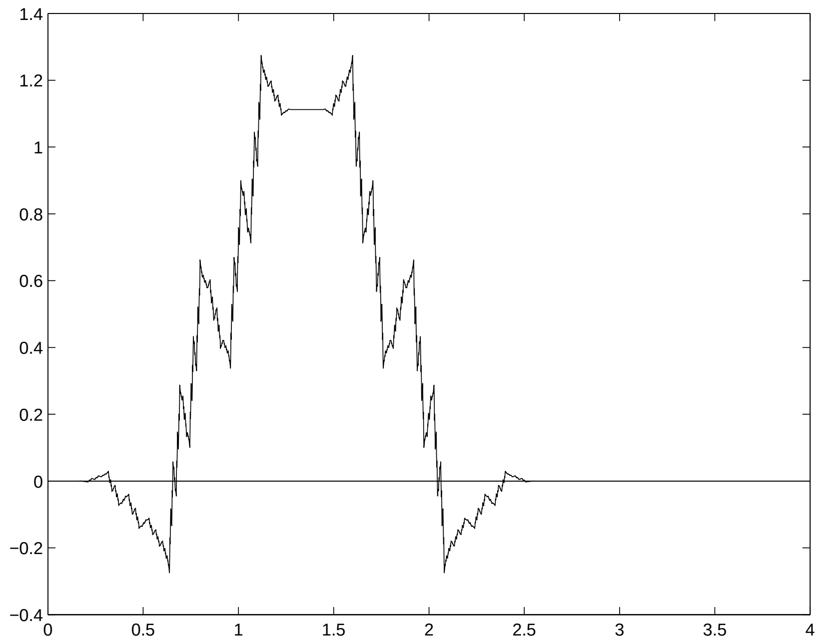

Figure 3.1: The symmetric scaling function ϕ.

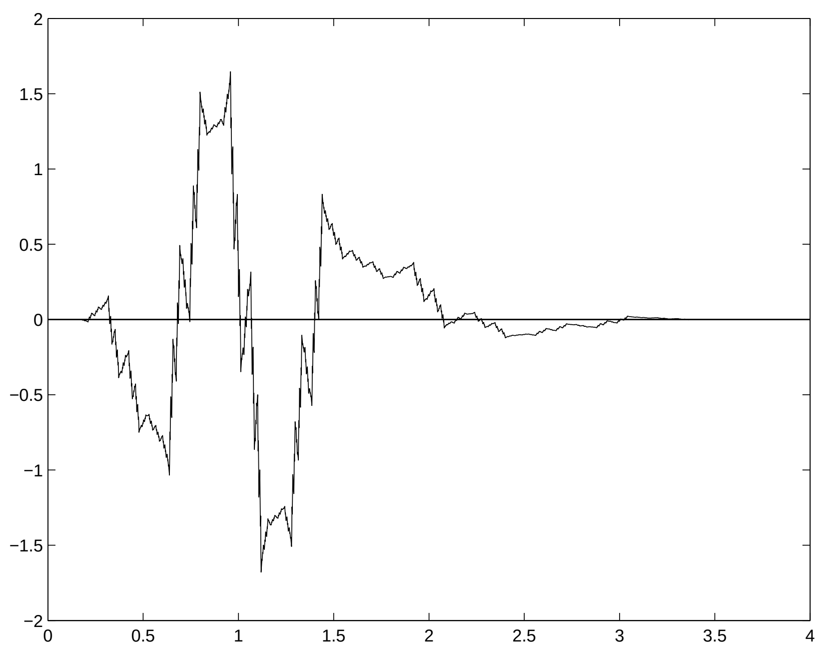

Figure 3.2: First wavelet function ψ_1 from symmetric ϕ.

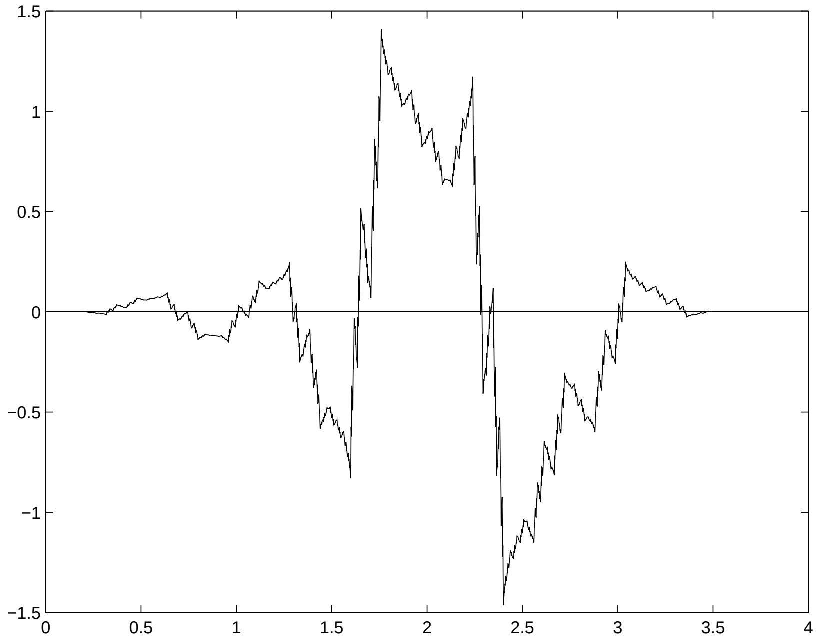

Figure 3.3: Second wavelet function ψ_2 from symmetric ϕ.

Table 3.2: The coefficients of $M(z), Q_1(z)$ and $Q_2(z)$ for Q3L6B.

$M(z)$	$Q_1(z)$	$Q_2(z)$
-0.0863012821413293	-0.2201454421779753	-0.0244485452793175
0.0849231913490577	0.4363664050675779	0.0484612522844910
0.2791769082103053	-0.2957579387259072	-0.0328457917871051
0.4196346154746627	0.0662645736752800	-0.1754034504767627
0.2484101419842756	0.0031969927650426	0.2205922233147196
0.0541564251230280	-0.0385688131554503	0.4016582616757073
	0.0282646998063002	-0.2545077740126141
	0.0167317895929153	-0.1506603839320132
	0.0036477331522170	-0.0328457917871051

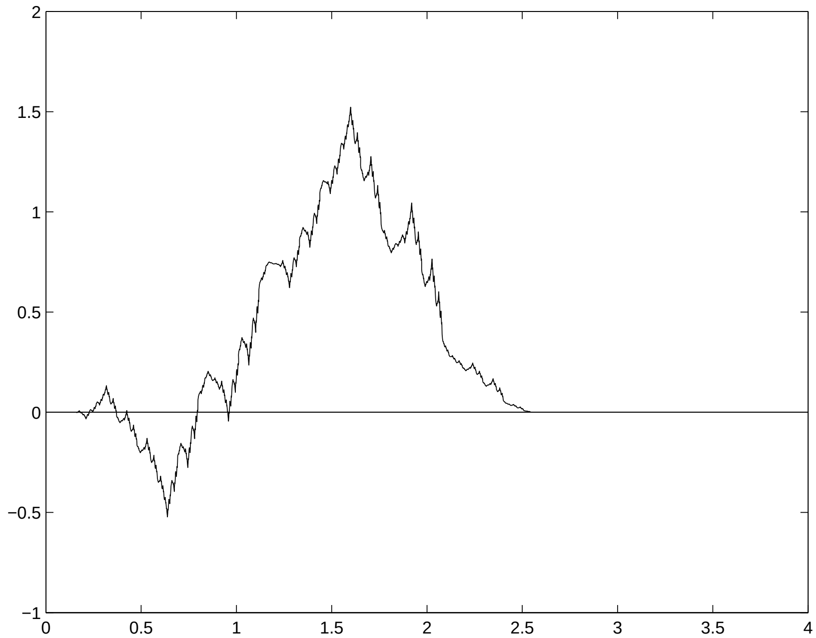

Figure 3.4: The scaling function ϕ for Q3L6B.

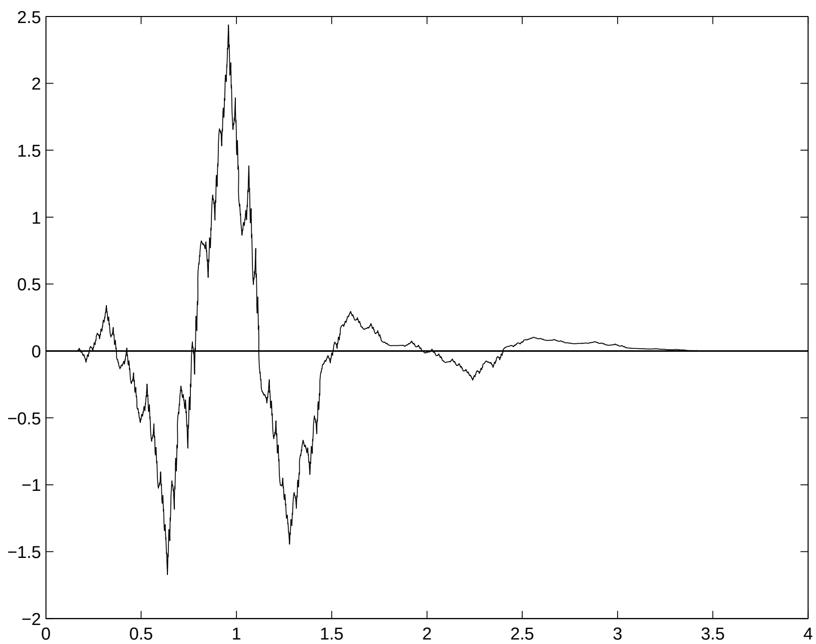

Figure 3.5: First wavelet function ψ_1 for Q3L6B.

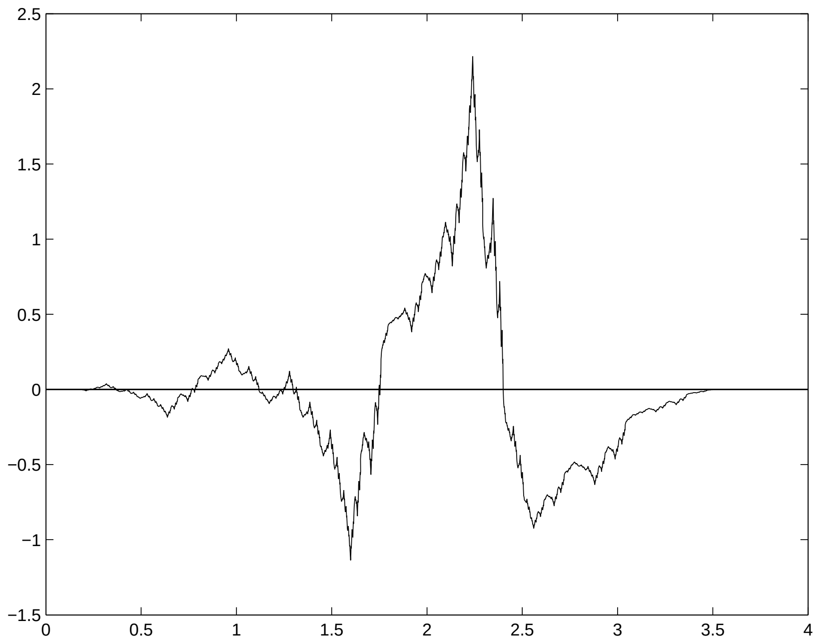

Figure 3.6: Second wavelet function ψ_2 for Q3L6B.

Chapter 4

APPLICATIONS TO IMAGE COMPRESSION

Digital images are bi-dimensional signals, image compression is the data compression of digital images. Gray-scale images are the signals with intensity value of each pixel at position (x, y). In this chapter, we shall use the seperable bivariate orthonormal wavelet constructed by the tensor product of one dimensional wavelets in Chapter 3 to compress the images.

4.1 Wavelet-based image compression

Data compression is the technique for storing and/transmitting the data using as few bits as possible by encoding the original data [14]. And the original data is to be reconstucted later. The compression can be lossy or lossless, the reconstruction will be perfectly or approximately. *Compression ratio* is the ratio of the size of the original data to the size of the compressed data; it is usually presented in discussions of data compression. Data compression is necessary despite of rapid progress in mass-storage technology, processor speed and broadband network, since the demand for data storage capacity and data transmission bandwith always outpaces the capability of available technologies.

The objective of image compression is to reduce the redundancy of the image data. The redundant and nonessential information are allowed to be lost in lossy compression schemes. The lossy compression schemes get rid of the data from the image that is hard to be noticed by human eyes. It works for images viewed by humans.

Transform coding is the most commonly used method for lossy compression. The success of transform coding schemes is attributed to simplicity of implementation, good performance and sound theoretical foundation[26].

Figure 4.1: A wavelet transform based encoder.

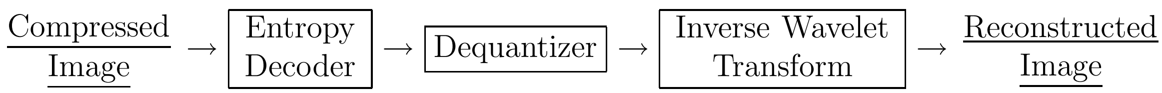

Figure 4.2: A wavelet transform based decoder.

The procedure of transform coding is as follows: transform the image into coefficients, quantize the coefficients and encode the quantized coefficients. A generic transform encoder and decoder are shown in Figure 4.1 and Figure 4.2, respectively.

Subband coding is a form of transform coding that breaks a signal into a number of different frequency bands and encodes each one independently. It has been used extensively first in speech coding [9] and later in image coding [31]. In the analysis stage of subband coding, the incoming signal is filtered into M subband components with almost no overlap in the frequency domain first, each having smaller bandwith than the original signal. Due to the limited bandwith, each component is downsampled such that the subband transformed data contains as much data as the original signal. Then subband components are quantized and compressed. The synthesis stage of subband coding performs inverse operations to those described above. It takes M subband signals as inputs, and upsamples them by inserting $M-1$ zero-valued samples between every adjacent pair of incoming signal samples. Subsequently, filtering with appropriate filters and summation operations is performed.

A subband coding scheme based on multiresolution wavelet bases is described by S. Mallat [21]. Because its basis functions have variable length, long basis functions represent flat background, while short basis functions represent regions with texture, it avoids blocking at medium bit-rate,

Bit rate and distortion of the compressed representation are two key parameters of evaluating a coding scheme. *Bit rate* is the average number of bits per pixel required to represent an image. *Distortion* is a numeric measure of the difference between the original image and the image reconstructed from the compressed representation.

The most common distortion measure between the original signal X with N samples and the signal reconstructed from the compressed representation $\tilde{X}$ in the engineering term is the *mean-squared error* (MSE)

$$MSE = \frac{1}{N}\|X - \tilde{X}\|_2^2 = \frac{1}{N}\sum_{i=1}^{N}|X_i - \tilde{X}_i|^2.$$

The quality of a compression method is often measured by the peak signal-to-noise ratio(PSNR). It measures the amount of noise introduced through a lossy compression of the image. For 8-bit(256-gray-scale) image data, it is given by

$$PSNR = 10\log_{10}\frac{255^2}{MSE}.$$

However, the subjective judgement of the viewer is also regarded as an important, perhaps the most important measure. It is well known that minimizing MSE does not guarantee optimal results in the perceptual sense. At lower bit rate, perceptual quality is more important.

For a given two-dimensional scaling function ϕ with dilation factor 3 and its eight companion orthonormal wavelets $\psi^1, \psi^2, \psi^3, \ldots, \psi^8$ we define

$$\phi_{j;k_1,k_2}(x,y) = 3^{-j}\phi(3^{-j}x - k_1, 3^{-j}y - k_2), \quad k_1, k_2 \in \mathbb{Z},$$

and

$$\psi^l_{j;k_1,k_2}(x,y) = 3^{-j}\psi^l(3^{-j}x - k_1, 3^{-j}y - k_2), \quad k_1, k_2 \in \mathbb{Z}, \quad l = 1, 2, 3, \ldots, \text{ or } 8.$$

The space $V_{j-1} = \overline{\text{span}_{L^2}\{\phi_{j-1;k_1,k_2}; k_1, k_2 \in \mathbb{Z}\}}$ can be decomposed as

$$V_{j-1} = V_j \oplus W_j = V_j \bigoplus_{l=1}^{8} W_j^l,$$

where the subspaces $W_j^l = \overline{\text{span}_{L^2}\{\psi_{j;k_1,k_2}^l; k_1, k_2 \in \mathbb{Z}\}}$ for $l = 1, 2, \ldots, 8$.

Suppose f is the representation of an image $\{c_{n_1,n_2}^{j-1}\}_{n_1,n_2 \in \mathbb{Z}} \in l^2(\mathbb{Z}^2)$ in V_{j-1}, i.e.,

$$f = \sum_{n_1,n_2 \in \mathbb{Z}} c_{n_1,n_2}^{j-1} \phi_{j-1;n_1,n_2}.$$

The orthogonal projections of f onto the subspaces $V_j, W_j^1, W_j^2, \ldots, W_j^8$ are given by

$$P_j f = \sum_{n_1,n_2 \in \mathbb{Z}} c_{n_1,n_2}^j \phi_{j;n_1,n_2}, \quad c_{n_1,n_2}^j =< f, \phi_{j;n_1,n_2} >,$$

and

$$Q_j^l f = \sum_{n_1,n_2 \in \mathbb{Z}} d_{n_1,n_2}^{l;j} \psi_{j;n_1,n_2}^l, \quad d_{n_1,n_2}^{l;j} =< f, \psi_{j;n_1,n_2}^l >, \quad l = 1, 2, \ldots, 8$$

respectively. These projections satisfy the relation

$$P_{j-1} f = P_j f + Q_j^1 f + Q_j^2 f + Q_j^3 f + Q_j^4 f + Q_j^5 f + Q_j^6 f + Q_j^7 f + Q_j^8 f.$$

An image can be decomposed into a low-pass subimage and eight high-pass subimages as described by the above relation. Like the original image, the low-pass subimage can be decomposed further into nine subimages, and the process can be repeated over and over again until a desired decomposition level is achieved. Figure 4.3 illustrates a two stage wavelet-based image decomposition.

From the dilation equations, it's easy to derive that

$$\phi_{j;n_1,n_2} = \sum_{k_1,k_2} h_{k_1-3n_1,k_2-3n_2} \phi_{j-1;k_1,k_2},$$

and

$$\psi_{j;n_1,n_2}^l = \sum_{k_1,k_2} g_{k_1-3n_1,k_2-3n_2}^l \phi_{j-1;k_1,k_2}, \quad l = 1, 2, \ldots, 8.$$

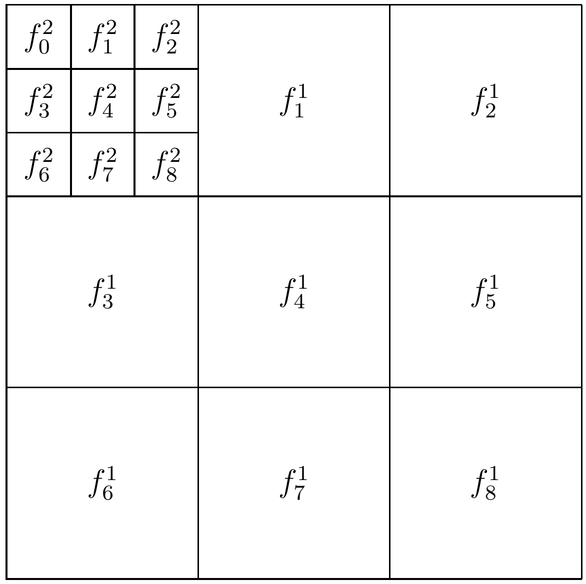

Figure 4.3: A two-level wavelet-based image decomposition. The image is divided into nine subbands using wavelet-based filters. The low-pass subimage is further decomposed into nine subimages.

It implies that

$$\begin{aligned} c^j_{n_1,n_2} &= < \sum_{m_1,m_2} c^{j-1}_{m_1,m_2} \phi_{j-1;m_1,m_2}, \sum_{k_1,k_2} h_{k_1-3n_1,k_2-3n_2} \phi_{j-1;k_1,k_2} > \\ &= \sum_{k_1,k_2} \bar{h}_{k_1-3n_1,k_2-3n_2} c^{j-1}_{k_1,k_2}, \end{aligned} \tag{4.1.1}$$

and

$$\begin{aligned} d^{l;j}_{n_1,n_2} &= < \sum_{m_1,m_2} c^{j-1}_{m_1,m_2} \phi_{j-1;m_1,m_2}, \sum_{k_1,k_2} g^l_{k_1-3n_1,k_2-3n_2} \phi_{j-1;k_1,k_2} > \\ &= \sum_{k_1,k_2} \bar{g}^l_{k_1-3n_1,k_2-3n_2} c^{j-1}_{k_1,k_2}, \quad l = 1, 2, \ldots, 8. \end{aligned} \tag{4.1.2}$$

Let H and G_l be *conjugate quadrature filters* (CQF) with impulse responses $\{\bar{h} = (\bar{h}_{k_1,k_2})$, $k_1, k_2 \in \mathbb{Z}\}$ and $\{\bar{g}^l = (\bar{g}^l_{k_1,k_2}), k_1, k_2 \in \mathbb{Z}\}$, respectively. From (4.1.1) and (4.1.2), we will see that c^j and $d^{l;j}$ are obtained by convoluting c^{j-1} with H and G_l, respectively, then downsampling by 3 in both x and y directions. Downsampling the signal by 3 means keeping every third entry and throwing out the rest.

