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# The convergence of three $L_1$ spline methods for scattered data interpolation and fitting

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Received 12 May 2006; received in revised form 1 September 2006; accepted 10 September 2006

Communicated by Günther Nürnberger  
Available online 30 October 2006

## Abstract

The convergences of three  $L_1$  spline methods for scattered data interpolation and fitting using bivariate spline spaces are studied in this paper. That is,  $L_1$  interpolatory splines, splines of least absolute deviation, and  $L_1$  smoothing splines are shown to converge to the given data function under some conditions and hence, the surfaces from these three methods will resemble the given data values.

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MSC: 41A15; 65D17

Keywords: Scattered data fitting;  $L_1$  spline methods; Bivariate splines; Convergence analysis

## 1. Introduction

Given a data set  $\{(x_i, y_i, f_i), i = 1, \dots, V\}$  with  $f_i = f(x_i, y_i)$ , we wish to find a smooth surface which interpolates or approximates the given data set so that the surface resembles the data function  $f$  as closely as possible. We will use bivariate splines to do so. That is, let  $\Delta$  be a triangulation of the data sites  $(x_i, y_i)$ ,  $i = 1, \dots, V$  and

$$S_d^r(\Delta) = \{s \in C^r(\Omega) : s|_t \in \mathbb{P}_d, \forall t \in \Delta\}$$

be the spline space of degree  $d$  and smoothness  $r \geq 1$  with  $r < d$ , where  $\Omega$  is the union of all triangles in  $\Delta$  and  $\mathbb{P}_d$  stands for the space of all polynomials of degree  $\leq d$ . Thus, we look for a

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<sup>1</sup> Supported by the National Science Foundation under Grant EAR 0327577.

spline function  $s$  in  $S_d^r(\Delta)$  to interpolate or fit the given data set. Usually, the well-known minimal energy method is used to construct interpolatory smooth surfaces (cf. e.g., [7] for a survey of the minimal energy methods). However, it may not give a desired surface. See, e.g., [13, Fig. 4.3]. It sometimes produces too many oscillations. Recently, the research works in [13–15] suggest to use  $L_1$  norm to replace the usual quadratic energy functional, i.e., the  $L_2$  norm, more precisely,

$$Q(s) = \sum_{t \in \Delta} \int_t \left( \left| \frac{\partial^2}{\partial x^2} s \right|^2 + 2 \left| \frac{\partial^2}{\partial x \partial y} s \right|^2 + \left| \frac{\partial^2}{\partial y^2} s \right|^2 \right) dx dy. \quad (1.1)$$

That is, instead of  $Q(s)$ , we use the following  $L_1$  norm of the second order derivatives of spline functions, i.e.,

$$E(s) = \sum_{t \in \Delta} \int_t \left( \left| \frac{\partial^2}{\partial x^2} s \right| + 2 \left| \frac{\partial^2}{\partial x \partial y} s \right| + \left| \frac{\partial^2}{\partial y^2} s \right| \right) dx dy. \quad (1.2)$$

The  $L_1$  spline interpolation and smoothing methods were first proposed in [14] using univariate  $C^1$  cubic splines and in [15] using bivariate  $C^1$  cubic Sibson's finite elements. Numerical experiments in these papers show that these methods have a good property of preserving shape. Due to the structure of Sibson's element, the method in [15] can only deal with rectangular grid data. A general version of  $L_1$  spline methods using bivariate splines of arbitrary degree  $d$  and smoothness  $r$  with  $d > r$  over arbitrary triangulation  $\Delta$  are described in [13] and can be given as follows: find  $s_f \in S_d^r(\Delta)$  such that  $s_f(x_i, y_i) = f_i, i = 1, \dots, V$  and

$$E(s_f) = \min\{E(s), s \in S_d^r(\Delta), s(x_i, y_i) = f_i, i = 1, \dots, V\}. \quad (1.3)$$

We will call  $s_f$  an  $L_1$  interpolatory spline of the given data set  $\{(x_i, y_i, f_i), i = 1, \dots, V\}$ . When the data values contain random errors, an interpolation is not suitable. We consider splines to fit instead interpolate the given data values. Let  $\Delta$  be an appropriate triangulation of  $\Omega$  which may not be a triangulation of the given data sites. We find  $s_f \in S_d^r(\Delta)$  such that

$$\ell(s_f - f) = \min\{\ell(s - f), s \in S_d^r(\Delta)\}, \quad (1.4)$$

where  $\ell(s - f) = \sum_{i=1}^V |s(x_i, y_i) - f_i|$ .  $s_f$  is the least absolute deviation from the given data (cf. [4]). We may call  $s_f$  a spline of least absolute deviation of  $f$ . In addition, we also consider the  $L_1$  smoothing spline method: to find  $s_f \in S_d^r(\Delta)$  such that

$$\ell(s_f - f) + \lambda E(s_f) = \min\{\ell(s - f) + \lambda E(s), s \in S_d^r(\Delta)\}, \quad (1.5)$$

where  $\lambda > 0$  is a parameter. An  $s_f$  which minimizes (1.5) is called an  $L_1$  smoothing spline of the give data.

The existence of such interpolatory and smoothing  $L_1$  spline functions for any given data is studied in [13]. That is, if a spline space  $S_d^r(\Delta)$  has an interpolatory spline function, then the  $L_1$  interpolatory spline exists. However, such solutions are not unique. The two  $L_1$  spline fitting methods always find solutions in  $S_d^r(\Delta)$  due to the fact that the minimization functionals are convex. However, they are not strictly convex and hence, the solution may not be unique. How to compute approximates of such spline solutions is also studied in [13]. That is, the authors in [13] discretized the integrals associated with the minimization functional, converted the minimization problems into linear programming problems, and then applied the well-known Karmarkar algorithm to compute a solution of these linear programming problems. Numerical examples were

presented in [13] to illustrate the advantage of the  $L_1$  spline methods over the minimal energy method. In this paper, we will show that the spline functions produced by these methods indeed converge to the data function  $f$  if  $f_i = f(x_i, y_i)$ ,  $i = 1, \dots, V$  for sufficiently smooth function  $f$  as the number of data sites increases. Hence, the  $L_1$  spline methods indeed provide an alternative methodology for surface designers.

It is worthy mentioning that the convergences of the usual minimal energy method, discrete least squares method, and penalized least squares method using bivariate splines for data interpolation and fitting were analyzed in [8,10,11]. However, I cannot generalize their analysis to study the convergence of the  $L_1$  spline methods since the  $L_1$  spline methods are nonlinear methods. Instead, I obtain the convergences in the  $L_1$  norm.

