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Method of virtual components for constructing redundant filter banks and wavelet frames

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Abstract

For a wavelet frame with polynomial symbols generated by an MRA with a given approximation order, we present the method for parametric description and construction of all dual analysis operators (filter banks) with the maximum number of vanishing moments whose symbols are polynomials. The similar problems related to the maximum frame approximation order and some other wavelet frame properties are considered.

Keywords: Wavelet frames; Redundant filter banks; Vanishing moments; Frame approximation order; Analysis operator

1. Introduction

The main goal of our paper is to present a filter bank tool which can be used for a decomposition in wavelet frames (in particular, tight wavelet frames) allowing to provide maximum approximation order, staying within a framework of mixed extension principle.

In the 80's, the interconnection between the rate of approximation and properties of scaling functions and wavelets was investigated in detail. In particular, it was known that approximation order of linear combinations of shifts of a scaling function is tightly connected with its ability to recover algebraic polynomials of a given degree. For orthogonal and bi-orthogonal wavelets the approximation order of a scaling function is equivalent to the existence of dual wavelets with the given number of vanishing moments.

This is not the case for wavelet frames. Neither approximation order of a multiresolution used for the expansions nor this for dual multiresolution have a direct impact on a number of vanishing moments of framelets even for the case of tight wavelet frames. This phenomenon was analyzed in [3] and [5].

The general theory of wavelet frames in $\mathbb{L}^2(\mathbb{R}^k)$ was developed by A. Ron and Z. Shen [14,15]. In particular, they found very handy tools for construction wavelet tight frames and bi-frames (for shortness, *framelets*) associated with a given scaling function. These tools were called the unitary extension principle (UEP) and mixed extension principle

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(MEP) correspondingly. Those principles give the opportunity to reduce the problem of finding framelets to solving a matrix equation (generally speaking, with non-square matrices), where the first columns of the matrices are defined by the known scaling functions. The level of the interest to framelet systems increased after the papers by C. Chui and W. He [2] and A. Petukhov [10] (see also [11] for the full version), where two different algorithms for solving the mentioned equation were found. However, it became clear very soon (actually, it has been clear since papers [14,15]) that, for many cases (e.g., for B-splines), a number of vanishing moments for UEP-wavelet frames is much less than multiresolution analysis (MRA) approximation order.

The MRA approximation order n means that smooth data can be well represented by linear combinations of shifts (integer translations) of a scaling function. In particular, polynomials of the degree $\leqslant n-1$ can be represented precisely. Unfortunately, a linear analysis operator obtained within UEP framework for computing the expansion coefficients does not have to provide the maximal approximation order.

For wavelet frames, there are two degrees of the violation of the maximum order approximation property. "The weak" form of violation consists in a phenomenon when framelet coefficients of polynomials of degree $n-1$ may not be equal to zero (not enough analysis operator vanishing moments), whereas coefficients for shifts of a scaling function allow to recover polynomials up to degree $n-1$, providing the maximum order of smooth functions approximation. This case was thoroughly studied in [5]. The property to recover polynomials up to the order $k-1$ with scaling function coefficients was called *the frame approximation order k* [5]. "The strong" violation leads to the impossibility to recover polynomials with low-pass coefficients.

Small wavelet (framelet) coefficients for smooth functions guaranteed by a high order of vanishing moments are very desirable. Thus, we need to satisfy some extra conditions to provide actual approximation order equal to the approximation order of the MRA. Otherwise, the low order of framelet vanishing moments necessarily leads to either the leak of the information from the low frequency components (a scaling function) to the high frequency (framelets) or just to unjustified increasing the magnitude of framelet coefficients. Since wavelet frames promise to be a very flexible tool for data representation, the development of linear methods of efficient data expansions in wavelet frames is an actual problem.

It should be mentioned that constructing wavelet frames satisfying too many nice properties may become very difficult. For example, if it is required to design UEP-based (anti)symmetric compactly supported tight wavelet frames with the minimal number of frame generators and an arbitrary number of vanishing moments. If to apply an additional requirement of implementation of the decomposition–reconstruction procedures with finite impulse response (FIR) filters (i.e., with polynomial filter banks), a general solution of the problem is unknown. While a few particular solutions were given by I. Selesnick [16], A. Petukhov [12] (2 vanishing moments), and Q. Jiang [8], I. Selesnick and A. Abdelnour [17] (3 vanishing moments). Those results rather emphasized the lack of the flexibility. Moreover, in all mentioned constructions a number of vanishing moments is behind the corresponding MRA approximation order.

For fairness sake, we have to mention that the compactness of the representation is not a universal requirement. For some of applied problems like de-noising, where wavelet frames deserved a good reputation, the compactness does not play any role. At the same time, usually, it does not contradict directly to other requirements. For this reason, this problem is among the most attractive problems of the wavelet frame theory.

Release from any of the mentioned conditions results in some problem simplification. The original beautiful solution of this problem allowing to overcome the disbalance between MRA approximation order and a number of vanishing moments was independently found by C. Chui et al. [3] and I. Daubechies et al. [5]. Shortly, it was proved that if an MRA is generated by a compactly supported scaling function φ and provides approximation order N, then there exists another compactly supported generator $\tilde{\varphi}$ of the same MRA such that the UEP-framelets associated with $\tilde{\varphi}$ have N vanishing moments. This method for constructing the framelets was called the oblique extension principle (OEP). OEP is easily realizable and effective tool. At the same time, the OEP trick results in the increase of the support of a scaling function and framelets. In addition, in spite of the compactness of the support the decomposition–reconstruction algorithms cannot be implemented any more with FIR filters. The involving of recursive infinite impulse response (IIR) filters or truncated filters becomes necessary. It may be restrictive for real time applications. In addition, application of filters with rational impulse response characteristic is even more problematic in multivariate case.

Another nice approaches to constructing compactly supported wavelet frames with more than two (anti)symmetric framelets were found in recent papers [6] and [7].

In this paper, we are going to present a new scheme of data representation with MEP-based technique (we sacrifice the tightness of the dual frame). The main idea of this scheme lies in the non-uniqueness of the decomposition in a given frame. We are going to parameterize and to use this uncertainty in the choice of the decomposition coefficients for fitting the analysis operator to given properties. Among these properties, the maximum number of vanishing moments and the approximation order provided by an analysis operator play a crucial role. While one of this paper objectives is to present the possible ways for constructing dual wavelet frames with the maximum approximation order and the maximum number of vanishing moments defined by the MRA approximation order, the framework of this paper does not assume the discussion of either $\mathbb{L}^2$-theory of wavelet frames or any other function space settings. Instead of that, we stay on a position of filter bank (FB) theory which deals rather with number sequences than with function spaces. This approach allows to avoid discussion about particular problems of function spaces and to concentrate attention on algebraic issues related to polynomial matrices. At the same time, not giving any recipe for solution of traditional problems of wavelet frames, we believe that the presented results may serve as constructive bricks for wide range of (not only $\mathbb{L}^2$) wavelet frames.

The structure of the paper is as follows. In Sections 2 and 3, we introduce main notions and formulate fundamental results motivating this research as well as the research goals. In Section 4, we describe all possible redundant FB expansions and linear analysis operators by means of a parametrization. Section 5 is the main part of this paper. All possible dual filter banks annihilating polynomials of the maximum degree will be described. In Section 6, we show how to solve a few problems of description of FBs providing some natural representation properties. In particular, constructing dual frames providing the maximum of the frame approximation order is considered. Section 7 is devoted to constructing examples of univariate bi-frames with polynomial symbols with the maximum number of vanishing moments.

2. Notations

We start with description of redundant filter bank transform on the sequence space $\mathcal{S} := \{\{x_i\}_{i\in\mathbb{Z}}\}$ of all possible real sequences. Since we are going to work only with operators based on finite convolutions which are well defined on any sequences, we do not need any restrictions on the growth or any other properties of sequence. In wavelet theory, those sequences correspond to expansion coefficients of a function in a system of shifts of a scaling function generating MRA. For this reason, considering applications of FBs to wavelet frames, the sequences of interest obey some conditions implied by function spaces. A detail discussion about FBs and their interconnection with wavelet theory can be found in [4] and [18].

The operation of convolution with a finite sequence $\{h_j\}$ (only a finite number of non-zero entries) can be represented either in the form $\sum_j x_j h_{k-j}$ or in z-domain as $X(z)H(z)$, where $X(z)$ is a formal Laurent series $X(z) := \sum_j x_j z^j$ and $H(z) := \sum_j h_j z^j$ is a Laurent polynomial.