Each resulting subimage is one-ninth of the size of the previous image since the process of decomposition downsamples the subimages by 3 in each dimension. Therefore, the total number of pixels doesn't change after each decomposition.

The original image can be reconstructed by repeated use of the relation

$$\begin{aligned} P_{j-1}f &= P_j f + Q_j^1 f + Q_j^2 f + Q_j^3 f + Q_j^4 f + Q_j^5 f + Q_j^6 f + Q_j^7 f + Q_j^8 f \\ &= \sum_{k_1,k_2\in\mathbb{Z}} c^j_{k_1,k_2}\phi_{j;k_1,k_2} + \sum_{l=1}^{8}\sum_{k_1,k_2\in\mathbb{Z}} d^{l;j}_{k_1,k_2}\psi^l_{j;k_1,k_2}. \end{aligned}$$

This implies

$$\begin{aligned} c^{j-1}_{n_1,n_2} &=< P_{j-1}f, \phi_{j-1;n_1,n_2} > \\ &= \sum_{k_1,k_2} c^j_{k_1,k_2}\langle\phi_{j;k_1,k_2}, \phi_{j-1;n_1,n_2}\rangle + \sum_{l=1}^{8}\sum_{k_1,k_2} d^{l;j}_{k_1,k_2}\langle\psi^l_{j;k_1,k_2}, \phi_{j-1;n_1,n_2}\rangle \\ &= \sum_{k_1,k_2} h_{n_1-3k_1,n_2-3k_2}c^j_{k_1,k_2} + \sum_{l=1}^{8}\sum_{k_1,k_2} g^l_{n_1-3k_1,n_2-3k_2}d^{l;j}_{k_1,k_2}. \end{aligned} \tag{4.1.3}$$

Recursion equation (4.1.3) gives the reconstruction algorithm. To get c^{j-1}, we upsample the c^j and $d^{l;j}$ by 3 in both x and y directions, that is inserting two rows(columns) of zeors between any two adjacent rows(columns), then convolve the interleaved image with filters $H^*$and G_l^*, where $H^*$and G_l^* are adjoints of H and G_l respectively, and finally add up the results. The two dimensional wavelet-based image decomposition and reconstruction schemes are illustrated in Figure 4.4.

4.2 Numerical experiments

FBIP is the wavelet with dilation factor 2, which is the best wavelet so far in image compression. It is used by the FBI to compress the images of finger prints, and was selected for the JPEG2000 standard. We try to find if there exist any wavelets with dilation factor other than 2 which will perform better than FBIP in image compression. In our experiment,

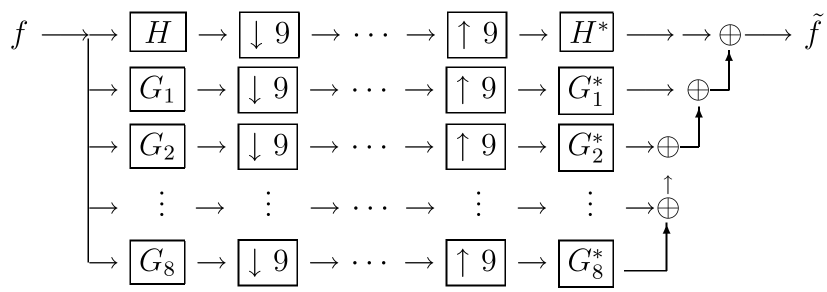

Figure 4.4: Image decomposition and reconstruction scheme. ↓ 9 stands for downsampling by 9, i.e. delete every two rows(columns) between any two adjacent rows(columns). ↑ 9 stands for upsampling by 9, i.e. insert two rows of zeros between every two adjacent rows and then insert two columns of zeros between every two adjacent columns.

we searched the wavelets generated by a scaling function supported on $[0, 5/2]$ with dilation factor 3, and picked the best one to compare with FBIP. The chosen one is called Q3L6B, the example shown in Chapter 3. According to our results, Q3L6B performs better than FBIP at a very high compression ratio.

For the coding part, we use the best wavelet coder called the Set Partitioning in Hierarchical Trees (SPIHT) algorithm developped by A. Said and W. A. Pearlman [27]. It is the improved variation of Embedded Zerotree Wavelet(EZW) provided by J.M.Shapiro [28] and it is the best wavelet coder available. We used the SPIHT encoding and decoding procedure without the entropy coding of the bit stream by the arithmetic code with small loss in performance (decreasing PSNR by 0.3 to 0.6 dB for the same bit rate). Non-expansive periodic extensions were employed at image boundaries for wavelet transforms.

The size of the signal has to be the multiple of power of 2 for FBIP, our Q3L6B only works for the signal of the length which is the multiple of power of 3. In order to compare, we develop experiments of two different classes of images. One class consists of the images of size 1296 × 1296(1296=$2^4 \times 3^4$), the other is of size 729 × 729(729=3^6).

The image compression scheme we adopted in the comparison is as follows:

- Decomposing the gray-scale images to a maximum number of levels. For example, the highest level to decompose an image of size 1296 × 1296 is 4 for Q3L6B, since the size of the top level will not be divisible by 3 after decomposing the image at level 4. Analogously, the highest level to decompose the same image is also 4 for FBIP.

- Encoding the decomposed image using SPIHT encoder (see [27] [28]) to a specified file size depending on our compression ratio.

- Decoding the compressed image file using SPIHT decoder.

- Reconstructing the image using wavelet inverse transform and rounding the values to the nearest integer.

- Calculating the peak signal to noise ratio (PSNR) which is a measure of the root mean squared error (RMSE). PSNR and RMSE are defined as

$$RMSE = \sqrt{\frac{1}{s^2}\sum_{i,j=1}^{s}(p_{i,j}-\hat{p}_{i,j})^2}$$
$$PSNR = 20\log_{10}(\frac{255}{RMSE})$$

  where s is the number of entries on each row(column) of the image, $p_{i,j}$ is the original gray-scale value and $\hat{p}_{i,j}$.

Case1:

We first choose the images of size 1296 × 1296, so both methods can decompose the image to the same level. As we explained in the image compression scheme, the highest level we can achieve is 4 for both wavelets. And the results are shown in Table 4.1. From the table, we see that at the low compression ratio, these two methods give the similar results. However, at very high compression ratio, Q3L6B has a far better performance than FBIP. And in this case, computation complexities are the same in terms of addition or multiplication.

Table 4.1: Coding results for different compression ratios showing Peak Signal-to-Noise Ratio (PSNR).

Image: Lotus 1296 × 1296					
Wavelets	8:1	16:1	32:1	64:1	128:1
Q3L6B	44.7257	41.4624	38.8632	35.9386	32.7791
FBIP	45.1248	41.7556	37.3076	29.7429	21.3917
Image: Field 1296 × 1296					
Wavelets	8:1	16:1	32:1	64:1	128:1
Q3L6B	39.3679	35.4619	32.0508	29.0901	26.3047
FBIP	40.8163	36.7085	31.6388	25.4171	21.5681
Image: Tower 1296 × 1296					
Wavelets	8:1	16:1	32:1	64:1	128:1
Q3L6B	44.4204	41.2039	38.2552	35.2284	32.0924
FBIP	44.7822	41.2443	36.2484	28.5168	19.6440

The original images and images reconstructed from the compressed file by Q3L6B are shown in the figures 4.5-4.10. From these pictures, we see that although the compression ratio is quite high, the image reconstructed is still acceptable by our eyes.

Case2:

Next, we use the images of size 729 × 729. We compress each image with Q3L6B as in Case 1. Since the size of images is not the product of power of 2, FBIP is not able to decompose it without special treatment. So we pad the image with black color on the right and the bottom so that the image is enlarged to the size of 1024 × 1024, thus it can be decomposed by FBIP. In order to get compressed files of the same size, the compression ratio for FBIP is twice the compression ratio for Q3L6B, since size of image file after padding is almost twice that of the original one. The padded part are thrown out before computing PSNR for FBIP. The result is shown in Table 4.2. The compression ratio listed in the table is for Q3L6B, the compression ratio for FBIP in each column is simply twice the ratio for Q3L6B. From the result, it shows that FBIP has an edge over Q3L6B. But Q3L6B is still slighly better that

Figure 4.5: Original "Lotus" 1296 × 1296 image.

Figure 4.6: Compressed "Lotus" 1296 × 1296 image at 128:1.

Figure 4.7: Original “Field” 1296 × 1296 image.

Figure 4.8: Compressed “field” 1296 × 1296 image at 128:1.

Figure 4.9: Original “Tower” 1296 $\times$ 1296 image.

Figure 4.10: Compressed “Tower” 1296 $\times$ 1296 image at 128:1.

FBIP at 128:1. In this case, the computation complexity of applying FBIP is more than that of applying Q3L6B.

Table 4.2: Coding results for different compression ratios showing Peak Signal-to-Noise Ratio(PSNR).

Image: Lotus 729 × 729					
Wavelets	8:1	16:1	32:1	64:1	128:1
Q3L6B	44.2402	40.8907	37.8211	34.7396	31.9963
FBIP	44.9269	41.4811	38.6205	35.0254	31.5536
Image: Tower 729 × 729					
Wavelets	8:1	16:1	32:1	64:1	128:1
Q3L6B	44.7341	41.4998	38.7396	36.0766	33.5088
FBIP	44.9319	41.8843	39.1086	36.1596	32.4003
Image: House 729 × 729					
Wavelets	8:1	16:1	32:1	64:1	128:1
Q3L6B	33.6120	29.2364	25.9603	23.3822	21.7214
FBIP	34.3664	29.9803	26.4572	23.7338	21.6179

From the experiment, it is hard to see that Q3L6B has any advantage over FBIP in image compression at low compression ratio. But it shows us that for images of size which is the product of power 3, Q3L6B can do better at very high compression ratio.

Figure 4.11: Original “Lotus” 729×729 image.

Figure 4.12: Compressed “Lotus” 729×729 image by Q3L6B at 128:1.

Figure 4.13: Compressed “Lotus” 729 × 729 image by FBIP at 128:1.

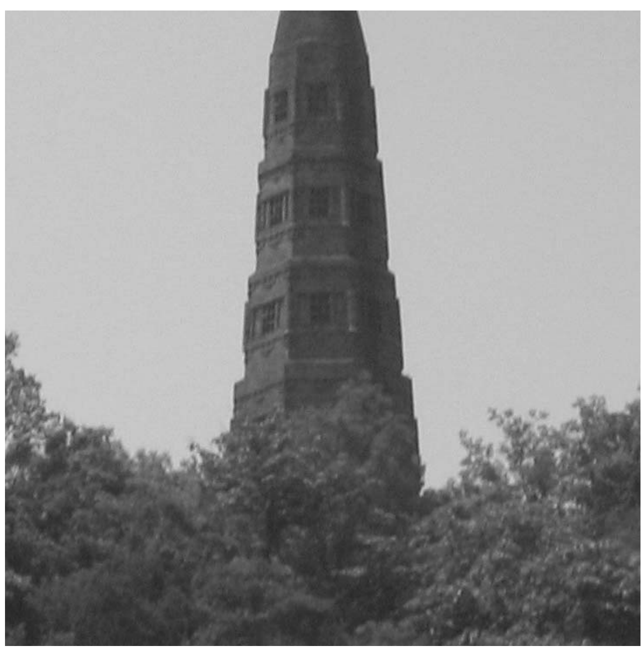

Figure 4.14: Original “Tower” 729 × 729 image.

Figure 4.15: Compressed "Tower" 729×729 image by Q3L6B at 128:1.

Figure 4.16: Compressed "Tower" 729×729 image by FBIP at 128:1.

Figure 4.17: Original “House” 729 × 729 image.

Figure 4.18: Compressed “House” 729 × 729 image by Q3L6B at 128:1.

Figure 4.19: Compressed "House" 729 × 729 image by FBIP at 128:1.

Chapter 5

CONSTRUCTION OF NONSEPERABLE BIVARIATE ORTHONORMAL WAVELETS

In this chapter, we will show how to construct nonseperable bivariate scaling functions. The scaling functions we are interested in are compactly supported on $[0,5/2]\times[0,5/2]$. Then we will present an algorithm to compute the corresponding orthonormal wavelets, which are also compactly supported.

5.1 Construction of scaling function

Let $m(x,y) = \sum\limits_{0\le i\le 5}\sum\limits_{0\le j\le 5} c_{ij}x^iy^j$ be a trigonometric polynomial with $x = e^{\sqrt{-1}\omega_1}$ and $y = e^{\sqrt{-1}\omega_2}$, $\omega_1,\omega_2\in\mathbb{Z}$. We want to find real-valued c_{ij}'s such that $m(x,y)$ satisfies the following three properties:

$$m(1,1) = 1, \tag{5.1.1}$$

$$\begin{aligned} &|m(x,y)|^2 + |m(xe^{i2\pi/3},y)|^2 + |m(x,ye^{i2\pi/3})|^2 + |m(xe^{i2\pi/3},ye^{i2\pi/3})|^2 \\ &+ |m(xe^{i4\pi/3},y)|^2 + |m(x,ye^{i4\pi/3})|^2 + |m(xe^{i2\pi/3},ye^{i4\pi/3})|^2 \\ &+ |m(xe^{i4\pi/3},ye^{i2\pi/3})|^2 + |m(xe^{i4\pi/3},ye^{i4\pi/3})|^2 = 1, \end{aligned} \tag{5.1.2}$$

$$m(x,y) = \frac{(1+x+x^2)}{3}\frac{(1+y+y^2)}{3}\tilde{m}(x,y), \tag{5.1.3}$$

for another trigonometric polynomial $\tilde{m}(x,y)$.

Throughout this section, we only focus on $m(x,y) = \sum_{0\le j\le 5, 0\le k\le 5} c_{j,k}x^jy^k$. Here we will write $m(x,y)$ in the polyphase form first,

$$\begin{aligned} m(x,y) =& f_0(x^3,y^3) + xf_1(x^3,y^3) + yf_2(x^3,y^3) + xyf_3(x^3,y^3) + x^2f_4(x^3,y^3) \\ &+ y^2f_5(x^3,y^3) + x^2yf_6(x^3,y^3) + xy^2f_7(x^3,y^3) + x^2y^2f_8(x^3,y^3), \end{aligned}$$

where $f_l(x,y) = a_l + b_l x + c_l y + d_l xy$, for $l = 0, 1, \ldots, 8$. The polyphase form of $c_{i,j}$'s is shown below

$$[c_{ij}]_{0\le i\le 3, 0\le j\le 5} = \begin{bmatrix} a_0 & a_1 & a_4 & b_0 & b_1 & b_4 \\ a_2 & a_3 & a_6 & b_2 & b_3 & b_6 \\ a_5 & a_7 & a_8 & b_5 & b_7 & b_8 \\ c_0 & c_1 & c_4 & d_0 & d_1 & d_4 \\ c_2 & c_3 & c_6 & d_2 & d_3 & d_6 \\ c_5 & c_7 & c_8 & d_5 & d_7 & d_8 \end{bmatrix}.$$

LEMMA 5.1.1. *$m(x,y)$ satisfies equation* (5.1.2) *if and only if* $\sum_{l=0}^{8} |f_l(x,y)|^2 = \frac{1}{9}$.

Proof: To simplify the proof, we need the following identity

$$|a+b+c|^2 + |a + e^{i2\pi/3}b + e^{i4\pi/3}c|^2 + |a + e^{i4\pi/3}b + e^{i2\pi/3}c|^2 = 3(|a|^2 + |b|^2 + |c|^2).$$

It can be verified by straightforward computation. Then we compute the left hand side of equation (5.1.2) in the polyphase form. With above identity, We will see that

$$\begin{aligned}
&|m(x,y)|^2 + |m(xe^{i2\pi/3}, y)|^2 + |m(xe^{i4\pi/3}, y)|^2 \\
&= |f_0 + yf_2 + y^2 f_5 + (xf_1 + xyf_3 + xy^2 f_7) + (x^2 f_4 + x^2 y f_6 + x^2 y^2 f_8)|^2 \\
&\quad + |f_0 + yf_2 + y^2 f_5 + e^{i2\pi/3}(xf_1 + xyf_3 + xy^2 f_7) + e^{i4\pi/3}(x^2 f_4 + x^2 y f_6 + x^2 y^2 f_8)|^2 \\
&\quad + |f_0 + yf_2 + y^2 f_5 + e^{i4\pi/3}(xf_1 + xyf_3 + xy^2 f_7) + e^{i2\pi/3}(x^2 f_4 + x^2 y f_6 + x^2 y^2 f_8)|^2 \\
&= 3(|f_0 + yf_2 + y^2 f_5|^2 + |xf_1 + xyf_3 + xy^2 f_7|^2 + |x^2 f_4 + x^2 y f_6 + x^2 y^2 f_8|^2),
\end{aligned}$$

where f_l represents $f_l(x^3, y^3)$ above for $l = 1, 2, \ldots, 8$. Similarily,

$$\begin{aligned}
&|m(x, ye^{i2\pi/3})|^2 + |m(xe^{i2\pi/3}, ye^{i2\pi/3})|^2 + |m(xe^{i4\pi/3}, ye^{i2\pi/3})|^2 \\
&= 3(|f_0 + ye^{i2\pi/3} f_2 + (ye^{i2\pi/3})^2 f_5|^2 + |xf_1 + x(ye^{i2\pi/3}) f_3 + x(ye^{i2\pi/3})^2 f_7|^2 \\
&\quad + |x^2 f_4 + x^2 (ye^{i2\pi/3}) f_6 + x^2 (ye^{i2\pi/3})^2 f_8|^2),
\end{aligned}$$

and

$$\begin{aligned}
&|m(x, ye^{i4\pi/3})|^2 + |m(xe^{i2\pi/3}, ye^{i4\pi/3})|^2 + |m(xe^{i4\pi/3}, ye^{i4\pi/3})|^2 \\
&= 3(|f_0 + ye^{i4\pi/3} f_2 + (ye^{i4\pi/3})^2 f_5|^2 + |xf_1 + x(ye^{i4\pi/3}) f_3 + x(ye^{i4\pi/3})^2 f_7|^2 \\
&\quad + |x^2 f_4 + x^2 (ye^{i4\pi/3}) f_6 + x^2 (ye^{i4\pi/3})^2 f_8|^2).
\end{aligned}$$

Therefore, the left hand side of equation (5.1.2) is equal to

$$
\begin{aligned}
&3(|f_0 + yf_2 + y^2 f_5|^2 + |xf_1 + xyf_3 + xy^2 f_7|^2 + |x^2 f_4 + x^2 y f_6 + x^2 y^2 f_8|^2) \\
&+ 3(|f_0 + ye^{i2\pi/3} f_2 + (ye^{i2\pi/3})^2 f_5|^2 + |xf_1 + x(ye^{i2\pi/3}) f_3 + x(ye^{i2\pi/3})^2 f_7|^2 \\
&+ |x^2 f_4 + x^2 (ye^{i2\pi/3}) f_6 + x^2 (ye^{i2\pi/3})^2 f_8|^2) + 3(|f_0 + ye^{i4\pi/3} f_2 + (ye^{i4\pi/3})^2 f_5|^2 \\
&+ |xf_1 + x(ye^{i4\pi/3}) f_3 + x(ye^{i4\pi/3})^2 f_7|^2 + |x^2 f_4 + x^2 (ye^{i4\pi/3}) f_6 + x^2 (ye^{i4\pi/3})^2 f_8|^2) \\
&= 9 \sum_{l=0}^{8} |f_l|^2
\end{aligned}
$$

by applying the identity again.

LEMMA 5.1.2. $\sum_{l=0}^{8} |f_l(x,y)|^2 = \frac{1}{9}$ *if and only if*

$$\sum_{l=0}^{8} (a_l^2 + b_l^2 + c_l^2 + d_l^2) = \frac{1}{9} \tag{5.1.4}$$

$$\sum_{l=0}^{8} (a_l b_l + c_l d_l) = 0 \tag{5.1.5}$$

$$\sum_{l=0}^{8} (a_l c_l + b_l d_l) = 0 \tag{5.1.6}$$

$$\sum_{l=0}^{8} a_l d_l = 0 \tag{5.1.7}$$

$$\sum_{l=0}^{8} b_l c_l = 0 \tag{5.1.8}$$

Proof:

$$
\begin{aligned}
\frac{1}{9} &= \sum_{l=0}^{8} |f_l(x,y)|^2 = \sum_{l=0}^{8} |a_l + b_l x + c_l y + d_l xy|^2 \\
&= \sum_{l=0}^{8} (a_l^2 + b_l^2 + c_l^2 + d_l^2 + (a_l b_l + c_l d_l)(x + \overline{x}) + (a_l c_l + b_l d_l)(y + \overline{y})(a_l d_l)(xy + \overline{xy}) \\
&\quad + (b_l c_l)(\overline{x} y + x \overline{y})).
\end{aligned}
$$

Since it has to be true for all x and y, the coefficients of $(x + \overline{x})$,$(y + \overline{y})$, $(xy + \overline{xy})$ and $(\overline{x} y + x \overline{y})$ have to be zeros.