Throughout the paper, we assume that  $d \geq 3r + 2$ . Let  $|\Delta|$  be the maximum of the diameters of the triangles in  $\Delta$ . For any  $\alpha = (\alpha_1, \alpha_2)$  with nonnegative integers  $\alpha_1, \alpha_2$  with  $|\alpha| = \alpha_1 + \alpha_2$ , we denote by  $D^\alpha = D_x^{\alpha_1} D_y^{\alpha_2}$ . Then it is known (cf. [12]) that the spline space  $S_d^r(\Delta)$  possesses optimal approximation property:

**Theorem 1.1.** Fix  $d \geq 3r + 2$  and  $0 \leq m \leq d$ . Then there exists a linear quasi-interpolation operator  $Q_m$  mapping  $L_1(\Omega)$  into  $S_d^r(\Delta)$  and a constant  $C$  such that if  $f$  is in the Sobolev space  $W_p^{m+1}(\Omega)$  with  $1 \leq p \leq \infty$ ,

$$\|D_x^{\alpha_1} D_y^{\alpha_2} (f - Q_m f)\|_{p,\Omega} \leq C |\Delta|^{m+1-\alpha_1-\alpha_2} |f|_{m+1,p,\Omega}, \quad (1.6)$$

for all  $0 \leq \alpha_1 + \alpha_2 \leq m$ . If  $\Omega$  is convex, then the constant  $C$  depends only on  $d, p, m$ , and on the smallest angle  $\theta_\Delta$  in  $\Delta$ . If  $\Omega$  is nonconvex,  $C$  also depends on the Lipschitz constant  $L_{\partial\Omega}$  associated with the boundary of  $\Omega$ .

In particular,  $Qf$  may be chosen to be an interpolatory spline with optimal approximation property (cf. [5,6]). Such quasi-interpolatory operator  $Q$  will be used in the following sections.

The rest of the paper is organized as follows. We first prove the convergence of the  $L_1$  interpolatory splines in Section 2. We shall point out that the proof of the convergence may be generalized to the setting of  $L_p$  spline interpolation. In Section 3, we establish the convergence for splines of least absolute deviation under some suitable conditions. We shall remark on the conditions and extend the convergence analysis. In Section 4 we give a convergence analysis for the  $L_1$  smoothing splines. We present some numerical results on the convergence of the  $L_1$  spline interpolations in Section 5. Finally we conclude the paper with several remarks.

## 2. Convergence of the $L_1$ interpolatory splines

In this section we let  $\Delta$  be a triangulation of the given data sites  $(x_i, y_i)$ ,  $i = 1, \dots, V$ . We assume that  $f_i = f(x_i, y_i)$  for a sufficiently smooth function  $f$  defined over  $\Omega$ . As explained in the introduction section, there always exists at least one spline  $s \in S_d^r(\Delta)$  satisfying the interpolation conditions:  $s(x_i, y_i) = f_i$ ,  $i = 1, \dots, V$  since  $d \geq 3r + 2$ . Denote by  $Sf$  an  $L_1$  interpolatory spline satisfying (1.3).

**Theorem 2.1.** Let  $Sf$  be the  $L_1$  spline interpolating  $f$  at the vertices of  $\Delta$ . Suppose that  $f \in C^2(\Omega)$ . Then there exists a constant  $C$  dependent on  $d$  and  $\theta_\Delta$  as well as the Lipschitz constant associated with the boundary  $\partial\Omega$  if  $\Omega$  is not convex such that

$$\|f - Sf\|_{L_1(\Omega)} \leq C |\Delta|^2 |f|_{2,\infty,\Omega}. \quad (2.1)$$

In order to prove this Theorem 2.1, we need [8, Lemma 6.1]. For convenience, we state it here without proof.

**Lemma 2.2.** *Suppose that  $g$  is continuously twice differentiable over a triangle  $T$ . Suppose that  $g$  is zero at three vertices of  $T$ . Then*

$$\|g\|_{L_\infty(T)} \leq C_1 |T|^2 |g|_{2,\infty,T}$$

for a positive constant  $C_1$  independent of  $g$  and  $T$ .

**Proof of Theorem 2.1.** Fix each triangle  $T \in \Delta$ . Since  $Sf - f$  is zero at the vertices of  $T$ , Lemma 2.2 can be applied to have

$$|Sf - f| \leq C_1 |T|^2 |Sf - f|_{2,\infty,T}.$$

Using the stability property of the B-coefficients of  $Sf$  over  $T$  (cf. [12, Lemma 4.1]), we have

$$|Sf|_{2,\infty,T} \leq \frac{C_2}{A_T} |Sf|_{2,1,T},$$

where  $A_T$  denotes the area of triangle  $T$  and

$$|Sf|_{2,1,T} := \int_T \left( |D_x^2 Sf| + 2|D_x D_y Sf| + |D_y^2 Sf| \right) dx dy.$$

Thus, we have

$$\begin{aligned} \int_\Omega |Sf - f| dx dy &= \sum_{T \in \Delta} \int_T |Sf - f| dx dy \leq C_1 \sum_{T \in \Delta} |T|^2 A_T |Sf - f|_{2,\infty,T} \\ &\leq C_1 |\Delta|^2 \sum_{T \in \Delta} (A_T |f|_{2,\infty,T} + C_2 |Sf|_{2,1,T}) \\ &\leq C_1 |\Delta|^2 \left( A_\Omega |f|_{2,\infty,\Omega} + C_2 \sum_{T \in \Delta} |Sf|_{2,1,T} \right) \\ &= C_1 |\Delta|^2 (A_\Omega |f|_{2,\infty,\Omega} + C_2 E(Sf)) \\ &\leq C_1 |\Delta|^2 (A_\Omega |f|_{2,\infty,\Omega} + C_2 E(Qf)), \end{aligned}$$

where  $A_\Omega$  denotes the area of  $\Omega$  and we have used the extremal property:  $E(Sf) \leq E(Qf)$ . Note that by Theorem 1.1, i.e., using (1.6) with  $m = 1$ ,  $p = \infty$ , and  $|\alpha| = 2$ ,

$$E(Qf) = |Qf|_{2,1,\Omega} \leq C_3 A_\Omega |f|_{2,\infty,\Omega}$$

for a constant  $C_3$  dependent only on  $d$  and  $\theta_\Delta$  and possible Lipschitz constant  $L_{\partial\Omega}$  if  $\Omega$  is not convex. Hence,

$$\int_\Omega |Sf - f| dx dy \leq C_1 (1 + C_2 C_3) A_\Omega |\Delta|^2 |f|_{2,\infty,\Omega}.$$

This completes the proof of (2.1) with  $C = C_1 (1 + C_2 C_3) A_\Omega$ .  $\square$

We observe that the proof above can be generalized to prove the convergence of the  $L_p$  minimal energy spline method for interpolation. That is, let  $Sf \in S_d^r(\Delta)$  be the interpolatory spline satisfying

$$E_p(Sf) = \min\{E_p(s), s(x_i, y_i) = f_i, i = 1, \dots, V, s \in S_d^r(\Delta)\},$$

where

$$E_p(s) = \sum_{t \in \Delta} \int_t \left( \left| \frac{\partial^2}{\partial x^2} s \right|^p + 2 \left| \frac{\partial^2}{\partial x \partial y} s \right|^p + \left| \frac{\partial^2}{\partial y^2} s \right|^p \right) dx dy.$$

Then we have

$$\|Sf - f\|_{L_p(\Omega)} \leq C|\Delta|^2 \|f\|_{2,\infty,\Omega} \quad (2.2)$$

if  $f \in W_\infty^2(\Omega)$ . The details are omitted here.