A polynomial filter bank operator is defined by its modulation matrix as

$$\mathbf{M}(z) = \begin{pmatrix} H_0(z) & H_1(z) & \dots & H_r(z) \\ H_0(ze_d) & H_1(ze_d) & \dots & H_r(ze_d) \\ \dots & \dots & \dots & \dots \\ H_0(ze_d^{d-1}) & H_1(ze_d^{d-1}) & \dots & H_r(ze_d^{d-1}) \end{pmatrix},$$

where $H_k(z)$, $k = 0, 1, 2, \dots, r$ are Laurent polynomials, $e_d := e^{i2\pi/d}$. A modulation matrix $\tilde{\mathbf{M}}(z)$ generated by polynomials $\tilde{H}_k(z)$, $k = 0, 1, 2, \dots, r$ is called dual (or inverse) if it satisfies the identity

$$\mathbf{M}(z)\tilde{\mathbf{M}}^*(z) \equiv I_d, \quad |z| = 1, \tag{1}$$

where $*$ means the Hermite conjugation, I_d is the $d \times d$ identity matrix. When $r \geqslant d$, the choice of a dual matrix is not unique. This fact gives an opportunity for optimization of that choice, according to the needs of applied problems.

The operation

$$\vec{\mathbf{Y}}(z) = \tilde{\mathbf{M}}^*(z)\vec{\mathbf{X}}(z), \tag{2}$$

where $\vec{\mathbf{X}}(z) := (X(z), X(ze_d^1), \dots, X(ze_d^{d-1}))^T$ is called *decomposition* and the operator $\tilde{\mathbf{M}}(z)$ is called *analysis operator*. The inverse operation

$$\vec{\mathbf{X}}(z) = \mathbf{M}(z)\vec{\mathbf{Y}}(z) \tag{3}$$

is called *reconstruction* and the operator $\mathbf{M}(z)$ is called *synthesis operator*.

We note that $\vec{\mathbf{Y}}(z)$ depends only on z^d, whereas the matrix $\mathbf{M}(z)$ and the vector $\vec{\mathbf{X}}(z)$ contain a lot of repeating information. To avoid this redundancy the polyphase representation of the transform is used. Let us introduce the $d \times d$ matrix of discrete Fourier transform $\mathcal{F} = (\mathcal{F}_{k,j})_{0 \leqslant k,j<d}$, $\mathcal{F}_{k,j} = e_d^{-jk}$, $j,k = 0,\dots,d-1$, and the diagonal matrix $\mathbf{D}(z)$ with elements z^{-k}, $k = 0,\dots,d-1$, on the diagonal. It is well known that the matrix $\mathbf{D}(z)\mathcal{F}\mathbf{M}(z)$ consists of polynomials depending on z^d. So this matrix can be represented in another form $\mathbb{M}(z^d) = \mathbf{D}(z)\mathcal{F}\mathbf{M}(z)$, where components of the matrix $\mathbb{M}(z)$ are Laurent polynomials. The matrix $\mathbb{M}(z)$ is called *a polyphase matrix* of FB-transform. The polyphase matrix $\tilde{\mathbb{M}}(z)$ is defined in the same way from the modulation matrix $\tilde{\mathbf{M}}(z)$. Polyphase forms of $\vec{\mathbf{X}}(z)$ and $\vec{\mathbf{Y}}(z)$ are defined by formulas $\vec{\mathbb{X}}(z^d) := \mathbf{D}(z)\mathcal{F}\vec{\mathbf{X}}(z)$ and $\vec{\mathbb{Y}}(z^d) := \vec{\mathbf{Y}}(z)$. Because of the unitarity of the matrixes $\mathcal{F}$ and $\mathcal{D}(\omega)$, the equalities (1), (2), and (3) may be rewritten in the polyphase form

$$\mathbb{M}(z)\tilde{\mathbb{M}}^T\big(z^{-1}\big) = I_k, \tag{4}$$

$$\vec{\mathbb{Y}}(z) = \tilde{\mathbb{M}}^T\big(z^{-1}\big)\vec{\mathbb{X}}(z), \tag{5}$$

$$\vec{\mathbb{X}}(z) = \mathbb{M}(z)\vec{\mathbb{Y}}(z). \tag{6}$$

We note that the last relations are valid for any complex $z \neq 0$.

We denote the entries of $\mathbb{M}(z)$ and $\vec{\mathbb{X}}(z)$ by $H_{j,k}(z)$ and $X_k(z)$, $j = 0,1,\dots,r$, $k = 0,1,\dots,d-1$, respectively. Then $H_j(z) = \sum_k z^k H_{j,k}(z^d)$, $X(z) = \sum_k z^k X_k(z^d)$.

Interconnection between redundant FB and wavelet frame transform can be briefly described as follows. Suppose we have an MRA $\cdots \subset V^0 \subset V^1 \cdots V^k \subset \cdots$ generated by dilations and translates of a scaling function φ associated with a symbol $H_0(e^{i\omega})$. The symbols $H_l(e^{i\omega})$ generate framelets ψ_l and framelet spaces W_l^k with $W^k := W_1^k + \cdots + W_r^k$ and $V^{k+1} = V^k + W^k$, where sums of the spaces are not necessarily direct and elements of the spaces are not necessarily from $\mathbb{L}^2(\mathbb{R})$. If $\{x_k\}$ is a sequence of expansion coefficients in the space V^{k+1}. The first entry of $\mathbb{Y}(z)$ contains expansion coefficients in V^k, whereas remaining entries give decomposition in framelets.

In the follow-up discussion, we mainly deal either with algebraic polynomials or with basic scaling functions φ. In (2) and (3), in z-domain, any polynomial $p(t) \in V^k$ has a representation

$$X(z) = \sum_j q(j) z^j, \tag{7}$$

where $q(j)$ is some polynomial of the same degree as p. It follows from (3), that a scaling function $\varphi \in V^0$ obviously has z-representation $X(z) = H_0(z)$ in V^1.

In what follows, we keep using boldface and "blackboard bold" fonts to distinct modulation matrices and vectors and their polyphase forms correspondingly.

3. Objectives

Transformations (5) and (6) are main constructive blocks of data representation in wavelet frames. The perfect reconstruction property (4) is a component of the mixed extension principle. Its special case when $\tilde{\mathbb{M}}(z) = \mathbb{M}(z)$ corresponds to the unitary extension principle.

We start with reformulating main MRA principles in FB language. It is a well-known fact that any Laurent polynomial $H_0(z)$, $H_0(1) = 1$ generates a multiresolution consisting of compactly supported tempered distributions. For this reason $H_0(z)$ is called *a symbol* of MRA. A given MRA approximation order, i.e., belonging all polynomials of the degree up to $n-1$ to MRA, can be guaranteed by a multiplicity n of the factor $S(z) := 1 + z^{-1} + \cdots + z^{d-1}$ in the symbol $H_0(z)$. In spite the fact that necessary and sufficient condition in terms of zeros of the polynomial $H_0(z)$ is formulated in more sophisticated form, for scaling functions of interest in applications, the assumption that $S^n(z)$ is a factor of $H_0(z)$ for MRAs reproducing polynomials of the order $n-1$ is quite usual. In particular, it can be justified by the known fact that for an MRA with the polynomial reproduction property, the generating scaling function with the mentioned factorization property exists. Moreover, such a function has the minimum size of the support.

For wavelet frames the fact that the space of polynomials $\mathcal{P}_{n-1}$ (or any other subspace of distributions $\mathcal{D}$) belongs to the spaces V^i forming MRA does not mean that the analysis operator is really a projector. Moreover, even the analysis operator is a projector from V_1 to $\mathcal{D} \subset V_0$ it does not mean that it annihilates framelet coefficients of elements of $\mathcal{D} \subset V_0$. In [5], for the case when polynomials from $\mathcal{P}_{n-1}$ can be recovered with synthesis operator, using only coefficients of the decomposition in scaling function shifts, the notion *frame approximation order n* was introduced.

In what follows, we consider the methods for constructing an analysis operators dual to a fixed redundant FB, providing a perfect recovery for $\mathcal{D} \subset V^0 \subset V^1$ when either $\mathcal{D} = V^0$ or $\mathcal{D} = \mathcal{P}_{n-1}$. For each of those two cases we consider two options when an analysis operator annihilates framelet coefficients and when it does not. Thus, assuming that MRA reproduced polynomials from $\mathcal{P}_{n-1}$, we are going to describe analysis operators satisfying the following properties:

Property 1 *(Frame approximation order n).* If $\vec{\mathbb{X}}(z)$ corresponds to $p \in \mathcal{P}_{n-1}$, then $\vec{\mathbb{X}}(z) = (H_{0,0}(z), \ldots, H_{0,d-1}(z))^T \times \mathbb{Y}_0(z)$, where $\mathbb{Y}_0(z)$ is the first component of $\vec{\mathbb{Y}}(z)$ defined by (5).

Property 2 *(n frame vanishing moments).* If $\vec{\mathbb{X}}(z)$ corresponds to $p \in \mathcal{P}_{n-1}$, then $\mathbb{Y}_k(z) = 0$, $k = 1, \ldots, r$, where $\mathbb{Y}_k(z)$ are components of $\vec{\mathbb{Y}}(z)$ defined by (5).

Property 3. If $\vec{\mathbb{X}}(z)$ corresponds to $f \in V^0$, then $\vec{\mathbb{X}}(z) = (H_{0,0}(z), \ldots, H_{0,d-1}(z))^T \mathbb{Y}_0(z)$, where $\mathbb{Y}_0(z)$ is defined by (5).