LEMMA 5.1.3. *$m(x, y)$ satisfies equations* (5.1.1) *and* (5.1.2) *if and only if* $a_l+b_l+c_l+d_l = \frac{1}{9}$, *for* $l = 0, 1, \ldots, 8$.

Proof: Let $s_l = a_l + b_l + c_l + d_l$, for $l = 0, 1, \ldots, 8$. For equation (5.1.2), (5.1.1) implies that

$$s_0 + s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 + s_8 = 1 \tag{5.1.9}$$

and

$$\begin{aligned} &m(e^{i2\pi/2}, 1) = m(e^{i4\pi/2}, 1) = m(1, e^{i2\pi/2}) = m(1, e^{i4\pi/2}) = m(e^{i2\pi/3}, e^{i2\pi/2}) \\ &= m(e^{i2\pi/3}, e^{i4\pi/2}) = m(e^{i4\pi/3}, e^{i2\pi/2}) = m(e^{i4\pi/3}, e^{i4\pi/2}) = 0. \end{aligned} \tag{5.1.10}$$

$m(e^{i2\pi/2}, 1) = 0$ is equivalent to

$$s_0 + e^{i2\pi/3}s_1 + s_2 + e^{i2\pi/3}s_3 + e^{i4\pi/3}s_4 + s_5 + e^{i4\pi/3}s_6 + e^{i2\pi/3}s_7 + e^{i4\pi/3}s_8 = 0. \tag{5.1.11}$$

Similarily, with other elements in equation (5.1.10), we get the following equations.

$$s_0 + s_1 + e^{i2\pi/3}s_2 + e^{i2\pi/3}s_3 + s_4 + e^{i4\pi/3}s_5 + e^{i2\pi/3}s_6 + e^{i4\pi/3}s_7 + e^{i4\pi/3}s_8 = 0. \tag{5.1.12}$$

$$s_0 + e^{i2\pi/3}s_1 + e^{i2\pi/3}s_2 + e^{i4\pi/3}s_3 + e^{i4\pi/3}s_4 + e^{i4\pi/3}s_5 + s_6 + s_7 + e^{i2\pi/3}s_8 = 0. \tag{5.1.13}$$

$$s_0 + e^{i4\pi/3}s_1 + s_2 + e^{i4\pi/3}s_3 + e^{i2\pi/3}s_4 + s_5 + e^{i2\pi/3}s_6 + e^{i4\pi/3}s_7 + e^{i2\pi/3}s_8 = 0. \tag{5.1.14}$$

$$s_0 + s_1 + e^{i4\pi/3}s_2 + e^{i4\pi/3}s_3 + s_4 + e^{i2\pi/3}s_5 + e^{i4\pi/3}s_6 + e^{i2\pi/3}s_7 + e^{i2\pi/3}s_8 = 0. \tag{5.1.15}$$

$$s_0 + e^{i2\pi/3}s_1 + e^{i4\pi/3}s_2 + s_3 + e^{i4\pi/3}s_4 + e^{i2\pi/3}s_5 + e^{i2\pi/3}s_6 + e^{i4\pi/3}s_7 + s_8 = 0. \tag{5.1.16}$$

$$s_0 + e^{i4\pi/3}s_1 + e^{i2\pi/3}s_2 + s_3 + e^{i2\pi/3}s_4 + e^{i4\pi/3}s_5 + e^{i4\pi/3}s_6 + e^{i2\pi/3}s_7 + s_8 = 0. \tag{5.1.17}$$

$$s_0 + e^{i4\pi/3}s_1 + e^{i4\pi/3}s_2 + e^{i2\pi/3}s_3 + e^{i2\pi/3}s_4 + e^{i2\pi/3}s_5 + s_6 + s_7 + e^{i4\pi/3}s_8 = 0. \tag{5.1.18}$$

Adding equations (5.1.9),(5.1.11) and (5.1.14) yields

$$3s_0 + 3s_2 + 3s_5 = 1. \tag{5.1.19}$$

Similarily, (5.1.9) + (5.1.12) + (5.1.15) yields

$$3s_0 + 3s_1 + 3s_4 = 1. \tag{5.1.20}$$

(5.1.9) + (5.1.12) + (5.1.15) yields

$$3s_0 + 3s_6 + 3s_7 = 1. \tag{5.1.21}$$

(5.1.9) + (5.1.16) + (5.1.17) yields

$$s_0 + s_3 + s_8 = \frac{1}{3}. \tag{5.1.22}$$

Also, with (5.1.9),(5.1.11)-(5.1.14) yields

$$s_1 + s_3 + s_7 = s_4 + s_6 + s_8 = \frac{1}{3}, \tag{5.1.23}$$

(5.1.12)-(5.1.15) yields

$$s_2 + s_3 + s_6 = s_5 + s_7 + s_8 = \frac{1}{3}, \tag{5.1.24}$$

(5.1.13)-(5.1.18) yields

$$s_3 + s_4 + s_5 = s_1 + s_2 + s_8 = \frac{1}{3}, \tag{5.1.25}$$

(5.1.16)-(5.1.17) yields

$$s_1 + s_5 + s_6 = s_2 + s_4 + s_7 = \frac{1}{3}. \tag{5.1.26}$$

From (5.1.9),(5.1.19),(5.1.20),(5.1.21) and (5.1.22), we have

$$s_0 = \frac{1}{3} \tag{5.1.27}$$

$$s_2 + s_5 = \frac{1}{3} - s_0 = \frac{2}{9} \tag{5.1.28}$$

$$s_1 + s_4 = s_6 + s_7 = s_3 + s_8 = \frac{2}{9} \tag{5.1.29}$$

(5.1.23) + (5.1.25)+(5.1.26) with equation (5.1.24) implies $3s_1 = 3s_4$, so $s_1 = s_4 = \frac{1}{9}$. Hence,

$$s_3 + s_7 = s_6 + s_8 = \frac{2}{9} \tag{5.1.30}$$

$$s_2 + s_7 = s_5 + s_6 = \frac{2}{9} \tag{5.1.31}$$

$$s_3 + s_5 = s_2 + s_8 = \frac{2}{9}. \tag{5.1.32}$$

Add (5.1.30) and (5.1.31) together, we get

$$s_2 + s_3 + 2s_7 = 2s_6 + s_5 + s_8$$

Thus

$$\frac{1}{3} - s_6 + 2s_7 = \frac{1}{3} + 2s_6 - s_7 \Longleftrightarrow s_6 = s_7 = \frac{1}{9}$$

Similarily, $s_2 = s_3 = s_5 = s_8 = \dfrac{1}{9}$.

LEMMA 5.1.4. *$m(x, y)$ satisfies (5.1.3) if and only if*

$$a_0 + b_0 = a_1 + b_1 = a_4 + b_4 \qquad c_0 + d_0 = c_1 + d_1 = c_4 + d_4 \tag{5.1.33}$$

$$a_2 + b_2 = a_3 + b_3 = a_6 + b_6 \qquad c_2 + d_2 = c_3 + d_3 = c_6 + d_6 \tag{5.1.34}$$

$$a_5 + b_5 = a_7 + b_7 = a_8 + b_8 \qquad c_5 + d_5 = c_7 + d_7 = c_8 + d_8 \tag{5.1.35}$$

$$a_0 + c_0 = a_2 + c_2 = a_5 + c_5 \qquad b_0 + d_0 = b_2 + d_1 = b_5 + d_5 \tag{5.1.36}$$

$$a_1 + c_1 = a_3 + c_3 = a_7 + c_7 \qquad b_1 + d_1 = b_3 + d_3 = b_7 + d_7 \tag{5.1.37}$$

$$a_4 + c_4 = a_6 + c_6 = a_8 + c_8 \qquad b_4 + d_4 = b_6 + d_6 = b_8 + d_8 \tag{5.1.38}$$

(5.1.39)

Proof: (5.1.3) is equivalent to

$$m(e^{i2\pi/3}, y) = m(e^{i4\pi/3}, y) = m(x, e^{i2\pi/3}) = m(x, e^{i2\pi/3}) = 0.$$

We start to look at $m(e^{i2\pi/3}, y)) = 0$ in the polyphase form first.

$$\begin{aligned}
m(e^{i2\pi/3}, y)) = 0 \Leftrightarrow\ & f_0(1, y^3) + e^{i2\pi/3} f_1(1, y^3) + y f_2(1, y^3) + e^{i2\pi/3} y f_3(1, y^3) + e^{i4\pi/3} f_4(1, y^3) \\
& y^2 f_5(1, y^3) + e^{i4\pi/3} y f_6(1, y^3) + e^{i2\pi/3} y^2 f_7(1, y^3) + e^{i4\pi/3} y^2 f_8(1, y^3) = 0 \\
\Leftrightarrow\ & a_0 + b_0 + c_0 y^3 + d_0 y^3 + e^{i2\pi/3}(a_1 + b_1 + c_1 y^3 + d_1 y^3) \\
& + y(a_2 + b_2 + c_2 y^3 + d_2 y^3) + e^{i2\pi/3} y(a_3 + b_3 + c_3 y^3 + d_3 y^3) \\
& + e^{i4\pi/3}(a_4 + b_4 + c_4 y^3 + d_4 y^3) + y^2(a_5 + b_5 + c_5 y^3 + d_5 y^3) \\
& + e^{i4\pi/3} y(a_6 + b_6 + c_6 y^3 + d_6 y^3) + e^{i2\pi/3} y^2(a_7 + b_7 + c_7 y^3 + d_7 y^3) \\
& + e^{i4\pi/3} y^2(a_8 + b_8 + c_8 y^3 + d_8 y^3) = 0
\end{aligned}$$

Because this equation is true for any y, we get the desired result

$$a_0+b_0=a_1+b_1=a_4+b_4 \qquad c_0+d_0=c_1+d_1=c_4+d_4$$
$$a_2+b_2=a_3+b_3=a_6+b_6 \qquad c_2+d_2=c_3+d_3=c_6+d_6$$
$$a_5+b_5=a_7+b_7=a_8+b_8 \qquad c_5+d_5=c_7+d_7=c_8+d_8$$

$m(e^{i4\pi/3}, y)=0$ will show the same result, $m(x, e^{i2\pi/3})=m(x, e^{i2\pi/3})=0$ will get the rest equation.

With 5.1.1 - 5.1.3, we are able to conclude the following theorem.

THEOREM 5.1.5. *$m(x,y)$ satisfies equations* (5.1.1),(5.1.2) *and* (5.1.3) *if and only if*

$$\begin{aligned}
a_0 &= \frac{1}{36}+\frac{1}{12}\cos\gamma\cos\theta\cos t_1\cos t_2+\frac{\sqrt{3}}{36}(\cos\xi\cos\eta+\cos\beta\cos\alpha),\\
a_1 &= \frac{1}{36}+\frac{1}{12}\cos\gamma\cos\theta\cos t_1\sin t_2+\frac{\sqrt{3}}{36}(\cos\xi\cos\eta+\sin\beta\cos\alpha),\\
a_2 &= \frac{1}{36}+\frac{1}{12}\cos\gamma\cos\theta\sin t_1+\frac{\sqrt{3}}{36}(\cos\xi\sin\eta+\cos\beta\cos\alpha),\\
a_3 &= \frac{1}{36}+\frac{1}{12}\cos\gamma\sin\theta\cos t_3\cos t_4+\frac{\sqrt{3}}{36}(\cos\xi\sin\eta+\sin\beta\cos\alpha),\\
a_4 &= \frac{1}{36}+\frac{1}{12}\cos\gamma\sin\theta\cos t_3\sin t_4+\frac{\sqrt{3}}{36}(\cos\xi\cos\eta+\sin\alpha),\\
a_5 &= \frac{1}{36}+\frac{1}{12}\cos\gamma\cos\theta\sin t_3+\frac{\sqrt{3}}{36}(\sin\xi+\cos\beta\cos\alpha),\\
a_6 &= \frac{1}{36}+\frac{1}{12}\sin\gamma\cos t_5\cos t_6+\frac{\sqrt{3}}{36}(\cos\xi\sin\eta+\sin\alpha),\\
a_7 &= \frac{1}{36}+\frac{1}{12}\sin\gamma\cos t_5\sin t_6+\frac{\sqrt{3}}{36}(\sin\xi+\cos\beta\cos\alpha),\\
a_8 &= \frac{1}{36}+\frac{1}{12}\sin\gamma\sin t_5+\frac{\sqrt{3}}{36}(\sin\xi+\sin\alpha),\\
b_0 &= \frac{1}{36}-\frac{1}{12}\cos\gamma\cos\theta\cos t_1\cos t_2+\frac{\sqrt{3}}{36}(\cos\xi\cos\eta-\cos\beta\cos\alpha),\\
b_1 &= \frac{1}{36}-\frac{1}{12}\cos\gamma\cos\theta\cos t_1\sin t_2+\frac{\sqrt{3}}{36}(\cos\xi\cos\eta-\sin\beta\cos\alpha),\\
b_2 &= \frac{1}{36}-\frac{1}{12}\cos\gamma\cos\theta\sin t_1+\frac{\sqrt{3}}{36}(\cos\xi\sin\eta-\cos\beta\cos\alpha),
\end{aligned}$$

$$b_3 = \frac{1}{36} - \frac{1}{12}\cos\gamma\sin\theta\cos t_3\cos t_4 + \frac{\sqrt{3}}{36}(\cos\xi\sin\eta - \sin\beta\cos\alpha),$$
$$b_4 = \frac{1}{36} - \frac{1}{12}\cos\gamma\sin\theta\cos t_3\sin t_4 + \frac{\sqrt{3}}{36}(\cos\xi\cos\eta - \sin\alpha),$$
$$b_5 = \frac{1}{36} - \frac{1}{12}\cos\gamma\cos\theta\sin t_3 + \frac{\sqrt{3}}{36}(\sin\xi - \cos\beta\cos\alpha),$$
$$b_6 = \frac{1}{36} - \frac{1}{12}\sin\gamma\cos t_5\cos t_6 + \frac{\sqrt{3}}{36}(\cos\xi\sin\eta - \sin\alpha),$$
$$b_7 = \frac{1}{36} - \frac{1}{12}\sin\gamma\cos t_5\sin t_6 + \frac{\sqrt{3}}{36}(\sin\xi - \cos\beta\cos\alpha),$$
$$b_8 = \frac{1}{36} - \frac{1}{12}\sin\gamma\sin t_5 + \frac{\sqrt{3}}{36}(\sin\xi - \sin\alpha),$$
$$c_0 = \frac{1}{36} - \frac{1}{12}\cos\gamma\cos\theta\cos t_1\cos t_2 + \frac{\sqrt{3}}{36}(-\cos\xi\cos\eta + \cos\beta\cos\alpha),$$
$$c_1 = \frac{1}{36} - \frac{1}{12}\cos\gamma\cos\theta\cos t_1\sin t_2 + \frac{\sqrt{3}}{36}(-\cos\xi\cos\eta + \sin\beta\cos\alpha),$$
$$c_2 = \frac{1}{36} - \frac{1}{12}\cos\gamma\cos\theta\sin t_1 + \frac{\sqrt{3}}{36}(-\cos\xi\sin\eta + \cos\beta\cos\alpha),$$
$$c_3 = \frac{1}{36} - \frac{1}{12}\cos\gamma\sin\theta\cos t_3\cos t_4 + \frac{\sqrt{3}}{36}(-\cos\xi\sin\eta + \sin\beta\cos\alpha),$$
$$c_4 = \frac{1}{36} - \frac{1}{12}\cos\gamma\sin\theta\cos t_3\sin t_4 + \frac{\sqrt{3}}{36}(-\cos\xi\cos\eta + \sin\alpha),$$
$$c_5 = \frac{1}{36} - \frac{1}{12}\cos\gamma\cos\theta\sin t_3 + \frac{\sqrt{3}}{36}(-\sin\xi + \cos\beta\cos\alpha),$$
$$c_6 = \frac{1}{36} - \frac{1}{12}\sin\gamma\cos t_5\cos t_6 + \frac{\sqrt{3}}{36}(-\cos\xi\sin\eta + \sin\alpha),$$
$$c_7 = \frac{1}{36} - \frac{1}{12}\sin\gamma\cos t_5\sin t_6 + \frac{\sqrt{3}}{36}(-\sin\xi + \cos\beta\cos\alpha),$$
$$c_8 = \frac{1}{36} - \frac{1}{12}\sin\gamma\sin t_5 + \frac{\sqrt{3}}{36}(-\sin\xi + \sin\alpha),$$
$$d_0 = \frac{1}{36} + \frac{1}{12}\cos\gamma\cos\theta\cos t_1\cos t_2 - \frac{\sqrt{3}}{36}(\cos\xi\cos\eta + \cos\beta\cos\alpha),$$
$$d_1 = \frac{1}{36} + \frac{1}{12}\cos\gamma\cos\theta\cos t_1\sin t_2 - \frac{\sqrt{3}}{36}(\cos\xi\cos\eta + \sin\beta\cos\alpha),$$
$$d_2 = \frac{1}{36} + \frac{1}{12}\cos\gamma\cos\theta\sin t_1 - \frac{\sqrt{3}}{36}(\cos\xi\sin\eta + \cos\beta\cos\alpha),$$
$$d_3 = \frac{1}{36} + \frac{1}{12}\cos\gamma\sin\theta\cos t_3\cos t_4 - \frac{\sqrt{3}}{36}(\cos\xi\sin\eta + \sin\beta\cos\alpha),$$
$$d_4 = \frac{1}{36} + \frac{1}{12}\cos\gamma\sin\theta\cos t_3\sin t_4 - \frac{\sqrt{3}}{36}(\cos\xi\cos\eta + \sin\alpha),$$

$$d_5 = \frac{1}{36} + \frac{1}{12}\cos\gamma\cos\theta\sin t_3 - \frac{\sqrt{3}}{36}(\sin\xi + \cos\beta\cos\alpha),$$
$$d_6 = \frac{1}{36} + \frac{1}{12}\sin\gamma\cos t_5\cos t_6 - \frac{\sqrt{3}}{36}(\cos\xi\sin\eta + \sin\alpha),$$
$$d_7 = \frac{1}{36} + \frac{1}{12}\sin\gamma\cos t_5\sin t_6 - \frac{\sqrt{3}}{36}(\sin\xi + \cos\beta\cos\alpha),$$
$$d_8 = \frac{1}{36} + \frac{1}{12}\sin\gamma\sin t_5 - \frac{\sqrt{3}}{36}(\sin\xi + \sin\alpha)$$

where the parameters γ,θ,α,β,ξ,η,t_1,t_2,t_3,t_4,t_5 *and* t_6 *have to satisfy the following equations:*

$$\begin{aligned} & cos\gamma\cos\theta\cos t_1\sin t_2 + \cos\gamma\sin\theta\cos t_3\cos t_4 + \cos\gamma\sin\theta\cos t_3\sin t_4 \\ & + cos\gamma\cos\theta\cos t_1\cos t_2 + \sin\xi\cos\beta\cos\alpha + \cos\gamma\cos\theta\sin t_1 + \sin\gamma\sin t_5 \\ & + \cos\gamma\cos\theta\sin t_3 + \sin\gamma\cos t_5\cos t_6 + \sin\gamma\cos t_5\sin t_6 + \cos\xi\cos\eta\cos\beta\cos\alpha \\ & + \cos\xi\cos\eta\sin\beta\cos\alpha + \cos\xi\sin\eta\cos\beta\cos\alpha + \cos\xi\sin\eta\sin\beta\cos\alpha + \sin\xi\sin\alpha \\ & + \cos\xi\cos\eta\sin\alpha + \cos\xi\sin\eta\sin\alpha + \sin\xi\sin\beta\cos\alpha - \cos\beta\sin\beta\cos^2\alpha) \\ & + \frac{\sqrt{3}}{3}(\sin\beta\cos\alpha - \cos\beta\cos\alpha) = 0 \end{aligned} \tag{5.1.40}$$

$$\begin{aligned} & \sqrt{3}(\sin\gamma\cos t_5\sin t_6\cos\beta\cos\alpha - \sin\gamma\cos t_5\sin t_6\sin\beta\sin\alpha) - \frac{3}{2}\cos^2\gamma\sin^2 t_3 \\ & + \frac{1}{2}\cos^2\alpha + 3\cos^2\gamma\cos^2\theta\sin^2 t_3 = 0 \end{aligned} \tag{5.1.41}$$

