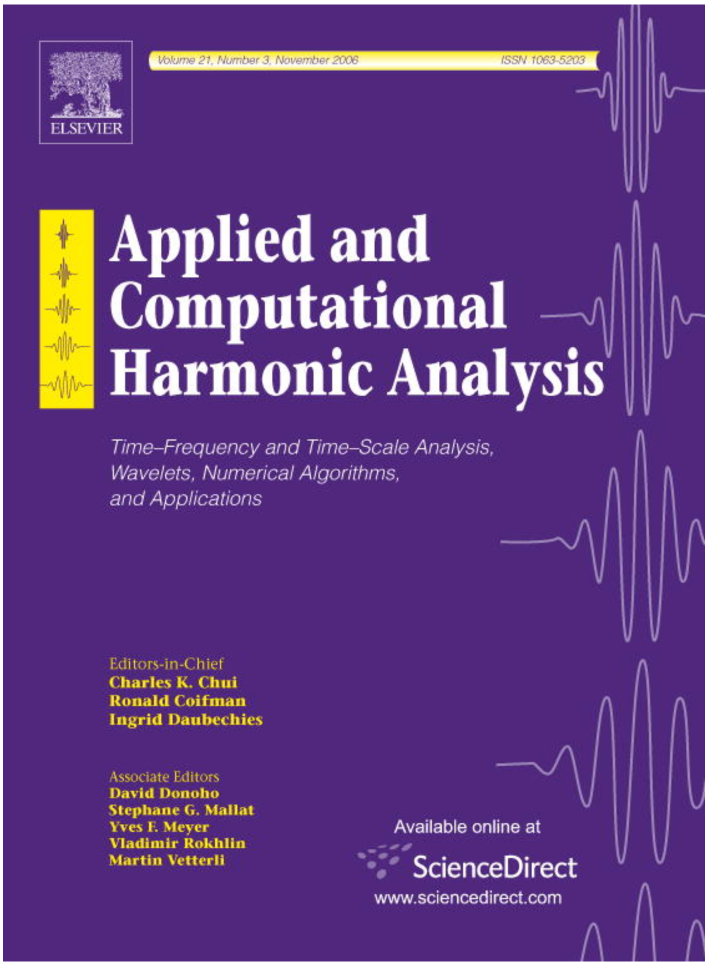
ELSEVIER
Volume 21, Number 3, November 2006
ISSN 1063-5203
Applied and Computational Harmonic Analysis
Time–Frequency and Time–Scale Analysis, Wavelets, Numerical Algorithms, and Applications
Editors-in-Chief
Charles K. Chui
Ronald Coifman
Ingrid Daubechies
Associate Editors
David Donoho
Stephane G. Mallat
Yves F. Meyer
Vladimir Rokhlin
Martin Vetterli
Available online at
ScienceDirect
www.sciencedirect.com

Available online at www.sciencedirect.com
ScienceDirect
Appl. Comput. Harmon. Anal. 21 (2006) 324–348
Applied and Computational Harmonic Analysis
www.elsevier.com/locate/acha

Construction of multivariate compactly supported tight wavelet frames

Ming-Jun Lai [a,*,1], Joachim Stöckler [b]

[a] *Department of Mathematics, The University of Georgia, Athens, GA 30602,USA*
[b] *Institut für Angewandte Mathematik, Universität Dortmund, D-44221 Dortmund, Germany*

Received 5 May 2004; revised 6 April 2006; accepted 7 April 2006

Available online 19 June 2006

Communicated by Charles K. Chui

Abstract

Two simple constructive methods are presented to compute compactly supported tight wavelet frames for any given refinable function whose mask satisfies the QMF or sub-QMF conditions in the multivariate setting. We use one of our constructive methods in order to find tight wavelet frames associated with multivariate box splines, e.g., bivariate box splines on a three or four directional mesh. Moreover, a construction of tight wavelet frames with maximum vanishing moments is given, based on rational masks for the generators. For compactly supported bi-frame pairs, another simple constructive method is presented.

Keywords: Compactly supported wavelets; Tight-frames; Box splines; Multi-resolution analysis; Quadrature mirror filters

1. Introduction

By sacrificing the orthonormality and allowing redundant representations, tight wavelet frames become much easier to construct than orthonormal wavelets. Furthermore, two remarkable properties of tight wavelet frames are: (1) they have the same computational complexity as orthonormal wavelets, and (2) they can be applied to image processing in exactly the same way as orthonormal wavelets. The theory of wavelet frames and wavelets has been developed in parallel. The expositions by Coifman and Meyer [24] and Daubechies [11] give necessary and sufficient conditions on an L_2-function, so that its integer translates and dilations form a frame. In [20], Hernández and Weiss give a new set of necessary and sufficient conditions which make the integer translates and dilations of an L_2 function an orthonormal basis. (See also [14].) A similar set of necessary and sufficient conditions for tight wavelet frames was given by Han in [17]. Independently, such conditions were obtained by Ron and Shen [29]. In addition, Ron and Shen formulated the so-called unitary extension principle (UEP) which allows them to construct many examples of B-spline and box spline wavelet frames (cf. [29–32]) and to construct compactly supported tight wavelet frames of any smoothness based on any dilation matrix in [16]. In addition, Benedetto and Li [4] developed a theory of frames parallel to the

* Corresponding author.
E-mail addresses: mjlai@math.uga.edu (M.-J. Lai), joachim.stoeckler@math.uni-dortmund.de (J. Stöckler).
1 This research is partly supported by the National Science Foundation under grant EAR-0327577.

1063-5203/$ – see front matter
doi:10.1016/j.acha.2006.04.001

orthonormal wavelets and used their frames for signal processing. It is worthwhile to point out that the number of B-spline and box spline wavelet frame generators constructed in [31,32] depends on the smoothness of the B-splines and box splines. The smoother the tight wavelet frame, the bigger is the number of tight wavelet frame generators. This problem was resolved by Chui and He [8,9] and Petukhov [26]. Chui and He showed that the number of tight wavelet frame generators is 2 for B-splines of any smoothness (cf. [26] for another proof), 7 for box splines on a three direction mesh and 15 for box splines on a four direction mesh. See [27] for constructing symmetric tight wavelet frames. In order to increase the order of vanishing moments of tight wavelet frames, the notion of vanishing moment recovery functions was introduced in [10] and, in parallel, the mixed extension principle was introduced in [13] for the construction of many examples of bi-frames in the univariate setting.

In this paper, we shall provide another method of constructing compactly supported tight wavelet frames in the multivariate setting. We first consider the unitary extension principle (UEP) and relate it to the problem of decomposing a nonnegative trigonometric polynomial into a finite sum of squares (sos) of absolute values of trigonometric polynomials. That is, let $P(\omega)$ be the trigonometric polynomial mask associated with a refinable function and suppose that P satisfies the sub-QMF condition

$$\sum_{j\in\{0,1\}^d\pi} \left|P(\omega+j)\right|^2 \leqslant 1,$$

under the standard dilation matrix $2I_{d\times d}$, where $I_{d\times d}$ is the identity matrix in $\mathbb{R}^d$. If we can find the decomposition

$$1-\sum_{j\in\{0,1\}^d\pi} \left|P(\omega+j)\right|^2 = \sum_{k=1}^{N}\left|\tilde{P}_k(2\omega)\right|^2, \tag{1.1}$$

where each $\tilde{P}_\ell$ is a trigonometric polynomial, then we shall show how to construct a set of finitely many tight wavelet frame generators based on this decomposition. Several necessary and sufficient conditions for nonnegative trigonometric (or Laurent) polynomials to be sos will be discussed. When our method is applied to the Laurent polynomials associated with multivariate box splines, we can show that the nonnegative Laurent polynomials can be indeed decomposed as in (1.1). Many examples of tight wavelet frames associated with box splines on three and four directional meshes will be presented. The number of tight wavelet frame generators is less than the number obtained by using the Kronecker product method given in [9]. This demonstrates an advantage of our method.

Since the number of vanishing moments plays a significant role in the application of wavelet frames, we shall also discuss a method for the construction of tight wavelet frames with rational masks and maximum vanishing moments. Moreover, the construction of compactly supported bi-frames will be considered. An explicit formula for the masks of bi-frames will be given. The formula yields bi-frames based on bivariate and trivariate box splines.

The paper is organized as follows. We start with a preliminary section which describes the well-known sufficient condition for tight wavelet frames, i.e., the oblique extension principle (OEP) which includes the UEP as a special case. In Section 3, we give an explicit formula for masks of tight wavelet frames which are derived from any given refinable function ϕ whose mask is a quadrature mirror filter (QMF). Let us point out that there are many constructive methods available in the literature for those refinable functions ϕ. (See [23] for several methods besides the trivial tensor product of univariate scaling functions.) Since the masks associated with certain refinable functions satisfy the so-called sub-QMF condition, we give another method for constructing tight wavelet frames from those refinable functions. The precise construction depends on whether a multivariate positive Laurent polynomial can be written as a sum of squares of absolute values of Laurent polynomials. We discuss some necessary and sufficient conditions of this algebraic property in Section 4. In Section 5, we mainly focus on the construction of tight wavelet frames by using bivariate box splines on three and four directional meshes. From there, we can easily conclude that the construction can be generalized to computing tight wavelet frames by using any given multivariate box spline. In order to increase the order of vanishing moments for tight frames, we show in Section 6 how to construct tight frames whose masks are rational functions which correspond to ARMA filters. In Section 7, we present a constructive method for bi-frames. Using box spline functions in the bivariate and trivariate setting, we give an explicit formula for the masks of these bi-frames.

2. Preliminaries

Let us first recall some notation. For $f, g \in L_2(\mathbb{R}^d)$, we denote the inner product by

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(y)\overline{g(y)}\,\mathrm{d}y$$

and the Fourier transform by

$$\hat{f}(\omega) = \int_{\mathbf{R}^d} f(y)e^{-i\omega y}\,\mathrm{d}y.$$

Given a function $\psi \in L_2(\mathbb{R}^d)$, we let

$$\psi_{j,k}(y) = 2^{jd/2}\psi\big(2^j y - k\big).$$

Let Ψ be a finite subset of $L_2(\mathbb{R}^d)$ and

$$\Lambda(\Psi) := \big\{\psi_{j,k};\ \psi \in \Psi,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\big\},$$

where $\mathbb{Z}$ is the set of all integers.

Definition 2.1. *We say that $\Lambda(\Psi)$ is a frame if there exist two positive numbers A and B such that*

$$A\|f\|^2_{L_2(\mathbb{R}^d)} \leqslant \sum_{g\in\Lambda(\Psi)} \big|\langle f, g\rangle\big|^2 \leqslant B\|f\|^2_{L_2(\mathbb{R}^d)}$$

for all $f \in L_2(\mathbb{R}^d)$. $\Lambda(\Psi)$ is a tight frame if it is a frame with $A = B$. In this case, after a renormalization of the g's in Ψ, we have

$$\sum_{g\in\Lambda(\Psi)} \big|\langle f, g\rangle\big|^2 = \|f\|^2_{L_2(\mathbb{R}^d)}$$

for all $f \in L_2(\mathbb{R}^d)$

It is known (cf. [11]) that when $\Lambda(\Psi)$ is a tight frame, any $f \in L_2(\mathbb{R}^d)$ can be represented as

$$f = \frac{1}{A}\sum_{g\in\Lambda(\Psi)} \langle f, g\rangle g.$$

Let $\phi \in L_2(\mathbb{R}^d)$ be a compactly supported refinable function, i.e.,

$$\hat{\phi}(\omega) = P(\omega/2)\hat{\phi}(\omega/2),$$

where $P(\omega)$ is a trigonometric polynomial. Let $S(\omega)$ be another trigonometric polynomial such that $S(\omega) \geqslant 0$ and $S(0) = 1$. We look for Q_i (trigonometric polynomial or rational function) such that

$$S(2\omega)P(\omega)\overline{P(\omega+\ell)} + \sum_{i=0}^{r} Q_i(\omega)\overline{Q_i(\omega+\ell)} = \begin{cases} S(\omega), & \text{if } \ell = 0,\\ 0, & \ell \in \{0,1\}^d\pi\backslash\{0\}.\end{cases} \tag{2.1}$$

The conditions (2.1) are called oblique extension principle (OEP) in [13]. $S(\omega)$ is called the vanishing moment recovery function in [10]. The OEP describes a less restrictive condition than the unitary extension principle (UEP) in [29].

With these Q_i's we can define wavelet frame generators $\psi^{(i)}$, defined in terms of their Fourier transforms, by

$$\hat{\psi}^{(i)}(\omega) = Q_i(\omega/2)\hat{\phi}(\omega/2), \quad i = 1, \ldots, r. \tag{2.2}$$

Then, if ϕ is continuous and Lip α, with $\alpha > 0$, and the OEP is satisfied, the family $\Psi = \{\psi^{(i)},\ i = 1, \ldots, r\}$ generates a tight frame, i.e., $\Lambda(\Psi)$ is a tight wavelet frame. (See [13] and [10] for different proofs.)

For convenience, we rewrite (2.1) in an equivalent matrix form as follows:

Lemma 2.2. *Let $\mathcal{P} = (P(\omega+\ell);\ \ell \in \{0,1\}^d\pi)^T$ be a vector of size $2^d \times 1$ and $\mathcal{Q} = (Q_i(\omega+\ell);\ \ell \in \{0,1\}^d\pi,\ i = 1,\dots,r)$ be a matrix of size $2^d \times r$. Then* (2.1) *is equivalent to*

$$\mathcal{Q}\mathcal{Q}^* = \operatorname{diag}\big(S(\omega+\ell);\ \ell \in \{0,1\}^d\pi\big) - S(2\omega)\mathcal{P}\mathcal{P}^*, \tag{2.3}$$

where $\mathcal{P}^*$ denotes the complex conjugate transpose of the column vector $\mathcal{P}$.