Property 4. If $\vec{\mathbb{X}}(z)$ corresponds to $f \in V^0$, then $\mathbb{Y}_k(z) = 0$, $k = 1, \ldots, r$, where $\mathbb{Y}_k(z)$ are defined by (5).

4. Method of virtual components for parametrization of expansions in frames and analysis operators

We consider an arbitrary MRA generated by a compactly supported refinable distribution φ with a dilation factor $d \geqslant 2$ and a polynomial symbol $H_0(e^{i\omega})$. We assume that we have a set of framelets $\{\psi^k\}_{k=1}^r$, $r \geqslant d$, with polynomial symbols $\{H_k\}_{k=1}^r$ and the dual MRA generated by $\tilde{\varphi}$, $\{\tilde{\psi}^k\}_{k=1}^r$, $r \geqslant d$, with polynomial symbols $\{\tilde{H}_k\}_{k=0}^r$. Their modulation and polyphase matrices satisfy identities (1) and (4).

Taking into account that matrices in (4) are rectangular, the choice of $\tilde{\mathbb{M}}$ which defines expansion coefficients in the fixed frame is not unique. For a fixed $\mathbb{M}(z)$, we are going to describe a simple convenient method to obtain all possible linear analysis operators $\tilde{\mathbb{M}}(z)$ satisfying (4). Moreover, this method allows to find operators $\tilde{\mathbb{M}}(z)$ providing the maximum number of vanishing moments corresponding to the approximation order of the MRA generated by φ and some variations of that property formulated in Section 3.

Let us assume that we are given arbitrary modulation rectangular polynomial matrices $\mathbf{M}(z)$ and $\tilde{\mathbf{M}}(z)$ with the square extensions $\mathbf{M}_e(z)$ and $\tilde{\mathbf{M}}_e(z)$ satisfying the relation

$$\mathbf{M}_e(z)\tilde{\mathbf{M}}_e^*(z) = I, \quad |z| = 1. \tag{8}$$

Probably, the simplest constructive way to get those matrices is as follows. Introduce the polyphase matrix $\mathbb{M}(z)$ associated with $\mathbf{M}(z)$. Using the algorithm from [9], find the extension matrix $\mathbb{M}_e(z)$ with the determinant z^k, where k is an arbitrary integer number (for instance, $k = 0$). Assign $\tilde{\mathbb{M}}_e(z) := (\mathbb{M}_e^{-1}(z^{-1}))^T$. Define the matrix $\tilde{\mathbb{M}}(z)$ as the first d rows of the matrix $\tilde{\mathbb{M}}_e(z)$. Define the matrices

$$\mathbb{G}(z) := \begin{pmatrix} G_{0,1}(z) & G_{1,1}(z) & \ldots & G_{r,1}(z) \\ G_{0,2}(z) & G_{1,2}(z) & \ldots & G_{r,2}(z) \\ \ldots & \ldots & \ldots & \ldots \\ G_{0,r+1-d}(z) & G_{1,r+1-d}(z) & \ldots & G_{r,r+1-d}(z) \end{pmatrix}$$

and $\tilde{\mathbb{G}}(z)$ as the last $r + 1 - d$ rows of the matrices $\mathbb{M}_e(z)$ and $\tilde{\mathbb{M}}_e(z)$, respectively. Thus, we have

$$\mathbf{M}_e(z) = \begin{bmatrix} \mathbf{M}(z) \\ \mathbb{G}(z^d) \end{bmatrix}, \qquad \tilde{\mathbf{M}}_e(z) = \begin{bmatrix} \tilde{\mathbf{M}}(z) \\ \tilde{\mathbb{G}}(z^d) \end{bmatrix}, \qquad \mathbb{M}_e(z) = \begin{bmatrix} \mathbb{M}(z) \\ \mathbb{G}(z) \end{bmatrix}, \qquad \tilde{\mathbb{M}}_e(z) = \begin{bmatrix} \tilde{\mathbb{M}}(z) \\ \tilde{\mathbb{G}}(z) \end{bmatrix}.$$

We note that the existence of the extension $\mathbb{M}_e(z)$ follows from the fact that $\mathbb{M}(z)$ is a polyphase matrix of a wavelet frame, hence, its minors of the size $d \times d$ do not vanish simultaneously. The algorithm from [9] can be applied for constructing the extension. Here and in what follows, we use the subscript e for the extensions of a given matrix or a vector.

As mentioned above, due to the redundancy, $\vec{\mathbb{Y}}(z)$ is not a unique vector satisfying (6). It is clear that a vector $\vec{\mathcal{Y}}(z^d)$ satisfies (6) if and only if the difference $\Delta\vec{\mathbb{Y}}(z^{-d}) := \vec{\mathbb{Y}}(z^{-d}) - \vec{\mathcal{Y}}(z^{-d})$ is orthogonal to rows of the matrix $\mathbf{M}(z)$ (or $\Delta\vec{\mathbb{Y}}(z^{-1})$ is orthogonal to rows of the matrix $\mathbb{M}(z)$). Thus, it is easy to see that the vector $\Delta\vec{\mathbb{Y}}(z)$ can be represented as a linear combination of the extension rows of the matrix $\tilde{\mathbb{M}}(z^{-1})$ introduced above. The coefficients of the linear combinations either can be considered as given at the beginning or can be found as an inner products of $\Delta\vec{\mathbb{Y}}(z)$ with the extension rows of the matrix $\mathbb{M}(z)$. We note that since the components of $\mathbb{M}(z)$ are Laurent polynomials, then the components of $\Delta\vec{\mathbb{Y}}(z)$ can be any formal Laurent series with an arbitrary (even not necessarily bounded) sequence of coefficients. If we denote those coefficients by $\mathbb{X}_d(z), \ldots, \mathbb{X}_r(z)$ and extend the vector $\vec{\mathbb{X}}(z)$ with these coefficients up to the vector $\vec{\mathbb{X}}_e(z)$, we have a vector $\vec{\mathcal{Y}}(z) = \mathbb{M}_e^T(z)\vec{\mathbb{X}}_e(z)$ satisfying (6). Moreover, all sequences $\mathbb{X}_d(z), \ldots, \mathbb{X}_r(z)$ generate vectors $\vec{\mathcal{Y}}(z)$ satisfying (6).

We will call this approach *the virtual components* (*VC*) *method*.

Thus, we found the description all possible vectors $\vec{\mathcal{Y}}(z)$ in a parametric form through all possible extensions of the vector $\vec{\mathbb{X}}(z)$ up to the vector $\vec{\mathbb{X}}_e(z)$ of the dimension $r+1$. As it will be shown below, this parametrization is extremely convenient for optimization of decompositions in wavelet frames.

While that parametrization can be used for (non-linear) optimization of decompositions of individual functions, we put aside this problems for the future research and concentrate our attention on linear optimization which can be useful for function classes.

In what follows, we are interested in the choice of VCs linearly depended on the input. Because of the natural requirement that those components depend only on z^d, an arbitrary linear operator can be represented in the form

$$\big(\mathbb{X}_d\big(z^d\big), \ldots, \mathbb{X}_r\big(z^d\big)\big)^T = \mathbf{F}(z)\vec{\mathbf{X}}(z) = \mathbb{F}\big(z^d\big)\vec{\mathbb{X}}\big(z^d\big),$$

where $\mathbf{F}(z)$ is a matrix of the size $(r-d+1)\times(r+1)$ whose elements are defined by the formula $\mathbf{F}_{k,j}(z) = \mathbf{F}_k(ze_d^j)$, $\mathbf{F}_k$ are arbitrary Laurent polynomials. The more general case when $\mathbf{F}_k$ may belong to some subclass of Laurent series deserves the consideration as well. However, within the framework of our paper we restrict ourselves with the polynomial case providing an opportunity to work with convolutions of the data with finite sequences.

5. Analysis operators with maximum number of vanishing moments

We give a method for "the correction" of a dual FB allowing to construct a dual filter bank with the maximum number of vanishing moments defined by the MRA approximation order.

Theorem 1. *Let $\mathbf{M}(z)$ be a modulation matrix of the size $(r+1)\times d$ and of the rank d, $d < r+1$, generated by Laurent polynomials $H_i(z)$, $i = 0, 1, \ldots r$, where $H_0(z) = (1 + z + \cdots + z^{d-1})^n R(z)$, $R(1) = d^{-n}$. Then there exists a dual polynomial matrix $\tilde{\mathbf{M}}(\omega)$ defining an analysis operator with n vanishing moments.*

The statement of Theorem 1 is quite elementary and probably may be proved with different methods. However, the VC method gives a convenient constructive method for computationally efficient implementation of analysis operators for the fixed $\mathbf{M}(\omega)$ in a parametric form.