### 3. Convergence of splines of least absolute deviation

In this section we derive error bounds for splines of least absolute deviation. For convenience, let  $\mathcal{V} = \{(x_i, y_i), i = 1, \dots, V\}$ . Let  $\Delta$  be a triangulation of  $\Omega$ . Note that any data site in  $\mathcal{V}$  may not be a vertex of  $\Delta$ . For convenience, let us assume that no data site in  $\mathcal{V}$  lies on the edges of  $\Delta$ . Also, the number  $V$  of data sites in  $\mathcal{V}$  is much larger than the number of vertices of  $\Delta$ .

We need to introduce the following two quantities related to the data sites with respect to triangulation  $\Delta$  and an integer  $d$ . These two constants play an important role in the analysis of the convergence of discrete least squares spline approximation in [10]. They are also key constants in our analysis. Let  $F_1$  be a positive number such that

$$F_1 \|s\|_{L_\infty(T)} \leq \sum_{v \in \mathcal{V} \cap T} |s(v)| \quad (3.1)$$

for all  $T \in \Delta$  and  $s \in \mathbb{P}_d$ . Let  $F_2$  be the maximum of the numbers of locations in triangle  $T$  for all  $T \in \Delta$ . That is,

$$\sum_{v \in \mathcal{V} \cap T} |s(v)| \leq F_2 \|s\|_{L_\infty(T)} \quad (3.2)$$

for all  $T \in \Delta$  and any  $s \in C(\Omega)$ .

In addition, we need another constant regarding  $\Delta$ . Denote by  $\rho_\Delta$  the smallest of the radii of the inscribed circles of triangles in  $\Delta$ . Let  $\beta$  be the smallest positive constant such that

$$\frac{|\Delta|}{\rho_\Delta} \leq \beta.$$

The number  $\beta$  is called the quasi-uniformity of triangulation  $\Delta$ . Note that the smallest angle  $\theta_\Delta$  can be bounded below by the constant times  $1/\beta$ .

Let  $Sf$  be a spline of least absolute deviation from the given data values  $f_i = f(x_i, y_i)$ . That is  $Sf$  satisfies (1.4). Thus,  $S$  defines a nonlinear map from any  $f \in C(\Omega)$  to  $S_d^r(\Delta)$ . Clearly,  $S$  is a projection. It is easy to see that

$$\ell(Sf) \leq 2\ell(f). \quad (3.3)$$

Indeed,  $\ell(Sf) \leq \ell(Sf - f) + \ell(f) \leq 2\ell(f)$ .



**Theorem 3.1.** Suppose that  $F_1 > 0$  and  $F_2 < +\infty$  are two constants such that  $F_2/F_1$  is independent of  $\Delta$ . Suppose that  $f \in C^{d+1}(\Omega)$ . Then

$$\|f - Sf\|_{L_1(\Omega)} \leq C_1 |\Delta|^{d+1} |f|_{d+1, \infty, \Omega} \quad (3.4)$$

for a positive constant  $C_1 = CA_\Omega(1 + \beta^2 F_2/(\pi F_1))$ , where  $C$  is the same positive constant appeared in (1.6) and  $A_\Omega$  is the area of domain  $\Omega$ .

**Proof.** First of all, let  $Qf$  be the quasi-interpolatory spline as in (1.6) with  $p = \infty$  and  $m = d$ . Then

$$\|f - Sf\|_{L_1(\Omega)} \leq \|f - Qf\|_{L_1(\Omega)} + \|Qf - Sf\|_{L_1(\Omega)}.$$

We now estimate the second term on the right in the above inequality.

$$\begin{aligned} \|Qf - Sf\|_{L_1(\Omega)} &= \sum_{T \in \Delta} \int_T |Qf - Sf| \leq \sum_{T \in \Delta} A_T \|Qf - Sf\|_{L_\infty(T)} \\ &\leq \frac{1}{2F_1} |\Delta|^2 \sum_{T \in \Delta} \sum_{v \in \mathcal{V} \cap T} |Qf(v) - Sf(v)| = \frac{|\Delta|^2}{2F_1} \ell(Qf - Sf) \end{aligned}$$

since  $A_T \leq |\Delta|^2/2$ . Note that

$$\ell(Qf - Sf) \leq \ell(Qf - f) + \ell(Sf - f) \leq \ell(Qf - f) + \ell(Qf - f) = 2\ell(Qf - f).$$

Here we have used the extremal property:  $\ell(f - Sf) \leq \ell(f - Qf)$ . Thus,  $\|Qf - Sf\|_{L_1(\Omega)} \leq \frac{|\Delta|^2}{F_1} \ell(Qf - f)$ . Next we have

$$\begin{aligned} \|Qf - Sf\|_{L_1(\Omega)} &\leq \frac{|\Delta|^2}{F_1} \sum_{T \in \Delta} \sum_{v \in \mathcal{V} \cap T} |Qf(v) - f(v)| \leq \frac{F_2 |\Delta|^2}{F_1} \sum_{T \in \Delta} C |\Delta|^{d+1} |f|_{d+1, \infty, \Omega} \\ &\leq \frac{CF_2}{\pi F_1} \beta^2 A_\Omega |\Delta|^{d+1} |f|_{d+1, \infty, \Omega}, \end{aligned} \quad (3.5)$$

where we have used (1.6) and the following fact:

$$\sum_{T \in \Delta} |\Delta|^2 \leq |\Delta|^2 A_\Omega / (\pi \rho_\Delta^2) \leq A_\Omega \beta^2 / \pi. \quad (3.6)$$

Therefore,

$$\|f - Sf\|_{L_1(\Omega)} \leq A_\Omega C |\Delta|^{d+1} |f|_{d+1, \infty, \Omega} + 2C \beta^2 A_\Omega \frac{F_2}{\pi F_1} |\Delta|^{d+1} |f|_{d+1, \infty, \Omega}.$$

This completes the proof of (3.4) with  $C_1 = CA_\Omega(1 + \beta^2 F_2/(\pi F_1))$ .  $\square$

**Theorem 3.2.** Under the assumptions in Theorem 3.1, there exists a positive constant  $C$  depending only on  $d$ ,  $A_\Omega$ ,  $\beta$  and the ratio  $F_2/F_1$  such that for every function  $f$  in  $W^{d+1, \infty}(\Omega)$

$$\|D^\alpha(f - Sf)\|_{L_1(\Omega)} \leq C |\Delta|^{d+1-|\alpha|} |f|_{d+1, \infty, \Omega}$$

for any  $\alpha = (\alpha_1, \alpha_2)$  with nonnegative integers  $\alpha_1, \alpha_2$  satisfying  $|\alpha| = \alpha_1 + \alpha_2 \leq d$ .