Proof. This can be verified directly. □

For example, when $d = 2$, $r = 4$ and $\omega = (\xi,\eta)$, we have

$$\mathcal{Q} = \begin{bmatrix} Q_1(\xi,\eta) & Q_1(\xi+\pi,\eta) & Q_1(\xi,\eta+\pi) & Q_1(\xi+\pi,\eta+\pi) \\ Q_2(\xi,\eta) & Q_2(\xi+\pi,\eta) & Q_2(\xi,\eta+\pi) & Q_2(\xi+\pi,\eta+\pi) \\ Q_3(\xi,\eta) & Q_3(\xi+\pi,\eta) & Q_3(\xi,\eta+\pi) & Q_3(\xi+\pi,\eta+\pi) \\ Q_4(\xi,\eta) & Q_4(\xi+\pi,\eta) & Q_4(\xi,\eta+\pi) & Q_4(\xi+\pi,\eta+\pi) \end{bmatrix}^T,$$

$\mathcal{P} = (P(\xi,\eta), P(\xi+\pi,\eta), P(\xi,\eta+\pi), P(\xi+\pi,\eta+\pi))^T$, and

$$\mathcal{Q}\mathcal{Q}^* = \begin{bmatrix} S(\xi,\eta) & & & \\ & S(\xi+\pi,\eta) & & \\ & & S(\xi,\eta+\pi) & \\ & & & S(\xi+\pi,\eta+\pi) \end{bmatrix} - S(2\xi,2\eta)\mathcal{P}\mathcal{P}^*.$$

For another example, when $S \equiv 1$, (2.3) is simply

$$\mathcal{Q}\mathcal{Q}^* = I_{2^d\times 2^d} - \mathcal{P}\mathcal{P}^*. \tag{2.4}$$

Our construction of tight wavelet frames is mainly based on the matrix form (2.3) or (2.4).

3. Frame construction: QMF and sub-QMF cases

In this section we first consider those refinable functions ϕ whose mask P satisfies the QMF condition

$$\sum_{\ell\in\{0,1\}^d\pi} \big|P(\omega+\ell)\big|^2 = 1, \quad \omega \in \mathbb{R}, \tag{3.1}$$

along with $P(0) = 1$. For simplicity, we further restrict to the case when $S(\omega) \equiv 1$. Let

$$\mathcal{M} = 2^{-d/2}\big(e^{im\cdot(\omega+\ell)}\big)_{\substack{\ell\in\{0,1\}^d\pi \\ m\in\{0,1\}^d}} \tag{3.2}$$

be the polyphase matrix, where ℓ denotes the row index and m denotes the column index of $\mathcal{M}$. Clearly, $\mathcal{M}$ is a unitary matrix. Up to the normalization factor $2^{-d/2}$, the polyphase components of the trigonometric polynomial P are defined by the column vector

$$\hat{\mathcal{P}} := \big(\hat{P}_m(2\omega);\ m \in \{0,1\}^d\big)^T = \mathcal{M}^*\mathcal{P},$$

where each $\hat{P}_m$ is a trigonometric polynomial. Hence, we obtain the polyphase decomposition of P by inspecting the first row of the identity $\mathcal{P} = \mathcal{M}\hat{\mathcal{P}}$, which gives

$$P(\omega) = 2^{-d/2} \sum_{m\in\{0,1\}^d} e^{im\cdot\omega}\hat{P}_m(2\omega). \tag{3.3}$$

Our first result applies to masks P which satisfy the QMF condition (3.1).

Theorem 3.1. *Suppose that a trigonometric polynomial P satisfies the QMF condition* (3.1). *Define $Q_1,\dots,Q_{2^d}$ by*

$$\mathcal{Q} := \big(Q_i(\omega+\ell)\big)_{\substack{\ell\in\{0,1\}^d\pi \\ i=1,\dots,2^d}} = \mathcal{M}(I_{2^d\times 2^d} - \hat{\mathcal{P}}\hat{\mathcal{P}}^*). \tag{3.4}$$

Then P and Q_i, $i = 1,\dots,2^d$, satisfy (2.1) *with $S(\omega) \equiv 1$.*

Proof. The QMF condition leads to the identity $\hat{\mathcal{P}}^*\hat{\mathcal{P}} = \mathcal{P}^*\mathcal{P} = 1$. Hence, it follows that

$$\mathcal{Q}\mathcal{Q}^* = I_{2^d\times 2^d} - \mathcal{M}\hat{\mathcal{P}}\hat{\mathcal{P}}^*\mathcal{M}^* = I_{2^d\times 2^d} - \mathcal{P}\mathcal{P}^*,$$

which is (2.4). Note that both $\mathcal{M}$ and $\mathcal{P} = \mathcal{M}\hat{\mathcal{P}}$ in (3.4) have the desired form, and so does $\mathcal{Q}$. This completes the proof. □

This shows that, for refinable functions ϕ whose mask P satisfies (3.1), we can easily construct tight wavelet frames associated with ϕ. We remark that there are many constructive methods available for finding refinable functions ϕ whose masks satisfy the QMF condition (3.1). See [23] for a survey on several constructive methods for QMF filters.

Since the matrix $\mathcal{Q}$ in (3.4) has 2^d columns, for convenience, we make use of the notation

$$\mathcal{Q} = \big(Q_m(\omega+\ell)\big)_{\substack{\ell\in\{0,1\}^d\pi \\ m\in\{0,1\}^d}}.$$

We now examine the vanishing moment property.

Corollary 3.2. $Q_m(0) = 0$ *for all* $m \in \{0,1\}^d$.

Proof. Since $\mathcal{Q} = \mathcal{M} - \mathcal{P}\hat{\mathcal{P}}^*$, the entries in the first row of $\mathcal{Q}$ give

$$Q_m(\omega) = 2^{-d/2}e^{im\cdot\omega} - P(\omega)\overline{\hat{P}_m(2\omega)}, \quad m \in \{0,1\}^d. \tag{3.5}$$

Since $P(0) = 1$ holds, the QMF condition implies $\mathcal{P}(0) = e_1$, the first canonical unit vector in $\mathbb{R}^{2^d}$. Hence, $\hat{P}_m(0) = 2^{-d/2}$ and $Q_m(0) = 0$ for all $m \in \{0,1\}^d$. □

Next, we note that the tight frame generators $\psi^{(m)}$, which are defined by their Fourier transform

$$\hat{\psi}^{(m)}(\omega) = Q_m\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right), \quad m \in \{0,1\}^d,$$

are linearly dependent. We have the following result.

Corollary 3.3. $\sum_{m\in\{0,1\}^d} \hat{P}_m(2\omega)Q_m(\omega) \equiv 0.$

Proof. By (3.1), we have $\hat{\mathcal{P}}^*\hat{\mathcal{P}} = \sum_{m\in\{0,1\}^d} |\hat{P}_m(2\omega)|^2 \equiv 1$. Now, (3.3) and (3.5) give

$$\sum_{m\in\{0,1\}^d} \hat{P}_m(2\omega)Q_m(\omega) = \sum_{m\in\{0,1\}^d} 2^{-d/2}e^{im\cdot\omega}\hat{P}_m(2\omega) - P(\omega)\sum_{m\in\{0,1\}^d}\big|\hat{P}_m(2\omega)\big|^2 = P(\omega) - P(\omega) = 0.$$

This completes the proof. □

We remark that, in general, the mask P of a scaling function does not satisfy the QMF condition. Instead, it may satisfy the following sub-QMF condition

$$\sum_{\ell\in\{0,1\}^d\pi} \big|P(\omega+\ell)\big|^2 \leqslant 1. \tag{3.6}$$

We now explain how to construct tight frames associated with the standard dilation matrix $2I_{d\times d}$ for such scaling functions ϕ. As before, we let $\hat{\mathcal{P}} = (\hat{P}_m(2\omega);\ m \in \{0,1\}^d)^T = \mathcal{M}^*\mathcal{P}$, where $\mathcal{M}$ is the polyphase matrix in (3.2) and $\mathcal{P} = (P(\omega+\ell);\ \ell \in \{0,1\}^d\pi)$. Then (3.6) is equivalent to

$$\hat{\mathcal{P}}^*\hat{\mathcal{P}} = \sum_{m\in\{0,1\}^d} \big|\hat{P}_m(2\omega)\big|^2 \leqslant 1. \tag{3.7}$$

Theorem 3.4. *Suppose that P satisfies the sub-QMF condition* (3.6). *Suppose that there exist Laurent polynomials $\tilde P_1, \dots, \tilde P_N$ such that*

$$\sum_{m\in\{0,1\}^d} \left|\hat P_m(\omega)\right|^2 + \sum_{i=1}^N \left|\tilde P_i(\omega)\right|^2 = 1. \tag{3.8}$$

Then there exist $2^d + N$ compactly supported tight frame generators with wavelet masks Q_m, $m = 1, \dots, 2^d + N$, such that P, Q_m, $m = 1, \dots, 2^d + N$, satisfy (2.1) *with $S(\omega) \equiv 1$.*

Proof. We define the combined column vector $\tilde{\mathcal P} = (\hat P_m(2\omega);\ m \in \{0,1\}^d,\ \tilde P_i(2\omega);\ 1 \leqslant i \leqslant N)^T$ of size $(2^d + N)$ and the matrix

$$\tilde{\mathcal Q} := I_{(2^d+N)\times(2^d+N)} - \tilde{\mathcal P}\tilde{\mathcal P}^*.$$

Note that all entries of $\tilde{\mathcal P}$ and $\tilde{\mathcal Q}$ are π-periodic. Identity (3.8) implies that $\tilde{\mathcal Q}\tilde{\mathcal Q}^* = \tilde{\mathcal Q}$ and this gives

$$\tilde{\mathcal P}\tilde{\mathcal P}^* + \tilde{\mathcal Q}\tilde{\mathcal Q}^* = I_{(2^d+N)\times(2^d+N)}.$$

Restricting to the first principle $2^d \times 2^d$ blocks in the above matrices, we have

$$\hat{\mathcal P}\hat{\mathcal P}^* + \hat{\mathcal Q}\hat{\mathcal Q}^* = I_{2^d\times 2^d}, \tag{3.9}$$

where $\hat{\mathcal P} = \mathcal M^*\mathcal P$ was already defined before and $\hat{\mathcal Q}$ denotes the first $2^d \times (2^d + N)$ block matrix of $\tilde{\mathcal Q}$. By (3.3), we have $\mathcal P = \mathcal M\hat{\mathcal P}$, and (3.9) yields

$$\mathcal P\mathcal P^* + \mathcal M\hat{\mathcal Q}(\mathcal M\hat{\mathcal Q})^* = I_{2^d\times 2^d},$$

which is (2.4). Thus we let

$$\mathcal Q = \mathcal M\hat{\mathcal Q}.$$

Then the first row $[Q_1, \dots, Q_{2^d+N}]$ of $\mathcal Q$ gives the desired trigonometric functions for compactly supported tight wavelet frame generators. The form $\mathcal Q = [Q_i(\omega + \ell)]$ is inherited from $\mathcal M$, since the entries of $\hat{\mathcal Q}$ are π-periodic. This completes the proof. □

We shall use the constructive scheme above to find compactly supported tight wavelet frames based on multivariate box splines, in particular, bivariate box splines on three and four directional meshes. This will be done in Section 5, after we discuss the existence of $\tilde P_j$'s for completing (3.8) in the next section.

It is well known that the tight frames constructed above with $S(\omega) \equiv 1$ may not have more than 1 vanishing moment. Thus, we will also employ certain trigonometric polynomials S in Section 6, in order to increase the order of vanishing moments.

4. Sufficient and necessary conditions for nonnegative Laurent polynomials to be sos

In this section we discuss the problem if (3.8) holds for every Laurent polynomial P satisfying (3.6). In other words, when a Laurent polynomial $1 - \sum_{k\in\{0,1\}^d\pi} |P(\omega + k)|^2$ is nonnegative, we want to know if there exist Laurent polynomials $\tilde P_j$ such that

$$1 - \sum_{\ell\in\{0,1\}^d\pi} \left|P(\omega+\ell)\right|^2 = \sum_j \left|\tilde P_j(2\omega)\right|^2.$$

In terms of the polyphase components $\hat P_m$ of P, the above equation can be equivalently written as

$$1 - \sum_{m\in\{0,1\}^d} \left|\hat P_m(\omega)\right|^2 = \sum_j \left|\tilde P_j(\omega)\right|^2.$$

This problem is related to a well-known problem in modern real algebra: If a real polynomial $p(x)$ is positive at every point in $\mathbb R^d$, must p then be a finite sum of polynomial squares? The answer to this question is "no," as conjectured by Minkowski and confirmed by Hilbert in 1888 (cf. [21]). This is related to the 17th of Hilbert's famous 23 problems (cf. [33]). It poses the question if any real positive polynomial can be written as a finite sum of squares of rational functions. The problem was settled (cf. [1,28]). Similarly, in the setting of Laurent polynomials, (3.8) holds for rational Laurent polynomials in $\mathbb{R}^d$. That is, if P satisfies (3.6), there exists a finite number of rational Laurent polynomials $\tilde{P}_i$ such that (3.8) holds. We refer to [2] for a constructive algorithm for finding these $\tilde{P}_i$'s in $\mathbb{R}^2$.

Since we are interested in compactly supported tight frames, we need to find Laurent polynomials $\tilde{P}_i$ to satisfy (3.8). Thus, we look for some sufficient and necessary conditions on $1-\sum_{k\in\{0,1\}^d\pi}|P(\omega+k)|^2$ such that (3.8) holds. For simplicity, let P stand for a nonnegative Laurent polynomial. We will find some conditions on P to ensure that P can be written as a sum of squares of Laurent polynomials with real coefficients, in short, P is sos. We begin with the following elementary formula.

Lemma 4.1. *For any $\omega\in[0,2\pi]$, we have*

$$1\pm\cos\omega=\frac{1}{2}\left|1\pm e^{i\omega}\right|^2. \tag{4.1}$$

With the above formula, we have the following simple sufficient condition (cf. [3]).

Theorem 4.2. *Suppose that a Laurent polynomial $P(\omega)=\sum_k c_k e^{ik\omega}$, with real coefficients c_k, is nonnegative for all $\omega\in[0,2\pi]^d$ If*

$$c_0\geqslant\sum_{k\neq 0}|c_k|, \tag{4.2}$$

then P is sos.