Proof. Since minors of the size $d\times d$ of the matrix $\mathbf{M}(z)$ cannot turn into 0 simultaneously for any $z \neq 0$, there exist (cf. Section 4) polynomial matrices $\mathbf{M}_e(z)$, $\tilde{\mathbf{M}}_e(z)$ satisfying (8).

We now find an algorithm for the correction of the matrix $\tilde{\mathbf{M}}(z)$ increasing the number of the vanishing moments up to the optimal value n.

Suppose the polynomials $\{H_0(ze_d^k)\}_{k=0}^{d-1}$ do not have common roots. Then the equations

$$\sum_{k=0}^{d-1} F_j\big(ze_d^k\big)H_0\big(ze_d^k\big) = G_{0,j}\big(z^d\big) + \big(1-z^d\big)^n A_j\big(z^d\big), \quad j = 0, 1, \ldots, r-d+1, \tag{9}$$

have (not unique) solutions $F_j(z)$ for any $G_{0,j}(z)$and $A_j(z)$. Of course, solutions to (9) can be more easily found in the polyphase form

$$\sum_{k=0}^{d-1} F_{j,k}(z)H_{0,k}(z) = G_{0,j}(z) + (1-z)^n A_j(z), \quad j = 0, 1, \ldots, r-d+1, \tag{10}$$

where $F_{j,k}$ and $H_{0,k}$ are polyphase components of F_j and H_0. In the case of the common roots, A_j cannot be chosen absolutely free. However, if the polynomials $G_{0,j}(z) + (1-z)^n A_j(z)$ has the same roots with the same multiplicity, then Eqs. (10) still have solutions.

We assign the formal Laurent series

$$\mathbb{X}_j\big(z^d\big) := \sum_{k=0}^{d-1} F_j\big(ze_d^k\big)\mathbf{X}\big(ze_d^k\big) \tag{11}$$

or

$$\mathbb{X}_j(z) := \sum_{k=0}^{d-1} F_{j,k}(z)\mathbb{X}_k(z), \quad j = d, \ldots, r, \tag{12}$$

to be the "signal" extension components.

Let $p(x)$ be an algebraic polynomial of the degree $n-1$. Then in z-domain, it can be represented as (7). In this section, we suppose that $\mathbf{X}(z)$ is a polynomial input for an analysis operator.

We note that $L(z)\mathbf{X}(z) = 0$ for a Laurent polynomial $L(z)$ if and only if $L(z)$ has a root $z = 1$ of the multiplicity n. The polynomial $H_0(z)$ has a factor $1 + z + \cdots + z^{d-1} = (1-z^d)/(1-z)$ of the multiplicity n. So $H_0(z)$ has roots of the multiplicity n at the points $z = e_d^k$, $k = 1, \ldots, d-1$. In particular, it means that

$$H_0\big(ze_d^k\big)\mathbf{X}\big(ze_d^l\big) = 0, \quad k \neq l \bmod d. \tag{13}$$

We need to prove that only the first component of the vector $\tilde{\mathbf{M}}_e(1/z)\vec{\mathbf{X}}_e(z)$ is not equal to 0. Instead of verifying this property for the remaining components, we choose a different strategy. First we compose a vector $\vec{\mathbf{Y}}(z)$ whose first component coincides with the first component of the vector $\tilde{\mathbf{M}}_e^*(z)\vec{\mathbf{X}}_e(z)$ and remaining components are equal to 0. Then we show that $\mathbf{M}_e(z)\vec{\mathbf{Y}}(z) = \vec{\mathbf{X}}_e(z)$. Since $\mathbf{M}_e(z)$ is a non-degenerate matrix, this implies the equality $\vec{\mathbf{Y}}(z) = \tilde{\mathbf{M}}_e(1/z)\vec{\mathbf{X}}_e(z)$.

The first component of the vector $\mathbf{Y}(z)$ can be represented in the form

$$\begin{aligned}
\mathbb{Y}_1\big(z^d\big) = \mathbf{Y}_1(z) &= \sum_{k=0}^{d-1} \tilde{H}_0\big(z^{-1}e_d^{-k}\big)\mathbf{X}\big(ze_d^k\big) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}\big(z^{-d}\big)\mathbb{X}_{d+j-1}\big(z^d\big) \\
&= \sum_{k=0}^{d-1} \tilde{H}_0\big(z^{-1}e_d^{-k}\big)\mathbf{X}\big(ze_d^k\big) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}\big(z^{-d}\big)\sum_{k=0}^{d-1} F_j\big(ze_d^k\big)\mathbf{X}\big(ze_d^k\big) \\
&= \sum_{k=0}^{d-1}\left(\tilde{H}_0\big(z^{-1}e_d^{-k}\big) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}\big(z^{-d}\big)F_j\big(ze_d^k\big)\right)\mathbf{X}\big(ze_d^k\big).
\end{aligned} \tag{14}$$

We need to check the validity of the equalities

$$\mathbf{X}\big(ze_d^l\big) = H_0\big(ze_d^l\big)\sum_{k=0}^{d-1}\left(\tilde{H}_0\big(z^{-1}e_d^{-k}\big) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}\big(z^{-d}\big)F_j\big(ze_d^k\big)\right)\mathbf{X}\big(ze_d^k\big), \quad l = 0, \ldots, d-1, \tag{15}$$

$$\mathbb{X}_{d+q-1}\big(z^d\big) = G_{0,q}\big(z^d\big)\sum_{k=0}^{d-1}\left(\tilde{H}_0\big(z^{-1}e_d^{-k}\big) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}\big(z^{-d}\big)F_j\big(ze_d^k\big)\right)\mathbf{X}\big(ze_d^k\big), \quad q = 1, \ldots, r+1-d. \tag{16}$$

Note that since the factor with $H_0(ze_d^l)$ in (15) depends only on z^d, we need to prove (15) only for $l = 0$.

Due to (13), we have the elementary identities

$$\mathbf{X}(z)H_0(z)\tilde{H}_0\big(z^{-1}\big) = \mathbf{X}(z)\sum_{k=0}^{d-1} H_0\big(ze_d^k\big)\tilde{H}_0\big(z^{-1}e_d^{-k}\big)$$

and

$$\mathbf{X}(z)H_0(z)F_j(z) = \mathbf{X}(z)\sum_{k=0}^{d-1} H_0\big(ze_d^k\big)F_j\big(ze_d^k\big).$$

Indeed, by (13), the products of $\mathbf{X}(z)$ with all components of the sums in the right part are equal to 0 once $k \neq 0$. Thus, we have

$$H_0(z)\sum_{k=0}^{d-1}\left(\tilde{H}_0\left(z^{-1}e_d^{-k}\right)+\sum_{j=1}^{r+1-d}\tilde{G}_{0,j}\left(z^{-d}\right)F_j\left(ze_d^k\right)\right)\mathbf{X}\left(ze_d^k\right)$$
$$=H_0(z)\left(\tilde{H}_0\left(z^{-1}\right)+\sum_{j=1}^{r+1-d}\tilde{G}_{0,j}\left(z^{-d}\right)F_j(z)\right)\mathbf{X}(z)$$
$$=\left(\sum_{k=0}^{d-1}H\left(ze_d^k\right)\tilde{H}_0\left(z^{-1}e_d^{-k}\right)+\sum_{j=1}^{r+1-d}\tilde{G}_{0,j}\left(z^{-d}\right)\sum_{k=0}^{d-1}H_0\left(ze_d^k\right)F_j\left(ze_d^k\right)\right)\mathbf{X}(z)$$
$$=\left(\sum_{k=0}^{d-1}H\left(ze_d^k\right)\tilde{H}_0\left(z^{-1}e_d^{-k}\right)+\sum_{j=1}^{r+1-d}\tilde{G}_{0,j}\left(z^{-d}\right)\left(G_{0,j}\left(z^d\right)+\left(1-z^d\right)^nA_j\left(z^d\right)\right)\right)\mathbf{X}(z)$$
$$=\mathbf{X}(z)+\sum_{j=1}^{r+1-d}\tilde{G}_{0,j}\left(z^{-d}\right)\left(1-z^d\right)^nA_j\left(z^d\right)\mathbf{X}(z)=\mathbf{X}(z).$$

To prove equalities (16) we transform them to equivalent ones. We subtract linear combinations of (15) with the coefficients $F_q(ze_d^l)$ from the equality (16) with the number q. The equivalent form is

$$0=-\left(1-z^d\right)^nA_q\left(z^d\right)\sum_{k=0}^{d-1}\left(\tilde{H}_0\left(z^{-1}e_d^{-k}\right)+\sum_{j=1}^{r+1-d}\tilde{G}_{0,j}\left(z^{-d}\right)F_j\left(ze_d^k\right)\right)\mathbf{X}\left(ze_d^k\right),\quad q=1,\dots,r+1-d. \tag{17}$$

So their validity follows just from the fact that $1-z^d$ has the roots e_d^k, $k=0,\dots,d-1$ and $\mathbf{X}(z)$ is the z-transform of a polynomial of the degree $n-1$. □

In the next section we show that formulas (11) and (9) describe all possible analysis operators with n vanishing moments.