**Proof.** Let  $Qf$  be the quasi-interpolatory spline in  $S_d^r(\Delta)$  which achieves the optimal approximation as in (1.6) with  $m = d$  and  $p = +\infty$ . For convenience, we use  $C$  to denote positive constants which may be different in different lines. Then

$$\begin{aligned} & \|D^\alpha(f - Sf)\|_{L_1(\Omega)} \\ & \leq \|D^\alpha(f - Qf)\|_{L_1(\Omega)} + \|D^\alpha(Qf - Sf)\|_{L_1(\Omega)} \\ & \leq CA_\Omega |\Delta|^{d+1-|\alpha|} \|f\|_{d+1,\infty,\Omega} + \sum_{T \in \Delta} C|T|^{-|\alpha|} \|Qf - Sf\|_{L_1(T)} \\ & \leq CA_\Omega |\Delta|^{d+1-|\alpha|} \|f\|_{d+1,\infty,\Omega} + C|\Delta|^{-|\alpha|} \|Qf - Sf\|_{L_1(\Omega)} \leq C|\Delta|^{d+1-|\alpha|} \|f\|_{d+1,\infty,\Omega} \end{aligned}$$

for a positive constant  $C$  dependent on  $\beta$ ,  $A_\Omega$ , and  $d$ . Here, we have used the Markov inequality (cf. [12]) and the proof of Theorem 3.1, i.e., (3.5).  $\square$

Let us remark on these two constants  $F_1$  and  $F_2$ . It is easy to see that  $F_2$  can be bounded by the maximum of the numbers of the data locations in triangles. That is,

$$F_2 \leq \max_{T \in \Delta} \#\{(x_i, y_i) \in T, i = 1, \dots, V\}.$$

To make  $F_1$  positive, we need to have a set of data sites in each triangle  $T$  which admits unique interpolation by polynomials of degree  $d$  for  $T \in \Delta$ . More precisely, fix a triangle  $T$  and write

$$\{(x_{T,i}, y_{T,i}), i = 1, \dots, n_T\} := \{(x_i, y_i) \in T, i = 1, \dots, V\}.$$

For simplicity, let us assume that  $n_T = (d+1)(d+2)/2$ . Suppose that these  $n_T$  data sites admit a unique polynomial interpolation in the following sense: for any given  $f \in C(T)$ , there exists a unique polynomial  $p_f$  of degree  $d$  such that

$$p_f(x_{T,i}, y_{T,i}) = f(x_{T,i}, y_{T,i}), \quad i = 1, \dots, n_T. \quad (3.7)$$

Then there exists a constant  $F_{T,1}$  such that

$$F_{T,1} \|s\|_{L_\infty(T)} \leq \sum_{i=1}^{n_T} |s(x_{T,i}, y_{T,i})| \quad \forall s \in \mathbb{P}_d. \quad (3.8)$$

Indeed, write  $s \in \mathbb{P}_d$  in terms of Bernstein–Bézier polynomial form:

$$s = \sum_{i+j+k=d} c_{ijk} B_{ijk},$$

where  $B_{ijk} = \frac{d!}{i!j!k!} \lambda_1^i \lambda_2^j \lambda_3^k$  and  $\lambda_1, \lambda_2, \lambda_3$  are the barycentric coordinates of  $(x, y)$  with respect to  $T$ . The uniqueness of the interpolation implies that the coefficient matrix of linear system:

$$\sum_{i+j+k=d} c_{ijk} B_{ijk}(x_{T,m}, y_{T,m}) = f(x_{T,m}, y_{T,m}), \quad m = 1, \dots, n_T$$



is invertible and hence, the vector  $\{c_{ijk}, i + j + k = d\}$  can be bounded in terms of the vector  $\{(f(x_{T,m}, y_{T,m}), m = 1, \dots, n_T)\}$ . That is,

$$\begin{aligned} \max\{|c_{ijk}|, i + j + k = d\} &\leq b_T \max\{|f(x_{T,m}, y_{T,m})|, m = 1, \dots, n_T\} \\ &\leq b_T \sum_{m=1}^{n_T} |f(x_{T,m}, y_{T,m})| \end{aligned}$$

for a positive constant  $b_T$ . Let  $F_{T,1} = 1/b_T$ . Then (3.8) follows. Indeed,

$$\begin{aligned} \|s\|_{L_\infty(T)} &\leq \max_{(x,y) \in T} \sum_{i+j+k=d} |c_{ijk}| B_{ijk}(x, y) \\ &\leq \max_{(x,y) \in T} \max\{|c_{ijk}|, i + j + k = d\} \sum_{i+j+k=d} B_{ijk}(x, y) \\ &\leq b_T \sum_{m=1}^{n_T} |s(x_{T,m}, y_{T,m})|. \end{aligned}$$

A sufficient condition for  $F_1$  to be positive follows immediately.

**Lemma 3.3.** Suppose that for each  $T$ , there exists a subset  $\{(x_{T,m}, y_{T,m}), m = 1, \dots, n_T\}$  of data sites which lie in  $T$  such that the space  $\mathbb{P}_d$  admits a unique interpolation at these data sites, i.e., for any  $f \in C(T)$ , there exists  $p_f \in \mathbb{P}_d$  satisfying (3.7). Then there exists a positive number  $F_1$  satisfying (3.1).

**Proof.** Let  $F_1 = \min\{F_{T,1}, T \in \Delta\}$ . Then it follows that

$$F_1 \|s\|_{L_\infty(T)} \leq F_{T,1} \sum_{m=1}^{n_T} |s(x_{T,m}, y_{T,m})| \leq F_{T,1} \sum_{(x_i, y_i) \in T} |s(x_i, y_i)|$$

for any  $s \in \mathbb{P}_d$  and  $T \in \Delta$ .  $\square$

When  $F_1$  is zero because that some triangles are lack of enough data sites or data sites are not located in a general position, we can still prove the convergence under some assumptions and adding extra smoothness conditions. See Remark 6.4.