Proof. Since $P(\omega)$ is real, we have

$$P(\omega)=\frac{1}{2}\big(P(\omega)+\overline{P(\omega)}\big)=c_0+\sum_{k\neq 0}\frac{c_k}{2}\big(e^{ik\omega}+e^{-ik\omega}\big).$$

Writing $\tilde{c}_0=c_0-\sum_{k\neq 0}|c_k|\geqslant 0$, we use Lemma 4.1 to get

$$P(\omega)=\tilde{c}_0+\sum_{k\neq 0}\big(|c_k|+c_k\cos(k\omega)\big)=\tilde{c}_0+\sum_{k\neq 0}|c_k|\big(1+\operatorname{sign}(c_k)\cos(k\omega)\big)=\tilde{c}_0+\sum_{k\neq 0}\frac{|c_k|}{2}\left|1+\operatorname{sign}(c_k)e^{ik\omega}\right|^2.$$

Since P is a Laurent polynomial, only finitely many c_k's can be nonzero. Thus, the summation is finite and P is sos. □

However, the condition 4.2 is not necessary as we can see from the following example.

Example 4.3. Consider

$$P(\omega)=62+20\big(e^{i\omega_1}+e^{-i\omega_1}\big)+\frac{23}{2}\big(e^{i\omega_2}+e^{-i\omega_2}\big)+\frac{23}{2}\big(e^{i(\omega_1+\omega_2)}+e^{-i(\omega_1+\omega_2)}\big).$$

Clearly, the condition (4.2) does not hold. However, it is easy to check that

$$\begin{aligned}P(\omega)&=21(1-\sin\omega_1\sin\omega_2)+(1-\sin\omega_1\sin\omega_2)^2\\&\quad+17(1+\cos\omega_1)+19(1+\cos\omega_1)(1+\cos\omega_2)+(1+\cos\omega_1)^2(1+\cos\omega_2)\\&\quad+(1+\cos\omega_1)(1+\cos\omega_2)^2+(1+\cos\omega_1)(1+\cos\omega_2)(1-\cos\omega_1\cos\omega_2).\end{aligned}$$

Note that

$$1-\cos\omega_1\cos\omega_2=\frac{1}{4}\left|1-e^{i(\omega_1+\omega_2)}\right|^2+\frac{1}{4}\left|1-e^{i(\omega_1-\omega_2)}\right|^2 \tag{4.3}$$

and

$$1-\sin\omega_1\sin\omega_2=\frac{1}{4}\left|1+e^{i(\omega_1+\omega_2)}\right|^2+\frac{1}{4}\left|1-e^{i(\omega_1-\omega_2)}\right|^2. \tag{4.4}$$

Based on (4.1), we know that P is sos. This shows that the condition (4.2) is not necessary.

Next we give a complete characterization on positive Laurent polynomials to be sos.

Theorem 4.4. *Let $P(\omega)=\sum_{k\in[-n,n]^d\cap\mathbb{Z}^d} c_k e^{ik\omega}$ be a Laurent polynomial of coordinate degree n which is positive for all $\omega\in\mathbb{R}^d$ and whose coefficients c_k are real. Then P is sos of real polynomials in $e^{i\omega}$ of coordinate degree $\leqslant n$, if and only if there exists a real, symmetric, positive semi-definite matrix $\mathcal{P}$ such that*

$$P(\omega)=\mathbf{x}^*\mathcal{P}\mathbf{x}, \tag{4.5}$$

where $\mathbf{x}=(e^{ik\omega};\ k\in[0,n]^d\cap\mathbb{Z}^d)$ is a column vector of size $(n+1)^d$.

Proof. Suppose that there exists a real, symmetric, positive semi-definite matrix $\mathcal{P}$ such that (4.5) holds. Then there exists a real matrix L and a diagonal matrix

$$D=\operatorname{diag}(d_1,\dots,d_r,0,\dots,0)$$

such that $\mathcal{P}=LDL^T$ with $d_i>0$, $i=1,\dots,r$, where r is the rank of $\mathcal{P}$. Let p_i, $i=1,\dots,r$, denote the first r components of $\mathbf{x}^*L$. Then

$$P(\omega)=\mathbf{x}^*\mathcal{P}\mathbf{x}=\sum_{i=1}^{r} d_r|p_i|^2.$$

That is, $P(\omega)$ is sos of finitely many polynomials with real coefficients.

On the other hand, if $P(\omega)=\sum_{i=1}^{r}|p_i|^2$ for some real polynomials p_i of coordinate degree $\leqslant n$, then $[p_1^*,\dots,p_r^*]=\mathbf{x}^*L$ for some real matrix L. Thus, we have

$$P(\omega)=\mathbf{x}^*LL^T\mathbf{x}$$

and, hence, $\mathcal{P}=LL^T$ defines the real, symmetric and positive semi-definite matrix in (4.5). This completes the proof. □

Unfortunately, finding such a real, symmetric and positive semi-definite matrix $\mathcal{P}$ is not easy. One of the reasons is that the size of the matrix $\mathcal{P}$ grows quickly as the number of variables and the degree of the Laurent polynomial increase.

We shall use positive definite functions to derive another necessary and sufficient condition (cf. [35]). We begin with two definitions.

Definition 4.5. *Let D be a point set in $\mathbb{R}^d$. A function $K(x,y)$ defined on $D\times D$ is a positive definite kernel* (PDK) *if, for every set $\{x_1,x_2,\dots,x_n\}\subset D$, the matrix*

$$\big(K(x_i,x_j)\big)_{i,j=1,\dots,n}$$

is positive semi-definite. When $K(x,y)=f(x-y)$, then f is called a positive definite function.

Definition 4.6. *Let H be a Hilbert space of functions defined on D with an inner product $\langle\cdot,\cdot\rangle_H$. If for any $x\in D$, there is an element $K_x\in H$ such that*

$$f(x)=\langle f,K_x\rangle_H,\quad \forall f\in H,$$

then H is called a reproducing kernel Hilbert space. The function K on $D\times D$, defined by

$$K(x,y)=\langle K_y,K_x\rangle_H,$$

is called the reproducing kernel.

It is known that for any PDK function K defined on $D\times D$, there exists a reproducing kernel Hilbert space H consisting of functions defined on D whose reproducing kernel is K (cf. [36, p. 19]). In fact, H is the completion of the linear span of $\{K(y,x),\ y\in D\}$ under the semi-norm $\|\sum_i c_iK(y_i,\cdot)\|:=(\sum_i|c_i|^2)^{1/2}$ which is defined for finite linear combinations. Then H is the desired reproducing kernel Hilbert space with reproducing kernel K. If K is a polynomial, then H is of finite dimension. Furthermore, let $\{\phi_i;\ i = 1, \dots, N = (\dim(H))\}$ be an orthonormal basis for H. Then

$$K(x, y) = \sum_{i=1}^{N} c_i(y)\phi_i(x)$$

with

$$c_i(y) = \langle K(\cdot, y), \phi_i\rangle = \overline{\langle \phi_i, K(\cdot, y)\rangle} = \overline{\phi_i(y)}, \quad \forall i = 1, \dots, N.$$

Therefore, $K(x, x) = \sum_{i=1}^{N} |\phi_i(x)|^2$. That is, $K(x, x)$ is an sos. This leads to the following

Theorem 4.7. *Let $P(x)$ be a positive Laurent polynomial for $x = e^{i\omega} \in D$, where $D := \{e^{i\omega},\ \omega \in [0, 2\pi]^d\}$. If there exists a* PDK *$K(x, y)$ such that $K(x, x) = P(x)$, then P is an sos. The converse is also true. That is, if $P(x)$ is a sum of squares of polynomials with real coefficients, then there exists a* PDK *$K(x, y)$ such that $K(x, x) = P(x)$.*

Proof. Based on the discussion above, we only need to prove the converse. Writing $P(x) = \sum_{i=1}^{n} |\phi_i(x)|^2$ with polynomials ϕ_i whose coefficients are real, we define

$$K(x, y) = \sum_{i=1}^{n} \phi_i(x)\overline{\phi_i(y)}.$$

Then it is easy to check that $K(x, y)$ is a PDK. This completes the proof. □

Our next step is to characterize which functions are PDK. We have

Theorem 4.8. *Suppose that $K(x, y)$ is a continuous function defined on $D \times D$ with $D = [-\pi, \pi]^d$. Let*

$$K(x, y) = \sum_{j,k} \hat{K}(j, k)e^{-ijx}e^{iky}$$

be the Fourier series expansion of $K(x, y)$. Then $K(x, y)$ is PDK if and only if the matrix $[\hat{K}(j, k)]$ is positive semi-definite.

Proof. Since $K(x, y)$ is continuous, the positive semi-definiteness of K is equivalent to

$$\int_D \int_D K(x, y) f(x)\overline{f(y)}\,\mathrm{d}x\,\mathrm{d}y \geqslant 0$$

for any continuous functions f. It follows that

$$\sum_{j,k} \hat{K}(j, k) \int_D \int_D f(x)\overline{f(y)}e^{-ijx}e^{iky}\,\mathrm{d}x\,\mathrm{d}y = (2\pi)^{2d} \sum_{j,k} \hat{K}(j, k)\hat{f}(j)\overline{\hat{f}(k)} \geqslant 0,$$

where $\hat{f}(j)$ denotes the Fourier coefficients of f. Hence, the matrix $[\hat{K}(j, k)]$ is positive semi-definite. □

There are several other simple properties of PDK functions (cf. [34]):

(1) If $f_i(x, y)$, $i = 1, \dots, N$, are PDK functions, so is the sum $\sum_{i=1}^{N} c_i f_i(x, y)$ for any positive constants c_i.
(2) If $f_i(x, y)$, $i = 1, 2, \dots,$ are PDK functions and they are convergent to $f(x, y)$, then the limit function is a PDK function.
(3) If $f_1(x, y)$ and $f_2(x, y)$ are PDK functions on $D \times D$, then their product $F(x_1, x_2; y_1, y_2) := f_1(x_1, y_1) f_2(x_2, y_2)$ is a PDK function on $D^2 \times D^2$.

Example 4.9. Let

$$B(\omega) = 1 + \frac{1}{2}\big(\cos(\omega_1) + \cos(\omega_2)\big).$$

It is easy to see that $B(\omega)$ is a positive Laurent polynomial. With $x_1 = \exp(i\omega_1)$ and $x_2 = \exp(i\omega_2)$, we write

$$B(\omega) = 1 + \frac{1}{4}(x_1 + 1/x_1 + x_2 + 1/x_2).$$

Let

$$K(x, y) = \frac{1}{2} + \frac{1}{4}(x_1/y_1 + x_2/y_2 + x_1 + 1/y_1 + x_2 + 1/y_2).$$

Then we can verify that $K(x, x) = B(\omega)$ and $K(x, y)$ is PDK since the matrix consisting of its Fourier coefficients

$$\begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 \end{bmatrix}$$

is positive semi-definite. On the other hand, if

$$K(x, y) = 1 + \frac{1}{4}(x_1 + 1/y_1 + x_2 + 1/y_2)$$

then the matrix of nonzero Fourier coefficients

$$\begin{bmatrix} 1 & 1/4 & 1/4 \\ 1/4 & 0 & 0 \\ 1/4 & 0 & 0 \end{bmatrix}$$

is not positive semi-definite, although $K(x, x) = B(\omega)$.

Given a positive Laurent polynomial $P(x)$, we can conclude from Example 4.9 that the problem of finding a PDK function $K(x, y)$ such that $K(x, x) = P(x)$ is not easy.

Finally let us mention that the above study is further continued in [15]. There a new and constructive proof of Dritschel's theorem is presented: if a multivariate Laurent polynomial $P(x)$ with $x = e^{i\omega}$ is strictly positive for all $|x| = 1$, then $P(x)$ is an sos. However in our situation, the corresponding Laurent polynomial is nonnegative. A new sufficient condition for nonnegative Laurent polynomials to be sos is given in [15].

5. Frame construction: multivariate box spline case

Multivariate box splines are a very important class of refinable functions. It is interesting to know how to construct compactly supported tight frames based on multivariate box splines. There are several methods available in the literature to compute tight wavelet frames using box splines, e.g., [9,31] as mentioned in Section 1. We shall employ the constructive procedure discussed in Section 3 in order to present our new method. One of the advantages is that the number of tight wavelet frame generators is smaller than that in [9] for many examples we present here.

Let us first recall the definition of box splines. Let D be a set of nonzero vectors in $\mathbb{R}^d$ (allowing multiples of the same vector) which span $\mathbb{R}^d$. The box spline ϕ_D associated with the direction set D is the function whose Fourier transform is defined by

$$\hat{\phi}_D(\omega) = \prod_{\xi \in D} \frac{1 - e^{-i\xi\cdot\omega}}{i\xi \cdot \omega}.$$

It is well known that the box spline ϕ_D is a piecewise polynomial function of degree $\leqslant \#D - d$, where $\#D$ denotes the cardinality of D. For more properties of box splines, see [5,6]. In particular, for $d = 2$ and $e_1 = (1, 0)^T$, $e_2 = (0, 1)^T$, and

$$D = \{\underbrace{e_1, \ldots, e_1}_{\ell}, \underbrace{e_2, \ldots, e_2}_{m}, \underbrace{e_1 + e_2, \ldots, e_1 + e_2}_{n}\},$$

the bivariate box spline $\phi_{\ell,m,n}$ based on this direction set D is called 3-direction box spline. Its Fourier transform is

$$\hat{\phi}_{\ell,m,n}(\xi,\eta)=\left(\frac{1-e^{-i\xi}}{i\xi}\right)^{\ell}\left(\frac{1-e^{-i\eta}}{i\eta}\right)^{m}\left(\frac{1-e^{-i(\xi+\eta)}}{i(\xi+\eta)}\right)^{n}.$$

Similarly, the box spline $\phi_{\ell,m,n,k}$ based on the 4-directional mesh is defined in terms of its Fourier transform by

$$\hat{\phi}_{\ell,m,n,k}(\xi,\eta)=\hat{\phi}_{\ell,m,n}(\xi,\eta)\left(\frac{1-e^{-i(\xi-\eta)}}{i(\xi-\eta)}\right)^{k}.$$

(For computation of 3-directional and 4-directional box splines, see [22].)