6. Related results

Theorem 1 guarantees the existence of the analysis operator with the following properties:

(1) *The approximation order of the analysis operator is equal to* n, i.e., polynomials up to the order $n-1$ can be recovered using the coefficients of $\mathbb{Y}_0(z)$ by the formula $\mathbf{X}(z)=H_0(z)\mathbb{Y}_0(z^d)$.
(2) *The analysis operator has vanishing moments of order* n, i.e., polynomials up to the order $n-1$ have zero framelet coefficients.

Now we give necessary and sufficient conditions within the VC method for each of 4 properties listed in Section 3. In particular, we prove that our choice of the coefficients in (11) satisfying (9) is not only sufficient but also necessary to provide the maximum number of vanishing moments.

We introduce the vectors $\vec{\mathbb{G}}_0(z)$ and $\vec{\tilde{\mathbb{G}}}_0(z)$ which are the first columns of the matrices $\mathbb{G}(z)$ and $\tilde{\mathbb{G}}(z)$ and the vectors $\vec{\mathbf{H}}_0(z):=(H_0(z),\dots,H_0(ze_d^{d-1}))^T$ and $\vec{\tilde{\mathbf{H}}}_0(z):=(\tilde{H}_0(z),\dots,\tilde{H}_0(ze_d^{d-1}))^T$.

Let us suppose that the polynomials $F_j(z)$ generating the matrix $\mathbf{F}(z)$ are arbitrary and, generally speaking, do not satisfy (9). Then the right part of (11) still can be represented in the matrix form as $\mathbf{F}(z)\vec{\mathbf{X}}(z)$, where the (j,k)th component of the matrix $\mathbf{F}(z)$ is $F_j(ze_d^k)$. Then, instead of (9), we have

$$\mathbf{F}(z)\vec{\mathbf{H}}_0(z)=\vec{\mathbb{G}}\left(z^d\right)+\vec{\mathbb{P}}\left(z^d\right),$$

for some vector $\vec{\mathbb{P}}(z)$ with polynomial entries.

We start with finding a criterion for Property 3 in terms of the vector $\vec{\mathbb{P}}(z)$.

Let a formal Laurent series $\mathbf{X}(z)$ be the z-transform of coefficients of the decomposition of some distribution $f(t)$ in the translates of the scaling function in V^1. Obviously, if this distribution belongs to V^0, then we have the representation

$$\mathbf{X}(z) = H_0(z)\mathbb{Q}(z^d),$$

for some formal Laurent series $\mathbb{Q}(z)$. In particular, if $f(z) = \varphi(x)$, we have $\mathbb{Q}(z) \equiv 1$.

Representing (14) in the matrix form, we have

$$\begin{aligned}
\mathbb{Y}_0(z^d) &= \vec{\mathbf{X}}^T(z)\big(\vec{\tilde{\mathbf{H}}}_0(1/z) + \mathbf{F}^T(z)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\big) = \mathbb{Q}(z^d)\vec{\mathbf{H}}_0^T(z)\big(\vec{\tilde{\mathbf{H}}}_0(1/z) + \mathbf{F}^T(z)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\big) \\
&= \mathbb{Q}(z^d)\big(\vec{\mathbf{H}}_0^T(z)\vec{\tilde{\mathbf{H}}}_0(1/z) + \vec{\mathbf{H}}_0^T(z)\mathbf{F}^T(z)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\big) \\
&= \mathbb{Q}(z^d)\big(\vec{\mathbf{H}}_0^T(z)\vec{\tilde{\mathbf{H}}}_0(1/z) + \big(\vec{\mathbb{G}}_0^T(z^d) + \vec{\mathbb{P}}^T(z^d)\big)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\big) \\
&= \mathbb{Q}(z^d)\big(1 + \vec{\mathbb{P}}^T(z^d)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\big).
\end{aligned}$$

Thus, (15) can be rewritten in the form

$$\vec{\mathbf{H}}_0(z)\mathbb{Q}(z^d) = \vec{\mathbf{H}}_0(z)\mathbb{Q}(z^d)\big(1 + \vec{\mathbb{P}}^T(z^d)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\big)$$

or

$$\vec{\mathbf{H}}_0(z)\mathbb{Q}(z^d)\vec{\mathbb{P}}^T(z^d)\vec{\tilde{\mathbb{G}}}_0(z^{-d}) = 0. \tag{18}$$

Recall that the validity of the relation (18) means that a distribution from V^0 represented by $\mathbb{Q}(z)$ is mapped to itself by the first component of the analysis operator. Let $\mathbb{Q}(z)$ be a function summable on the unit circle. Since $H_0(z)$ and $\vec{\mathbb{P}}^T(z^d)\vec{\tilde{\mathbb{G}}}_0(z^{-d})$ are Laurent polynomials, (18) takes place only if

$$\vec{\mathbb{P}}^T(z)\vec{\tilde{\mathbb{G}}}_0(1/z) = 0, \tag{19}$$

i.e., $\vec{\mathbb{P}}(z)$ has to be orthogonal to $\vec{\tilde{\mathbb{G}}}_0(z)$. However, such a choice of $\vec{\mathbb{P}}(z)$ guarantees the same projection property for any distribution $\mathbb{Q}(z)$, i.e., the first component of the analysis operator provides us with a projector from V^1 to V^0. Thus, (19) is necessary and sufficient for Property 3. Of course, the condition (19) is sufficient to provide the maximum approximation order for the analysis operator.

At the same time, (19) is not necessary for Property 1 since the analysis operator provides the approximation order n if the operator above is a projector only on algebraic polynomials up to the order $n-1$. For polynomials of the order $n-1$, the periodic distribution $\mathbb{Q}(e^{i\omega})$ is a linear combination of the δ-function and its derivatives up to the order $n-1$ with the support at $\omega = 0$. Since $\mathbf{H}_0(1) \neq 0$, (18) takes place if and only if the polynomial $\vec{\mathbb{P}}^T(e^{i\omega})\vec{\tilde{\mathbb{G}}}(e^{-i\omega})$ and its derivatives up to the order $n-1$ are equal to 0 at the point $\omega = 0$. Thus,

$$\vec{\mathbb{P}}^T(z)\vec{\tilde{\mathbb{G}}}_0(1/z) = (1-z)^n P(z), \tag{20}$$

where $P(z)$ is an arbitrary Laurent polynomial. Condition (20) is necessary and sufficient to provide the frame approximation order n, i.e., for Property 1. If the associated framelets form a Bessel system, (20) means that the approximation order with truncated expansions in the obtained bi-frame is equal to n.

In Section 5, we found a sufficient condition for n vanishing moments of dual framelets provided that the approximation order of the MRA is not less than n. Now we prove the necessity of that condition.

The condition (20) is necessary and sufficient to satisfy the group of equalities (15). In other words, it guarantees the recovery of a polynomial using only $\mathbb{Y}_0(z^d)$. Conditions (16) has no influence on the recovery properties. However, they prohibit from the leakage of coefficients to framelet spaces. The essence of this effect is as follows. If for elements of V^0 condition (15) is valid but (16) is not, then, applying the analysis operator to V^0, we may have non-zero framelet coefficients. At the same time, using those non-zero coefficients as an input for the synthesis operator, we have the zero output. This effect is undesirable when a compact representation of a function is a priority.

If $\vec{\mathbf{X}}(z) = \vec{\mathbf{H}}_0(z)\mathbb{Q}(z^d)$ the condition (16) can be rewritten in the form

$$\mathbf{F}(z)\vec{\mathbf{H}}_0(z)\mathbb{Q}(z^d) = \vec{\mathbb{G}}_0(z^d)\mathbb{Q}(z^d)\big(1 + \vec{\mathbb{P}}^T(z^d)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\big)$$

or

$$\big(\vec{\mathbb{G}}_0\big(z^d\big)+\vec{\mathbb{P}}\big(z^d\big)\big)\mathbb{Q}\big(z^d\big)=\vec{\mathbb{G}}_0\big(z^d\big)\mathbb{Q}\big(z^d\big)\big(1+\vec{\mathbb{P}}^T\big(z^d\big)\vec{\tilde{\mathbb{G}}}_0\big(z^{-d}\big)\big).$$

Hence,

$$\vec{\mathbb{P}}(z)\mathbb{Q}(z)=\vec{\mathbb{G}}_0(z)\mathbb{Q}(z)\vec{\mathbb{P}}^T(z)\vec{\tilde{\mathbb{G}}}_0(1/z). \tag{21}$$

Let $\mathbb{Q}(z)\not\equiv 0$ represent a function summable on the unit circle with. Due to (19), the right part of (21) is equal to 0. Hence, $\vec{\mathbb{P}}(z)\equiv\vec{0}$ is a necessary (and sufficient) condition for Property 4 under which the analysis operator is the identity operator on the spaces V^j.