We again observe that the proof of Theorem 3.1 can be generalized to prove the convergence of the  $L_p$  splines of least absolute deviation in the following sense: let  $Sf \in S_d^r(\Delta)$  be a spline satisfying

$$\ell_p(Sf - f) = \min\{\ell_p(s - f), s \in S_d^r(\Delta)\},$$

where

$$\ell_p(s - f) = \sum_{i=1}^V |s(x_i, y_i) - f_i|^p.$$

Similarly, let

$$F_{1,p} \|s\|_{L_\infty(T)}^p \leq \sum_{(x_i, y_i) \in T} |s(x_i, y_i)|^p$$

for all  $s \in \mathbb{P}_d$  and  $T \in \Delta$  and

$$\sum_{(x_i, y_i) \in T} |s(x_i, y_i)|^p \leq F_{2,p} \|s\|_{L_\infty(T)}^p$$

for all  $s \in C(T)$ . Then we have

**Theorem 3.4.** Suppose that  $F_{1,p} > 0$  and  $F_{2,p} < +\infty$  are two constants such that  $F_{2,p}/F_{1,p}$  is independent of  $\Delta$ . Suppose that  $f \in C^{d+1}(\Omega)$ . Then

$$\|D^\alpha(f - Sf)\|_{L_p(\Omega)} \leq C \frac{F_{2,p}}{F_{1,p}} |\Delta|^{d+1-|\alpha|} |f|_{d+1,\infty,\Omega}$$

for  $|\alpha| \leq d+1$ , where  $C$  is a positive constant independent of  $f$ , but dependent on the quasi-uniformity  $\beta$  of  $\Delta$ .

The proof is similar to that of Theorems 3.1 and 3.2. We leave the detail to the reader.

#### 4. Convergence of $L_1$ smoothing fitting

In this section we shall investigate the convergence of our  $L_1$  smoothing spline for data fitting. Let  $S_{\lambda,f}$  be a solution of the  $L_1$  smoothing spline corresponding to the parameter  $\lambda$ . We would like to know  $\|f - S_{\lambda,f}\|_{L_1(\Omega)}$ . That is, we will show that  $S_{\lambda,f}$  converges to  $f$  as the size  $|\Delta|$  of triangulation  $\Delta$  and  $\lambda$  go to zero. We first note that  $S_{0,f}$  is just a spline of least absolute deviation from  $f$  as discussed in the previous section. Since we have already known that  $S_{0,f}$  approximates  $f$ , we mainly estimate  $\|S_{0,f} - S_{\lambda,f}\|_{L_1(\Omega)}$  as the study in [11].

Define a functional:

$$\langle s, g \rangle_{\mathcal{P}} = \langle s - f, g - f \rangle_{\ell} + \lambda \langle s, g \rangle_{\mathcal{E}}$$

with  $\lambda > 0$  being a fixed parameter, where

$$\begin{aligned} \langle s, g \rangle_{\ell} &:= \sum_{i=1}^n f(v_i) \operatorname{sign}(g(v_i)), \\ \operatorname{sign}(g(v_i)) &= \begin{cases} 1 & \text{if } g(v_i) > 0, \\ 0 & \text{if } g(v_i) = 0, \\ -1 & \text{if } g(v_i) < 0. \end{cases} \end{aligned}$$

Note that  $\ell(s) = \langle s, s \rangle_{\ell}$  and

$$\langle s, g \rangle_{\mathcal{E}} := \int_{\Omega} \sum_{|\alpha|=2} D^\alpha s \operatorname{sign}(D^\alpha g) dx dy,$$

with  $\mathcal{E}(s) = \langle s, s \rangle_{\mathcal{E}}$ .

We begin with some elementary properties of  $S_{\lambda,f}$ .

**Lemma 4.1.** Let  $S_{0,f}$  be a spline of least absolute deviation from the data values obtained from  $f$ . Then

$$\langle s, S_{0,f} - f \rangle_{\ell} = 0 \quad \forall s \in S_d^r(\Delta). \quad (4.1)$$

**Proof.** Let  $s \in S_d^r(\Delta)$  and consider  $\langle S_{0,f} + \alpha s - f, S_{0,f} + \alpha s - f \rangle_\ell$  which achieves the minimal when  $\alpha = 0$ . Note that  $g(\alpha) = S_{0,f}(x_i, y_i) + \alpha s(x_i, y_i) - f_i$  is a linear function of  $\alpha$  and  $\text{sign}(g(\alpha))$  is a piecewise constant function. Thus,

$$\frac{d}{d\alpha} \langle S_{0,f} + \alpha s - f, S_{0,f} + \alpha s - f \rangle_\ell = \sum_{i=1}^V s(x_i, y_i) \text{sign}(S_{0,f}(x_i, y_i) + \alpha s(x_i, y_i) - f_i).$$

(4.1) follows from the fact  $0 = \frac{d}{d\alpha} \langle S_{0,f} + \alpha s - f, S_{0,f} + \alpha s - f \rangle_\ell|_{\alpha=0}$ .  $\square$

Similarly we have

**Lemma 4.2.** The  $L_1$  smoothing spline  $S_{\lambda,f} \in \mathcal{S}$  satisfies

$$\langle s, f - S_{\lambda,f} \rangle_\ell = \lambda \langle s, S_{\lambda,f} \rangle_{\mathcal{E}} \quad \forall s \in S_d^r(\Delta). \quad (4.2)$$

**Proof.** We use a similar argument as in the proof of Lemma 4.1. Since  $S_{\lambda,f}$  is a minimizer, for any fixed  $s \in S_d^r(\Delta)$ , we have

$$\frac{d}{d\alpha} (\langle S_{\lambda,f} + \alpha s - f, S_{\lambda,f} + \alpha s - f \rangle_\ell + \lambda E(S_{\lambda,f} + \alpha s)) \Big|_{\alpha=0} = 0.$$

For each triangle  $T$ ,  $s$  and  $S_{\lambda,f}$  are polynomials and hence,  $\text{sign}(D_x^2(S_{\lambda,f} + \alpha s))$  is a piecewise constant function.  $\frac{d}{d\alpha} \text{sign}(D_x^2(S_{\lambda,f} + \alpha s))$  is zero almost everywhere, except for a measure zero set. The boundedness of  $D_x^2 s$  and  $D_x^2 S_{\lambda,f}$  implies that

$$\int_T (D_x^2(S_{\lambda,f} + \alpha s)) \frac{d}{d\alpha} \text{sign}(D_x^2(S_{\lambda,f} + \alpha s)) dx dy = 0.$$

Hence, we have

$$0 = \frac{d}{d\alpha} E(S_{\lambda,f} + \alpha s) \Big|_{\alpha=0} = \langle s, S_{\lambda,f} \rangle_{\mathcal{E}}.$$

Combining with the result in Lemma 4.1 we conclude (4.2). This completes the proof.  $\square$

We next show that  $\langle S_{0,f}, S_{0,f} \rangle_{\mathcal{E}}$  is bounded from the above.