Let us note that the mask of any multivariate box spline ϕ_D, with $D\subset\mathbb{Z}^d$ and the standard dilation matrix $2I_{d\times d}$, satisfies (3.6). Indeed, we infer from the identity $\hat{\phi}_D(\omega)=P_D\big(\frac{\omega}{2}\big)\hat{\phi}_D\big(\frac{\omega}{2}\big)$ that

$$P_D(\omega)=\prod_{\xi\in D}\frac{1+e^{-i\xi\cdot\omega}}{2}.$$

This is, indeed, a trigonometric polynomial and $|P_D(\omega)|^2=\prod_{\xi\in D}(\cos\frac{\xi\cdot\omega}{2})^2$. Moreover, we have the following result.

Lemma 5.1. *Suppose that a given direction set $D\subset\mathbb{Z}^d$ contains all of the standard unit vectors e_i of $\mathbb{R}^d$, $i=1,\dots,d$. Then P_D satisfies* (3.6).

Proof. Since $|P_D(\omega)|^2\leqslant\prod_{i=1}^d\cos^2\frac{\omega_i}{2}$, with $\omega=(\omega_1,\dots,\omega_s)^T\in\mathbb{R}^d$, we have

$$\sum_{\ell\in\{0,1\}^d\pi}\big|P_D(\omega+\ell)\big|^2\leqslant\prod_{i=1}^d\left(\cos^2\frac{\omega_i}{2}+\sin^2\frac{\omega_i}{2}\right)=1.$$

This completes the proof. □

To show that the constructive steps in the proof of Theorem 3.4 can be applied to construct tight frames using box splines, we begin with the following examples (cf. [25]).

Example 5.2. Consider the 3-directional box spline $\phi_{1,1,1}$. It is easy to see that

$$1-\sum_{\ell\in\{0,1\}^2\pi}\big|P_{1,1,1}(\omega+\ell)\big|^2=\frac{3}{8}-\frac{1}{8}\cos(2\omega_1)-\frac{1}{8}\cos(2\omega_2)-\frac{1}{8}\cos(2\omega_1+2\omega_2).$$

Thus, we let

$$\tilde{P}_1(\omega)=\frac{\sqrt{6}}{8}\big(1-e^{i\omega_1}\big)\quad\text{and}\quad\tilde{P}_2(\omega)=\frac{\sqrt{2}}{8}\big(2-e^{i\omega_2}-e^{i(\omega_1+\omega_2)}\big).$$

Clearly, we have

$$\sum_{\ell\in\{0,1\}^2\pi}\big|P_{1,1,1}(\omega+\ell)\big|^2+\sum_{i=1}^2\big|\tilde{P}_i(2\omega)\big|^2=1.$$

Thus, we can apply the constructive steps in the proof of Theorem 3.4 to get 6 tight frame masks Q_i, $i=1,\dots,6$. We have implemented the constructive steps in a symbolic algebra software Maple and found these Q_i's. We note that the constructive procedure in [9] yields 7 tight frame generators.

Example 5.3. Consider the box spline $\phi_{2,2,1}$. We find that

$$1-\sum_{\ell\in\{0,1\}^2\pi}\big|P_{2,2,1}(\omega+\ell)\big|^2=\frac{19}{32}-\frac{7}{32}\cos(2\omega_1)-\frac{7}{32}\cos(2\omega_2)-\frac{1}{64}\cos(2\omega_1-2\omega_2)-\frac{9}{64}\cos(2\omega_1+2\omega_2).$$

Let

$$\tilde{P}_1(\omega) = \frac{\sqrt{21}}{12} - \frac{\sqrt{102} + 2\sqrt{21}}{48} e^{i\omega_1} + \frac{\sqrt{102} - 2\sqrt{21}}{48} e^{i\omega_2},$$
$$\tilde{P}_2(\omega) = -\frac{\sqrt{42} + 2\sqrt{51}}{48} + \frac{\sqrt{42}}{24} e^{i\omega_2} - \frac{\sqrt{42} - 2\sqrt{51}}{48} e^{i(\omega_1+\omega_2)}.$$

It is easy to check that

$$\sum_{\ell \in \{0,1\}^2 \pi} = \left| P_{2,2,1}(\omega + \ell) \right|^2 + \sum_{i=1}^{2} \left| \tilde{P}_i(2\omega) \right|^2 = 1.$$

Hence, the constructive steps in the proof of Theorem 3.4 yield 6 tight frame masks and thus, 6 tight frame generators. We note that using the constructive procedure in [9], one will get 7 tight frame generators.

Example 5.4. For the box spline $\phi_{2,2,2}$, we have

$$\begin{aligned}
1 - \sum_{\ell \in \{0,1\}^2 \pi} \left| P_{2,2,2}(\omega + \ell) \right|^2 &= \bigl(339 - 106\bigl(\cos(2\omega_1) + \cos(2\omega_2) + \cos(2\omega_1 + 2\omega_2)\bigr) \\
&\quad - \bigl(\cos(4\omega_1) + \cos(4\omega_2) + \cos\bigl(4(\omega_1 + \omega_2)\bigr)\bigr) \\
&\quad - 6\bigl(\cos(4\omega_1 + 2\omega_2) + \cos(2\omega_1 + 4\omega_2) + \cos(2\omega_1 - 2\omega_2)\bigr)\bigr)/512 \\
&= \sum_{i=1}^{3} \left| \tilde{P}_i(2\omega) \right|^2,
\end{aligned}$$

where

$$\tilde{P}_1(\omega) = \frac{\sqrt{14}}{96} + \frac{\sqrt{14}}{16} e^{i\omega_1} + \frac{\sqrt{14}}{16} e^{i\omega_2} - \frac{173}{1344}\sqrt{14} e^{i(\omega_1+\omega_2)} - \frac{3}{448}\sqrt{14} e^{2i(\omega_1+\omega_2)},$$

$$\begin{aligned}
\tilde{P}_2(\omega) &= \frac{1}{5376}\Bigl(\sqrt{713608 + 42\sqrt{178402}} + \sqrt{713608 - 42\sqrt{178402}}\Bigr) \\
&\quad - \frac{1}{2854432}\Bigl(\sqrt{713608 + 42\sqrt{178402}} - \sqrt{713608 - 42\sqrt{178402}}\Bigr)\sqrt{178402}\, e^{i2\omega_2} \\
&\quad - \frac{1}{40281744384}\Bigl(14112\Bigl(\sqrt{713608 - 42\sqrt{178402}} - \sqrt{713608 + 42\sqrt{178402}}\Bigr) \\
&\quad + 42\sqrt{178402}\Bigl(\sqrt{713608 - 42\sqrt{178402}} + \sqrt{713608 + 42\sqrt{178402}}\Bigr)\Bigr)\sqrt{178402}\, e^{i(\omega_1+\omega_2)},
\end{aligned}$$

$$\begin{aligned}
\tilde{P}_3(\omega) &= \frac{1}{5376}\Bigl(\sqrt{713608 + 42\sqrt{178402}} - \sqrt{713608 - 42\sqrt{178402}}\Bigr) \\
&\quad - \frac{1}{2854432}\Bigl(\sqrt{713608 + 42\sqrt{178402}} + \sqrt{713608 - 42\sqrt{178402}}\Bigr)\sqrt{178402} e^{2i\omega_1} \\
&\quad + \frac{1}{40281744384}\Bigl(14112\Bigl(\sqrt{713608 - 42\sqrt{178402}} + \sqrt{713608 + 42\sqrt{178402}}\Bigr) \\
&\quad + 42\sqrt{178402}\Bigl(\sqrt{713608 - 42\sqrt{178402}} - \sqrt{713608 + 42\sqrt{178402}}\Bigr)\Bigr)\sqrt{178402}\, e^{i(\omega_1+\omega_2)}.
\end{aligned}$$

Thus, we only need 7 frame generators for $\phi_{2,2,2}$.

Example 5.5. For the box spline $\phi_{1,1,1,1}$, we have

$$1 - \sum_{\ell \in \{0,1\}^2 \pi} \left| P_{1,1,1,1}(\omega + \ell) \right|^2 = \frac{5}{8} - \frac{1}{8}\bigl(e^{i2\omega_1} + e^{-i2\omega_1}\bigr) - \frac{1}{8}\bigl(e^{2i\omega_2} + e^{-2i\omega_2}\bigr)$$

$$- \frac{1}{32}\left(e^{2i(\omega_1+\omega_2)} + e^{-2i(\omega_1+\omega_2)}\right) - \frac{1}{32}\left(e^{2i(\omega_1-\omega_2)} + e^{-2i(\omega_1-\omega_2)}\right)$$
$$= \sum_{i=1}^{2} \left|\tilde{P}_i(2\omega)\right|^2,$$

where $\tilde{P}_1(\omega) = \frac{\sqrt{6}}{8}(1 - e^{i(\omega_1-\omega_2)})$ and

$$\tilde{P}_2(\omega) = -\frac{1}{4} + \frac{\sqrt{6}}{8} + \frac{1}{4}\left(e^{i\omega_1} + e^{i\omega_2}\right) - \frac{2+\sqrt{6}}{8} e^{i(\omega_1+\omega_2)}.$$

Hence, the constructive steps in the proof of Theorem 3.4 yield 6 tight frame masks and hence, 6 tight frame generators. For this particular example, the construction in [9] requires 15 generators.

Example 5.6. For the box spline $\phi_{2,2,1,1}$, we have

$$1 - \sum_{\ell\in\{0,1\}^2\pi} \left|P_{2,2,1,1}(\omega+\ell)\right|^2 = \sum_{i=1}^{4} \left|\tilde{P}_i(2\omega)\right|^2,$$

where $\tilde{P}_1(\omega) = \frac{\sqrt{1886}}{224}(1 - e^{2i\omega_1})$,

$$\tilde{P}_2(\omega) = -\frac{3\sqrt{14}}{64} + \frac{\sqrt{40531922}}{25472} + \frac{3\sqrt{14}}{32} e^{i\omega_2} - \left(\frac{3\sqrt{14}}{64} + \frac{\sqrt{40531922}}{25472}\right) e^{2i\omega_2},$$

$$\tilde{P}_3(\omega) = \frac{7\sqrt{2}}{64} + \frac{7\sqrt{2}}{64} e^{2i\omega_2} - \frac{\sqrt{2}}{224} e^{i(2\omega_1+\omega_2)} - \frac{3\sqrt{2}}{14} e^{i(\omega_1+\omega_2)},$$

and

$$\tilde{P}_4(\omega) = \frac{\sqrt{398}}{112} + \frac{\sqrt{398}}{112} e^{2i\omega_1} - \frac{3135\sqrt{398}}{178304} e^{2i\omega_1} - \frac{7\sqrt{398}}{25472} e^{i(\omega_1+2\omega_2)}.$$

Hence, we will have 8 tight frame generators. For this particular example, the construction in [9] requires 15 generators.

We refer to [25] for the detailed computation of these tight frame generators in the above examples and for their applications in image processing. We have the following general result for 3- and 4-directional box splines.

Lemma 5.7. *For $P_{\ell,m,n}$ there exist $\tilde{P}_1, \ldots, \tilde{P}_N$ with $N = 9$ such that* (3.8) *holds. That is,*

$$1 - \sum_{j\in\{0,1\}^2\pi} \left|P_{\ell,m,n}(\omega+j)\right|^2 = \sum_{i=1}^{9} \left|\tilde{P}_i(2\omega)\right|^2 \tag{5.1}$$

for some trigonometric polynomials $\tilde{P}_i$. Similarly, for $P_{\ell,m,n,r}$ there exists a collection of at most 22 *additional Laurent polynomials $\tilde{P}_i$ such that* (3.8) *holds. That is,*

$$1 - \sum_{j\in\{0,1\}^2\pi} \left|P_{\ell,m,n,r}(\omega+j)\right|^2 = \sum_{i=1}^{22} \left|\tilde{P}_i(2\omega)\right|^2 \tag{5.2}$$

for some trigonometric polynomials $\tilde{P}_i$.

Proof. We first consider box splines on the 3-directional mesh. Let

$$P_{\ell,m,n}(\omega) = \left(\frac{1+e^{i\omega_1}}{2}\right)^{\ell} \left(\frac{1+e^{i\omega_2}}{2}\right)^{m} \left(\frac{1+e^{i(\omega_1+\omega_2)}}{2}\right)^{n}, \quad \ell, m, n \geqslant 1,$$

be the mask associated with the box spline $\phi_{\ell,m,n}$. For convenience, we use 2ω instead of ω, i.e., $|P_{\ell,m,n}(2\omega)|^2 = \cos^{2\ell}(\omega_1)\cos^{2m}(\omega_2)\cos^{2n}(\omega_1+\omega_2)$ and

$$\sum_{j\in\{0,1\}^2\pi}\left|P_{\ell,m,n}(2\omega+j)\right|^2=\cos^{2n}(\omega_1+\omega_2)\big(\cos^{2\ell}(\omega_1)\cos^{2m}(\omega_2)+\sin^{2\ell}(\omega_1)\sin^{2m}(\omega_2)\big)$$
$$+\sin^{2n}(\omega_1+\omega_2)\big(\sin^{2\ell}(\omega_1)\cos^{2m}(\omega_2)+\cos^{2\ell}(\omega_1)\sin^{2m}(\omega_2)\big).$$

Note that

$$\begin{aligned}
&\cos^{2\ell}(\omega_1)\cos^{2m}(\omega_2)+\sin^{2\ell}(\omega_1)\sin^{2m}(\omega_2)\\
&=\left(\frac{1+\cos(2\omega_1)}{2}\right)^{\ell}\left(\frac{1+\cos(2\omega_2)}{2}\right)^{m}+\left(\frac{1-\cos(2\omega_1)}{2}\right)^{\ell}\left(\frac{1-\cos(2\omega_1)}{2}\right)^{m}\\
&=\frac{2}{2^{\ell+m}}\sum_{0\leqslant j\leqslant \ell/2}\sum_{0\leqslant k\leqslant m/2}\binom{\ell}{2j}\binom{m}{2k}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\\
&\quad+\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\cos(2\omega_1)\cos(2\omega_2).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\sin^{2\ell}(\omega_1)\cos^{2m}(\omega_2)+\cos^{2\ell}(\omega_1)\sin^{2m}(\omega_2)\\
&=\frac{2}{2^{\ell+m}}\sum_{0\leqslant j\leqslant \ell/2}\sum_{0\leqslant k\leqslant m/2}\binom{\ell}{2j}\binom{m}{2k}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\\
&\quad-\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\cos(2\omega_1)\cos(2\omega_2).
\end{aligned}$$