Proof: Lemma5.1.1-5.1.4 provide that we need to solve the following nonlinear system,

$$a_l + b_l + c_l + d_l = \frac{1}{9}, \quad \text{for } l = 0, 1, \ldots, 8 \tag{5.1.42}$$

$$\sum_{l=0}^{8}(a_l^2 + b_l^2 + c_l^2 + d_l^2) = \frac{1}{9} \tag{5.1.43}$$

$$\sum_{l=0}^{8}(a_l b_l + c_l d_l) = 0 \tag{5.1.44}$$

$$\sum_{l=0}^{8}(a_l c_l + b_l d_l) = 0 \tag{5.1.45}$$

$$\sum_{l=0}^{8} a_l d_l = 0 \tag{5.1.46}$$

$$\sum_{l=0}^{8} b_l c_l = 0 \tag{5.1.47}$$

By (5.1.43)-(5.1.47),

$$\sum_{l=0}^{8}[(a_l-b_l)^2+(c_l-d_l)^2=\frac{1}{9} \text{ and } \sum_{l=0}^{8}(a_l-c_l)(b_l-d_l)=0.$$

Hence

$$\sum_{l=0}^{8}(a_l-b_l+c_l-d_l)^2=\frac{1}{9}.$$

By (5.1.42), it turns that

$$\sum_{l=0}^{8}[2(a_l+c_l)-\frac{1}{9}]^2=\frac{1}{9}.$$

Therefore, Lemma5.1.4 implies

$$[2(a_0+c_0)-\frac{1}{9}]^2+[2(a_1+c_1)-\frac{1}{9}]^2+[2(a_4+c_4)-\frac{1}{9}]^2=\frac{1}{27}$$

$$\begin{aligned}
c_i &= \frac{1}{18}+\frac{\sqrt{3}}{18}\cos\beta\cos\alpha-a_i \text{ for } i=0,2,5,\\
c_i &= \frac{1}{18}+\frac{\sqrt{3}}{18}\sin\beta\cos\alpha-a_i \text{ for } i=1,3,7,\\
c_i &= \frac{1}{18}+\frac{\sqrt{3}}{18}\sin\alpha-a_i \text{ for } i=4,6,8.
\end{aligned}$$

Similarily, we can solve b_i, for $i=0,1,\ldots,8$ as

$$\begin{aligned}
b_i &= \frac{1}{18}+\frac{\sqrt{3}}{18}\cos\xi\cos\eta-a_i \text{ for } i=0,2,5,\\
b_i &= \frac{1}{18}+\frac{\sqrt{3}}{18}\sin\xi\cos\eta-a_i \text{ for } i=1,3,7,\\
b_i &= \frac{1}{18}+\frac{\sqrt{3}}{18}\sin\eta-a_i \text{ for } i=4,6,8.
\end{aligned}$$

Using (5.1.42) again, $d_i,\ i=0,1,\ldots,8$ can be defined as

$$\begin{aligned}
d_0 &= -\frac{\sqrt{3}}{18}(\cos\xi\cos\eta+\cos\beta\cos\alpha)+a_0,\\
d_1 &= -\frac{\sqrt{3}}{18}(\cos\xi\cos\eta+\sin\beta\cos\alpha)+a_1,
\end{aligned}$$

$$d_2 = -\frac{\sqrt{3}}{18}(\cos\xi\sin\eta + \cos\beta\cos\alpha) + a_2,$$
$$d_3 = -\frac{\sqrt{3}}{18}(\cos\xi\sin\eta + \sin\beta\cos\alpha) + a_3,$$
$$d_4 = -\frac{\sqrt{3}}{18}(\cos\xi\cos\eta + \sin\alpha) + a_4,$$
$$d_5 = -\frac{\sqrt{3}}{18}(\sin\xi + \cos\beta\cos\alpha) + a_5,$$
$$d_6 = -\frac{\sqrt{3}}{18}(\cos\xi\sin\eta + \sin\alpha) + a_6,$$
$$d_7 = -\frac{\sqrt{3}}{18}(\sin\xi + \sin\beta\cos\alpha) + a_7,$$
$$d_8 = -\frac{\sqrt{3}}{18}(\sin\xi + \sin\alpha) + a_8.$$

Now we only need to solve $a_i,\ i = 0, 1, \ldots, 8$. Let

$$\widetilde{a_0} = a_0 - \frac{3}{36}(\cos\xi\cos\eta + \cos\beta\cos\alpha),$$
$$\widetilde{a_1} = a_1 - \frac{3}{36}(\cos\xi\cos\eta + \sin\beta\cos\alpha),$$
$$\widetilde{a_2} = a_2 - \frac{3}{36}(\cos\xi\sin\eta + \cos\beta\cos\alpha),$$
$$\widetilde{a_3} = a_3 - \frac{3}{36}(\cos\xi\sin\eta + \sin\beta\cos\alpha),$$
$$\widetilde{a_4} = a_4 - \frac{3}{36}(\cos\xi\cos\eta + \sin\alpha),$$
$$\widetilde{a_5} = a_5 - \frac{3}{36}(\sin\xi + \cos\beta\cos\alpha),$$
$$\widetilde{a_6} = a_6 - \frac{3}{36}(\cos\xi\sin\eta + \sin\alpha),$$
$$\widetilde{a_7} = a_7 - \frac{3}{36}(\sin\xi + \cos\beta\cos\alpha),$$
$$\widetilde{a_8} = a_8 - \frac{3}{36}(\sin\xi + \sin\alpha).$$

Using d_i's defined above, (5.1.46) gives

$$\sum_{l=0}^{8} \widetilde{a_l}^2 = \frac{1}{432}\{6 + 2[\sin\xi\cos\beta\cos\alpha + \cos\xi\sin\eta\sin\alpha + \sin\xi\sin\beta\cos\alpha + \cos\xi\cos\eta\sin\alpha$$
$$+ \cos\xi\cos\eta\cos\beta\cos\alpha + \cos\xi\cos\eta\sin\beta\cos\alpha + \cos\xi\sin\eta\cos\beta\cos\alpha$$
$$+ \cos\xi\sin\eta\sin\beta\cos\alpha + \sin\xi\sin\alpha]\},$$

with (5.1.47), it also yields

$$\sum_{l=0}^{8}\widetilde{a_l} = \frac{1}{24}\{6+2[\sin\xi\cos\beta\cos\alpha+\cos\xi\sin\eta\sin\alpha+\sin\xi\sin\beta\cos\alpha+\cos\xi\cos\eta\sin\alpha$$

$$+\cos\xi\cos\eta\cos\beta\cos\alpha+\cos\xi\cos\eta\sin\beta\cos\alpha+\cos\xi\sin\eta\cos\beta\cos\alpha$$

$$+\cos\xi\sin\eta\sin\beta\cos\alpha+\sin\xi\sin\alpha]\}.$$

It's easy to see that

$$\sum_{l=0}^{8}\widetilde{a_l} = \frac{1}{18}\sum_{l=0}^{8}\widetilde{a_l}.$$

It implies

$$\sum_{l=0}^{8}(\widetilde{a_l}-\frac{1}{36})^2 = (\frac{1}{12})^2.$$

Therefore,

$$\begin{aligned}
\widetilde{a_0} &= \frac{1}{36}+\frac{1}{12}\cos\gamma\cos\theta\cos t_1\cos t_2,\\
\widetilde{a_1} &= \frac{1}{36}+\frac{1}{12}\cos\gamma\cos\theta\cos t_1\sin t_2,\\
\widetilde{a_2} &= \frac{1}{36}+\frac{1}{12}\cos\gamma\cos\theta\sin t_1,\\
\widetilde{a_3} &= \frac{1}{36}+\frac{1}{12}\cos\gamma\sin\theta\cos t_3\cos t_4,\\
\widetilde{a_4} &= \frac{1}{36}+\frac{1}{12}\cos\gamma\sin\theta\cos t_3\sin t_4,\\
\widetilde{a_5} &= \frac{1}{36}+\frac{1}{12}\cos\gamma\sin\theta\sin t_3,\\
\widetilde{a_6} &= \frac{1}{36}+\frac{1}{12}\sin\gamma\cos t_5\cos t_6,\\
\widetilde{a_7} &= \frac{1}{36}+\frac{1}{12}\sin\gamma\cos t_5\sin t_6,\\
\widetilde{a_8} &= \frac{1}{36}+\frac{1}{12}\sin\gamma\sin t_5.
\end{aligned}$$

Now we are able to define $a_i,\ b_i,\ c_i,\ d_i$'s. After plugging the coefficients back into (5.1.42)-(5.1.47), we find thatγ,θ,α,β,ξ,η,t_1,t_2,t_3,t_4,t_5 and t_6 have to satisfy

$$
\begin{aligned}
&-\frac{1}{216}(cos\gamma\cos\theta\cos t_1\sin t_2+\cos\gamma\sin\theta\cos t_3\cos t_4+\cos\gamma\sin\theta\cos t_3\sin t_4\\
&+cos\gamma\cos\theta\cos t_1\cos t_2+\sin\xi\cos\beta\cos\alpha+\cos\gamma\cos\theta\sin t_1+\sin\gamma\sin t_5\\
&+\cos\gamma\cos\theta\sin t_3+\sin\gamma\cos t_5\cos t_6+\sin\gamma\cos t_5\sin t_6+\cos\xi\cos\eta\cos\beta\cos\alpha\\
&+\cos\xi\cos\eta\sin\beta\cos\alpha+\cos\xi\sin\eta\cos\beta\cos\alpha+\cos\xi\sin\eta\sin\beta\cos\alpha+\sin\xi\sin\alpha\\
&+\cos\xi\cos\eta\sin\alpha+\cos\xi\sin\eta\sin\alpha+\sin\xi\sin\beta\cos\alpha-\cos\beta\sin\beta\cos^2\alpha)\\
&+\frac{3}{216}\sqrt{3}(\sin\gamma\cos t_5\sin t_6\cos\beta\cos\alpha-\sin\gamma\cos t_5\sin t_6\sin\beta\sin\alpha)-\frac{3}{2}\cos^2\gamma\sin^2 t_3\\
&-\frac{1}{532}\cos^2\alpha-\frac{1}{72}\cos^2\gamma\cos^2\theta\sin^2 t_3+\frac{\sqrt{3}}{648}(\sin\beta\cos\alpha-\cos\beta\cos\alpha)=0
\end{aligned}
$$

and

$$
\begin{aligned}
&\sqrt{3}(\sin\gamma\cos t_5\sin t_6\cos\beta\cos\alpha-\sin\gamma\cos t_5\sin t_6\sin\beta\sin\alpha)-\frac{3}{2}\cos^2\gamma\sin^2 t_3\\
&+\frac{1}{2}\cos^2\alpha+3\cos^2\gamma\cos^2\theta\sin^2 t_3=0
\end{aligned}
$$

After simplifying above two equations, we derive the constraint equations (5.1.40) and (5.1.41).

5.2 Construction of wavelets

Once the scaling function is determined, we can use the similar methoad as in [15] to construct bivariate compactly supported orthonormal wavelets.

We start with the polyphase components of $m(x,y)$.

Lemma 5.2.1. *$m(x,y)$ satisfies (5.1.2) if and only if its polyphase components f_0,f_1,..., f_8 satisfy* $\sum_{l=0}^{8}|f_l(x,y)|^2=\frac{1}{9}$.

Write $[f_0,f_1,\dots,f_8]^T=\mathbf{a}+x\mathbf{b}+y\mathbf{c}+xy\mathbf{d}$ with $a=[a_0,a_1,\dots,a_8]^T$ and etc. Hence it is easy to see that $L=[\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}]$ is a 9×4 matrix, and we define matrix L_1 as the 4×4 block on the top of this matrix. Then there exists an orthonormal matrix H_1(by Householder transform) such that H_1L_1 is a lower triangle matrix. Let H be a 9×9 matrix, such that

$$H = \begin{bmatrix} H_1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

i.e. the top left 4×4 block is H_1, 1's are on the diagonal line and zeros otherwise. Clearly H is also an orthonormal matrix. Then

$$\begin{bmatrix} \widetilde{f_0} \\ \widetilde{f_1} \\ \widetilde{f_2} \\ \widetilde{f_3} \\ \widetilde{f_4} \\ \widetilde{f_5} \\ \widetilde{f_6} \\ \widetilde{f_7} \\ \widetilde{f_8} \end{bmatrix} = H \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix} = HL \begin{bmatrix} 1 \\ x \\ y \\ xy \end{bmatrix} = \begin{bmatrix} \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \\ \times & \times & \times & 0 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ xy \end{bmatrix}$$

Since H is orthonormal, $\sum_{l=0}^{8} |\widetilde{f_l}|^2 = \sum_{l=0}^{8} |f_l|^2 = \frac{1}{9}$. If $|\widetilde{f_0}| = \frac{1}{3}$, then $\widetilde{f_l} = 0$, for $l = 1, 2, \ldots, 8$. Otherwise, Let

$$v = [\widetilde{f_0}, \widetilde{f_1}, \ldots, \widetilde{f_8}]^T - \frac{1}{3}[1, 0, 0, \ldots, 0]^T \text{ and } H(v) = I_9 - \frac{2vv^*}{v^*v}$$

be a Householder matrix such that

$$H(v) \begin{bmatrix} \widetilde{f_0} \\ \widetilde{f_1} \\ \widetilde{f_2} \\ \widetilde{f_3} \\ \widetilde{f_4} \\ \widetilde{f_5} \\ \widetilde{f_6} \\ \widetilde{f_7} \\ \widetilde{f_8} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If $|\widetilde{f_0}| = \frac{1}{3}$, then simply let $H(v)$ be a 9×9 identity matrix. Therefore

$$[f_0, f_1, \ldots, f_8] = [\frac{1}{3}, 0, 0, \ldots, 0]\overline{H(v)}H.$$

By choosing $M(x,y) = \frac{1}{3}\overline{H(v)}H$, we have

$$M(x,y)M^*(x,y) = \frac{1}{9}I_9$$

Now we can define the polynomials m_l, $l = 0, 1, \ldots, 8$ with $m_0(x,y) = m(\omega_1, \omega_2)$ as follows:

$$\begin{aligned}&[m_l(\omega_1,\omega_2), m_l(\omega_1+\sigma_1,\omega_2), m_l(\omega_1,\omega_2+\sigma_1), m_l(\omega_1+\sigma_1,\omega_2+\sigma_1), m_l(\omega_1+\sigma_2,\omega_2),\\ &m_l(\omega_1,\omega_2+\sigma_2), m_l(\omega_1+\sigma_2,\omega_2+\sigma_1), m_l(\omega_1+\sigma_1,\omega_2+\sigma_2), m_l(\omega_1+\sigma_2,\omega_2+\sigma_2)]_{l=0,1,\ldots,8}\\ &= M(e^{3i\omega_1}, e^{3i\omega_2})DT \quad (5.2.1)\end{aligned}$$

where $\sigma_1 = \frac{2\pi}{3}$,$\sigma_2 = \frac{4\pi}{3}$ and matrices D,T defined as

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{i\sigma_1} & 1 & e^{i\sigma_1} & e^{i\sigma_2} & 1 & e^{i\sigma_2} & e^{i\sigma_1} & e^{i\sigma_2} \\ 1 & 1 & e^{i\sigma_1} & e^{i\sigma_1} & 1 & e^{i\sigma_2} & e^{i\sigma_1} & e^{i\sigma_2} & e^{i\sigma_2} \\ 1 & e^{i\sigma_1} & e^{i\sigma_1} & e^{i\sigma_2} & e^{i\sigma_2} & e^{i\sigma_2} & 1 & 1 & e^{i\sigma_1} \\ 1 & e^{i\sigma_2} & 1 & e^{i\sigma_2} & e^{i\sigma_1} & 1 & e^{i\sigma_1} & e^{i\sigma_2} & e^{i\sigma_1} \\ 1 & 1 & e^{i\sigma_2} & e^{i\sigma_2} & 1 & e^{i\sigma_1} & e^{i\sigma_2} & e^{i\sigma_1} & e^{i\sigma_1} \\ 1 & e^{i\sigma_2} & e^{i\sigma_1} & 1 & e^{i\sigma_1} & e^{i\sigma_2} & e^{i\sigma_2} & e^{i\sigma_1} & 1 \\ 1 & e^{i\sigma_1} & e^{i\sigma_2} & 1 & e^{i\sigma_2} & e^{i\sigma_1} & e^{i\sigma_1} & e^{i\sigma_2} & 1 \\ 1 & e^{i\sigma_2} & e^{i\sigma_2} & e^{i\sigma_1} & e^{i\sigma_1} & e^{i\sigma_1} & 1 & 1 & e^{i\sigma_2} \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\omega_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\omega_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i(\omega_1+\omega_2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2i\omega_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{2i\omega_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i(2\omega_1+\omega_2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i(\omega_1+2\omega_2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{2i(\omega_1+\omega_2)} \end{bmatrix}$$

THEOREM 5.2.2. *Let m_l's be the trignometric polynomials defined above. Let ϕ be the scaling function defined as* $\frac{1}{2\pi}\prod_{n=1}^{\infty} m_0(3^{-n}\omega_1, 3^{-n}\omega_2)$. *Then ψ_l are the wavelets defined by*

$$\tilde{\psi}_l(\omega_1,\omega_2) = m_l(\frac{\omega_1}{3}, \frac{\omega_2}{3})\hat{\phi}(\frac{\omega_1}{3}, \frac{\omega_2}{3}), l = 1, 2, \ldots, 8,$$

in $L^2(\mathbb{R}^2)$

Chapter 6

CONSTRUCTION OF COMPACTLY SUPPORTED TIGHT WAVELET FRAMES

In this chapter, we aim to construct multivariate compactly supported tight frames from any given refinable space.

6.1 Brief introduction

Definition 6.1.1. *$\phi(x)$ is called* refinable function *with respect to mask h if $\phi(x) = \sum_{l=0}^{N} h_l \phi(2x-l)$*

Definition 6.1.2. *Let $s \geq 1$ be an integer and $L_2(\mathbb{R}^s)$ be the usual Hilbert space of $\mathbb{R}^s$. $\{\psi^i\}$, $i = 1, 2, \ldots, n$ in $L_2(\mathbb{R}^s)$ are called* tight wavelet frames*, if*

$$\|f\|^2 = \sum_{l=1}^{n} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^s} |\langle f, \psi_{j,k}^l \rangle|^2$$

where $\psi_{j,k}^l = 2^{j/2}\psi^l(2^j \ \cdot - k)$, for $l = 1, 2, \ldots, n$.

We start with a refinable fucntion $\phi \in L_2(\mathbb{R}^s)$ which generates a multiresolution approximation of $L_2(\mathbb{R}^2)$, that is $\hat{\phi}(2\omega) = P(\omega)\hat{\phi}(\omega)$ where $P(\omega)$ is the associated trigonometric polynomial. We assume that

$$\sum_{k=1}^{2^s} |P(\omega + \pi n_k)|^2 \leq 1,$$

where n_k, $k = 1, 2, \ldots, 2^s$ denote all the vertices of the s dimensional cube $[0, 1]^s$. There exist a finite number of Laurent polynomials $\widetilde{P}_m, m = 1, \cdots, m_0$ such that

$$\sum_{k=1}^{2^s} \left(|P(\omega + \pi n_k)|^2 + \sum_{m=1}^{m_0} |\widetilde{P}_m(\omega + \pi n_k)|^2 \right) = 1. \tag{6.1.1}$$

by generalize Dritschel theorem to multivariate setting [10]. Then, we shall construct a refinable vector with trigonometric polynomial entries based on $P(\omega)$ and $\widetilde{P}(\omega)$.

6.2 Tight frames in univariate case

Let $\phi \in L_2(\mathbb{R})$ be a refinable function whose Fourier transform satisfies

$$\widehat{\phi}(2\omega) = P(\omega)\widehat{\phi}(\omega).$$

Let p_1 and p_2 be the polyphase components of P, i.e.,

$$P(\omega) = p_1(2\omega) + zp_2(2\omega).$$

with $z = e^{i\omega}$. Assume that $|P(\omega)|^2 + |P(\omega+\pi)|^2 \leq 1$, in other words, $|p_1(\omega)|^2 + |p_2(\omega)|^2 \leq \frac{1}{2}$.

By the Riesz Lemma below, there exists a Laurent polynomial $\tilde{p}$ such that

$$|p_1(2\omega)|^2 + |p_2(2\omega)|^2 + |\tilde{p}(2\omega)|^2 = 1/2.$$

Lemma 6.2.1. *(Reisz Lemma)Let A be a positive trigonometric polynomial invariant under the substitution $\omega \to -\omega$; A is necessarily of the form*

$$A(\omega) = \sum_{k=0}^{M} a_k \cos k\omega, \quad a_k \in \mathbb{R}.$$

Then there exists a trigonometric polynomial B of order M, i.e.

$$B(\omega) = \sum_{k=0}^{M} b_k e^{ik\omega}, \quad b_k \in \mathbb{R}.$$

such that $|B(\omega)|^2 = A(\omega)$.