**Lemma 4.3.** Suppose that  $f \in C^{m+1}(\Omega)$  with  $1 \leq m \leq d$ . Then there exists a positive constant  $C_f$  dependent on  $f$  only such that

$$\langle S_{0,f}, S_{0,f} \rangle_{\mathcal{E}} \leq C_f.$$

**Proof.** By using Theorem 3.2, we have

$$\begin{aligned} \langle S_{0,f}, S_{0,f} \rangle_{\mathcal{E}} &\leq \langle S_{0,f} - f, S_{0,f} - f \rangle_{\mathcal{E}} + \langle f, f \rangle_{\mathcal{E}} \\ &\leq C(|\Delta|^{m-1} \|f\|_{m+1,\infty,\Omega} + C \|f\|_{2,\infty,\Omega}) =: C_f. \end{aligned}$$

This completes the proof.  $\square$

We are now ready to state and prove the main result in this section.

**Theorem 4.4.** Suppose that the data sites satisfy (3.1) and (3.2). Let  $S_{\lambda,f}$  be the spline minimizing (1.5). Then

$$\|f - S_{\lambda,f}\|_{L_1(\Omega)} \leq C_3 |\Delta|^{d+1} |f|_{d+1,\infty,\Omega} + \lambda \frac{C_f |\Delta|^2}{F_1} \quad (4.3)$$

for every function  $f$  in  $C^{d+1}(\Omega)$ , where  $C_f$  is a constant dependent on  $f$  as in Lemma 4.3 and  $C_3$  is a constant dependent on  $\beta$  and the ratio  $F_2/F_1$ .

**Proof.** We start with

$$\|f - S_{\lambda,f}\|_{L_1(\Omega)} \leq \|f - S_{0,f}\|_{L_1(\Omega)} + \|S_{0,f} - S_{\lambda,f}\|_{L_1(\Omega)}.$$

The first term on the right of the above inequality can be estimated using Theorem 3.1. That is, there exists a constant  $C_2$  depending only on  $d$  and  $\beta$  such that for every function  $f$  in  $W^{d+1,\infty}(\Omega)$

$$\|S_{0,f} - f\|_{L_1(\Omega)} \leq C_2 |\Delta|^{d+1} |f|_{d+1,\infty,\Omega}.$$

Let us work on the second term. We have

$$\begin{aligned} \|S_{0,f} - S_{\lambda,f}\|_{L_1(\Omega)} &\leq \sum_{T \in \Delta} A_T \|S_{0,f} - S_{\lambda,f}\|_{L_\infty(T)} \leq \frac{1}{F_1} |\Delta|^2 \ell(S_{0,f} - S_{\lambda,f}) \\ &\leq \frac{|\Delta|^2}{F_1} (\ell(S_{0,f} - f) + \ell(S_{\lambda,f} - f)) \\ &\leq \frac{|\Delta|^2}{F_1} (\ell(S_{0,f} - f) + \ell(S_{\lambda,f} - f) + \lambda \mathcal{E}(S_{\lambda,f})) \\ &\leq \frac{|\Delta|^2}{F_1} (2\ell(S_{0,f} - f) + \lambda \mathcal{E}(S_{0,f})). \end{aligned}$$

Here we have used the extremal property:

$$\ell(S_{\lambda,f} - f) + \lambda \mathcal{E}(S_{\lambda,f}) \leq \ell(S_{0,f} - f) + \lambda \mathcal{E}(S_{0,f}).$$

Since  $\mathcal{E}(S_{0,f}) = \langle S_{0,f}, S_{0,f} \rangle_{\mathcal{E}}$ , we can use Lemma 4.3. Let  $Qf$  be the quasi-interpolatory spline of  $f$  in  $S_d^r(\Delta)$  as in (1.6). Since  $\ell(S_{0,f} - f) \leq \ell(Qf - f)$ , we have

$$\ell(Qf - f) \leq \sum_{T \in \Delta} F_2 \|Qf - f\|_{L_\infty(T)} \leq F_2 \frac{A_\Omega}{\pi \rho_\Delta^2} C |\Delta|^{d+1} |f|_{d+1,\infty,\Omega}$$

by using (1.6) and (3.6). Hence it follows

$$\|S_{0,f} - S_{\lambda,f}\|_{L_1(\Omega)} \leq \frac{|\Delta|^2}{F_1} \frac{2F_2 A_\Omega}{\pi \rho_\Delta^2} C |\Delta|^{d+1} |f|_{d+1,\infty,\Omega} + \lambda \frac{|\Delta|^2}{F_1} C_f,$$

where we have used Lemma 4.3. Hence,

$$\|f - S_{\lambda,f}\|_{L_1(\Omega)} \leq \left( C_2 + \frac{F_2}{F_1} C \frac{A_\Omega \beta^2}{\pi} \right) |\Delta|^{d+1} |f|_{d+1,\infty,\Omega} + \lambda \frac{C_f |\Delta|^2}{F_1}.$$

This completes the proof.  $\square$

We observe again that the above analysis can be extended to the following  $L_p$  smoothing splines situation. Let  $Sf \in S_d^r(\Delta)$  be a spline satisfying

$$\ell_p(Sf - f) + \lambda E_p(Sf) = \min\{\ell_p(s - f) + \lambda E_p(s), s \in S_d^r(\Delta)\},$$

where  $\ell_p(s)$  and  $E_p(s)$  were defined in the previous two sections. Then we have

**Theorem 4.5.** *Suppose that  $F_{1,p} > 0$  and  $F_{2,p} < +\infty$  defined in the end of the previous section are two constants such that  $F_{2,p}/F_{1,p}$  is independent of  $\Delta$ . Suppose that  $f \in C^{d+1}(\Omega)$ . Then*

$$\|f - Sf\|_{L_p(\Omega)} \leq C|\Delta|^{d+1-|\alpha|}|f|_{d+1,\infty,\Omega} + \lambda C_{f,p}|\Delta|^2/F_{1,p},$$

where  $C$  is a positive constant independent of  $f$ , but dependent on the quasi-uniformity  $\beta$  of  $\Delta$  and the ratio of  $F_{2,p}$  and  $F_{1,p}$ . Here  $C_{f,p}$  is a positive constant dependent on  $f$  and  $p$  only.

Proof of Theorem 4.5 is similar to that of Theorem 4.4. The details are omitted here.

## 5. Numerical experiments

We implemented these three  $L_1$  spline methods as described in [13]. Let us present several examples to demonstrate that the surfaces of  $L_1$  spline interpolation indeed approximate the surface that the data set represents. We shall first give an example based on scattered data locations as shown in Fig. 1 to show that  $L_1$  interpolatory splines resemble the surface of a given data set. Next we present a table of the maximum errors between three testing functions and corresponding  $L_1$  interpolatory splines to demonstrate the convergence of the  $L_1$  spline interpolation method. For comparison, we also show the maximum errors of the standard minimal energy interpolatory splines. Finally we demonstrate our  $L_1$  spline method for a set of real data.