Next we can see

$$\cos^{2n}(\omega_1+\omega_2)+\sin^{2n}(\omega_1+\omega_2)=\frac{2}{2^n}\sum_{0\leqslant 2i\leqslant n}\binom{n}{2i}\cos^{2i}(2\omega_1+2\omega_2)=:f(4\omega_1+4\omega_2)\tag{5.3}$$

and

$$\begin{aligned}
\cos^{2n}(\omega_1+\omega_2)-\sin^{2n}(\omega_1+\omega_2)&=\frac{2}{2^n}\sum_{0\leqslant 2i+1\leqslant n}\binom{n}{2i+1}\cos^{2i}(2\omega_1+2\omega_2)\cos(2\omega_1+2\omega_2)\\
&=:g(4\omega_1+4\omega_2)\cos(2\omega_1+2\omega_2),
\end{aligned}$$

where f and g are univariate trigonometric polynomials. Note that $0\leqslant f\leqslant 1$ and $0\leqslant g\leqslant 1$. Hence,

$$\begin{aligned}
&\sum_{j\in\{0,1\}^2\pi}\left|P_{\ell,m,n}(2\omega+j)\right|^2\\
&=\frac{2}{2^{\ell+m}}\sum_{0\leqslant j\leqslant \ell/2}\sum_{0\leqslant k\leqslant m/2}\binom{\ell}{2j}\binom{m}{2k}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\\
&\quad\times\big(\cos^{2n}(\omega_1+\omega_2)+\sin^{2n}(\omega_1+\omega_2)\big)\\
&\quad+\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\\
&\quad\times\cos(2\omega_1)\cos(2\omega_2)\big(\cos^{2n}(\omega_1+\omega_2)-\sin^{2n}(\omega_1+\omega_2)\big)\\
&=\frac{2}{2^{\ell+m}}\sum_{0\leqslant j\leqslant \ell/2}\sum_{0\leqslant k\leqslant m/2}\binom{\ell}{2j}\binom{m}{2k}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)f(4\omega_1+4\omega_2)\\
&\quad+\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\\
&\quad\times\cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_1+2\omega_2)g(4\omega_1+4\omega_2).
\end{aligned}$$

We are ready to show (5.1). Indeed,

$$
\begin{aligned}
&1-\sum_{j\in\{0,1\}^2\pi}\left|P_{\ell,m,n}(2\omega+j)\right|^2\\
&=1-\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j\leqslant \ell}\sum_{0\leqslant 2k\leqslant m}\binom{\ell}{2j}\binom{m}{2k}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)f(4\omega_1+4\omega_2)\\
&\quad-\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\\
&\quad\times\cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_1+2\omega_2)g(4\omega_1+4\omega_2)\\
&=1-\frac{2}{2^{\ell+m}}\sum_{0\leqslant j\leqslant \ell/2}\sum_{0\leqslant k\leqslant m/2}\binom{\ell}{2j}\binom{m}{2k}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\\
&\quad-\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\\
&\quad+\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j\leqslant \ell}\sum_{0\leqslant 2k\leqslant m}\binom{\ell}{2j}\binom{m}{2k}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\bigl(1-f(4\omega_1+4\omega_2)\bigr)\\
&\quad+\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\\
&\quad\times\bigl(1-\cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_1+2\omega_2)g(4\omega_1+4\omega_2)\bigr).
\end{aligned}
$$

Note that

$$
1=\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j\leqslant \ell}\sum_{0\leqslant 2k\leqslant m}\binom{\ell}{2j}\binom{m}{2k}+\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}.
$$

It follows that

$$
\begin{aligned}
&1-\frac{2}{2^{\ell+m}}\sum_{0\leqslant j\leqslant \ell/2}\sum_{0\leqslant k\leqslant m/2}\binom{\ell}{2j}\binom{m}{2k}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\\
&\quad-\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\\
&=\frac{2}{2^{\ell+m}}\sum_{0\leqslant j\leqslant \ell/2}\sum_{0\leqslant k\leqslant m/2}\binom{\ell}{2j}\binom{m}{2k}\bigl(1-\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\bigr)\\
&\quad+\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\bigl(1-\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\bigr)\\
&=\frac{2}{2^{\ell+m}}\sum_{0\leqslant j\leqslant \ell/2}\sum_{0\leqslant k\leqslant m/2}\binom{\ell}{2j}\binom{m}{2k}\bigl(1-\cos^{2j}(2\omega_1)+\cos^{2j}(2\omega_1)\bigl(1-\cos^{2k}(2\omega_2)\bigr)\bigr)\\
&\quad+\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\bigl(1-\cos^{2j}(2\omega_1)+\cos^{2j}(2\omega_1)\bigl(1-\cos^{2k}(2\omega_2)\bigr)\bigr).
\end{aligned}
$$

It is easy to see that

$$
\frac{2}{2^{\ell+m}}\sum_{0\leqslant j\leqslant \ell/2}\sum_{0\leqslant k\leqslant m/2}\binom{\ell}{2j}\binom{m}{2k}\bigl(1-\cos^{2j}(2\omega_1)\bigr)
$$

is a trigonometric polynomial $f_1(4\omega_1)$ and is nonnegative. By using Fejér–Riesz factorization, there exists a trigonometric polynomial $\tilde{p}_1$ such that

$$\frac{2}{2^{\ell+m}}\sum_{0\leqslant j\leqslant \ell/2}\sum_{0\leqslant k\leqslant m/2}\binom{\ell}{2j}\binom{m}{2k}\big(1-\cos^{2j}(2\omega_1)\big)=\big|\tilde{p}_1(4\omega_1)\big|^2.$$

Similarly, we have

$$\frac{2}{2^{\ell+m}}\sum_{0\leqslant j\leqslant \ell/2}\sum_{0\leqslant k\leqslant m/2}\binom{\ell}{2j}\binom{m}{2k}\cos^{2j}(2\omega_1)\big(1-\cos^{2k}(2\omega_2)\big)=\big|\tilde{p}_{2,1}(4\omega_1)\big|^2\big|\tilde{p}_{2,2}(4\omega_2)\big|^2,$$

$$\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\big(1-\cos^{2j}(2\omega_1)\big)=\big|\tilde{p}_3(4\omega_1)\big|^2,$$

and

$$\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\cos^{2j}(2\omega_1)\big(1-\cos^{2k}(2\omega_2)\big)=\big|\tilde{p}_{4,1}(4\omega_1)\big|^2\big|\tilde{p}_{4,2}(4\omega_2)\big|^2.$$

Next we consider the expression

$$\begin{aligned}&\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j\leqslant \ell}\sum_{0\leqslant 2k\leqslant m}\binom{\ell}{2j}\binom{m}{2k}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\big(1-f(4\omega_1+4\omega_2)\big)\\&\quad=\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j\leqslant \ell}\binom{\ell}{2j}\cos^{2j}(2\omega_1)\sum_{0\leqslant 2k\leqslant m}\binom{m}{2k}\cos^{2k}(2\omega_2)\big(1-f(4\omega_1+4\omega_2)\big)\\&\quad=\big|\tilde{p}_{5,1}(4\omega_1)\big|^2\big|\tilde{p}_{5,2}(4\omega_2)\big|^2\big|\tilde{p}_{5,3}(4\omega_1+4\omega_2)\big|^2,\end{aligned}$$

where the last step makes use of $1-f(4\omega_1+4\omega_2)\geqslant 0$. Finally, we have

$$\begin{aligned}&\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\sum_{0\leqslant 2k+1\leqslant m}\binom{\ell}{2j+1}\binom{m}{2k+1}\cos^{2j}(2\omega_1)\cos^{2k}(2\omega_2)\\&\qquad\times\big(1-\cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_1+2\omega_2)g(4\omega_1+4\omega_2)\big)\\&\quad=\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\binom{\ell}{2j+1}\cos^{2j}(2\omega_1)\sum_{0\leqslant 2k+1\leqslant m}\binom{m}{2k+1}\cos^{2k}(2\omega_2)\big(1-g(4\omega_1+4\omega_2)\big)\\&\qquad+\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\binom{\ell}{2j+1}\cos^{2j}(2\omega_1)\sum_{0\leqslant 2k+1\leqslant m}\binom{m}{2k+1}\cos^{2k}(2\omega_2)g(4\omega_1+4\omega_2)\\&\qquad\times\big(1-\cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_1+2\omega_2)\big).\end{aligned}$$

Note that

$$1-\cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_1+2\omega_2)=\frac{3}{4}-\frac{\cos(4\omega_1+4\omega_2)}{4}-\frac{\cos(4\omega_1)}{4}-\frac{\cos(4\omega_2)}{4}.$$

By Fejér–Riesz factorization again, we have

$$\begin{aligned}&\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\binom{\ell}{2j+1}\cos^{2j}(2\omega_1)\sum_{0\leqslant 2k+1\leqslant m}\binom{m}{2k+1}\cos^{2k}(2\omega_2)\big(1-g(4\omega_1+4\omega_2)\big)\\&\quad=\big|\tilde{p}_{6,1}(4\omega_1)\big|^2\big|\tilde{p}_{6,2}(4\omega_2)\big|^2\big|\tilde{p}_{6,3}(4\omega_1+4\omega_2)\big|^2\end{aligned}$$

and

$$\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\binom{\ell}{2j+1}\cos^{2j}(2\omega_1)\sum_{0\leqslant 2k+1\leqslant m}\binom{m}{2k+1}\cos^{2k}(2\omega_2)g(4\omega_1+4\omega_2)$$
$$\times\big(1-\cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_1+2\omega_2)\big)$$
$$=\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\cos^{2j}\binom{\ell}{2j+1}(2\omega_1)\sum_{0\leqslant 2k+1\leqslant m}\binom{m}{2k+1}\cos^{2k}(2\omega_2)g(4\omega_1+4\omega_2)\frac{1-\cos(4\omega_1+4\omega_2)}{4}$$
$$+\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\binom{\ell}{2j+1}\cos^{2j}(2\omega_1)\sum_{0\leqslant 2k+1\leqslant m}\binom{m}{2k+1}\cos^{2k}(2\omega_2)g(4\omega_1+4\omega_2)\frac{1-\cos(4\omega_1)}{4}$$
$$+\frac{2}{2^{\ell+m}}\sum_{0\leqslant 2j+1\leqslant \ell}\binom{\ell}{2j+1}\cos^{2j}(2\omega_1)\sum_{0\leqslant 2k+1\leqslant m}\binom{m}{2k+1}\cos^{2k}(2\omega_2)g(4\omega_1+4\omega_2)\frac{1-\cos(4\omega_2)}{4}$$
$$=\big|\tilde p_{7,1}(4\omega_1)\big|^2\big|\tilde p_{7,2}(4\omega_2)\big|^2\big|\tilde p_{7,3}(4\omega_1+4\omega_2)\big|^2+\big|\tilde p_{8,1}(4\omega_1)\big|^2\big|\tilde p_{8,2}(4\omega_2)\big|^2\big|\tilde p_{8,3}(4\omega_1+4\omega_2)\big|^2$$
$$+\big|\tilde p_{9,1}(4\omega_1)\big|^2\big|\tilde p_{9,2}(4\omega_2)\big|^2\big|\tilde p_{9,3}(4\omega_1+4\omega_2)\big|^2.$$

Therefore, we have established the result for box splines on a three direction mesh.

Next we consider the mask associated with box splines on a four direction mesh. We shall use the same ideas as above to obtain the desired result. Let

$$P_{\ell,m,n,r}(\omega)=\left(\frac{1+e^{i\omega_1}}{2}\right)^{\ell}\left(\frac{1+e^{i\omega_2}}{2}\right)^{m}\left(\frac{1+e^{i(\omega_1+\omega_2)}}{2}\right)^{n}\left(\frac{1+e^{i(\omega_1-\omega_2)}}{2}\right)^{r}$$

be the mask associated with the box spline function $\phi_{\ell,m,n,r}$. For convenience, we use 2ω instead of ω, i.e., $|P_{\ell,m,n,r}(2\omega)|^2=\cos^{2\ell}(\omega_1)\cos^{2m}(\omega_2)\cos^{2n}(\omega_1+\omega_2)\cos^{2r}(\omega_1-\omega_2)$. Thus,

$$\sum_{j\in\{0,1\}^2\pi}\big|P_{\ell,m,n,r}(2\omega+j)\big|^2=\cos^{2n}(\omega_1+\omega_2)\cos^{2r}(\omega_1-\omega_2)\big(\cos^{2\ell}(\omega_1)\cos^{2m}(\omega_2)+\sin^{2\ell}(\omega_1)\sin^{2m}(\omega_2)\big)$$
$$+\sin^{2n}(\omega_1+\omega_2)\sin^{2r}(\omega_1-\omega_2)\big(\sin^{2\ell}(\omega_1)\cos^{2m}(\omega_2)+\cos^{2\ell}(\omega_1)\sin^{2m}(\omega_2)\big).$$

As in the discussion of the 3-directional box spline, we have

$$\cos^{2n}(\omega_1+\omega_2)\cos^{2r}(\omega_1-\omega_2)+\sin^{2n}(\omega_1+\omega_2)\sin^{2r}(\omega_1-\omega_2)$$
$$=\frac{2}{2^{n+r}}\sum_{0\leqslant 2i+1\leqslant n}\sum_{0\leqslant 2q+1\leqslant r}\binom{n}{2i+1}\binom{r}{2q+1}$$
$$\times\cos^{2i}(2\omega_1+2\omega_2)\cos^{2q}(2\omega_1-2\omega_2)\cos(2\omega_1+2\omega_2)\cos(2\omega_1-2\omega_2)$$
$$+\frac{2}{2^{n+r}}\sum_{0\leqslant 2i\leqslant n}\sum_{0\leqslant 2q\leqslant r}\binom{n}{2i}\binom{r}{2q}\cos^{2i}(2\omega_1+2\omega_2)\cos^{2q}(2\omega_1-2\omega_2)$$
$$=: f_1(4\omega_1,4\omega_2)\cos(2\omega_1+2\omega_2)\cos(2\omega_1-2\omega_2)+f_2(4\omega_1,4\omega_2);$$

the last expression plays the same role as the function $f(4\omega_1+4\omega_2)$ in (5.3). Similarly, we have

$$\cos^{2n}(\omega_1+\omega_2)\cos^{2r}(\omega_1-\omega_2)-\sin^{2n}(\omega_1+\omega_2)\sin^{2r}(\omega_1-\omega_2)$$
$$=\frac{2}{2^{n+r}}\sum_{0\leqslant 2i\leqslant n}\sum_{0\leqslant 2q+1\leqslant r}\binom{n}{2i}\binom{r}{2q+1}\cos^{2i}(2\omega_1+2\omega_2)\cos^{2q}(2\omega_1-2\omega_2)\cos(2\omega_1-2\omega_2)$$
$$+\frac{2}{2^{n+r}}\sum_{0\leqslant 2i+1\leqslant n}\sum_{0\leqslant 2q\leqslant r}\binom{n}{2i+1}\binom{r}{2q}\cos^{2i}(2\omega_1+2\omega_2)\cos^{2q}(2\omega_1-2\omega_2)\cos(2\omega_1+2\omega_2)$$
$$=: g_1(4\omega_1,4\omega_2)\cos(2\omega_1-2\omega_2)+g_2(4\omega_1,4\omega_2)\cos(2\omega_1+2\omega_2).$$