If $\mathbb{Q}(z)$ is a distribution representing a polynomial of the order n, then, by (20), the right part of (21) is equal to zero again. Therefore, the components of the vector $\vec{\mathbb{P}}(z)$ have to have zeros of the multiplicity n at the point $z=1$. Thus, the method we used in the proof of Theorem 1 gives all possible analysis operators providing n vanishing moments.

It means that for at least one $j=J$ we have

$$\sum_{k=0}^{d-1}F_J\big(ze_d^k\big)H_0\big(ze_d^k\big)=G_{0,J}\big(z^d\big)+\big(1-z^d\big)^N B\big(z^d\big),$$

where $N<n$, $B(1)\neq 0$. Thus, (17) turns into

$$0=-\big(1-z^d\big)^N B\big(z^d\big)\sum_{k=0}^{d-1}\Bigg(\tilde{H}_0\big(z^{-1}e_d^{-k}\big)+\sum_{j=1}^{r+1-d}\tilde{G}_{0,j}\big(z^{-d}\big)F_j\big(ze_d^k\big)\Bigg)\mathbf{X}\big(ze_d^k\big).$$

Summarizing the reasonings above, we can formulate the following theorem.

Theorem 2. *Let $\mathbb{M}_e(z)$, $\det\mathbb{M}_e(z)=z^k$, be an extension of the polyphase matrix of a wavelet frame with an MRA approximation order n,*

$$\tilde{\mathbb{M}}_e:=\begin{bmatrix}\tilde{\mathbb{M}}(z)\\ \tilde{\mathbb{G}}(z)\end{bmatrix}:=\mathbb{M}_e^{-1}\big(z^{-1}\big).$$

Then all possible polynomial analysis operators $\tilde{\mathbb{M}}^\#(z)$ linearly depending on the input can be represented in the form

$$\tilde{\mathbb{M}}^\#(z)=\tilde{\mathbb{M}}(z)+\mathbb{F}^T\big(z^{-1}\big)\tilde{\mathbb{G}}(z) \tag{22}$$

with an arbitrary polynomial matrix $\mathbb{F}(z)$.

Let $\vec{\mathbb{P}}(z)=\mathbb{F}(z)\vec{\mathbb{H}}_0(z)-\vec{\mathbb{G}}_0(z)$. The analysis operator $\tilde{\mathbb{M}}^\#(z)$ satisfies:

- (i) *Property 1 if and only if $\vec{\mathbb{P}}^T(z)\vec{\tilde{\mathbb{G}}}_0(z)=(1-z)^nP(z)$, where $P(z)$ is an arbitrary Laurent polynomial;*
- (ii) *Property 2 if and only if $\vec{\mathbb{P}}(z)=(1-z)^n\vec{P}(z)$, where $\vec{P}(z)$ is an arbitrary Laurent polynomial vector;*
- (iii) *Property 3 if and only if $\vec{\mathbb{P}}^T(z)\vec{\tilde{\mathbb{G}}}_0(z)=0$;*
- (iv) *Property 4 if and only if $\vec{\mathbb{P}}(z)\equiv\vec{0}$.*

Note that, while the matrix $\tilde{\mathbb{M}}^\#(z)$ may have a greater degree than the original matrix $\tilde{\mathbb{M}}(z)$, in many practically important cases, the matrices $\mathbb{F}(z)$ and $\tilde{\mathbb{G}}(z)$ have low dimensions. For this reason, sometimes formula (22) allows to reduce the computational costs significantly if to compute $\vec{\mathbb{Y}}(z)$ as $\tilde{\mathbb{M}}^T(1/z)\vec{\mathbb{X}}(z)+\tilde{\mathbb{G}}^T(1/z)\mathbb{F}(z)\vec{\mathbb{X}}(z)$ instead of the direct computation $\tilde{\mathbb{M}}^{\#T}(1/z)\vec{\mathbb{X}}(z)$.

7. Examples

In this section, for piecewise linear and piecewise parabolic tight frames, we design parametric families of analysis operators having 2 and three vanishing moments correspondingly. All presented results combine the symmetry, polynomial FB implementation, and the minimum number (for diadic framelets, this number is 2) of frame generators.

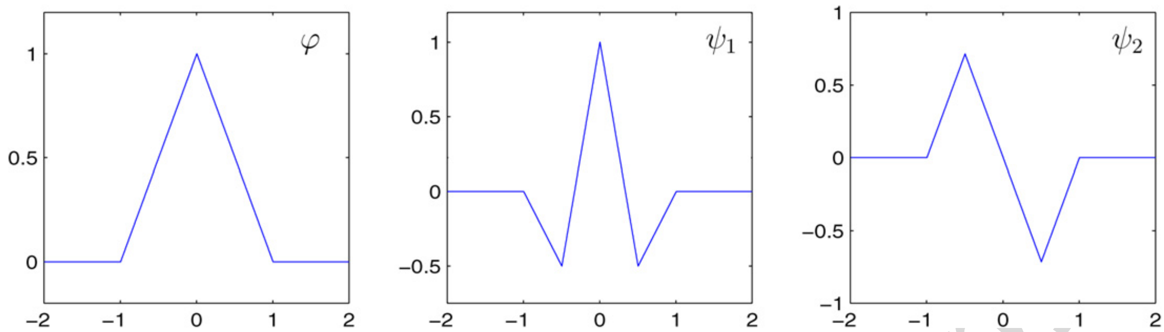

Fig. 1. Piecewise linear tight frame.

Example 1. We consider the tight frame generated by the piecewise linear B-spline ("the hat-function") with the symbol $H_0(z) = (z^{-1} + 2 + z)/4$. In the polyphase form

$$H_0(z) = \frac{1}{\sqrt{2}}\big(H_{0,0}(z^2) + z^{-1}H_{0,1}(z^2)\big),$$

where $H_{0,0}(z) = 1/\sqrt{2}$, $H_{0,1}(z) = 1 + z/2\sqrt{2}$. The standard choice of framelets gives a polyphase matrix

$$\mathbb{M}(z) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1+z}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1+z}{2\sqrt{2}} \\ 0 & \frac{1-z}{2} \end{pmatrix}$$

of a tight frame (cf. [14]). The graphs of the scaling function and framelets are given in Fig. 1.

Let us extend the matrix $\mathbb{M}(z)$ up to the matrix

$$\mathbb{M}_e(z) := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1+z}{2\sqrt{2}} & -\frac{1-z}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1+z}{2\sqrt{2}} & \frac{1-z}{2\sqrt{2}} \\ 0 & \frac{1-z}{2} & -\frac{1+z}{2} \end{pmatrix}.$$

The matrix $\mathbb{M}_e(z)$ is paraunitary. Therefore, $\tilde{\mathbb{M}}(z) = \mathbb{M}(z)$, $\tilde{\mathbb{G}}_0(z) = \mathbb{G}_0(z) := -(1-z)/(2\sqrt{2})$, where we use the notation from Section 4, turning down the vector symbols for $\mathbb{G}_0$ and $\tilde{\mathbb{G}}_0$.

We are going to describe polyphase matrices of minimal degrees defining dual filter banks providing two vanishing moments. We mention that even for $\tilde{\mathbb{M}}(z) = \mathbb{M}(z)$ the frame approximation order is equal to 2 (cf. [5]).

We need to solve Eq. (10) for a various choice of $A_0(z)$. (Anti)symmetric solutions of the minimal degree is a special subject of our interest. For this reason, we consider a special choice $A_0(z) = A(1 - z^{-1})/2\sqrt{2}$, where A is an arbitrary constant. The norming factor $1/2\sqrt{2}$ is chosen for our convenience. Thus, we have the equation

$$2F_{0,0}(z) + F_{0,1}(z)(1+z) = z - 1 + A \cdot (1-z)^2\big(1 - z^{-1}\big). \tag{23}$$

First let us assume $A = 0$. Then we have $\mathbb{P}(z) \equiv 0$, i.e., Property 4 takes place. The simplest solution to (23) is $F_{0,0}(z) \equiv -1$, $F_{0,1}(z) \equiv 1$. This solution gives the VCs component $\mathbb{X}_3(z) = \mathbb{X}_2(z) - \mathbb{X}_1(z)$. It is clear that the degree of the corrected dual modulation matrix is greater by one than the degree of the initial matrix $\tilde{\mathbf{M}}(z)$. However, the components of the corrected matrix are not symmetric any more. The simplest (anti)symmetric filter banks are generated by the solution $F_{0,0}(z) = -\frac{1}{2}(1-z)$, $F_{0,1}(z) \equiv 0$. Then the corrected filters can be represented by formulas

$$\tilde{H}_0^{\#}(z) = \tilde{H}_0(z) + \frac{2 - z^{-2} - z^2}{8} = \frac{3}{4} + \frac{z^{-1} + z}{4} - \frac{z^{-2} + z^2}{8},$$

$$\tilde{H}_1^{\#}(z) = \tilde{H}_1(z) - \frac{2 - z^{-2} - z^2}{8} = \frac{1}{4} - \frac{z^{-1} + z}{4} + \frac{z^{-2} + z^2}{8} = \frac{1}{8}(1-z)^2\big(z^{-2} + 1\big),$$

$$\tilde{H}_2^{\#}(z) = \tilde{H}_2(z) + \frac{-z^{-2} + z^2}{4\sqrt{2}} = \frac{z^{-1} - z}{2\sqrt{2}} + \frac{-z^{-2} + z^2}{4\sqrt{2}} = \frac{1}{4\sqrt{2}}(1-z)^2\big(-z^{-2} + 1\big).$$

The scaling function and the framelets are shown in Fig. 2.