Let $p_3 = \tilde{p}/r_1$ and $p_4 = \tilde{p}/r_2$, where $\frac{1}{r_1^2} + \frac{1}{r_2^2} = 1$. For convenience, we can let $r_1 = r_2 = \sqrt{2}$. Rewrite $p_i(\omega)$ as $p_i(\omega) = \sum_{j=0}^{m} c_{i,j} z^j$ with $m < \infty$ for $i = 1, 2, 3$. Then we have

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = [c_{ij}] \begin{bmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^m \end{bmatrix}$$

Find a unitary matrix L such that

$$L\,[c_{ij}] = \begin{bmatrix} \tilde{c}_{11} & 0 & \cdots & 0 & 0 & 0 \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2,m-2} & 0 & 0 \\ \tilde{c}_{31} & \tilde{c}_{32} & \cdots & \tilde{c}_{3,m-2} & \tilde{c}_{3,m-1} & 0 \\ \tilde{c}_{41} & \tilde{c}_{42} & \cdots & \tilde{c}_{4,m-2} & \tilde{c}_{4,m-1} & \tilde{c}_{4,m} \end{bmatrix}$$

Note that

$$(L \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix})^T L \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \frac{1}{2}.$$

We can find a unitary Householder matrix H, $H = I_2 - \dfrac{2vv^*}{v^*v}$ with I_2 being the identity matrix of size 2×2 and

$$v = L \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} - \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

such that

$$HL \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

That is,

$$[p_1, p_2, p_3, p_4] = [1/\sqrt{2}, 0, 0, 0]\overline{H}L$$

Let matrix $M(x, y) = \overline{H}L$, and

$$N(\omega) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ z & 0 & -z & 0 \\ 0 & 1 & 0 & 1 \\ 0 & z & 0 & -z \end{bmatrix}$$

with $z = e^{i\omega}$. It is easy to see that $N(\omega)$ is an unitary matrix.

Let us denote

$$M(2\omega)N(\omega) = \begin{bmatrix} \mathcal{P}(\omega) & \mathcal{P}(\omega+\pi) \\ \mathcal{Q}(\omega) & \mathcal{Q}(\omega+\pi) \end{bmatrix} \tag{6.2.1}$$

with

$$\mathcal{P}(\omega) = \begin{bmatrix} P(\omega) & P_1(\omega) \\ P_2(\omega) & P_3(\omega) \end{bmatrix} \text{ and } \mathcal{Q}(\omega) = \begin{bmatrix} Q_0(\omega) & Q_1(\omega) \\ Q_2(\omega) & Q_3(\omega) \end{bmatrix}. \tag{6.2.2}$$

Here P is the mask associated with the given refinable function ϕ while $P_i, i = 1, 2, 3$ as well as $Q_i, i = 1, 2, 3, 4$ are new masks acquired by using $M(2\omega)N(\omega)$. Since $M(2\omega)N(\omega)$ is unitary, we have

$$\mathcal{P}(\omega)\mathcal{P}^*(\omega) + \mathcal{P}(\omega+\pi)\mathcal{P}^*(\omega+\pi) = I_2 \tag{6.2.3}$$

and

$$\mathcal{P}(\omega)\mathcal{Q}^*(\omega) + \mathcal{P}(\omega+\pi)\mathcal{Q}^*(\omega+\pi) = 0_2 \tag{6.2.4}$$

where 0_2 denotes of zero matrix of size 2×2.

Since $P(0) = 1$, $P_1(0) = 0$ and $P_2(0) = 0$ due to the fact that each row or column of the unitary matrix $M(2\omega)H$ has the norm 1. By the same reason, $|P_3(0)| < 1$. The following lemma proved by C. Cabrelli, C. Heil and U. Molter [3] will ensure the infinite product

$$\prod_{j=1}^{\infty} \mathcal{P}(\omega/2^j)$$

converges and function vector $\hat{\Phi}(\omega)$ defined by

$$\widehat{\Phi}(2\omega) = \mathcal{P}(\omega)\hat{\Phi}(\omega) \text{ hence } \widehat{\Phi}(\omega) = \prod_{j=1}^{\infty} \mathcal{P}(\omega/2^j) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{6.2.5}$$

is refinable.

Lemma 6.2.2. . *The infinite matrix product* $\prod_{j=1}^{\infty} \mathcal{P}(\omega/2^j)$ *converges uniformly on any compact set to a continuous matrix-valued function if and only if* $s \times s$ *matrix* $\mathcal{P}(0)$ *is similar to* $\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix}$, *where* I_r *is the identity matrix of size* $r \times r$ *with* $r < s$, *and the magnitudes of all eigenvalues of* J *are strictly less than* 1.

And a multiscale generization of Mallet's Lemma [22] will ensure the refinable function vector is in $(L_2(\mathbb{R}))^2$, its proof can be found in [13]. we include it here for the convenience.

Lemma 6.2.3. *If the mask* $\mathcal{P}(\omega)$ *with trigonometric polynomial entries satisfying* (6.2.3) *and the condition in Lemma 6.2.2, then* $\prod_{j=1}^{\infty} \mathcal{P}(\omega/2^j)$ *converges in* $L^2(\mathbb{R})^{2\times 2}$, *i.e. each entry converges in* $L^2(\mathbb{R})$.

Proof: By Lemma 6.2.2, $\prod_{j=1}^{\infty} \mathcal{P}(\omega/2^j)$ is pointwisely converging to, say $\mathcal{P}_0(\omega)$. Define

$$\mathcal{P}_k(\omega) = \prod_{j=1}^{k} \mathcal{P}(\omega/2^j)$$

$$\mathcal{M}_k(\omega) = \mathcal{P}_k(\omega)\chi_{[-\pi,\pi]}(\omega/2^k).$$

Then $\mathcal{M}_k(\omega) \longrightarrow \mathcal{P}_\infty(\omega)$ pointwise a.e.. We write

$$\mathcal{M}_k(\omega) = [m_{k;ij}(\omega)]_{1\le i,j\le 2},$$

$$\mathcal{P}_k(\omega) = [p_{k;ij}(\omega)]_{1\le i,j\le 2}.$$

Notice that

$$\mathcal{P}_k(\omega) = \mathcal{P}_{k-1}(\omega)\mathcal{P}(\omega/2^k).$$

It is easy to see that $\mathcal{P}_k(\omega)$ is a $2^{k+1}\pi \times 2^{k+1}\pi$ periodic matrix-valued function. Denote

$$\mathbf{p}_k(\omega) = [p_{k;11}(\omega),\, p_{k;12}(\omega)] = [0,1]\mathcal{P}_k(\omega).$$

Then

$$\begin{aligned}\int_{\mathbb{R}} \sum_{j=1}^{2} |m_{k;1j}(\omega)|^2 d\omega &= \int_{[-2^k\pi,2^k\pi]} \sum_{j=1}^{2} |p_{k;1j}(\omega)|^2 d\omega \\ &= \int_{[0,2^{k+1}\pi]} \mathbf{p}_k(\omega)\cdot\mathbf{p}_k(\omega)^* d\omega \\ &= \int_{[0,2^{k+1}\pi]} \mathbf{p}_{k-1}(\omega)\mathcal{P}(\omega/2^k)\mathcal{P}^*(\omega/2^k)\mathbf{p}_{k-1}(\omega)^* d\omega\end{aligned}$$

and hence,

$$\begin{aligned}&= \int_{[0,2^k\pi]} \mathbf{p}_{k-1}(\omega)\left(\mathcal{P}(\omega/2^k)\mathcal{P}^*(\omega/2^k) + \mathcal{P}(\omega/2^k+\pi)\mathcal{P}^*(\omega/2^k+\pi)\right)\mathbf{p}_{k-1}(\omega)^* d\omega \\ &= \int_{[0,2^k\pi]} \mathbf{p}_{k-1}\mathbf{p}_{k-1}(\omega)^* d\omega = \int_{\mathbb{R}} \sum_{j=1}^{2} |m_{k-1;1j}(\omega)|^2 d\omega.\end{aligned}$$

Consequently, by Fatou's lemma,

$$\int_{\mathbb{R}} \lim_{k\to\infty} \sum_{j=1}^{2} |m_{k;1j}(\omega)|^2 d\omega \le \lim_{k\to\infty} \sup \int_{\mathbb{R}} \sum_{j=1}^{2} |m_{k;1j}(\omega)|^2 d\omega = 2\pi.$$

Similarly we can show $\int_{\mathbb{R}} \lim_{k\to\infty} \sum_{j=1}^{2} |m_{k;2j}(\omega)|^2 d\omega \le 2\pi$. Thus by the definition (6.2.5),

$$\int_{\mathbb{R}} \widehat{\Phi}^*(\omega)\widehat{\Phi}(\omega)d\omega \le \lim_{k\to+\infty} \sup \int_{\mathbb{R}} \sum_{i,j=1}^{2} |m_{k;i,j}(\omega)|^2 d\omega \le 4\pi.$$

That is, $\Phi(\omega) \in (L_2(\mathbb{R}))^2$.

Since $\mathcal{P}$ is a trigonometric polynomial matrix, the refinable function vector Φ is compactly supported [13]. Therefore, we can use it to construct compactly supported wavelet frames as follows.

THEOREM 6.2.4. *Let $\Psi(x) = (\psi_1(x), \psi_2(x))^T$ be a vector of compactly supported functions constructed by*

$$\widehat{\Psi}(\omega) = \mathcal{Q}(\omega/2)\widehat{\Phi}(\omega/2),$$

where $\mathcal{Q}(\omega)$ is define by (6.2.2). *Then $\{\psi_1(2^j x - k), \psi_2(2^j x - k), j, k \in \mathbb{Z}\}$ constitutes a tight wavelet frame for $L_2(\mathbb{R})$ in the following sense: for all $f \in L_2(\mathbb{R})$,*

$$\sum_{j,k\in\mathbb{Z}} \left(|\langle f, \psi_1(2^j \cdot -k)\rangle|^2 + |\langle f, \psi_2(2^j \cdot -k)\rangle|^2 \right) = \|f\|^2.$$

Proof: This is a generalization of the tight wavelet frames in [18], also [6]. However, our proof is different. Take compactly supported function $f \in L_2(\mathbb{R})$. Then we claim that

$$\sum_{k\in\mathbb{Z}} |\langle f, \psi_i(2^j \cdot -k)\rangle|^2$$

converges for all $j \in \mathbb{Z}$ and $i = 1, 2$. Indeed, for $i = 1$ or $i = 2$, we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\langle f, \psi_i(2^j \cdot -k)\rangle|^2 &\le \left(\sum_{k\in\mathbb{Z}} \int |f(x)|\,|\psi_i(2^j x - k)|dx \right)^2 \\
&\le \|f\|^2 \sum_{k\in\mathbb{Z}} \int_{x\in\text{supp}(f)} |\psi_i(2^j x - k)|^2 dx \\
&\le \|f\|^2 2^{-j} \sum_{k\in\mathbb{Z}} \int_{y\in 2^{-j}\text{supp}(f)} |\psi_i(y - k)|^2 dy
\end{aligned}$$

For each j, there exists a positive integer K such that $2^{-j}\text{supp}(f)\cap[(2^{-j}\text{supp}(f)+k]$ is empty for $|k|\geq K$. Then

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\int_{y\in 2^{-j}\text{supp}(f)}|\psi_i(y-k)|^2dy\\
=&\sum_{m\in\mathbb{Z}}\sum_{\ell=0}^{K-1}\int_{y\in 2^{-j}\text{supp}(f)}|\psi_i(y-mK-\ell)|^2dy\\
\leq&\sum_{\ell=0}^{K-1}\int|\psi_i(y-\ell)|^2dy\leq K\|\psi_i\|^2.
\end{aligned}$$

Next we need to prove that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left(|\langle f,\phi_1(\cdot-k)\rangle|^2+|\langle f,\phi_2(\cdot-k)\rangle|^2\right)\\
+&\sum_{k\in\mathbb{Z}}\left(|\langle f,\psi_1(\cdot-k)\rangle|^2+|\langle f,\psi_2(\cdot-k)\rangle|^2\right)\\
=&\sum_{k\in\mathbb{Z}}\left(|\langle f,2\phi_1(2\cdot-k)\rangle|^2+|\langle f,2\phi_2(2\cdot-k)\rangle|^2\right).
\end{aligned}\tag{6.2.6}$$

It is standard to have

$$\begin{aligned}
\langle f,\phi_1(\cdot-k)\rangle=&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{f}(\omega)\overline{\widehat{\phi_1}(\omega)}e^{-ik\omega}d\omega\\
=&\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\sum_{\ell\in\mathbb{Z}}\widehat{f}(\omega+2\pi\ell)\overline{\widehat{\phi_1}(\omega+2\pi\ell)}e^{-ik\omega}d\omega
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}|\langle f,\phi_1(\cdot-k)\rangle|^2\\
=&\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{\ell\in\mathbb{Z}}\widehat{f}(\omega+2\pi\ell)\overline{\widehat{\phi_1}(\omega+2\pi\ell)}\right|^2d\omega.
\end{aligned}$$

For convenience, we let

$$\begin{aligned}
\alpha_j(\omega)=&\sum_{\ell\in\mathbb{Z}}\widehat{f}(2^j\omega+2^{j+1}\pi\ell)\overline{\widehat{\phi_1}(\omega+2\pi\ell)}\\
\beta_j(\omega)=&\sum_{\ell\in\mathbb{Z}}\widehat{f}(2^j\omega+2^{j+1}\pi\ell)\overline{\widehat{\phi_2}(\omega+2\pi\ell)}
\end{aligned}$$

By definition of the refinable function vector Φ, it is clear that

$$\widehat{\phi_1}(\omega) = P(\omega/2)\widehat{\phi_1}(\omega/2) + P_1(\omega/2)\widehat{\phi_2}(\omega/2).$$

Then,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} |\langle f, \phi_1(\cdot - k)\rangle|^2 \\
=&\frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{\ell\in\mathbb{Z}} \widehat{f}(\omega+2\pi\ell)\overline{\widehat{\phi_1}(\omega+2\pi\ell)}\right|^2 d\omega \\
=&\frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{\ell\in\mathbb{Z}} \widehat{f}(\omega+2\pi\ell)\overline{\left[P(\omega/2+\pi\ell)\widehat{\phi_1}(\omega/2+\pi\ell) + P_1(\omega/2+\pi\ell)\widehat{\phi_2}(\omega/2+\pi\ell)\right]}\right|^2 d\omega \\
=&\frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{\ell\in\mathbb{Z}} \widehat{f}(\omega+2\pi\ell)\overline{P(\omega/2+\pi\ell)\widehat{\phi_1}(\omega/2+\pi\ell)}\right. \\
&\quad \left. + \sum_{\ell\in\mathbb{Z}} \widehat{f}(\omega+2\pi\ell)\overline{P_1(\omega/2+\pi\ell)\widehat{\phi_2}(\omega/2+\pi\ell)}\right|^2 d\omega \\
=&\frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{m\in\mathbb{Z}} \widehat{f}(\omega+2\pi\cdot 2m)\overline{P(\omega/2+2\pi m)\widehat{\phi_1}(\omega/2+2\pi m)}\right. \\
&\quad + \sum_{m\in\mathbb{Z}} \widehat{f}(\omega+2\pi\cdot 2m)\overline{P_1(\omega/2+2\pi m)\widehat{\phi_2}(\omega/2+2\pi m)} \\
&\quad + \sum_{m\in\mathbb{Z}} \widehat{f}(\omega+2\pi\cdot 2m+2\pi)\overline{P(\omega/2+2\pi m+\pi)\widehat{\phi_1}(\omega/2+2\pi m+\pi)} \\
&\quad \left. + \sum_{m\in\mathbb{Z}} \widehat{f}(\omega+2\pi\cdot 2m+2\pi)\overline{P_1(\omega/2+2\pi m+\pi)\widehat{\phi_2}(\omega/2+2\pi m+\pi)}\right|^2 d\omega \\
=&\frac{1}{2\pi}\int_0^{2\pi} |\overline{P(\omega/2)}\alpha_1(\omega/2) + \overline{P(\omega/2+\pi)}\alpha_1(\omega/2+\pi) \\
&\qquad + \overline{P_1(\omega/2)}\beta_1(\omega/2) + \overline{P_1(\omega/2+\pi)}\beta_1(\omega/2+\pi)|^2 d\omega.
\end{aligned}$$

Similarly we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}|\langle f,\phi_2(\cdot-k)\rangle|^2\\=&\frac{1}{2\pi}\int_0^{2\pi}|\beta_0(\omega)|^2d\omega\\=&\frac{1}{2\pi}\int_0^{2\pi}|P_2(\omega/2)\alpha_1(\omega/2)+P_2(\omega/2+\pi)\alpha_1(\omega/2+\pi)\\&\qquad+P_3(\omega/2)\beta_1(\omega/2)+P_3(\omega/2+\pi)\beta_1(\omega/2+\pi)|^2d\omega.\end{aligned}$$

Also, we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}|\langle f,\psi_1(\cdot-k)\rangle|^2\\=&\frac{1}{2\pi}\int_0^{2\pi}|Q_0(\omega/2)\alpha_1(\omega/2)+Q_0(\omega/2+\pi)\alpha_1(\omega/2+\pi)\\&\qquad+Q_1(\omega/2)\beta_1(\omega/2)+Q_1(\omega/2+\pi)\beta_1(\omega/2+\pi)|^2d\omega\end{aligned}$$

and

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}|\langle f,\psi_2(\cdot-k)\rangle|^2\\=&\frac{1}{2\pi}\int_0^{2\pi}|Q_2(\omega/2)\alpha_1(\omega/2)+Q_2(\omega/2+\pi)\alpha_1(\omega/2+\pi)\\&\qquad+Q_3(\omega/2)\beta_1(\omega/2)+Q_3(\omega/2+\pi)\beta_1(\omega/2+\pi)|^2d\omega.\end{aligned}$$

To prove (6.2.6), we expand the squares in the integrals on the right hand sides of the above four equations and add altogether. Then we start to evaluate the coefficient of each term.

The coefficient of $|\alpha_1(\omega/2)|^2$ is

$$|P(\omega/2)|^2+|P_2(\omega/2)|^2+|Q_0(\omega/2)|^2+|Q_2(\omega/2)|^2,$$

which is one because $M(2\omega)N(\omega)$ is a unitary matrix. We show the matrix $M(2\omega)N(\omega)$ explicitly here for the convenience.

$$M(2\omega)N(\omega)=\begin{bmatrix}P(\omega)&P_1(\omega)&P(\omega+\pi)&P_1(\omega+\pi)\\P_2(\omega)&P_3(\omega)&P_2(\omega+\pi)&P_3(\omega+\pi)\\Q_0(\omega)&Q_1(\omega)&Q_0(\omega+\pi)&Q_1(\omega+\pi)\\Q_2(\omega)&Q_3(\omega)&Q_2(\omega+\pi)&Q_3(\omega+\pi)\end{bmatrix}\tag{6.2.7}$$

Similarily, the coefficients of $|\alpha_1(\omega/2+\pi)|^2$,$|\beta_1(\omega/2)|^2$ and $|\beta_1(\omega/2+\pi)|^2$ are also one. Next, we need to examine the coefficients of other terms. The coefficient of $\alpha_1(\omega/2)\overline{\alpha_1(\omega/2+\pi)}$ is

$$\overline{P(\omega/2)}P(\omega/2+\pi)+\overline{P_2(\omega/2)}P_2(\omega/2+\pi)+$$
$$\overline{Q_0(\omega/2)}Q_0(\omega/2+\pi)+\overline{Q_2(\omega/2)}Q_2(\omega/2+\pi),$$

which is the inner product of the first column vector and the third column vector of the matrix $M(2\omega)N(\omega)$ in (6.2.7). Since $M(2\omega)N(\omega)$ is unitary, this coefficient is zero. The coefficient of $\alpha_1(\omega/2)\overline{\beta_1(\omega/2)}$ is

$$\overline{P(\omega/2)}P_1(\omega/2)+\overline{P_2(\omega/2)}P_3(\omega/2)+\overline{Q_0(\omega/2)}Q_1(\omega/2)+\overline{Q_2(\omega/2)}Q_3(\omega/2),$$

which is the inner product of the first column vector and the second column vector of the matrix $M(2\omega)N(\omega)$, hence zero. In the same way, we can show that all coefficients associated with $\alpha_1(\omega/2)\overline{\beta_1(\omega/2+\pi)}$, $\alpha_1(\omega/2+\pi)\overline{\beta_1(\omega/2)}$,$\alpha_1(\omega/2+\pi)\overline{\beta_1(\omega/2+\pi)}$ and $\beta_1(\omega/2)\overline{\beta_1(\omega/2+\pi)}$ are zero. Hence we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left(|\langle f,\phi_1(\cdot-k)\rangle|^2+|\langle f,\phi_2(\cdot-k)\rangle|^2\right)\\
+&\sum_{k\in\mathbb{Z}}\left(|\langle f,\psi_1(\cdot-k)\rangle|^2+|\langle f,\psi_2(\cdot-k)\rangle|^2\right)\\
=&\frac{1}{2\pi}\int_0^{2\pi}\left(|\alpha_1(\omega/2)|^2+|\alpha_1(\omega/2+\pi)|^2\right)d\omega\\
&\quad+\frac{1}{2\pi}\int_0^{2\pi}\left(|\beta_1(\omega/2)|^2+|\beta_1(\omega/2+\pi)|^2\right)d\omega\\
=&\frac{1}{\pi}\int_0^{2\pi}\left(|\alpha_1(\omega)|^2+|\beta_1(\omega)|^2\right)d\omega.
\end{aligned}$$

Since

$$\sum_{k\in\mathbb{Z}}|\langle f,\sqrt{2}\phi_1(2\cdot-k)\rangle|^2=\frac{1}{2\pi}\int_0^{2\pi}|\alpha_1(\omega)|^2d\omega$$

and

$$\sum_{k\in\mathbb{Z}}|\langle f,\sqrt{2}\phi_2(2\cdot-k)\rangle|^2=\frac{1}{2\pi}\int_0^{2\pi}|\alpha_1(\omega)|^2d\omega,$$

we conclude (6.2.6). In general, we have

$$\begin{aligned}
&\sum_{j=M_0}^{M_1}\sum_{k\in\mathbb{Z}}\left(|\langle f, 2^{j/2}\psi_1(2^j\cdot -k)\rangle|^2 + |\langle f, 2^{j/2}\psi_2(2^j\cdot -k)\rangle|^2\right)\\
=&\sum_{k\in\mathbb{Z}}\left(|\langle f, 2^{(M_1+1)/2}\phi_1(2^{M_1+1}\cdot -k)\rangle|^2 + |\langle f, 2^{(M_1+1)/2}\phi_2(2^{M_1+1}\cdot -k)\rangle|^2\right)\\
&-\sum_{k\in\mathbb{Z}}\left(|\langle f, 2^{M_0/2}\phi_1(2^{M_0}\cdot -k)\rangle|^2 + |\langle f, 2^{M_0/2}\phi_2(2^{M_0}\cdot -k)\rangle|^2\right)
\end{aligned}$$

with $M_0 < M_1$. Let $M_0 \to -\infty$. The standard arguments as [6] show that the last two terms go to zero. Let $M_1 \to +\infty$. The standard arguments yield

$$\sum_{k\in\mathbb{Z}}|\langle f, 2^{M_1/2}\phi_1(2^{M_1}\cdot -k)\rangle|^2 = \int_{\mathbb{R}}|\widehat{f}(\omega)|^2|\widehat{\phi}_1(2^{-M_1}\omega)|^2 d\omega + R_1$$

with R_1 going to zero. Since $\widehat{\phi}_1(0) = 1$, the above terms converge to the norm $\|f\|^2$ of $f \in L_2(\mathbb{R})$. Similarly,

$$\sum_{k\in\mathbb{Z}}|\langle f, 2^{M_1/2}\phi_2(2^{M_1}\cdot -k)\rangle|^2 = \int_{\mathbb{R}}|\widehat{f}(\omega)|^2|\widehat{\phi}_2(2^{-M_1}\omega)|^2 d\omega + R_2$$

which converges to zero since $\widehat{\phi}_2(0) = 0$. We have thus established the results.