**Example 5.1.** We consider a triangulation  $\Delta$  with vertices being the data sites in Fig. 1 and use spline functions in  $S_5^1(\Delta)$ . We find the  $L_1$  interpolatory spline of function  $y = \sin(\pi(x^2 + y^2)) + 1$ . In Fig. 2, we can see that the  $L_1$  interpolatory spline is very close to the given function. In fact, the maximum error of the interpolatory spline function and test function  $y$  based on  $101 \times 101$  equally spaced points over  $[0, 1] \times [0, 1]$  is 0.08905. That is, the interpolatory spline surface approximates the test function  $y$ .

**Example 5.2.** In this example we consider a standard domain  $[0, 1] \times [0, 1]$  and let  $\Delta$  be a type I triangulation of  $4 \times 4$  equally spaced points. We find the  $L_1$  interpolatory splines in  $S_5^1(\Delta)$  of several test functions and compute the maximum errors instead of  $L_1$  norm just for convenience. (The errors in  $L_1$  norm behavior similar to the maximum errors.) The maximum errors of the spline functions against the test functions are computed based on  $201 \times 201$  equally spaced points of  $[0, 1] \times [0, 1]$ . As the underlying triangulation  $\Delta$  is refined, the maximum errors converge to zero in the same order as indicated in Theorem 2.1. For comparison, we also use the standard minimal energy method to find interpolatory splines in  $S_5^1(\Delta)$  of the test functions. The maximum errors of the interpolatory splines and the test functions are listed in Table 1. In Table 1,  $\Delta_n$  denotes the uniform refinement of  $\Delta_{n-1}$  for  $n = 1, \dots, 3$  with  $\Delta_0 = \Delta$ . The numerical results shown that they are in the same approximation order although the maximum errors from the minimal energy method are slightly better than that from the  $L_1$  spline method.

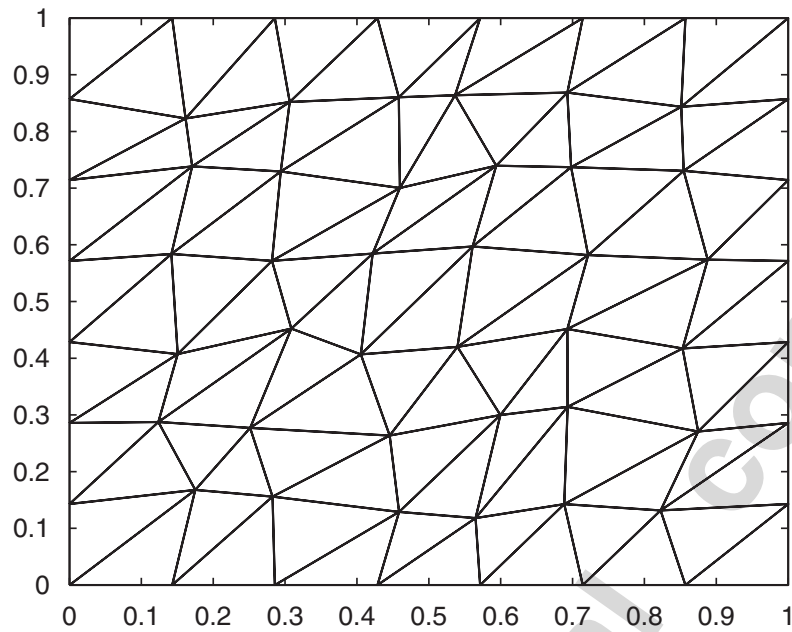
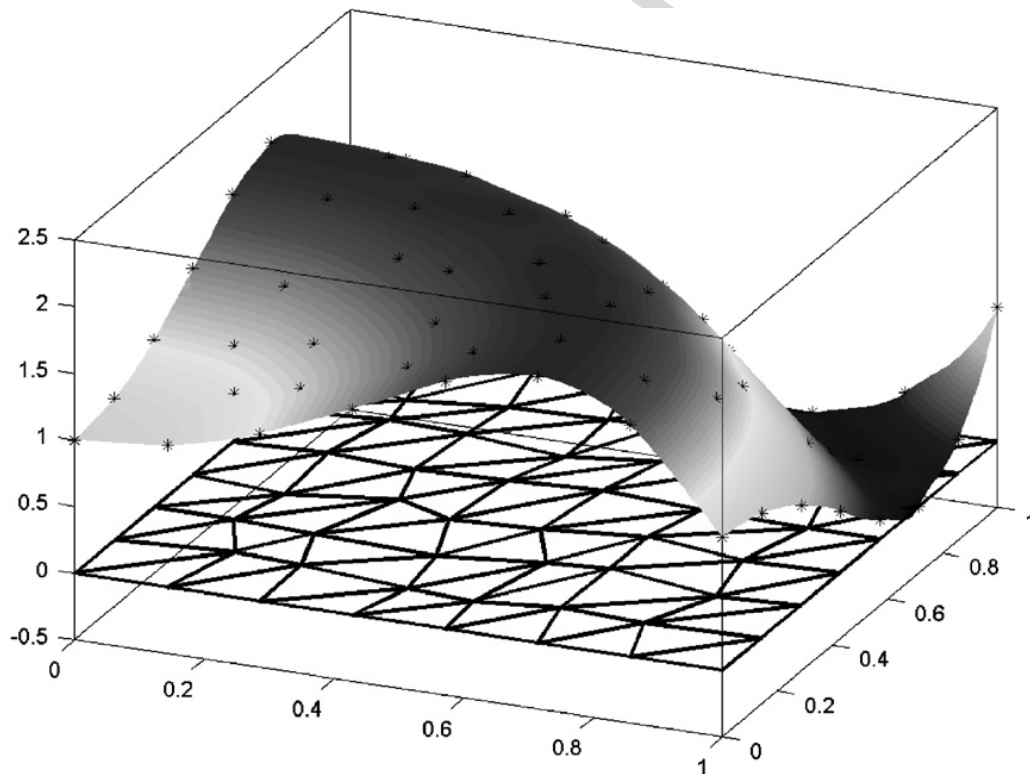


Fig. 1. A triangulation of the given data sites.