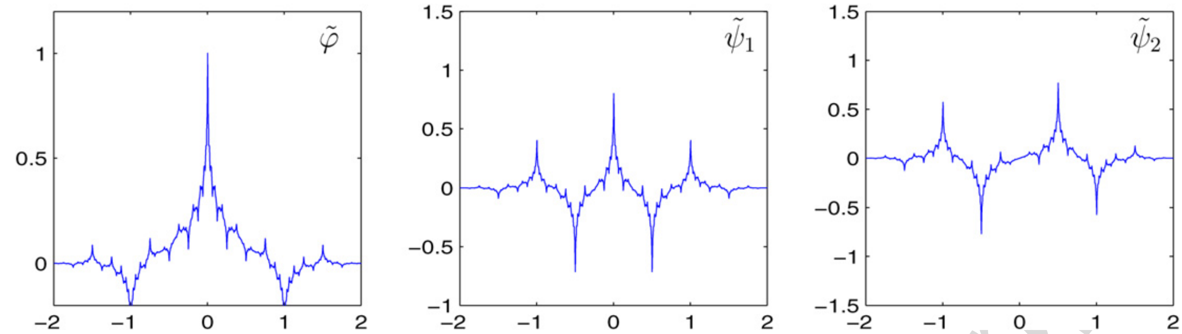

Fig. 2. Dual almost bi-orthogonal frame.

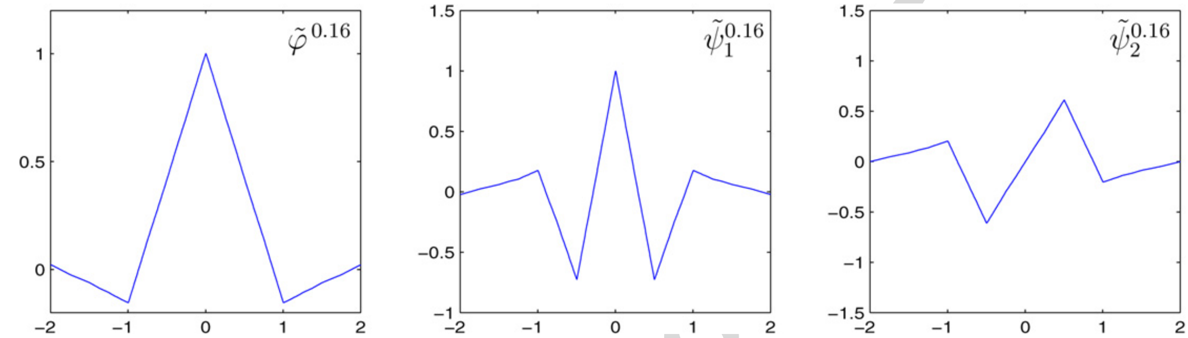

Fig. 3. Dual frame with 2 vanishing moments, $A = 0.16$.

Obviously, the obtained pair of the scaling functions coincides with the famous pair generating 5/3 bi-orthogonal wavelet basis (cf. [1,4]). This case is degenerate. The minor of $\tilde{\mathbb{M}}^{\#}(z)$ corresponding to framelets is equal to zero. We see that both $\tilde{H}_1^{\#}(z)$ and $\tilde{H}_2^{\#}(z)$ have the common factor $(1-z)^2$. At the same time, complementary factors depend on z^2. It means that wavelet spaces $\tilde{W}^1$ and $\tilde{W}^2$ coincide and, moreover, they coincide with the space W^* generated by a wavelet with a symbol $(1-z)^2$, i.e., the classical wavelet from the 5/3 pair. Note that the original piecewise linear tight frame does not possess that property.

If $A \neq 0$, the preserving symmetry solution to (23) of minimal degree has the form $F_{0,0} = -\frac{1}{2}(1-z) + 2(1-z)A$, $F_{0,1} = -(1-z)(1+z^{-1})A$. Thus, the corrected symbols of the dual scaling function and the framelets can be represented in the form

$$\begin{aligned}
\tilde{H}_0^A(z) &= \tilde{H}_0^{\#}(z) - A\frac{2-z^{-2}-z^2}{2} + A\frac{(2-z^{-2}-z^2)(z^{-1}+z)}{4}\\
&= \frac{3-4A}{4} + (1+A)\frac{z^{-1}+z}{4} - (1-4A)\frac{z^{-2}+z^2}{8} - A\frac{z^{-3}+z^3}{4},\\
\tilde{H}_1^A(z) &= \tilde{H}_1^{\#}(z) + A\frac{2-z^{-2}-z^2}{2} - A\frac{(2-z^{-2}-z^2)(z^{-1}+z)}{4}\\
&= \frac{1+4A}{4} - (1+A)\frac{z^{-1}+z}{4} + (1-4A)\frac{z^{-2}+z^2}{8} + A\frac{z^{-3}+z^3}{4},\\
\tilde{H}_2^A(z) &= \tilde{H}_2^{\#}(z) + A\frac{z^{-2}-z^2}{\sqrt{2}} + A\frac{(z-z^{-1})(2+z^{-2}+z^2)}{2\sqrt{2}}\\
&= (1-A)\frac{z^{-1}-z}{2\sqrt{2}} + (1-4A)\frac{-z^{-2}+z^2}{4\sqrt{2}} - A\frac{z^{-3}+z^3}{2\sqrt{2}}.
\end{aligned}$$

Examples of dual wavelet frames for $A = 0.16,\ 0.2$ are given in Figs. 3 and 4.

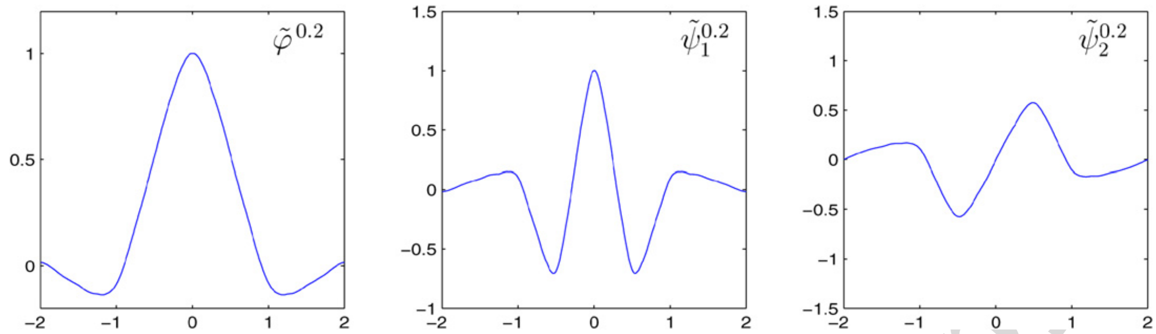

Fig. 4. Dual frame with 2 vanishing moments, $A = 0.2$.

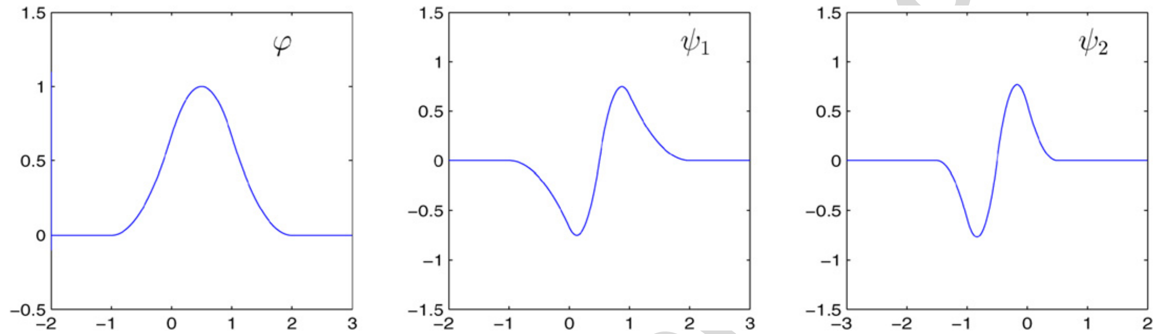

Fig. 5. Piecewise parabolic tight frame.

Example 2. Now we consider the tight frame generated by piecewise parabolic splines [2,13]. The corresponding symbols generate the extended polyphase matrix

$$\mathbb{M}_e(z) = \sqrt{2}\begin{pmatrix} \frac{3+z}{8} & \frac{1+3z}{8} & \sqrt{6}\frac{-1+z}{8} \\ \frac{-3+z}{8} & \frac{-1+3z}{8} & \sqrt{6}\frac{1+z}{8} \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{2}}{4} \end{pmatrix}.$$

The graphs of framelets are given in Fig. 5.