Here we will show some examples using the symbol of B-splines.

EXAMPLE 6.2.5. *We consider B-spline of degree 1, i.e. let its symbol* $\widehat{\phi}_1(z) = \left(\frac{1+z}{2}\right)^2$ *as* $P(z)$*, where* $z = e^{i\omega}$*. We find that*

$$\widetilde{P}(z) = -\frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}z + \frac{\sqrt{2}}{8}z^2 + \frac{\sqrt{2}}{8}z^3$$

which satisfies (6.1.1)*. The corresponding refinable matrix is as follows:*

$$\mathcal{P}(z) = \begin{bmatrix} P(z) & P_1(z)\\ P_2(z) & P_3(z)\end{bmatrix}$$

where $P(z)$ *is the original symbol,* $P_1(z)$ *is* $\widetilde{P}(z)$*. Using the method we explained earlier, we find*

$$P_2(z) = -\frac{1+\sqrt{2}}{8} - \frac{1}{4}z + \frac{\sqrt{2}-1}{4}z^2 + \frac{1}{4}z^3 + -\frac{1+\sqrt{2}}{8}z^4,$$

and

$$P_3(z) = \frac{2+\sqrt{2}}{16} + \frac{2+\sqrt{2}}{16}z - \frac{2+\sqrt{2}}{16}z^4 - \frac{2+\sqrt{2}}{16}z^5.$$

The entries of the extension matrix $Q(z)$ *defined in* (6.2.2) *are found as*

$$\begin{aligned}
Q_0(z) &= -\frac{2+\sqrt{2}}{16} + \frac{\sqrt{2}}{8}z - \frac{\sqrt{2}}{8}z^3 + \frac{\sqrt{2}}{16}z^4 + \frac{1}{8}z^4\\
Q_1(z) &= \frac{1+\sqrt{2}}{16} + \frac{1+\sqrt{2}}{16}z + \frac{3\sqrt{2}-1}{8}z^2 - \frac{1+\sqrt{2}}{8}z^3 + \frac{1+\sqrt{2}}{16}z^4 + \frac{1+\sqrt{2}}{16}z^5\\
Q_2(z) &= -\frac{2+\sqrt{2}}{16} + \frac{\sqrt{2}}{8}z - \frac{\sqrt{2}}{8}z^3 + \frac{2+\sqrt{2}}{16}z^4\\
Q_3(z) &= \frac{1+\sqrt{2}}{16} + \frac{1+\sqrt{2}}{16}z - \frac{1+\sqrt{2}}{8}z^2 + \frac{3\sqrt{2}-1}{8}z^3 + \frac{1+\sqrt{2}}{16}z^4 + \frac{1+\sqrt{2}}{16}z^5
\end{aligned}$$

EXAMPLE 6.2.6. *Using the symbol of B-spline of degree 2* $P(z) = \left(\frac{1+z}{2}\right)^3$, *then*

$$\widetilde{P} = -\frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8}z + \frac{\sqrt{3}}{8}z^2 + \frac{\sqrt{3}}{8}z^3.$$

As in the first example, $P_1(z) = \widetilde{P}(z)$. *The other entries of the corresponding matrix mask are*

$$P_2(z) = -\frac{5+\sqrt{5}}{640}(20+5\sqrt{5}-(60-5\sqrt{5})z+(60-30\sqrt{5})z^2-(20+2\sqrt{5})z^3+9\sqrt{5}z^4+3\sqrt{5}z^5),$$

and

$$P_3(z) = -\frac{5\sqrt{15}+5\sqrt{3}}{640}(1+z)(5+3z^2)(z^2-1).$$

Then the entries of $Q(z)$ *are:*

$$\begin{aligned}
Q_0(z) &= \frac{5\sqrt{6}+\sqrt{30}}{1280}(z^2-1)(4\sqrt{5}+5-(12\sqrt{5}-15)z+15z^2+5z^3)\\
Q_1(z) &= \frac{5\sqrt{2}+\sqrt{10}}{1280}(15+15z+(130-32\sqrt{5})z^2-30z^3+15z^4+15z^5)\\
Q_2(z) &= \frac{5\sqrt{6}+\sqrt{30}}{1280}(z^2-1)(4\sqrt{5}+5-(12\sqrt{5}-15)z+15z^2+5z^3)\\
Q_3(z) &= \frac{5\sqrt{2}+\sqrt{10}}{1280}(z^2-1)(15+15z-30z^2+(130-32\sqrt{5})z^3+15z^4+15z^5)
\end{aligned}$$

EXAMPLE 6.2.7. $P(z) = \left(\frac{1+z}{2}\right)^4$ *is the symbol of B-spline of degree 3, then*

$$\widetilde{P}(z) = \frac{2+\sqrt{10}}{32} + \frac{6+\sqrt{10}}{32}z - \frac{\sqrt{10}-2}{16}z^2 - \frac{2+\sqrt{10}}{16}z^3 + \frac{\sqrt{10}-6}{32}z^4 + \frac{\sqrt{10}-2}{32}z^5.$$

As usual, $P_1(z) = \widetilde{P}(z)$*. Denote,*

$$a = \sqrt{7680 - 2388\sqrt{10} - 251\sqrt{10}\sqrt{50 - 12\sqrt{10}} + 830\sqrt{50 - 12\sqrt{10}}},$$

we find out the other entries of $\mathcal{P}(z)$*. they are*

$$\begin{aligned}
P_2(z) = &-\frac{1}{84911189760}((1501\sqrt{10}\sqrt{50 - 12\sqrt{10}} - 2890\sqrt{2}a + 14926\sqrt{10} - 50530 \\
&- 5920\sqrt{50 - 12\sqrt{10}} + 56\sqrt{5}a)(-240090 + 221756\sqrt{10}\sqrt{50 - 12\sqrt{10}}z^5 \\
&+ 19932\sqrt{10}\sqrt{50 - 12\sqrt{10}}z - 61768\sqrt{10}\sqrt{50 - 12\sqrt{10}}z^3 + 195030\sqrt{50 - 12\sqrt{10}}z^4 \\
&+ 25710\sqrt{50 - 12\sqrt{10}}z^6 - 267590\sqrt{50 - 12\sqrt{10}}z^2 + 37617\sqrt{10}\sqrt{50 - 12\sqrt{10}}z^4 \\
&+ 5439\sqrt{10}\sqrt{50 - 12\sqrt{10}}z^6 - 32860\sqrt{10}z^2 + 67968\sqrt{2}\sqrt{50 - 12\sqrt{10}}az^3 \\
&- 113184\sqrt{2}\sqrt{50 - 12\sqrt{10}}a - 262880\sqrt{2}az^3 + 163080\sqrt{50 - 12\sqrt{10}}z \\
&+ 47488\sqrt{5}az^2 - 103456\sqrt{5}az^3 - 103456\sqrt{5}az^3 - 15900\sqrt{10}z^6 - 95400\sqrt{10}z^4 \\
&- 797120\sqrt{2}az + 8480\sqrt{5}az + 4983\sqrt{10}\sqrt{50 - 12\sqrt{10}} + 47488\sqrt{5}a \\
&+ 45504\sqrt{5}\sqrt{50 - 12\sqrt{10}}az - 131970z^6 - 527880z^5 + 158400\sqrt{2}\sqrt{50 - 12\sqrt{10}}az \\
&- 960360z + 1276240z^3 + 530000\sqrt{2}a - 113184\sqrt{2}\sqrt{50 - 12\sqrt{10}}a \\
&- 40896\sqrt{5}\sqrt{50 - 12\sqrt{10}}a + 1615970z^2 - 40896\sqrt{5}\sqrt{50 - 12\sqrt{10}}az^2 - 63600\sqrt{10}z^5 \\
&+ 207760\sqrt{10}z^3 - 259840\sqrt{50 - 12\sqrt{10}}z^3 + 102840\sqrt{50 - 12\sqrt{10}}z^5 - 1031910z^4 \\
&+ 53000\sqrt{2}az^2 - 27959\sqrt{10}\sqrt{50 - 12\sqrt{10}}z^2 + 40770\sqrt{50 - 12\sqrt{10}} \\
&+ 36288\sqrt{5}\sqrt{50 - 12\sqrt{10}}az^3))
\end{aligned}$$

$$\begin{aligned}
P_3(z) = &-\frac{1}{84911189760}((1501\sqrt{10}\sqrt{50 - 12\sqrt{10}} - 2890\sqrt{2}a + 14926\sqrt{10} - 50530 \\
&- 5920\sqrt{50 - 12\sqrt{10}} + 56\sqrt{5}a)(240090 + 21186\sqrt{10}\sqrt{50 - 12\sqrt{10}}z^5 \\
&- 35334\sqrt{10}\sqrt{50 - 12\sqrt{10}}z + 21564\sqrt{10}\sqrt{50 - 12\sqrt{10}}z^3 + 100365\sqrt{50 - 12\sqrt{10}}z^4 \\
&+ 49935\sqrt{50 - 12\sqrt{10}}z^6 + 212345\sqrt{50 - 12\sqrt{10}}z^2 + 9396\sqrt{10}\sqrt{50 - 12\sqrt{10}}z^4 \\
&+ 3462\sqrt{10}\sqrt{50 - 12\sqrt{10}}z^6 - 395645\sqrt{10}z^2 + 67968\sqrt{2}\sqrt{50 - 12\sqrt{10}}az^2
\end{aligned}$$

$$- 147332\sqrt{50-12\sqrt{10}}z - 1485\sqrt{50-12\sqrt{10}}z^7 - 7416\sqrt{10}\sqrt{50-12\sqrt{10}}z^7$$
$$- 103456\sqrt{5}az^2 + 18285\sqrt{10}z^6 + 19875\sqrt{10}z^4 + 50085\sqrt{10}z^7 + 120045\sqrt{10}z$$
$$+ 120045\sqrt{10} - 25368\sqrt{10}\sqrt{50-12\sqrt{10}} - 103456\sqrt{5}a - 316410z^6 - 663050z^5$$
$$- 52470z^7 + 720270z - 4770z^3 - 262880\sqrt{2}a + 67968\sqrt{2}\sqrt{50-12\sqrt{10}}a$$
$$+ 36288\sqrt{5}\sqrt{50-12\sqrt{10}}a - 766910z^2 + 36288\sqrt{5}\sqrt{50-12\sqrt{10}}az^2$$
$$- 43725\sqrt{10}z^5 - 126405\sqrt{10}z^3 + 27045\sqrt{50-12\sqrt{10}}z^3 + 121655\sqrt{50-12\sqrt{10}}z^5$$
$$- 615330z^4 - 262880\sqrt{2}az^2 + 83102\sqrt{10}\sqrt{50-12\sqrt{10}}z^2 - 65685\sqrt{50-12\sqrt{10}}))$$

And the entries of $\mathcal{Q}(z)$ are

$$Q_0(z) = \frac{-1445+14\sqrt{10}}{256431793075200}\sqrt{100-24\sqrt{10}}\Big(-120045 - 63420\sqrt{10}z^2$$
$$+ 78760\sqrt{2}\sqrt{50-12\sqrt{10}}az^3 - 62680\sqrt{2}\sqrt{50-12\sqrt{10}}az^2 - 167360\sqrt{2}az^3$$
$$+ 150304\sqrt{5}az^2 - 208720\sqrt{5}az^3 + 56172\sqrt{10}z^6 + 19932\sqrt{10}z^8 + 335220\sqrt{10}z^4$$
$$+ 79728\sqrt{10}z^7 + 3452\sqrt{5}\sqrt{50-12\sqrt{10}}az^4 - 140\sqrt{2}\sqrt{50-12\sqrt{10}}az^4$$
$$- 20992\sqrt{5}az^4 - 199280\sqrt{2}az + 2120\sqrt{5}az + 11872\sqrt{5}a$$
$$+ 14840\sqrt{5}\sqrt{50-12\sqrt{10}}az + 154020z^6 + 43035z^8 - 244620z^5 + 172140z^7$$
$$+ 45580\sqrt{2}\sqrt{50-12\sqrt{10}}az - 480180z - 896940z^3 + 18100\sqrt{2}az^4 + 132500\sqrt{2}a$$
$$- 32860\sqrt{2}\sqrt{50-12\sqrt{10}}a - 12932\sqrt{5}\sqrt{50-12\sqrt{10}}a - 824460z^2$$
$$- 45944\sqrt{5}\sqrt{50-12\sqrt{10}}az^2 - 173952\sqrt{10}z^5 - 253680\sqrt{10}z^3 + 2197050z^4$$
$$+ 146360\sqrt{2}az^2 - 28660\sqrt{2}\sqrt{50-12\sqrt{10}}az^5 - 20456\sqrt{5}\sqrt{50-12\sqrt{10}}az^5$$
$$+69680\sqrt{2}az^5 + 65416\sqrt{5}az^5 + 61040\sqrt{5}\sqrt{50-12\sqrt{10}}az^3\Big)(3020+$$
$$20\sqrt{2}\sqrt{50-12\sqrt{10}}a + 3\sqrt{5}\sqrt{50-12\sqrt{10}}a - 105\sqrt{2}a + 22\sqrt{5}a)$$

$$
\begin{aligned}
Q_1(z) =& \frac{-1445+14\sqrt{10}}{512863586150400}\sqrt{2}\sqrt{50-12\sqrt{10}}\Big(-240090+9060\sqrt{10}z^2 \\
&- 93200\sqrt{2}\sqrt{50-12\sqrt{10}}az^3 + 183776\sqrt{5}\sqrt{50-12\sqrt{10}}az^2 + 269988\sqrt{10}z^6 \\
&- 76557\sqrt{10}z^8 - 1358094\sqrt{10}z^4 - 143148\sqrt{10}z^7 - 120045\sqrt{10}z - 120045\sqrt{10} \\
&+ 48544\sqrt{5}\sqrt{50-12\sqrt{10}}az^4 + 108200\sqrt{2}\sqrt{50-12\sqrt{10}}az^4 - 74864\sqrt{5}az^4 \\
&+ 51728\sqrt{5}a - 235560z^6 - 58890z^8 + 2382780z^5 - 996600z^7 + 113250z^9 - 720270z \\
&- 779160z^3 - 406480\sqrt{2}az^4 + 131440\sqrt{2}a - 40280\sqrt{2}\sqrt{50-12\sqrt{10}}a \\
&- 22048\sqrt{5}\sqrt{50-12\sqrt{10}}a - 1322760z^2 - 64696\sqrt{5}\sqrt{50-12\sqrt{10}}az^2 \\
&+504642\sqrt{10}z^5 - 244620\sqrt{10}z^3 - 8579820z^4 + 250720\sqrt{2}az^2 + 3171\sqrt{10}z^9\Big) \\
&(3020 + 20\sqrt{2}\sqrt{50-12\sqrt{10}}a + 3\sqrt{5}\sqrt{50-12\sqrt{10}}a - 105\sqrt{2}a + 22\sqrt{5}a).
\end{aligned}
$$

$$
\begin{aligned}
Q_2(z) =& -\frac{-1445+14\sqrt{10}}{2419167859200}((7720a + 6560\sqrt{50-12\sqrt{10}}az^5 + 9060\sqrt{5}z^3 + 9060\sqrt{5}z^3 \\
&- 9060\sqrt{5}z + 27180\sqrt{5}z^3 - 9060\sqrt{2}z^5 + 9060\sqrt{2}z - 2280\sqrt{10}\sqrt{50-12\sqrt{10}}az^3 \\
&- 320\sqrt{50-12\sqrt{10}}az - 1352\sqrt{10}\sqrt{50-12\sqrt{10}}az^4 - 6795\sqrt{2}z^8 - 27180\sqrt{2}z^7 \\
&- 9060\sqrt{5}z^7 - 2265\sqrt{5}z^8 - 3640\sqrt{10}az^5 + 1700\sqrt{10}\sqrt{50-12\sqrt{10}}az^5 \\
&+ 580\sqrt{10}\sqrt{50-12\sqrt{10}}az - 3800\sqrt{10}az - 4940\sqrt{50-12\sqrt{10}}az^4 \\
&+ 4920\sqrt{50-12\sqrt{10}}az^2 + 2276\sqrt{10}a - 5448\sqrt{10}az^2 - 9060\sqrt{5}z^2 + 18120\sqrt{2}z^2 \\
&- 2265\sqrt{5} - 376\sqrt{10}\sqrt{50-12\sqrt{10}}az^5 - 22360az^5 - 12240az^2 + 16120az^4 \\
&+ 2265\sqrt{2} - 3880a + 22650\sqrt{5}z^4 + 14640az^3 + 3172\sqrt{10}az^4 - 36240\sqrt{2}z^6 \\
&+ 20\sqrt{50-12\sqrt{10}}az^5 - 6240\sqrt{50-12\sqrt{10}}az^3 + 22650\sqrt{2}z^4 + 1728az^2 \\
&+ 7440\sqrt{10}az^3)(3220 + 20\sqrt{2}\sqrt{50-12\sqrt{10}}a + 3\sqrt{5}\sqrt{50-12\sqrt{10}}a \\
&- 105\sqrt{2}a + 22\sqrt{5}a))
\end{aligned}
$$

$$
\begin{aligned}
Q_3(z) =& \frac{-1445+14\sqrt{10}}{4838335718400}((7680\sqrt{50-12\sqrt{10}}az^5 - 36240\sqrt{5}z^3 + 54360\sqrt{5}z^5 + 9060\sqrt{5}z \\
& - 63420\sqrt{2}z^3 - 448470\sqrt{2}z^5 - 2265\sqrt{2}z + 2008\sqrt{10}\sqrt{50-12\sqrt{10}}az^4 - 29445\sqrt{2}z^8 \\
& - 63420\sqrt{2}z^7 - 36240\sqrt{5}z^7 + 4224\sqrt{10}az^5 + 576\sqrt{10}\sqrt{50-12\sqrt{10}}az^5 \\
& + 6640\sqrt{50-12\sqrt{10}}az^4 - 7200\sqrt{50-12\sqrt{10}}az^2 - 1504\sqrt{10}a + 6912\sqrt{10}az^2 \\
& - 63420\sqrt{2}z^2 + 9060\sqrt{5}z^9 - 2265\sqrt{2}z^9 + 344\sqrt{10}\sqrt{50-12\sqrt{10}}a - 40320az^5 \\
& + 19680az^2 - 19760az^4 + 6795\sqrt{2} + 80a - 5408\sqrt{10}az^4 + 9060\sqrt{2}z^6 \\
& + 560\sqrt{50-12\sqrt{10}}a + 77010\sqrt{2}z^4 - 2352\sqrt{10}\sqrt{50-12\sqrt{10}}az^2)(3020+ \\
& 20\sqrt{2}\sqrt{50-12\sqrt{10}}a + 3\sqrt{5}\sqrt{50-12\sqrt{10}}a - 105\sqrt{2}a + 22\sqrt{5}a))
\end{aligned}
$$

6.3 Multivariate setting

To extend our construction to the multivariate setting, we let $\phi \in L_2(\mathbb{R}^s)$, $s > 1$ be a refinable function whose Fourier transform satisfies

$$\widehat{\phi}(2\omega) = P(\omega)\widehat{\phi}(\omega).$$

where $\omega \in \mathbb{R}^n$. Then we find the polyphase form of $P(\omega)$,

$$P(\omega) = \sum_j z_j p_j(2\omega),$$

where $j = j_1, j_2, \ldots, j_s$ with $j_k \in \{0,1\}$ for $k = 1,2,\ldots,s$, and $z^j = z_1^{j_1} z_2^{j_2} \cdots z_s^{j_s}$ with $z_k = e^{i\omega_k}$. It is easy to see there are 2^s components of the polyphase form.