By the observation that

$$f_1(4\omega_1,4\omega_2)=\frac{2}{2^{n+r}}\sum_{0\leqslant 2i+1\leqslant n}\binom{n}{2i+1}\cos^{2i}(2\omega_1+2\omega_2)\sum_{0\leqslant 2q+1\leqslant r}\binom{r}{2q+1}\cos^{2q}(2\omega_1-2\omega_2)\geqslant 0$$

and by Fejér–Riesz factorization, we can find a trigonometric polynomial $\hat{p}_1$ such that $f_1(4\omega_1, 4\omega_2) = |\hat{p}_1(4\omega_1, 4\omega_2)|^2$. Thus, using the same arguments as for the Laurent polynomials associated with box splines on the three direction mesh, we have

$$
\begin{aligned}
&1 - f_1(4\omega_1, 4\omega_2)\cos(2\omega_1 + 2\omega_2)\cos(2\omega_1 - 2\omega_2) - f_2(4\omega_1, 4\omega_2) \\
&\quad = 1 - f_1(4\omega_1, 4\omega_2) - f_2(4\omega_1, 4\omega_2) \\
&\qquad + f_1(4\omega_1, 4\omega_2)\bigl(1 - \cos(2\omega_1 + 2\omega_2)\cos(2\omega_1 - 2\omega_2)\bigr) \\
&\quad = \frac{2}{2^{n+r}} \sum_{0 \leqslant 2i \leqslant n} \sum_{0 \leqslant 2q \leqslant r} \binom{n}{2i}\binom{r}{2q}\bigl(1 - \cos^{2i}(2\omega_1 + 2\omega_2)\cos^{2q}(2\omega_1 - 2\omega_2)\bigr) \\
&\qquad + \frac{2}{2^{n+r}} \sum_{0 \leqslant 2i+1 \leqslant n} \sum_{0 \leqslant 2q+1 \leqslant r} \binom{n}{2i+1}\binom{r}{2q+1}\bigl(1 - \cos^{2i}(2\omega_1 + 2\omega_2)\cos^{2q}(2\omega_1 - 2\omega_2)\bigr) \\
&\qquad + f_1(4\omega_1, 4\omega_2)\left(1 - \frac{1}{2}\cos(4\omega_1) - \frac{1}{2}\cos(4\omega_2)\right) \\
&\quad = \bigl|\hat{p}_{2,1}(4\omega_1, 4\omega_2)\bigr|^2 + \bigl|\hat{p}_{2,2}(4\omega_1, 4\omega_2)\bigr|^2 + \bigl|\hat{p}_{2,3}(4\omega_1, 4\omega_2)\bigr|^2 + \bigl|\hat{p}_{2,4}(4\omega_1, 4\omega_2)\bigr|^2 \\
&\qquad + \bigl|\hat{p}_{2,5}(4\omega_1, 4\omega_2)\bigr|^2 + \bigl|\hat{p}_{2,6}(4\omega_1, 4\omega_2)\bigr|^2 + \bigl|\hat{p}_1(4\omega_1, 4\omega_2)\bigr|^2\left(\left|\frac{1 - e^{i4\omega_1}}{\sqrt{2}}\right|^2 + \left|\frac{1 - e^{i4\omega_2}}{\sqrt{2}}\right|^2\right)
\end{aligned}
$$

for some trigonometric polynomials $\hat{p}_{2,i}$, $i = 1, \ldots, 6$. Similarly,

$$
\begin{aligned}
&1 - \cos(2\omega_1)\cos(2\omega_2)\bigl(g_1(4\omega_1, 4\omega_2)\cos(2\omega_1 - 2\omega_2) + g_2(4\omega_1, 4\omega_2)\cos(2\omega_1 + 2\omega_2)\bigr) \\
&\quad = 1 - g_1(4\omega_1, 4\omega_2) - g_2(4\omega_1, 4\omega_2) + g_1(4\omega_1, 4\omega_2)\bigl(1 - \cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_1 - 2\omega_2)\bigr) \\
&\qquad + g_2(4\omega_1, 4\omega_2)\bigl(1 - \cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_1 + 2\omega_2)\bigr) \\
&\quad = \frac{2}{2^{n+r}} \sum_{0 \leqslant 2i \leqslant n} \sum_{0 \leqslant 2q+1 \leqslant r} \binom{n}{2i}\binom{r}{2q+1} \\
&\qquad \times \bigl(1 - \cos^{2i}(2\omega_1 + 2\omega_2) + \cos^{2i}(2\omega_1 + 2\omega_2)\bigl(1 - \cos^{2q}(2\omega_1 - 2\omega_2)\bigr)\bigr) \\
&\qquad + \frac{2}{2^{n+r}} \sum_{0 \leqslant 2i+1 \leqslant n} \sum_{0 \leqslant 2q \leqslant r} \binom{n}{2i+1}\binom{r}{2q} \\
&\qquad \times \bigl(1 - \cos^{2i}(2\omega_1 + 2\omega_2) + \cos^{2i}(2\omega_1 + 2\omega_2)\bigl(1 - \cos^{2q}(2\omega_1 - 2\omega_2)\bigr)\bigr) \\
&\qquad + g_1(4\omega_1, 4\omega_2)\left(\frac{3}{4} - \frac{\cos(4\omega_1 + 4\omega_2)}{4} - \frac{\cos(4\omega_1)}{4} - \frac{\cos(4\omega_2)}{4}\right) \\
&\qquad + g_2(4\omega_1, 4\omega_2)\left(\frac{3}{4} - \frac{\cos(4\omega_1 + 4\omega_2)}{4} - \frac{\cos(4\omega_1)}{4} - \frac{\cos(4\omega_2)}{4}\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\frac{2}{2^{n+r}} \sum_{0 \leqslant 2i \leqslant n} \sum_{0 \leqslant 2q+1 \leqslant r} \binom{n}{2i}\binom{r}{2q+1}\bigl(1 - \cos^{2i}(2\omega_1 + 2\omega_2) + \cos^{2i}(2\omega_1 + 2\omega_2)\bigl(1 - \cos^{2q}(2\omega_1 - 2\omega_2)\bigr)\bigr) \\
&\quad = \bigl|\hat{p}_{3,1}(4\omega_1, 4\omega_2)\bigr|^2 + \bigl|\hat{p}_{3,2}(4\omega_1, 4\omega_2)\bigr|^2
\end{aligned}
$$

and

$$
\begin{aligned}
&g_1(4\omega_1, 4\omega_2)\left(\frac{3}{4} - \frac{\cos(4\omega_1 + 4\omega_2)}{4} - \frac{\cos(4\omega_1)}{4} - \frac{\cos(4\omega_2)}{4}\right) \\
&\quad = \bigl|\hat{p}_{3,3}(4\omega_1, 4\omega_2)\bigr|^2 + \bigl|\hat{p}_{3,4}(4\omega_1, 4\omega_2)\bigr|^2 + \bigl|\hat{p}_{3,5}(4\omega_1, 4\omega_2)\bigr|^2.
\end{aligned}
$$

The other two terms are similar. Thus, we have

$$1-\cos(2\omega_1)\cos(2\omega_2)\big(g_1(4\omega_1,4\omega_2)\cos(2\omega_1-2\omega_2)+g_2(4\omega_1,4\omega_2)\cos(2\omega_1+2\omega_2)\big)$$
$$=\sum_{j=1}^{10}\big|\hat p_{3,j}(4\omega_1,4\omega_2)\big|^2$$

for some 10 trigonometric polynomials $\tilde p_{3,j}$.

Now we apply the same procedure as that for box splines on a three direction mesh with the replacement of $f(4\omega_1,4\omega_2)$ by the term involving f_1 and f_2 in (5.3) and the replacement of $g(4\omega_1,4\omega_2)$ by the term having g_1 and g_2. Indeed,

$$1-\sum_{j\in\{0,1\}^2\pi}\big|P_{\ell,m,n,r}(2\omega+j)\big|^2$$
$$=\big|\tilde p_1(4\omega_1)\big|^2+\big|\tilde p_{2,1}(4\omega_1)\big|^2\big|\tilde p_{2,2}(4\omega_2)\big|^2+\big|\tilde p_3(4\omega_1)\big|^2+\big|\tilde p_{4,1}(4\omega_1)\big|^2\big|\tilde p_{4,2}(4\omega_2)\big|^2$$
$$+\big|\tilde p_{5,1}(4\omega_1)\big|^2\big|\tilde p_{5,2}(4\omega_2)\big|^2\left(\sum_{j=1}^{8}\big|\hat p_{2,j}(4\omega_1,4\omega_2)\big|^2\right)$$
$$+\big|\tilde p_{6,1}(4\omega_1)\big|^2\big|\tilde p_{6,2}(4\omega_2)\big|^2\sum_{j=1}^{10}\big|\hat p_{3,j}(4\omega_1,4\omega_2)\big|^2,$$

where

$$\hat p_{2,7}=\hat p_1(4\omega_1,4\omega_2)\frac{1-e^{i4\omega_1}}{\sqrt2}\quad\text{and}\quad\hat p_{2,8}=\hat p_1(4\omega_1,4\omega_2)\frac{1-e^{i4\omega_2}}{\sqrt2}.$$

Therefore, we have established (3.8) with $N=22$ for the case of box splines on the four direction mesh. This completes the proof. □

By combining Lemma 5.7 and Theorem 3.4, we obtain the following result.

Theorem 5.8. *For each* 3*-direction* (*or* 4*-direction*) *box spline, there exists a set of compactly supported frame generators* $\Psi=\{\psi^{(i)},\ i=1,\dots,N\}$ *with* $N\leqslant 13$ (*or* 26) *generating a tight wavelet frame.*

We first note that the number of tight wavelet framelets is fixed for box splines of any smoothness. The numbers of tight wavelet framelets for box splines on three and four directional meshes in Lemma 5.7 are not as good as those found in [9], although many examples from above show that the actual number may be smaller. We also note that the order of vanishing moments of the tight wavelet frames constructed above is 1. The next section provides a method in order to increase this order.

6. Frame construction: maximum vanishing moments

We now consider how to increase the order of vanishing moments. Based on the developments in [10,13], we will use rational trigonometric functions S to do so. But we must first explain the maximum vanishing moments in the multivariate setting. Recall that $\hat\phi(0)=1$. Thus, $P(0)=1$ and, in general, $P(\ell)=0$ for all $\ell\in\{0,1\}^d\pi\backslash\{0\}$. Let us assume that P has the following zero property:

$$P(\omega+\ell)=O\big(|\omega|^m\big),\quad \ell\in\{0,1\}^d\pi\backslash\{0\},\tag{6.1}$$

where m is a positive integer. For example, for bivariate box splines $\phi_{j,k,\ell}$, the definition of P

$$P_{j,k,l}(\omega)=\left(\frac{1+e^{i\xi}}{2}\right)^j\left(\frac{1+e^{i\eta}}{2}\right)^k\left(\frac{1+e^{i(\xi+\eta)}}{2}\right)^l,$$

with $\omega=(\xi,\eta)$, reveals that $P_{j,k,l}(\ell+\omega)=O(|\omega|^m)$, where $m=\min\{j+k,j+l,k+l\}$ and $\ell\in\{0,1\}^2\pi\backslash\{0\}$. Let $\psi^{(i)}$, $i=1,\dots,r$, be tight wavelet frame generators defined in terms of their Fourier transform

$$\hat\psi^{(i)}(\omega)=Q_i(\omega/2)\hat\phi(\omega/2).$$

Then we say that $\psi^{(i)}$, $i = 1, \ldots, r$, have maximum vanishing moments, if

$$Q_i(\omega) = O\big(|\omega|^m\big), \quad i = 1, \ldots, r.$$

In this section, we consider a special case, where $S(\omega)$ is a rational trigonometric function satisfying the following properties:

$$S(0) = 1, \tag{6.2}$$

$$\sum_{\ell \in \{0,1\}^d \pi} \frac{|P(\omega+\ell)|^2 S(2\omega)}{S(\omega+\ell)} = 1, \tag{6.3}$$

$$1/S(\omega) = \sum_{k=1}^{n} s_k(\omega)\overline{s_k(\omega)} \tag{6.4}$$

for some rational Laurent polynomials s_k.

We remark that the existence of S satisfying (6.2) and (6.3) is trivial. For example, let

$$F_\phi(x) := \int_{\mathbb{R}^d} \phi(x+y)\phi(y)\,\mathrm{d}y$$

be the autocorrelation function associated with ϕ and

$$B_\phi(\omega) := \sum_{k \in \mathbb{Z}^d} F_\phi(k) e^{ik\omega}.$$

Then it is easy to show that

$$\sum_{\ell \in \{0,1\}^d \pi} \big|P(\omega+\ell)\big|^2 B_\phi(\omega+\ell) = B_\phi(2\omega) \tag{6.5}$$

(cf. [7] when $d = 1$). Moreover, the identities

$$\int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \sum_{k \in \mathbb{Z}^d} \phi(x-k) = 1$$

give $B_\phi(0) = 1$. If we let S be the rational trigonometric function $S = 1/B_\phi$, then S satisfies (6.2) and (6.3).