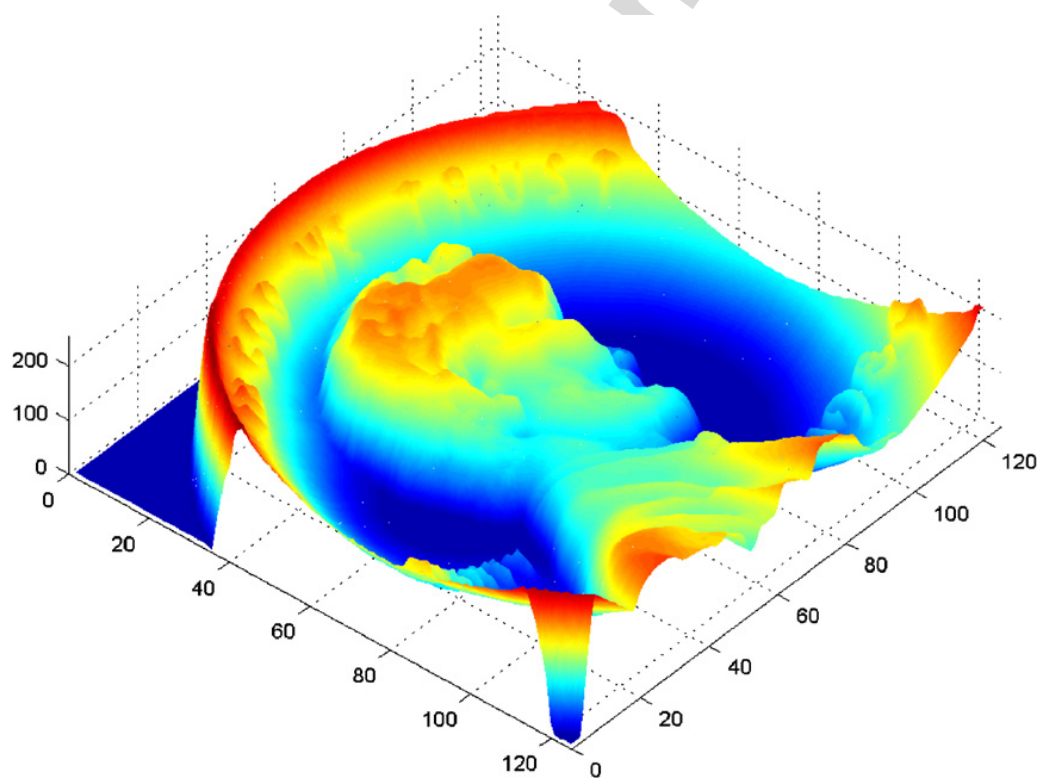
Fig. 2.  $L_1$  interpolatory spline surface.

**Example 5.3.** Consider a standard 2D image as shown in Fig. 3. It has  $116 \times 116$  data values. We use the integers between  $[1, 116] \times [1, 116]$  as data locations and triangulate them using the lines parallel to the  $x$ -axis,  $y$ -axis, and the line  $x = y$ . Let  $\Delta$  be the triangulation. Consider the  $C^1$  quintic spline space  $S_5^1(\Delta)$ . We find the interpolatory spline  $S_f \in S_5^1(\Delta)$  by using the  $L_1$  spline method. The interpolatory surface  $S_f$  shown as in Fig. 4 is an excellent representation of the penny data. It is clear to see that there are no overshoots over the edge ring of the penny.





Fig. 3. Pixel values of 2D image.

Fig. 4. 3D  $L_1$  interpolatory spline surface.

## 6. Remarks

We have the following remarks in order.

**Remark 6.1.** We have only proved the convergence in the  $L_1$  norm for three  $L_1$  spline methods. It is interesting to know the convergence in the maximum norm similar to the results in [8–11].

Table 1

Convergences of  $L_1$  spline interpolation

Test functions	Methods	$\Delta$	$\Delta_1$	$\Delta_2$	$\Delta_3$
$\frac{1}{1+x^2+y^2}$	$L_1$	0.01714	0.006953	0.001272	0.0003234
	$L_2$	0.01306	0.002954	0.0007099	0.0001748
$\sin(\pi(x^2 + y^2))$	$L_1$	0.11121	0.04031	0.011152	0.001965
	$L_2$	0.09357	0.01843	0.004470	0.001104
$10 \exp(-(x^2 + y^2))$	$L_1$	0.25152	0.07646	0.019381	0.003243
	$L_2$	0.12327	0.02883	0.007051	0.001745

I failed to generalize the techniques in these papers to establish the convergence in the maximum norm due to the nonlinearity of three  $L_1$  spline methods. However, the numerical evidence from Table 1 in Section 5 strongly suggests that the convergence of the  $L_1$  spline interpolation in the maximum norm is the same as the convergence of the standard minimal energy interpolation.

**Remark 6.2.** The  $L_1$  spline methods on spherical setting were studied in [16,17] using tensor product of univariate  $C^1$  cubic splines and some nonpolynomial functions. It is interesting and useful to continue the investigation to see if the  $L_1$  spline methods are good for shape preservation using other spline functions. Recently, triangulated spherical splines for scattered data interpolation and fitting are studied in [3]. The convergence of the minimal energy method for spherical spline interpolation using the usual quadratic energy functional is studied in [1]. Convergence of discrete and penalized least squares fitting for spherical scattered data is analyzed in [2]. It is possible to use the spherical splines to find interpolatory or fitting surfaces by using  $L_1$  spline methods. We certainly would like to know if the surfaces resemble the shape of the given data or not. More details may be reported elsewhere.

**Remark 6.3.** In Sections 2–4 we presented the convergence results on spline interpolation and fitting using energy functionals involved  $L_p$  and  $\ell_p$  norms with  $p > 0$ . It is interesting to develop a computational algorithm which efficiently solve such minimization problems and to learn numerical behaviors of spline minimizers which interpolate and fitting scattered data.

**Remark 6.4.** When the constant  $F_1$  in (3.1) fails to be positive, that is, some triangles do not have enough data sites and/or data sites are not located in a general position, the spline of least absolute deviation may not converge. For example, if a triangle on the boundary does not contain any data sites, the spline will not converge over this triangle and hence the convergence in  $L_1$  norm over the domain  $\Omega$  will be ruined. Assume that such “bad” triangles happen in a few triangles which are surrounded by “good” triangles in the sense that (3.1) holds for a positive constant  $F_1$ . Then one way to correct the problem is to impose extra smoothness conditions across edges of these “bad” triangles. For simplicity, let us say there is only one “bad” triangle. Then we add extra smoothness condition across one edge of the bad triangle so that the polynomial piece on the bad triangle is the same polynomial on the neighboring triangle sharing that edge with the bad triangle. Then  $F_1$  in (3.1) will be positive for all triangles except for “bad” triangle. Also there is a positive constant  $F'_1$  such that (3.1) holds over the union of two triangles (one bad and one good triangle). With appropriate modification in the proof, Theorem 3.1 still hold with a different constant  $C_1$ .

## Acknowledgment

The author would like to thank one of referees who raises an interesting question which is discussed in Remark 6.4.

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