If $\sum_{k=1}^{2^s} |P(\omega + \pi n_k)|^2 \leq 1$,where n_k, $k = 1,2,\ldots,2^s$ denote all the vertices of the s dimensional cube $[0,1]^s$, then $\sum_j |p_j(2\omega)|^2 \leq \frac{1}{2^s}$.

By generalized Ditschel theorem in multivariate setting, there exists finite number of Laurent polynomials $\tilde{p_k}$,$k = 1,2,\ldots,n$ such that

$$\sum_j |p_j(2\omega)|^2 + \sum_{k=1}^{n} |\tilde{p_k}(2\omega)|^2 = \frac{1}{2^s}.$$

Assume $\sum_{k=1}^{2^s}\left(|P(\omega+\pi n_k)|^2+\sum_{m=1}^{m_0}|\widetilde{P}_m(\omega+\pi n_k)|^2\right)=1$ for some integer $m_0>0$. Similar to the univariate setting, we want to find the unitary extension matrix given the first row vector as

$$\Big[P(\omega+\pi n_1),\widetilde{P_1}(\omega+\pi n_1),\widetilde{P_2}(\omega+\pi n_1),\cdots,\widetilde{P_{m_0}}(\omega+\pi n_1),$$
$$P(\omega+\pi n_2),\widetilde{P_1}(\omega+\pi n_2),\widetilde{P_2}(\omega+\pi n_2),\cdots,\widetilde{P_{m_0}}(\omega+\pi n_2),\cdots,$$
$$P(\omega+\pi n_{2^s}),\widetilde{P_1}(\omega+\pi n_{2^s}),\widetilde{P_2}(\omega+\pi n_{2^s}),\widetilde{P_{m_0}}(\omega+\pi n_{2^s})\Big]$$

where we let n_1 be zero in $\mathbb{R}^s$.

The way to find unitary extension matrix $M(z)$ here is similar to the method we used in univariate setting. Write the polyphase components in the matrix form,

$$\begin{bmatrix} p_{\{0,0,0,0,\dots,0\}} \\ p_{\{1,0,0,0,\dots,0\}} \\ p_{\{0,1,0,0,\dots,0\}} \\ \vdots \\ p_{\{1,1,1,1,\dots,1\}} \\ \tilde{p_1}_{\{0,0,0,0,\dots,0\}} \\ \tilde{p_1}_{\{1,0,0,0,\dots,0\}} \\ \tilde{p_1}_{\{0,1,0,0,\dots,0\}} \\ \vdots \\ \tilde{p_1}_{\{1,1,1,1,\dots,1\}} \\ \vdots \\ p_{\tilde{m_0}\{0,0,0,0,\dots,0\}} \\ p_{\tilde{m_0}\{1,0,0,0,\dots,0\}} \\ p_{\tilde{m_0}\{0,1,0,0,\dots,0\}} \\ \vdots \\ p_{\tilde{m_0}\{1,1,1,1,\dots,1\}} \end{bmatrix} = [c_{ij}] \begin{bmatrix} 1 \\ z_1 \\ z_2 \\ \vdots \\ z^m \end{bmatrix}$$

where $m=\{k_1,k_2,\dots,k_s\}$ with nonnegative integer k_i for $i=1,2,\dots,s$.

Find a unitary matrix L such that

$$L\,[c_{ij}] = \begin{bmatrix} \tilde{c}_{11} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \tilde{c}_{21} & \tilde{c}_{22} & 0 & \cdots & 0 & 0 & 0 \\ \tilde{c}_{31} & \tilde{c}_{32} & \tilde{c}_{33} & 0 & \cdots & 0 & 0 \\ \tilde{c}_{41} & \tilde{c}_{42} & \tilde{c}_{43} & \tilde{c}_{44} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{c}_{M1} & \tilde{c}_{M2} & \tilde{c}_{M3} & \cdots & \tilde{c}_{M,m-2} & \tilde{c}_{M,m-1} & \tilde{c}_{M,m} \end{bmatrix}$$

where $M = 2^s(m_0+1)$. Clearly,

$$(L \begin{bmatrix} p_{\{0,0,0,0,\ldots,0\}} \\ p_{\{1,0,0,0,\ldots,0\}} \\ p_{\{0,1,0,0,\ldots,0\}} \\ \vdots \\ p_{\{1,1,1,1,\ldots,1\}} \\ \tilde{p_1}_{\{0,0,0,0,\ldots,0\}} \\ \tilde{p_1}_{\{1,0,0,0,\ldots,0\}} \\ \tilde{p_1}_{\{0,1,0,0,\ldots,0\}} \\ \vdots \\ \tilde{p_1}_{\{1,1,1,1,\ldots,1\}} \\ \vdots \\ p_{\tilde{m_0}\{0,0,0,0,\ldots,0\}} \\ p_{\tilde{m_0}\{1,0,0,0,\ldots,0\}} \\ p_{\tilde{m_0}\{0,1,0,0,\ldots,0\}} \\ \vdots \\ p_{\tilde{m_0}\{1,1,1,1,\ldots,1\}} \end{bmatrix})^T L \begin{bmatrix} p_{\{0,0,0,0,\ldots,0\}} \\ p_{\{1,0,0,0,\ldots,0\}} \\ p_{\{0,1,0,0,\ldots,0\}} \\ \vdots \\ p_{\{1,1,1,1,\ldots,1\}} \\ \tilde{p_1}_{\{0,0,0,0,\ldots,0\}} \\ \tilde{p_1}_{\{1,0,0,0,\ldots,0\}} \\ \tilde{p_1}_{\{0,1,0,0,\ldots,0\}} \\ \vdots \\ \tilde{p_1}_{\{1,1,1,1,\ldots,1\}} \\ \vdots \\ p_{\tilde{m_0}\{0,0,0,0,\ldots,0\}} \\ p_{\tilde{m_0}\{1,0,0,0,\ldots,0\}} \\ p_{\tilde{m_0}\{0,1,0,0,\ldots,0\}} \\ \vdots \\ p_{\tilde{m_0}\{1,1,1,1,\ldots,1\}} \end{bmatrix} = \frac{1}{2^s}$$

Thus we define the unitary Householder matrix H as $H = I_M - \dfrac{2vv^*}{v^*v}$ with I_M being the identity matrix of size $M \times M$ and

$$v = L \begin{bmatrix} p_{\{0,0,0,0,\ldots,0\}} \\ p_{\{1,0,0,0,\ldots,0\}} \\ p_{\{0,1,0,0,\ldots,0\}} \\ \vdots \\ p_{\{1,1,1,1,\ldots,1\}} \\ \tilde{p_1}_{\{0,0,0,0,\ldots,0\}} \\ \tilde{p_1}_{\{1,0,0,0,\ldots,0\}} \\ \tilde{p_1}_{\{0,1,0,0,\ldots,0\}} \\ \vdots \\ \tilde{p_1}_{\{1,1,1,1,\ldots,1\}} \\ \vdots \\ p_{\tilde{m_0}\{0,0,0,0,\ldots,0\}} \\ p_{\tilde{m_0}\{1,0,0,0,\ldots,0\}} \\ p_{\tilde{m_0}\{0,1,0,0,\ldots,0\}} \\ \vdots \\ p_{\tilde{m_0}\{1,1,1,1,\ldots,1\}} \end{bmatrix} - \begin{bmatrix} \frac{1}{2^{s-1}} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

such that

$$HL\begin{bmatrix} p_{\{0,0,0,0,\ldots,0\}} \\ p_{\{1,0,0,0,\ldots,0\}} \\ p_{\{0,1,0,0,\ldots,0\}} \\ \vdots \\ p_{\{1,1,1,1,\ldots,1\}} \\ \tilde{p_1}_{\{0,0,0,0,\ldots,0\}} \\ \tilde{p_1}_{\{1,0,0,0,\ldots,0\}} \\ \tilde{p_1}_{\{0,1,0,0,\ldots,0\}} \\ \vdots \\ \tilde{p_1}_{\{1,1,1,1,\ldots,1\}} \\ \vdots \\ p_{\tilde{m_0}\{0,0,0,0,\ldots,0\}} \\ p_{\tilde{m_0}\{1,0,0,0,\ldots,0\}} \\ p_{\tilde{m_0}\{0,1,0,0,\ldots,0\}} \\ \vdots \\ p_{\tilde{m_0}\{1,1,1,1,\ldots,1\}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2^{s-1}} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

That is,

$$\begin{bmatrix} p_{\{0,0,0,0,\ldots,0\}} \\ p_{\{1,0,0,0,\ldots,0\}} \\ p_{\{0,1,0,0,\ldots,0\}} \\ \vdots \\ p_{\{1,1,1,1,\ldots,1\}} \\ \tilde{p_1}_{\{0,0,0,0,\ldots,0\}} \\ \tilde{p_1}_{\{1,0,0,0,\ldots,0\}} \\ \tilde{p_1}_{\{0,1,0,0,\ldots,0\}} \\ \vdots \\ \tilde{p_1}_{\{1,1,1,1,\ldots,1\}} \\ \vdots \\ p_{\tilde{m_0}\{0,0,0,0,\ldots,0\}} \\ p_{\tilde{m_0}\{1,0,0,0,\ldots,0\}} \\ p_{\tilde{m_0}\{0,1,0,0,\ldots,0\}} \\ \vdots \\ p_{\tilde{m_0}\{1,1,1,1,\ldots,1\}} \end{bmatrix}^T = [1/2^{s-1}, 0, 0, \ldots, 0]\overline{H}L$$

Let matrix $M(z) = \overline{H}L$, and $N(x)$ be the square matrix defined as follows: first, we find the order of $j = \{j_{i_1}, j_{i_2}, \ldots, j_{i_s}\}$ where $j_{i_k} \in \{0, 1\}$, name the ordered sequence as $j_1, j_2, \ldots, j_{2^s}$. Denote the vertex π_k of cube $[0,1]^s$ as $\pi[j_{k_1}, j_{k_2}, \ldots, j_{k_s}]$ for $k = 1, 2, \ldots, 2^s$. For convenience, we let j_1, π_1 be zeros. Then we define t_n as a vector with 2^s components, kth component is determined by j_k and π_m. Assume $j_k = [j_{k_1}, j_{k_2}, \ldots, j_{k_s}]$, and $\pi_m = \pi[j_{m_1}, j_{m_2}, \ldots, j_{m_s}]$. We set $U = 0$, then start to check j_{m_1}, if both j_{m_1} and j_{k_1} are ones, then let $U = U + 1$. Otherwise, we move on to compare j_{m_2} and j_{k_2}. We keep going until we finish comparing j_{m_s} and j_{k_s}. So we will get U as the number of positions where corresponding components are ones in above two sequences j_k and π_m. Then we define t_m as the vector with kth component defined as $(-1)^U z^{j_k}$. Before we assign values to entries of $N(x)$, we set $N(x)$ as zero matrix first. Then let the first 2^s components in first column as t_1, the second 2^s components in second column as t_1, $\ldots$, the $m_0 + 1$th 2^s components in $m_0 + 1$th column as t_1. Similarily, let the first 2^s components in $m_0 + 2$th column as t_2, the second 2^s components in $m_0 + 3$th column as t_2, $\ldots$, the $m_0 + 1$th 2^s components in $2m_0 + 2$th column as t_2. Repeat this process until we finish the last column vector. It can be verified that $N(x)$ is unitary.

Let us denote

$$M(2\omega)N(\omega) = \begin{bmatrix} \mathcal{P}(\omega+\pi_1) & \mathcal{P}(\omega+\pi_2) & \cdots & \mathcal{P}(\omega+\pi_{2^s}) \\ \mathcal{Q}_1(\omega+\pi_1) & \mathcal{Q}_1(\omega+\pi_2) & \cdots & \mathcal{Q}_1(\omega+\pi_{2^s}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Q}_{2^s-1}(\omega+\pi_1) & \mathcal{Q}_{2^s-1}(\omega+\pi_2) & \cdots & \mathcal{Q}_{2^s-1}(\omega+\pi_{2^s}) \end{bmatrix} \tag{6.3.1}$$

with

$$\mathcal{P}(\omega) = \begin{bmatrix} P(\omega) & \widetilde{P_1}(\omega) & \widetilde{P_2}(\omega) & \cdots & \widetilde{P_{m_0}}(\omega) \\ P_1(\omega) & P_2(\omega) & P_3(\omega) & \cdots & P_{m_0+1}(\omega) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{(m_0-1)(m_0+1)+1}(\omega) & P_{(m_0-1)(m_0+1)+2}(\omega) & P_{(m_0-1)(m_0+1)+3}(\omega) & \cdots & P_{m_0(m_0+1)}(\omega) \end{bmatrix} \tag{6.3.2}$$

and

$$\mathcal{Q}_i(\omega) = \begin{bmatrix} Q_{i,1}(\omega) & Q_{i,2}(\omega) & \cdots & Q_{i,m_0+1}(\omega) \\ Q_{i,m_0+2}(\omega) & Q_{i,m_0+3}(\omega) & \cdots & Q_{i,2m_0+2}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{i,m_0(m_0+1)+1}(\omega) & Q_{i,m_0(m_0+1)+2}(\omega) & \cdots & Q_{i,(m_0+1)^2}(\omega) \end{bmatrix}. \tag{6.3.3}$$

Since $M(2\omega)N(\omega)$ is unitary, we have

$$\sum_{l=1}^{2^s} \mathcal{P}(\omega+\pi_l)\mathcal{P}^*(\omega+\pi_l) = I_{m_0+1} \tag{6.3.4}$$

and

$$\sum_{l=1}^{2^s} \mathcal{P}(\omega+\pi_l)\mathcal{Q}^*(\omega+\pi_l) = 0_{m_0+1} \tag{6.3.5}$$

where 0_{m_0+1} denotes of zero matrix of size $(m_0+1)\times(m_0+1)$.

Define the square matrix of size $(m_0+1)\times(m_0+1)$ on top left of $M(z)$ as $\mathcal{P}(\omega)$. By Lemma 6.2.2, The infinite matrix product $\prod_{j=1}^{\infty} \mathcal{P}(\omega/2^j)$ converges uniformly on any compact set to a continuous matrix-valued function if and only if magnitudes of all eigenvalues of minor $\mathcal{P}_{[1,1]}(\omega)$ are strictly less than 1. However, it can not be proven yet that the magnitudes of eigenvalues of $\mathcal{P}_{[1,1]}(\omega)$ are strictly less than 1 when $s > 2$. When $s = 2$, $\mathcal{P}_{[1,1]}(\omega)$ is not a matrix, but a polynomial. Hence we can use the argument in univariate setting.

Similar to the univariate setting, we shall have the following Lemma

LEMMA 6.3.1. *If the mask $\mathcal{P}(\omega)$ with trigonometric polynomial entries satisfying* (6.3.4) *and the condition in Lemma 6.2.2, then $\prod_{j=1}^{\infty} \mathcal{P}(\omega/2^j)$ converges in $L^2(\mathbb{R}^s)^{(m_0+1)\times(m_0+1)}$, i.e. each entry converges in $L^2(\mathbb{R}^s)$.*

Clearly, the refinable function vector Φ is compactly supported [13]. Thus it can be used to construct multivariate compactly supported wavelet frames as follows.

THEOREM 6.3.2. *Let $\Psi_i(x) = (\psi_{i,1}(x), \psi_{i,2}(x), \ldots, \psi_{i,m_0+1})^T$ be a vector of compactly supported functions constructed by*

$$\widehat{\Psi}_i(\omega) = \mathcal{Q}_i(\omega/2)\widehat{\Phi}(\omega/2),$$

where $\mathcal{Q}_i(\omega)$ is defined by (6.3.3) *for $i = 1, 2, \ldots, 2^s-1$. $\mathcal{P}(\omega)$ with trigonometric polynomial entries satisfying* (6.3.4) *and the condition in Lemma 6.2.2. Then $\{\psi_{i,1}(2^jx-k), \psi_{i,2}(2^jx-k), \ldots, \psi_{i,m_0+1}(2^jx-k), i = 1, 2, \ldots, 2^s-1 j, k \in \mathbb{Z}^s\}$ constitutes a tight wavelet frame for $L_2(\mathbb{R}^s)$ in the following sense: for all $f \in L_2(\mathbb{R}^s)$,*

$$\sum_{j,k\in\mathbb{Z}} \sum_{i=1}^{2^s-1} \sum_{r=1}^{m_0+1} \left(|\langle f, \psi_{i,r}(2^j\cdot -k)\rangle|^2\right) = \|f\|^2.$$

For simplicity, we consider the construction in the bivariate setting here. We use some results in [17] to find $\mathcal{P}(x,y)$ such that

$$\sum_{\ell\in\{0,1\}^2\pi}\Big(|P(\omega+\ell)|^2+|\widetilde{P}(\omega+\ell)|^2\Big)=1,\ \text{where } \omega=(\omega_1,\omega_2)\in\mathbb{R}^2. \tag{6.3.6}$$

In the bivariate case, we will use Box splines as the refinable function $\phi\in L_2(\mathbb{R}^2)$, which generates a multiresolution approximation of $L_2(\mathbb{R}^2)$.