We next remark that for many bivariate box spline function $\phi_{j,k,l}$, $B_{\phi_{j,k,l}}$ can be written as a sum of squares of trigonometric polynomials. That is, $S = 1/B_\phi$ also satisfies (6.4) for these ϕ. See some examples below after we present our main result of this section.

Theorem 6.1. *Suppose that S satisfies* (6.2)–(6.4). *Then there exist $2^d n$ tight wavelet frame generators which have maximum vanishing moments.*

Proof. For matrices and vectors, whose row (or column) size is $2^d n$, we make use of the notation $(k, \ell) \in \{1, \ldots, n\} \times \{0, 1\}^d =: J$ in order to denote a row (or column) index. Let $\tilde{\mathcal{P}} = (\sqrt{S(2\omega)}P(\omega+\ell\pi)\overline{s_k(\omega+\ell\pi)};\ (k,\ell) \in J)^T$ be a vector of size $2^d n \times 1$. By (6.3) and (6.4), we obtain that $\tilde{\mathcal{P}}^*\tilde{\mathcal{P}} = 1$ and, therefore,

$$(I_{2^d n \times 2^d n} - \tilde{\mathcal{P}}\tilde{\mathcal{P}}^*)^2 = I_{2^d n \times 2^d n} - \tilde{\mathcal{P}}\tilde{\mathcal{P}}^*.$$

Moreover, let $\mathcal{R} = (\delta_{\lambda,\ell} S(\omega+\ell\pi) s_k(\omega+\ell\pi);\ \lambda \in \{0,1\}^d,\ (k,\ell) \in J)$ be a block diagonal matrix of size $2^d \times 2^d n$ with diagonal blocks of size $1 \times n$. Again, (6.3) and (6.4) give

$$\mathcal{R}\mathcal{R}^* = \operatorname{diag}\big(S(\omega+\ell\pi);\ \ell \in \{0,1\}^d\big), \qquad \mathcal{R}\tilde{\mathcal{P}} = \sqrt{S(2\omega)}\mathcal{P},$$

where $\mathcal{P} = (P(\omega+\lambda);\ \lambda \in \{0,1\}^d \pi)$.

Then we define

$$\mathcal{Q} = \mathcal{R}(I_{2^d n \times 2^d n} - \tilde{\mathcal{P}}\tilde{\mathcal{P}}^*)(\mathcal{M} \otimes I_{n \times n}), \tag{6.6}$$

where $\mathcal{M}$ is the polyphase matrix in (3.2) and $\mathcal{M} \otimes I_{n\times n} = 2^{-d/2}(e^{i\lambda(\omega+\ell\pi)}\delta_{k,j};\ (k,\ell)\in J,\ (j,\lambda)\in J)$ is the Kronecker product. The matrix $\mathcal{Q}$ satisfies (2.3); that is, we have

$$\mathcal{Q}\mathcal{Q}^* = \mathcal{R}\mathcal{R}^* - \mathcal{R}\tilde{\mathcal{P}}\tilde{\mathcal{P}}^*\mathcal{R}^* = \operatorname{diag}\bigl(S(\omega+\ell);\ \ell\in\{0,1\}^d\pi\bigr) - S(2\omega)\mathcal{P}\mathcal{P}^*.$$

Let $Q_{(k,\ell)}$, $(k,\ell)\in J$, denote the entries of the first row of $\mathcal{Q}$. Simple computations give

$$\mathcal{R}(\mathcal{M}\otimes I_{n\times n}) = 2^{-d/2}\bigl(e^{i\ell(\omega+\lambda)}S(\omega+\lambda)s_k(\omega+\lambda)\bigr)_{\substack{\lambda\in\{0,1\}^d\pi\\ (k,\ell)\in J}}$$

and

$$\tilde{\mathcal{P}}^*(\mathcal{M}\otimes I_{n\times n}) = 2^{-d/2}\sqrt{S(2\omega)}\left(\sum_{\lambda\in\{0,1\}^d\pi}\overline{P(\omega+\lambda)}s_k(\omega+\lambda)e^{i\ell(\omega+\lambda)}\right)_{(k,\ell)\in J}.$$

Note that the last expression is a row vector whose entries are π periodic. Hence, the matrix Q has the required form $\mathcal{Q} = (Q_{(k,\ell)}(\cdot+\lambda);\ \lambda\in\{0,1\}^d\pi,\ (k,\ell)\in J)$ as in Lemma 2.2. Moreover, we obtain that

$$Q_{(k,\ell)}(\omega) = 2^{-d/2}e^{i\ell\omega}S(\omega)s_k(\omega) - 2^{-d/2}S(2\omega)P(\omega)\sum_{\lambda\in\{0,1\}^d\pi}\overline{P(\omega+\lambda)}s_k(\omega+\lambda)e^{i\ell(\omega+\lambda)}.$$

This sum has the order $O(|\omega|^m)$ near $\omega=0$, since $S(\omega)-S(2\omega)|P(\omega)|^2 = O(|\omega|^{2m})$ by (6.3) and the remaining summands with $\lambda\neq 0$ contain a factor $P(\omega+\lambda)=O(|\omega|^m)$. Therefore, the wavelet frame generators $\psi^{(i)}$ associated with Q_i, $i=1,\ldots,2^d n$, have the maximum number of vanishing moments. This completes the proof. □

We should note that although the tight wavelet frame generators are not compactly supported, the filters Q_m are rational functions. Hence the computational methods for ARMA filters apply.

Next we present some examples of refinable functions for which (6.4) is satisfied.

Example 6.2. For the bivariate box spline function $\phi_{1,1,1}$, it can be easily shown that

$$B_{1,1,1}(\omega) = \frac{1}{4} + \frac{1}{12}\bigl|1+e^{i\xi}+e^{i(\xi+\eta)}\bigr|^2.$$

Note that $B_{1,1,1}(\omega)$ is the sum of squares (sos) of two Laurent polynomials of low degree. On the other hand, it is impossible to write $B_{1,1,1}$ as a modulus square of a single Laurent polynomial of coordinate degree 1. Indeed, this would imply that there exists a real 4×4 matrix $B=(b_{jk})_{1\leqslant j,k\leqslant 4}$, of rank 1, such that

$$B_{1,1,1}(\omega) = \frac{1}{2} + \frac{1}{12}\bigl(e^{i\xi}+e^{-i\xi}+e^{i\eta}+e^{-i\eta}+e^{i(\xi+\eta)}+e^{-i(\xi+\eta)}\bigr) = \bigl[1, e^{-\xi}, e^{-i\eta}, e^{-i(\xi+\eta)}\bigr]B\begin{bmatrix}1\\ e^{\xi}\\ e^{\eta}\\ e^{i(\xi+\eta)}\end{bmatrix}.$$

This would result in 5 linear constraints for the entries of B, namely

$$b_{11}+b_{22}+b_{33}+b_{44}=1/2,\qquad b_{12}+b_{34}=1/12,\qquad b_{13}+b_{24}=1/12,\qquad b_{14}=1/12,\qquad b_{23}=0.$$

Because $b_{23}=0$, $b_{14}=1/12$ and B has rank 1, then either $b_{13}=0$ or the second row is a zero vector. Similarly, since $b_{32}=0$ and $b_{41}=1/12$, either $b_{42}=0$ or the last row is a zero vector. All these four cases lead to $b_{11}+b_{22}+b_{33}+b_{44}=1/4$ which is a contradiction to the first linear constraint.

Example 6.3. Let us consider the box spline $\phi_{1,1,1,1}$ which is a C^1 quadratic spline function on the four directional mesh. Based on the algorithm in [22], we have

$$\begin{aligned}B_{1,1,1,1}(\omega) = \frac{1}{480}\bigl(&280+118e^{i\xi}+118e^{-i\xi}+118e^{i\eta}+118e^{-i\eta}\\ &+44e^{i(\xi+\eta)}+44e^{-i(\xi+\eta)}+44e^{i(\xi-\eta)}+44e^{-i(\xi-\eta)}\\ &+6e^{2i\xi}+6e^{2i\eta}+6e^{-2i\xi}+6e^{-2i\eta}+e^{i(\xi+2\eta)}+e^{i(\xi-2\eta)}\\ &+e^{i(2\xi+\eta)}+e^{i(2\xi-\eta)}+e^{-i(\xi+2\eta)}+e^{-i(\xi-2\eta)}e^{-i(2\xi+\eta)}+e^{-i(2\xi-\eta)}\bigr).\end{aligned}$$

In terms of $\cos\xi$ and $\cos\eta$, neglecting the common factor, the right-hand side can be written as

$$\begin{aligned}&140+118(\cos\xi+\cos\eta)+44\big(\cos(\xi+\eta)+\cos(\xi-\eta)\big)\\&\quad+6(\cos 2\xi+\cos 2\eta)+\cos(2\xi+\eta)+\cos(\xi+2\eta)+\cos(2\xi-\eta)+\cos(\xi-2\eta)\\&=88(1+\cos\xi)(1+\cos\eta)+2(1+\cos 2\xi)(1+\cos\eta)+2(1+\cos\xi)(1+\cos 2\eta)\\&\quad+4(1+\cos\xi)\big(3+2(1+\cos\xi)\big)+4(1+\cos\eta)\big(3+2(1+\cos\eta)\big).\end{aligned}$$

Note that $1+\cos\xi=\frac{1}{2}|1+e^{i\xi}|^2$ and similar relations hold for the remaining terms. Thus, $B_{1,1,1,1}$ is an sos of several Laurent polynomials.

Example 6.4. We also find that $B_{2,2,1}$ is an sos of several Laurent polynomials. Using the algorithm in [22], the expression for $B_{2,2,1}$ is

$$B_{2,2,1}(\omega)=\frac{1}{10080}\big[e^{-2i\xi},e^{-i\xi},1,e^{i\xi},e^{2i\xi}\big]\begin{bmatrix}0&1&31&47&5\\1&178&1144&814&47\\31&1144&3194&1144&31\\47&814&1144&178&1\\5&47&31&1&0\end{bmatrix}\begin{bmatrix}e^{-2i\eta}\\e^{-i\eta}\\1\\e^{i\eta}\\e^{2i\eta}\end{bmatrix}.$$

In fact, it can be directly verified that

$$\begin{aligned}10080B_{2,2,1}(\omega)&=990-\frac{103564}{113}+\left|\sqrt{226}+\frac{324}{\sqrt{226}}e^{i\omega_1}+\sqrt{226}e^{i(\omega_1+\omega_2)}\right|^2\\&\quad+5\big(1+\cos(2\omega_1+2\omega_2)\big)+29\big(1+\cos(2\omega_1)\big)+29\big(1+\cos(2\omega_2)\big)\\&\quad+356\big(1+\cos(\omega_1)\big)\big(1+\cos(\omega_2)\big)\\&\quad+2\big(1+\cos(2\omega_1)\big)\big(1+\cos(\omega_2)\big)+2\big(1+\cos(\omega_1)\big)\big(1+\cos(2\omega_2)\big)\\&\quad+92\big(1+\cos(\omega_1)\big)\big(1+\cos(\omega_1+\omega_2)\big)+92\big(1+\cos(\omega_2)\big)\big(1+\cos(\omega_1+\omega_2)\big)\end{aligned}$$

which can be easily seen to be an sos of 10 Laurent polynomials. We leave the verification to the interested reader.

Clearly, the symbol $B_{j,k,0}$ which is associated with the tensor product of B-splines $\phi_{j,k,0}$ is an sos of Laurent polynomials by the Riesz theorem. Based on these facts, we conjecture that $B_{jkl}(\omega)$ is sos of finitely many Laurent polynomials for any bivariate box spline ϕ_{jkl}.

7. Frame construction: compactly supported bi-frames

In this section, we construct compactly supported bi-frames. For refinable functions ϕ and ϕ^{dual}, let P and P^{dual} denote the symbols of the respective refinement masks. (We use the superscript dual in order to point to duality of the respective frames; we do not impose duality between the translates of ϕ and ϕ^{dual}.) Moreover, let ψ_j and ψ_j^{dual} be functions associated with ϕ and ϕ^{dual} defined by

$$\hat{\psi}_j(\omega)=Q_j(\omega/2)\hat{\phi}(\omega/2)\quad\text{and}\quad\widehat{\psi_j^{\mathrm{dual}}}(\omega)=Q_j^{\mathrm{dual}}(\omega/2)\widehat{\phi^{\mathrm{dual}}}(\omega/2),\tag{7.1}$$

where Q_j, Q_j^{dual}, $j=1,\dots,r$, are two families of masks. Let $\Lambda(\Psi)$ and $\Lambda(\Psi^{\mathrm{dual}})$ be the corresponding families of shifts and dilates:

$$\begin{aligned}\Lambda(\Psi)&:=\big\{2^{jd/2}\psi_i\big(2^jx-k\big);\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d,\ i=1,\dots,r\big\},\\\Lambda\big(\Psi^{\mathrm{dual}}\big)&:=\big\{2^{jd/2}\psi_i^{\mathrm{dual}}\big(2^jx-k\big);\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d,\ i=1,\dots,r\big\}.\end{aligned}\tag{7.2}$$

Definition 7.1. *The two families $\Lambda(\Psi)$ and $\Lambda(\Psi^{\mathrm{dual}})$ are called bi-frames if they are Bessel families and the duality relation*

$$\langle f,g\rangle=\sum_{i=1}^{n}\sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{Z}^d}}\langle f,\psi_{i;j,k}\rangle\big\langle\psi_{i;j,k}^{\mathrm{dual}},g\big\rangle\tag{7.3}$$

holds for all $f, g \in L_2(\mathbb{R}^d)$. The functions ψ_i and ψ_i^{dual} are called bi-framelets or bi-frame generators.

The concept of bi-frames was first introduced in [30]. See several examples of bi-frames in [10,13,32]. We shall present another method to construct bi-frames. Let us begin with the following well-known result called the mixed oblique extension principle (cf. [13, Proposition 5.2] and [12, Theorem 2.2]).