The correcting coefficients can be computed from the equation

$$F_{0,0}(z)(3+z) + F_{0,1}(z)(1+3z) = \sqrt{6}(-1+z) + A\cdot(1-z)^3. \tag{24}$$

If $A = 0$, the simplest solution of (24) is $F_{0,0}(z) = -\sqrt{6}/2$, $F_{0,1}(z) = \sqrt{6}/2$. This correction leads to a symmetric scaling function (distribution) and a framelet with the symbols

$$\tilde{H}_0^{\#}(z) = \frac{3}{4}(1+z) - \frac{1}{4}\big(z^{-1}+z^2\big), \qquad \tilde{H}_1^{\#}(z) = \frac{3}{4}(-1+z) + \frac{1}{4}\big(z^{-1}-z^2\big).$$

Unfortunately, the symbol $\tilde{H}_0^{\#}(z)$ above generates a scaling function with the non-integrable square (cf. [4, Section 8.3.4]). Note that since the symbols $H_1(z)$ and $\tilde{H}_0^{\#}(z)$ are orthogonal, we have $\tilde{H}_2^{\#}(z) = 0$. Thus, $H_0(z)$, $H_1(z)$, $\tilde{H}_0^{\#}(z)$, $\tilde{H}_1^{\#}(z)$ form a quadruple of biorthogonal symbols.

To construct a genuine symmetric dual frame we need to allow the larger support of framelets. All such dual frames with the mask support of the length 8 can be represented in the form

$$\tilde{H}_0^{\#}(z) = \left(\frac{3}{4} - A(2-B)\right)(1+z) + \left(A(3+3B) - \frac{1}{4}\right)\big(z^{-1}+z^2\big) - A(1+3B)\big(z^{-2}+z^3\big) + AB\big(z^{-3}+z^4\big),$$

$$\tilde{H}_1^{\#}(z) = \left(\frac{3}{4} - A(4+7B)\right)(-1+z) + \left(\frac{1}{4} - A(3+5B)\right)\big(z^{-1}-z^2\big) + A(1+3B)\big(z^{-2}-z^3\big) + AB\big(z^{-3}-z^4\big),$$

$$\tilde{H}_2^{\#}(z) = \frac{2A}{\sqrt{3}}\big((3+4B)\big(-z^{-1}+1\big) + (1+3B)\big(z^{-2}-z\big) - B\big(z^{-3}-z^2\big)\big),$$

where A and B are arbitrary parameters. We present a few relatively smooth examples.

We note that for $B = 0$ we have shorter supports. The choice $A = 0.08$ gives wavelets close to piecewise constant functions (Fig. 6). For $A = 0.09$ we have much smoother framelets (Fig. 7).

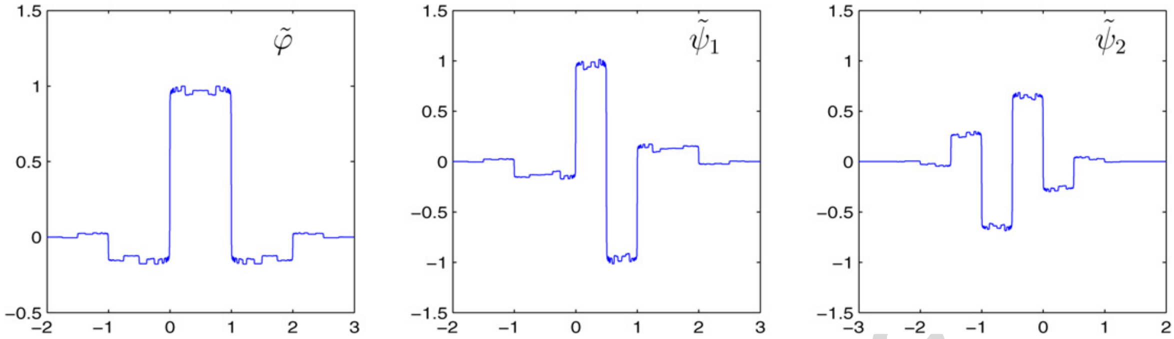

Fig. 6. Dual frame with 3 vanishing moments, $A = 0.08$, $B = 0$.

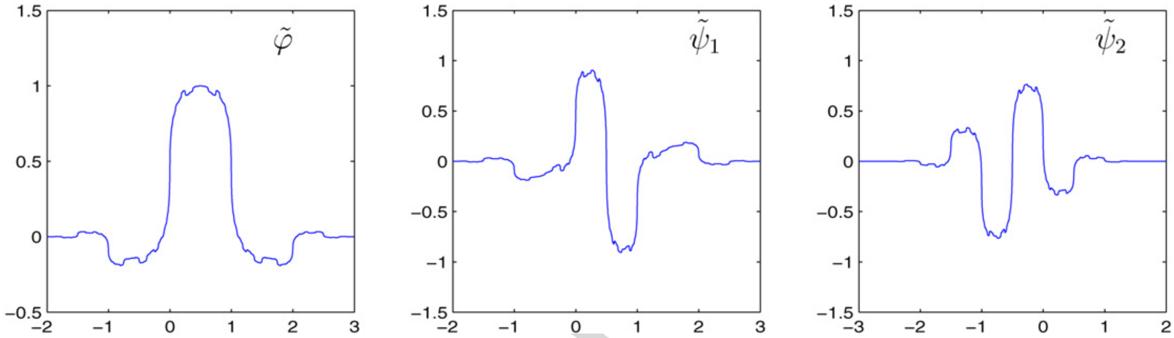

Fig. 7. Dual frame with 3 vanishing moments, $A = 0.09$, $B = 0$.

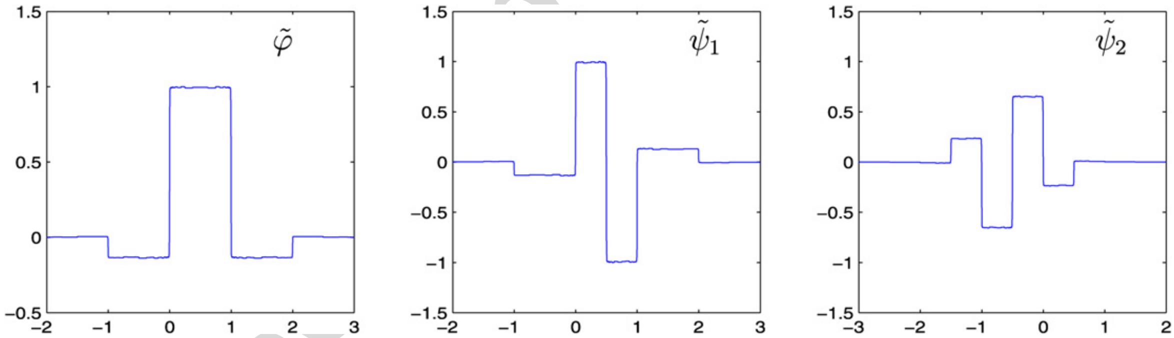

Fig. 8. Dual frame with 3 vanishing moments, $A = 0.0896$, $B = -0.075$.

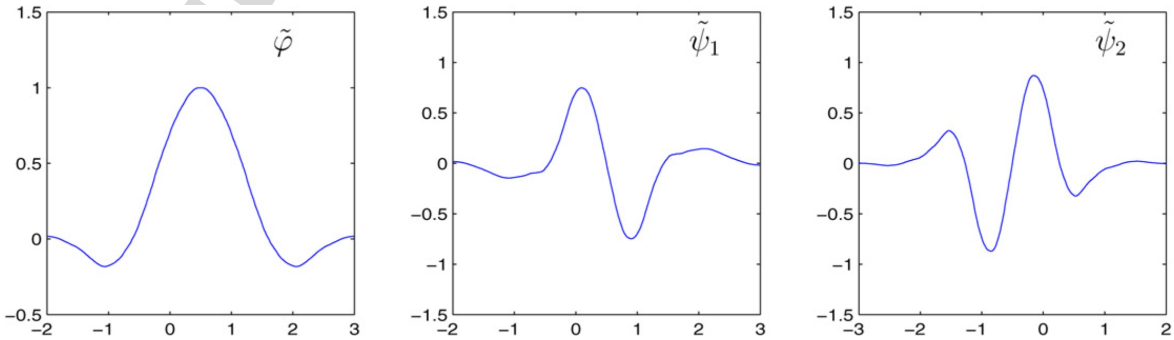

Fig. 9. Dual frame with 3 vanishing moments, $A = 0.2$, $B = -0.275$.

In the case of masks of the length 8 (i.e., $B \neq 0$), we have more flexibility. In particular, for $A = 0.0896$, $B = -0.075$ we have framelets closer to piecewise constant functions (Fig. 8) and for $A = 0.2$, $B = -0.275$ we have very smooth framelets (Fig. 9) with 3 vanishing moments.

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