Example 6.3.3. *Let us consider symbol of box-spline $\phi_{1,1,1}$*

$$P_{1,1,1}(x,y)=\left(\frac{1+x}{2}\right)\left(\frac{1+y}{2}\right)\left(\frac{1+xy}{2}\right)$$

on a three direction mesh where $x=e^{i\omega_1}$ and $y=e^{i\omega_2}$. It is easy to see that

$$1-\sum_{\ell\in\{0,1\}^2\pi}|P(\omega+\ell)|^2=\frac{3}{8}-\frac{1}{8}\cos(\omega_1)-\frac{1}{8}\cos(\omega_2)-\frac{1}{8}\cos(\omega_1+\omega_2).$$

Thus, we let

$$\widetilde{P}(\omega)=\frac{1}{8}+\frac{1}{8}x+\frac{\sqrt{3}}{16}y+\frac{\sqrt{3}}{16}xy-\frac{1}{16}y^2-\frac{\sqrt{3}}{16}x^2y-\frac{1}{16}xy^2-\frac{1}{16}x^2y^2-\frac{\sqrt{3}}{16}x^3y-\frac{1}{16}x^3y^2$$

Like our one-dimensional examples, the entries on the first row of the mask matrix $\mathcal{P}(x,y)$ are $P(x,y)$ and $\widetilde{P}(x,y)$. Other entries of the mask matrix are

$$P_2(x,y)=-\frac{1+x^2y^2+y^2+y^4x^2-6xy^2+x+xy^4+y+x^2y+y^3+x^2y^3-2xy-2xy^3}{16}$$

$$P_3(x,y)=\frac{(1+y^2)(-2+y^2+x^2y^2-2x+xy^2+x^3y^2-\sqrt{3}y+\sqrt{3}yx^2-\sqrt{3}xy+\sqrt{3}x^3y)}{32}$$

There are 3 extension matrices $\mathcal{Q}^1(x,y)$,$\mathcal{Q}^2(x,y)$ and $\mathcal{Q}^3(x,y)$, the entries of these matrices are as follows:

$$\mathcal{Q}^1_{[1,1]}(x,y)=-\frac{1+x^2y^2+x^2+x^4y^2+x+x^1+xy^2+x^3y^2-6x^2y+y+yx^4-2xy-2yx^3}{16}$$

$$\mathcal{Q}^1_{[1,2]}(x,y)=\frac{1}{32}(-2+y^2+2x^2y^2-2x^2+x^4y^2-2x+xy^2+2x^3y^2-2x^3+x^5y^2-\sqrt{3}y$$
$$+\sqrt{3}yx^4-\sqrt{3}xy+\sqrt{3}x^5y)$$

$$\mathcal{Q}^1_{[2,1]}(x,y)=-\frac{1}{16}(-6x^2y^2+1+x^4y^4+x+x^3y^2+xy^2+y^4x^3+y+x^2y^3+x^2y+x^4y^3$$
$$-2xy-2x^3y^3)$$

$$\mathcal{Q}^1_{[2,2]}(x,y) = \frac{1}{32}(1+x^2y^2)(-2+y^2+x^2y^2-2x+xy^2+x^3y^2-\sqrt{3}y+\sqrt{3}yx^2-\sqrt{3}xy + \sqrt{3}x^3y)$$

$$\mathcal{Q}^2_{[1,1]}(x,y) = -\frac{1}{32}(2x^2y^2-x^2-1)(1+x^2y^2+x+xy^2+y+x^2y-2xy)$$

$$\mathcal{Q}^2_{[1,2]}(x,y) = \frac{1}{64}(26x^2y^2+2y^4x^2+2x^4y^4+2x^2-x^4y^2+2-y^2-6x^3y^2+2y^4x^3+2x^5y^4 + 2x^3-x^5y^2+2x-xy^2-2\sqrt{3}y^3x^2+2\sqrt{3}y^3x^4-\sqrt{3}yx^4+\sqrt{3}y-2\sqrt{3}x^3y^3 + 2\sqrt{3}x^5y^3-\sqrt{3}x^5y+\sqrt{3}xy)$$

$$\mathcal{Q}^2_{[2,1]}(x,y) = -\frac{1}{32}(2x^2y^2-x^2-1)(1+x^2y^2+x+xy^2+y+x^2y-2xy)$$

$$\mathcal{Q}^2_{[2,2]}(x,y) = -\frac{1}{64}(-6x^2y^2+2y^4x^2+2x^4y^4+2x^2-x^4y^2+2-y^2+26x^3y^2+2y^4x^3+2x^5y^4 + 2x^3-x^5y^2+2x-xy^2-2\sqrt{3}y^3x^2+2\sqrt{3}y^3x^4-\sqrt{3}yx^4+\sqrt{3}y-2\sqrt{3}x^3y^3 + 2\sqrt{3}x^5y^3-\sqrt{3}x^5y+\sqrt{3}xy)$$

$$\mathcal{Q}^3_{[1,1]}(x,y) = -\frac{\sqrt{3}}{32}(-1+x^2-x^2y^2+x^4y^2-x+x^3-xy^2+x^3y^2-y+yx^4+2xy-2yx^3)$$

$$\mathcal{Q}^3_{[1,2]}(x,y) = \frac{1}{64}(2\sqrt{3}-\sqrt{3}y^2-2\sqrt{3}x^2+\sqrt{3}x^4y^2+2\sqrt{3}x-\sqrt{3}xy^2 - 2\sqrt{3}x^3+\sqrt{3}x^5y^2+26x^2y+3y+3yx^4+3xy-6yx^3+3x^5y)$$

$$\mathcal{Q}^3_{[2,1]}(x,y) = -\frac{3}{32}(-1+x^2-x^2y^2+x^4y^2-x+x^3-xy^2+x^3y^2-y+yx^4+2xy-2yx^3)$$

$$\mathcal{Q}^3_{[2,2]}(x,y) = \frac{1}{64}(2\sqrt{3}-\sqrt{3}y^2-2\sqrt{3}x^2+\sqrt{3}x^4y^2+2\sqrt{3}x-\sqrt{3}xy^2 - 2\sqrt{3}x^3+\sqrt{3}x^5y^2+3y-6x^2y+3yx^4+26yx^3+3xy+3x^5y)$$

EXAMPLE 6.3.4. *Finally, we start our construction with symbol of Box-spline* $\phi_{2,2,1}$

$$P_{2,2,1}(x,y) = \left(\frac{1+x}{2}\right)^2\left(\frac{1+y}{2}\right)^2\left(\frac{1+xy}{2}\right).$$

Then the corresponding $\widetilde{P}$ *to satisfy* (6.3.6) *is*

$$\widetilde{P}(x,y) = \frac{1}{96}(-\sqrt{21}-\sqrt{102}+\sqrt{84}y^2-\sqrt{21}x^2y^2+\sqrt{102}x^2y^2-\sqrt{21}x-\sqrt{102}x+\sqrt{84}xy^2 - \sqrt{21}x^3y^2+\sqrt{102}x^3y^2+\sqrt{168}y-\sqrt{51}x^2y-\sqrt{42}x^2y+\sqrt{51}y^3-\sqrt{42}y^3 + \sqrt{168}xy-\sqrt{51}x^3y-\sqrt{42}x^3y+\sqrt{51}xy^3-\sqrt{42}xy^3).$$

And other entries in mask matrix $\mathcal{P}(x,y)$ is

$$
\begin{aligned}
P_2(x,y) = {} & \frac{-6\sqrt{13}+\sqrt{26}}{28288}(9+9y^2+304x^2+14x^5y^2+18x+18y+48\sqrt{3}x^2-120\sqrt{2}yx^2 \\
& -24\sqrt{2}y+35x^4y^2+7x^5y+7x^4+52x^2y^2+46x^3y^2+96\sqrt{2}xy-558x^2y+36xy^2 \\
& +28yx^4+16x^3y^3+9xy^3+7x^5y^3+45xy-18x^3-24\sqrt{2}x+18y^3x^2-24\sqrt{2}x^3 \\
& +14x^4y^3+48\sqrt{2}x^3y-116x^3y) \\
P_3(x,y) = {} & -\frac{-6\sqrt{39}+\sqrt{84}}{84864}(7x^2+9)(\sqrt{7}+\sqrt{34}-2\sqrt{7}y^2+\sqrt{7}x^2y^2-\sqrt{34}x^2y^2+\sqrt{7}x \\
& +\sqrt{34}x-2\sqrt{7}xy^2+\sqrt{7}x^3y^2-\sqrt{34}x^3y^2-2\sqrt{34}y+\sqrt{17}yx^2+\sqrt{14}yx^2-\sqrt{17}y^3 \\
& +\sqrt{14}y^3-2\sqrt{14}xy+\sqrt{17}x^3y\quad+\sqrt{14}x^3y-\sqrt{17}xy^3+\sqrt{14}xy^3)
\end{aligned}
$$

Then we find the entries of three extension matrices as follows:

$$
\begin{aligned}
\mathcal{Q}^1_{[1,1]}(x,y) = {} & \frac{-6\sqrt{221}+\sqrt{442}}{480896}(-12y^2+39x^2+60x^3y^4+78x^5y^2+42x^5y^4-104\sqrt{2}yx^2 \\
& +56\sqrt{2}x^2y^3+32\sqrt{2}y^3+216x^4y^2-39x^2y^4+105x^4y^4+39x^5y-12xy^5+39x^4 \\
& +144\sqrt{2}x^2y^2+912x^2y^2-48xy^4+42x^4y^5-2394x^3y^2+21x^5y^5+78x^2y \\
& +144\sqrt{2}x^3y^3-348x^3y^3-60xy^3+60x^5y^3+78x^3+666y^3x^2-24y^3-12y^4 \\
& +416\sqrt{2}x^3y-128\sqrt{2}xy^3+9x^3y^5+195x^3y-24xy^2+156yx^4+32\sqrt{2}xy^2 \\
& -24x^2y^5+162x^4y^3-488\sqrt{2}x^3y^2-104\sqrt{2}x^3) \\
\mathcal{Q}^1_{[1,2]}(x,y) = {} & \frac{6\sqrt{663}-\sqrt{1326}}{480896}(7x^2y^2-4y^2+13x^2)(\sqrt{7}+\sqrt{34}-2\sqrt{7}y^2+\sqrt{7}x+\sqrt{34}x \\
& +\sqrt{7}x^2y^2-\sqrt{34}x^2y^2-2\sqrt{7}xy^2+\sqrt{7}x^3y^2-\sqrt{34}x^3y^2-2\sqrt{14}y+\sqrt{17}yx^2 \\
& +\sqrt{14}yx^2-\sqrt{17}y^3+\sqrt{14}y^3-2\sqrt{14}xy+\sqrt{17}x^3y+\sqrt{14}x^3y-\sqrt{7}xy^3 \\
& +\sqrt{14}xy^3)
\end{aligned}
$$

$$\mathcal{Q}^1_{[2,1]}(x,y) = \frac{-6\sqrt{17}+\sqrt{34}}{147968}(-272x - 272y + 72\sqrt{2}x^2 + 84\sqrt{2}yx^4 + 42\sqrt{2}y^5x^2 + 246\sqrt{2}yx^2$$
$$+ 150\sqrt{2}y^3x^4 - 528\sqrt{2}x^2y^3 + 42\sqrt{2}y^3 + 102\sqrt{2}y + 51\sqrt{2} + 21\sqrt{2}xy^5$$
$$- 1596\sqrt{2}x^2y^2 - 640x^2y^2 - 400x^3y^2 + 255\sqrt{2}xy - 112x^2y - 112xy^2$$
$$+ 150\sqrt{2}x^3y^4 + 42\sqrt{2}x^5y^2 + 246\sqrt{2}xy^2 + 84\sqrt{2}xy^4 - 528\sqrt{2}x^3y^2 + 54\sqrt{2}x^5y^4$$
$$+ 132\sqrt{2}y^4x^2 - 156\sqrt{2}x^3y^3 + 21\sqrt{2}y^4 + 448x^3y^3 + 448xy^3 + 1088xy - 112x^3$$
$$- 112y^3 + 72\sqrt{2}y^2 + 42\sqrt{2}x^3 + 156\sqrt{2}x^3y + 21\sqrt{2}x^5y + 48\sqrt{2}x^3y^5 + 156\sqrt{2}xy^3$$
$$+ 48\sqrt{2}x^5y^3 + 54\sqrt{2}x^4y^5 + 21\sqrt{2}x^4 + 448x^3y + 27\sqrt{2}x^5y^5 + 102\sqrt{2}x$$
$$- 400y^3x^2 + 132\sqrt{2}x^4y^2 + 135\sqrt{2}x^4y^4)$$

$$\mathcal{Q}^1_{[2,2]}(x,y) = \frac{6\sqrt{51}-\sqrt{102}}{147968}(9x^2y^2 + 7y^2 + 7x^2 + 17)(\sqrt{14} + 2\sqrt{17} - 2\sqrt{14}y^2$$
$$+ \sqrt{14}x^2y^2 - 2\sqrt{17}x^2y^2 + \sqrt{14}x + 2\sqrt{17}x - 2\sqrt{14}xy^2 + \sqrt{14}x^3y^2 - 2\sqrt{17}x^3y^2$$
$$- 4\sqrt{7}y + \sqrt{34}yx^2 + 2\sqrt{7}x^2y - \sqrt{34}y^3 + 2\sqrt{7}y^3 - 4\sqrt{7}xy + \sqrt{34}x^3y + 2\sqrt{7}x^3y$$
$$- \sqrt{34}xy^3 + 2\sqrt{7}xy^3)$$

$$\mathcal{Q}^2_{[1,1]}(x,y) = \frac{6\sqrt{3}-\sqrt{6}}{26112}(\sqrt{7}x^2y^2 + \sqrt{34}x^2y^2 - \sqrt{28}x^2 + \sqrt{7} - \sqrt{34})(3 + 3x^2 + 3y^2 + 15x^2y^2$$
$$- \sqrt{128}x + 6x + 12xy^2 + 6x^3y^2 - \sqrt{128}y + 6y + 12x^2y + 6y^3x^2 + 32\sqrt{2}xy$$
$$+ 15xy + 3x^3y + 3xy^3 + 3x^3y^3)$$

$$\mathcal{Q}^2_{[1,2]}(x,y) = -\frac{-6+\sqrt{2}}{8704}(27 + 14y^2 + 14x^2 + 14x^3y^4 + 14x^5y^2 + 27x^5y^4 + 27x - 3\sqrt{119}x^5y^3$$
$$+ 2\sqrt{119}x^5y - \sqrt{119}x^3y^5 + 2\sqrt{238}x^3y^4 - 2\sqrt{238}x^5 * y^2 + 14\sqrt{2}yx^4 + 14x^3$$
$$- 18\sqrt{2}yx^2 - 24\sqrt{2}y^3x^4 + 3\sqrt{119}y^3 - 4\sqrt{119}y + 2\sqrt{119}x^3y^3 + 3\sqrt{119}xy^3$$
$$- 2\sqrt{238}xy^2 + 28\sqrt{2}x^2y^3 - 24\sqrt{2}y^3 + 14\sqrt{2}y + 14x^4y^2 + 14x^2y^4 + 27x^4y^4$$
$$+ \sqrt{119}x^3y - 4\sqrt{119}xy + 2\sqrt{238}x^3 - 3\sqrt{119}y^3x^4 - \sqrt{119}y^5x^2 + 2\sqrt{119}yx^4$$
$$+ 2\sqrt{119}y^3x^2 + \sqrt{119}yx^2 + 128\sqrt{2}x^2y^2 - 2\sqrt{238}y^2 + 2\sqrt{238}x^2 + 658x^2y^2$$
$$- 110x^3y^2 + 14\sqrt{2}xy + 14xy^2 - 2\sqrt{238}x^4y^2 + 2\sqrt{238}x^2y^4 + 28\sqrt{2}x^3y^3$$
$$- 18\sqrt{2}x^3y + 14\sqrt{2}x^5y + 10\sqrt{2}x^3y^5 - 24\sqrt{2}xy^3 - 24\sqrt{2}x^5y^3 + 10\sqrt{2}y^5x^2)$$

$$\mathcal{Q}^{2}_{[2,1]}(x,y) = \frac{6\sqrt{3}-\sqrt{6}}{26112}(\sqrt{7}x^2y^2+\sqrt{34}x^2y^2-\sqrt{28}x^2+\sqrt{7}-\sqrt{34})(3+3x^2+3y^2+15x^2y^2$$
$$-\sqrt{128}x+6x+12xy^2+6x^3y^2-\sqrt{128}y+6y+12x^2y+6y^3x^2+32\sqrt{2}xy$$
$$+15xy+3x^3y+3xy^3+3x^3y^3)$$

$$\mathcal{Q}^{2}_{[2,2]}(x,y) = \frac{6-\sqrt{2}}{8704}(27+14y^2+14x^2+14x^3y^4+14x^5y^2+27x^5y^4+27x-3\sqrt{119}x^5y^3$$
$$+2\sqrt{238}x^3y^4-2\sqrt{238}x^5y^2+14\sqrt{2}yx^4+10\sqrt{2}y^5x^2-18\sqrt{2}yx^2+2\sqrt{119}x^5y$$
$$-24\sqrt{2}y^3x^4+3\sqrt{119}y^3-4\sqrt{119}y+2\sqrt{119}x^3y^3+3\sqrt{119}xy^3-\sqrt{119}x^3y^5$$
$$-2\sqrt{238}xy^2+28\sqrt{2}x^2y^3-24\sqrt{2}y^3+14\sqrt{2}y+14x^4y^2+14x^2y^4+27x^4y^4$$
$$+\sqrt{119}x^3y-4\sqrt{119}xy+2\sqrt{238}x^3-3\sqrt{119}y^3x^4-\sqrt{119}y^5x^2+2\sqrt{238}x^2y^4$$
$$+2\sqrt{119}yx^4+2\sqrt{119}y^3x^2+\sqrt{119}yx^2+110x^2y^2-2\sqrt{238}y^2-24\sqrt{2}xy^3$$
$$+2\sqrt{238}x^2+658x^2y^2+128\sqrt{2}x^3y^2+14\sqrt{2}xy+14xy^2-2\sqrt{238}x^4y^2$$
$$+28\sqrt{2}x^3y^3+14x^3-18\sqrt{2}x^3y+14\sqrt{2}x^5y+10\sqrt{2}x^3y^5-24\sqrt{2}x^5y^3)$$

$$\mathcal{Q}^{3}_{[1,1]}(x,y) = \frac{-6\sqrt{3}+\sqrt{6}}{52224}(4\sqrt{7}x^2y^2-y^2\sqrt{34}-2\sqrt{7}y^2+\sqrt{34}x^2-2\sqrt{7}x^2)(3\sqrt{2}+3\sqrt{2}x^2$$
$$+3\sqrt{2}y^2+15\sqrt{2}x^2y^2-16x+6\sqrt{2}x+12\sqrt{2}xy^2+6\sqrt{2}x^3y^2-16y+6\sqrt{2}y$$
$$+12\sqrt{2}yx^2+6\sqrt{2}x^2y^3+64xy+15\sqrt{2}xy+3\sqrt{2}x^3y+3\sqrt{2}xy^3+3\sqrt{2}x^3y^3)$$

$$\mathcal{Q}^{3}_{[1,2]}(x,y) = \frac{-6+\sqrt{2}}{8704}(-10\sqrt{2}x^2+2\sqrt{238}x^2y-128\sqrt{2}x^2y^3-3x^5y-3xy^5-28\sqrt{2}x^2y^2$$
$$+24\sqrt{2}x^4y^2+3\sqrt{119}y^2+\sqrt{119}x^2-14\sqrt{2}x^4y^4+18\sqrt{2}x^3y^4+24\sqrt{2}x^5y^2$$
$$+\sqrt{119}x^3+24\sqrt{2}xy^2-14\sqrt{2}xy^4-28\sqrt{2}x^3y^2-14\sqrt{2}x^5y^4-2\sqrt{238}xy^3$$
$$+2\sqrt{238}x^3y-2\sqrt{119}y^4+18\sqrt{2}y^4x^2-2\sqrt{238}y^3-3\sqrt{119}x^5y^2-28x^2y-3yx^4$$
$$-\sqrt{119}x^3y^4-2\sqrt{119}xy^4+3\sqrt{119}xy^2-2\sqrt{119}x^3y^2+4\sqrt{119}x^5y^4-28x^3y^5$$
$$-3\sqrt{119}x^4y^2-\sqrt{119}x^2y^4-2\sqrt{119}x^2y^2+4\sqrt{119}x^4y^4-3y^5-28x^4y^3$$
$$-2\sqrt{238}x^4y^3+2\sqrt{238}x^2y^5-2\sqrt{238}x^5y^3+2\sqrt{238}x^3y^5-14\sqrt{2}y^4-28x^3y$$
$$+118x^3y^3-28xy^3-28x^5y^3-650y^3x^2-28y^3+24\sqrt{2}y^2-10\sqrt{2}x^3-28x^2y^5)$$

$$\mathcal{Q}^3_{[2,1]}(x,y) = \frac{-6\sqrt{3}+\sqrt{6}}{52224}(4\sqrt{7}x^2y^2 - y^2\sqrt{34} - 2\sqrt{7}y^2 + \sqrt{34}x^2 - 2\sqrt{7}x^2)(3\sqrt{2} + 3\sqrt{2}x^2$$
$$+ 3\sqrt{2}y^2 + 15\sqrt{2}x^2y^2 - 16x + 6\sqrt{2}x + 12\sqrt{2}xy^2 + 6\sqrt{2}x^3y^2 - 16y + 6\sqrt{2}y$$
$$+ 12\sqrt{2}yx^2 + 6\sqrt{2}x^2y^3 + 64xy + 15\sqrt{2}xy + 3\sqrt{2}x^3y + 3\sqrt{2}xy^3 + 3\sqrt{2}x^3y^3)$$

$$\mathcal{Q}^3_{[2,2]}(x,y) = \frac{-6+\sqrt{2}}{8704}(-10\sqrt{2}x^2 + 2\sqrt{238}x^2y + 118x^2y^3 - 3x^5y - 3xy^5 - 28\sqrt{2}x^2y^2$$
$$+ 24\sqrt{2}x^4y^2 + 3\sqrt{119}y^2 + \sqrt{119}x^2 - 14\sqrt{2}x^4y^4 + 18\sqrt{2}x^3y^4 + 24\sqrt{2}x^5y^2$$
$$+ \sqrt{119}x^3 + 24\sqrt{2}xy^2 - 14\sqrt{2}xy^4 - 28\sqrt{2}x^3y^2 - 14\sqrt{2}x^5y^4 - 2\sqrt{238}xy^3$$
$$+ 2\sqrt{238}x^3y - 2\sqrt{119}y^4 + 18\sqrt{2}y^4x^2 - 2\sqrt{238}y^3 - 3\sqrt{119}x^5y^2 - 28x^2y - 3yx^4$$
$$- \sqrt{119}x^3y^4 - 2\sqrt{119}xy^4 + 3\sqrt{119}xy^2 - 2\sqrt{119}x^3y^2 + 4\sqrt{119}x^5y^4 - 28x^3y^5$$
$$- 3\sqrt{119}x^4y^2 - \sqrt{119}x^2y^4 - 2\sqrt{119}x^2y^2 + 4\sqrt{119}x^4y^4 - 3y^5 - 28x^4y^3 - 28y^3$$
$$- 2\sqrt{238}x^4y^3 + 2\sqrt{238}x^2y^5 - 2\sqrt{238}x^5y^3 + 2\sqrt{238}x^3y^5 - 14\sqrt{2}y^4 - 28x^3y$$
$$- 128\sqrt{2}x^3y^3 - 28xy^3 - 28x^5y^3 - 650y^3x^2 + 24\sqrt{2}y^2 - 10\sqrt{2}x^3 - 28x^2y^5)$$

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