Theorem 7.2. *Let $S(\omega)$ be a 2π periodic function which is essentially bounded and continuous at the origin with $S(0)=1$. Suppose that ϕ and ϕ^{dual} are compactly supported refinable functions. Suppose that there are Q_i, Q_i^{dual}, $i=1,\dots,r$, satisfying*

$$S(2\omega)P(\omega)\overline{P^{\mathrm{dual}}(\omega+\ell)}+\sum_{i=0}^{r}Q_i(\omega)\overline{Q_i^{\mathrm{dual}}(\omega+\ell)}=\begin{cases}S(\omega), & \text{if } \ell=0,\\ 0, & \text{otherwise}\end{cases}\tag{7.4}$$

for $\ell\in\{0,1\}^d\pi$. Suppose that $Q_i(\omega)$ and $Q_i^{\mathrm{dual}}(\omega)$ have a zero at $\omega=0$. Let ψ_i and ψ_i^{dual} be the functions defined by their Fourier transform in (7.1). Then the two families $\Lambda(\Phi)$ and $\Lambda(\Phi^{\mathrm{dual}})$ are bi-frames.

Also, similar to Lemma 2.2, we have

Lemma 7.3. *Let $\mathcal{P}=(P(\omega+\ell);\ \ell\in\{0,1\}^d\pi)$ be a vector of size $2^d\times 1$, $\mathcal{Q}=(Q_i(\omega+\ell);\ \ell\in\{0,1\}^d\pi,\ i=1,\dots,r)$ be a matrix of size $2^d\times r$, and $\mathcal{P}^{\mathrm{dual}}$, $\mathcal{Q}^{\mathrm{dual}}$ be given analogously. Then (7.4) is equivalent to*

$$\mathcal{Q}\big(\mathcal{Q}^{\mathrm{dual}}\big)^*=\mathrm{diag}\big(S(\omega+\ell);\ \ell\in\{0,1\}^d\pi\big)-S(2\omega)\mathcal{P}\big(\mathcal{P}^{\mathrm{dual}}\big)^*.\tag{7.5}$$

Proof. This can be verified directly. □

We first consider the case where $S(\omega)\equiv 1$. We will construct compactly supported bi-frames for those masks P and P^{dual} which satisfy

$$\sum_{\ell\in\{0,1\}^d\pi}P(\omega+\ell)\overline{P^{\mathrm{dual}}(\omega+\ell)}=1\tag{7.6}$$

and $P(0)=1=P^{\mathrm{dual}}(0)$. Let $\mathcal{P}$ and $\mathcal{P}^{\mathrm{dual}}$ be given as in Lemma 7.3. Recall the unitary matrix $\mathcal{M}$ defined in (3.2). Then we have

Theorem 7.4. *Define*

$$\mathcal{Q}:=\big(Q_i(\omega+\ell)\big)_{\substack{\ell\in\{0,1\}^d\pi\\ i=1,\dots,2^d}}=\big(I_{2^d\times 2^d}-\mathcal{P}\big(\mathcal{P}^{\mathrm{dual}}\big)^*\big)\mathcal{M}$$

and

$$\mathcal{Q}^{\mathrm{dual}}:=\big(Q_i^{\mathrm{dual}}(\omega+\ell)\big)_{\substack{\ell\in\{0,1\}^d\pi\\ i=1,\dots,2^d}}=\big(I_{2^d\times 2^d}-\mathcal{P}^{\mathrm{dual}}\mathcal{P}^*\big)\mathcal{M}.$$

Then $\mathcal{P}$, $\mathcal{P}^{\mathrm{dual}}$, $\mathcal{Q}$, and $\mathcal{Q}^{\mathrm{dual}}$ satisfy (7.5), with $S(\omega)\equiv 1$. Let ψ_i and ψ_i^{dual} be defined by (7.1), with these Q_i's and Q_i^{dual}'s. Then $\{\psi_i,\ i=1,\dots,2^d\}$ and $\{\psi_i^{\mathrm{dual}},\ i=1,\dots,2^d\}$ are bi-framelets.

Proof. It is trivial to verify that

$$\mathcal{Q}\big(\mathcal{Q}^{\mathrm{dual}}\big)^*=I_{2^d\times 2^d}-\mathcal{P}\big(\mathcal{P}^{\mathrm{dual}}\big)^*$$

which is (7.5) with $S(\omega)=1$. Since both $\mathcal{M}$ and $\mathcal{P}$ have the desired form and since $(\mathcal{P}^{\mathrm{dual}})^*\mathcal{M}$ is a row vector whose entries are π periodic, the matrix $\mathcal{Q}$ has the desired form as well. Analogous statements hold for $\mathcal{Q}^{\mathrm{dual}}$.

Next we need to verify the vanishing moment conditions for Q_i and Q_i^{dual}. Let $(\hat{P}_m(2\omega);\ m\in\{0,1\}^d)=\mathcal{M}^*\mathcal{P}$ be the polyphase components of P. Then

$$Q_m^{\mathrm{dual}}(\omega)=2^{-d/2}e^{im\cdot\omega}-P^{\mathrm{dual}}(\omega)\overline{\hat{P}_m(2\omega)}$$

and $\hat{P}_m(0) = 2^{-d/2}$. Note that $P^{\text{dual}}(0) = 1$. Therefore, $Q_m^{\text{dual}}(0) = 0$ for $m \in \{0,1\}^d$. Analogous statements show that $Q_m(0) = 0$ for $m \in \{0,1\}^d$. Using Theorem 7.2, we conclude that ψ_i and ψ_i^{dual} defined above, using these Q_m's and Q_m^{dual}'s, are bi-framelets. This completes the proof. □

We have the following examples of bi-frame generators based on bivariate and trivariate box splines.

Example 7.5. For the mask $P_{\ell,m,n}$, associated with the bivariate box spline $\phi_{\ell,m,n}$ on the three direction mesh, many dual masks $P_{\ell,m,n}^{\text{dual}}$ were given in [18] satisfying (7.6) with $d = 2$. Then the formulae for $\mathcal{Q}$ and $\mathcal{Q}^{\text{dual}}$ given in Theorem 7.4 provide an explicit representation of bi-framelets or bi-frame generators.

Example 7.6. For the mask $P_{\ell,m,n,p,q}$ of the trivariate box spline $\phi_{\ell,m,n,p,q}$, many dual masks $P_{\ell,m,n,p,q}^{\text{dual}}$ were given in [19] satisfying (7.6) with $d = 3$. Once again, the formulae for $\mathcal{Q}$ and $\mathcal{Q}^{\text{dual}}$ given in Theorem 7.4 provide an explicit representation of bi-framelets or bi-frame generators for trivariate box spline functions.

Next we consider a general refinable function ϕ. Let P be the mask associated with ϕ. Note that $P(0) = 1$. Assume that $P(\ell) = 0$ for $\ell \in \{0,1\}\pi^d \backslash \{0\}$. To ensure (7.6) for any given mask P, we may use the celebrated Hilbert Nullstellensatz. Indeed, we let $P_m(2\omega)$ be the polyphase components of P, i.e.,

$$\big(P_m(2\omega);\ m \in \{0,1\}^d\big) = \mathcal{M}^*\big(P(\omega + \ell);\ \ell \in \{0,1\}^d\big).$$

Similarly, for the dual mask P^{dual}, let $P_m^{\text{dual}}(2\omega)$ be the polyphase components of P^{dual}. Then (7.6) is equivalent to

$$\sum_{m \in \{0,1\}^d} P_m(\omega)\overline{P_m^{\text{dual}}}(\omega) = 1.$$

By the Hilbert Nullstellensatz, we have

Lemma 7.7. *Let P be the mask of a refinable function ϕ. Write $\hat{P}_m(z) := P_m(\omega)$ in terms of $z = e^{i\omega} := (e^{i\omega_1}, \ldots, e^{i\omega_d}) \in \mathbb{C}^d$ for $m \in \{0,1\}^d$. If the Laurent polynomials $\hat{P}_m$ have no common zero in $(\mathbb{C}\backslash\{0\})^d$, then there exist Laurent polynomials $\hat{Q}_m(z)$ such that*

$$\sum_{m \in \{0,1\}^d} \hat{P}_m(z)\hat{Q}_m(z) = 1. \tag{7.7}$$

Thus, we let $P^{\text{dual}}(\omega) = 2^{-d/2}\sum_{m \in \{0,1\}^d} e^{im\cdot\omega}\hat{Q}_m(e^{i2\omega})$. Then P and P^{dual} satisfy (7.6). In order to apply our Theorem 7.4, we only need to make sure that $P^{\text{dual}}(0) = 1$. Using the fact $P(0) = 1$ and the assumption $P(\ell) = 0$ for $\ell \in \{0,1\}^d\pi\backslash\{0\}$, we conclude from (7.6) that $P^{\text{dual}}(0) = 1$. Hence, we obtain the following

Theorem 7.8. *Given a mask P, suppose that $P(0) = 1$ and $P(\ell) = 0$ for $\ell \in \{0,1\}^d\pi\backslash\{0\}$. Let $P_m(\omega)$, $m \in \{0,1\}^d$, be the polyphase components of P. Writing $\hat{P}_m(z) := P_m(\omega)$ in terms of $z = e^{i\omega}$, suppose that the Laurent polynomials $\hat{P}_m$ have no common zero in $z \in \mathbf{C}^d\backslash\{0\}$. Then there exists a pair of bi-frames $\{\psi_i;\ i = 1,\ldots,2^d\}$ and $\{\psi_i^{\text{dual}};\ i = 1,\ldots,2^d\}$ associated with P.*

We remark that the computation of P^{dual} satisfying (7.7) is not easy in general.

References

[1] E. Artin, Über die Zerlegung definiter Funktionen in Quadrate, Hamb. Abh. 5 (1927) 100–115.
[2] S. Basu, A constructive algorithm for 2D spectral factorization with rational spectral factors, IEEE Trans. Circuits Systems I 47 (2000) 1309–1318.
[3] S. Basu, Private communication, 2001–2002.
[4] J. Benedetto, S.D. Li, The theory of multiresolution analysis frames and applications to filter banks, Appl. Comput. Harmon. Anal. 5 (1998) 389–427.
[5] C. de Boor, K. Hölig, S. Riememschneider, Box Splines, Springer-Verlag, New York, 1993.

[6] C.K. Chui, Multivariate Splines, SIAM, Philadelphia, 1988.
[7] C.K. Chui, An Introduction to Wavelets, Academic Press, San Diego, 1992.
[8] C.K. Chui, W. He, Compactly supported tight frames associated with refinable functions, Appl. Comput. Harmon. Anal. 8 (2000) 293–319.
[9] C.K. Chui, W. He, Construction of multivariate tight frames via Kronecker products, Appl. Comput. Harmon. Anal. 11 (2001) 305–312.
[10] C.K. Chui, W. He, J. Stöckler, Compactly supported tight and sibling frames with maximum vanishing moments, Appl. Comput. Harmon. Anal. 13 (2002) 224–262.
[11] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.
[12] I. Daubechies, B. Han, Pairs of dual wavelet frames from any two refinable functions, Constr. Approx. 20 (2004) 325–352.
[13] I. Daubechies, B. Han, A. Ron, Z.W. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal. 14 (2003) 1–46.
[14] M. Frazier, G. Garrig'os, X. Wang, G. Weiss, A characterization of functions that generate wavelet and related expansion, J. Fourier Anal. Appl. 2 (1997) 883–906.
[15] J. Geronimo, M.J. Lai, Factorization of multivariate positive Laurent polynomials, J. Approx. Theory 139 (2006) 327–345.
[16] K. Gröchenig, A. Ron, Tight compactly supported wavelet frames of arbitrarily high smoothness, Proc. Amer. Math. Soc. 126 (1998) 1101–1107.
[17] B. Han, On dual wavelet tight frames, Appl. Comput. Harmon. Anal. 4 (1997) 380–413.
[18] W. He, M.J. Lai, Construction of bivariate compactly supported biorthogonal box spline wavelets with arbitrarily high regularities, J. Appl. Comput. Harmon. Anal. 6 (1999) 53–74.
[19] W. He, M.J. Lai, Construction of trivariate compactly supported biorthogonal box wavelets, J. Approx. Theory 120 (2003) 1–19.
[20] E. Hernández, G. Weiss, A First Course on Wavelets, CRC Press, Boca Raton, FL, 1996.
[21] D. Hilbert, Über die Darstellung definiter Formen als Summe von Formenquadraten, Math. Ann. 32 (1888) 342–350.
[22] M.J. Lai, Fortran subroutines for B-nets of box splines on three and four directional meshes, Numer. Algorithms 2 (1992) 33–38.
[23] M.J. Lai, Methods for constructing nonseparable compactly supported orthonormal wavelets, in: D.X. Zhou (Ed.), Wavelet Analysis: Twenty Year's Development, World Scientific, Singapore, 2002, pp. 231–251.
[24] Y. Meyer, R. Coifman, Wavelets, Calderon–Zygmund and Multilinear Operators, Hermann, Paris, 1990; and Cambridge Univ. Press, 1997.
[25] K. Nam, Box spline tight frames and their applications for image processing, Ph.D. dissertation, University of Georgia, Athens, GA, 2005.
[26] A. Petukhov, Explicit construction of framelets, Appl. Comput. Harmon. Anal. 11 (2001) 313–327.
[27] A. Petukhov, Symmetric framelets, Constr. Approx. 19 (2003) 309–328.
[28] A. Pfister, Zur Darstellung definiter Funktionen als Summe von Quadraten, Invent. Math. 4 (1967) 229–237.
[29] A. Ron, Z.W. Shen, Affine systems in $L_2(R^d)$: The analysis of the analysis operator, J. Funct. Anal. 148 (1997) 408–447.
[30] A. Ron, Z.W. Shen, Affine systems in $L_2(R^d)$. II. Dual systems, J. Fourier Anal. Appl. 3 (1997) 617–637.
[31] A. Ron, Z.W. Shen, Compactly supported tight affine spline frames in $L_2(R^d)$, Math. Comp. 67 (1998) 191–207.
[32] A. Ron, Z.W. Shen, Construction of compactly supported affine frames in $L_2(R^d)$, in: K.S. Lau (Ed.), Advances in Wavelets, Springer-Verlag, New York, 1998, pp. 27–49.
[33] W. Rudin, Sums of squares of polynomials, Amer. Math. Monthly 107 (2000) 813–821.
[34] J. Stewart, Positive definite functions and generalization, an historical survey, Rocky Mountain J. Math. 6 (1976) 409–434.
[35] R.W. Yang, Private communication, 2001.
[36] R.M. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, New York, 1980.