Maximum Norm Estimate for Bivariate Spline Solutions to Second Order Elliptic Partial Differential Equations in Non-divergence Form

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Abstract

The convergence of the bivariate spline solution to the solution of the second order elliptic PDE in non-divergence form in the maximum norm is presented in this paper. Mainly, the L_∞ norm of the spline projection in the Sobolev space $H_0^2(\Omega) \cap H_0^1(\Omega)$ is shown to be bounded, where Ω is a polygonal domain. With the boundedness of the projection, one can establish the error of the spline solution to the weak solution in the L_∞ norm. The ideas of the proof can be extended to deal with other linear elliptic PDEs.

Keywords: maximum norm estimate, spline approximation, boundedness of projection
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1 Introduction

We are interested in L_∞ estimate of the error from a bivariate spline solution to the second elliptic partial differential equations in non-divergence form:

$$\sum_{i,j=1}^{2} a_{ij}\partial_{ij}^2 u = f \quad \text{in} \quad \Omega,$$
$$u = g, \quad \text{on} \quad \partial\Omega, \tag{1}$$

where Ω is an open bounded domain in $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\Omega$, $\partial_{ij}^2 = \dfrac{\partial^2}{\partial x_i \partial x_j}$ is the second order partial derivative operator with respect to x_i and x_j for $i,j = 1,2$. Assume that the tensor $a(x) = \{a_{ij}(x_1, x_2)\}_{2\times2}$ is symmetric positive definite and uniformly bounded over Ω. Note that the tensor $a(x)$ is essentially bounded and hence, the PDE in (1) cannot be rewritten in a divergence form.

Numerical solutions to (1) have been studied recently in [28] and [29] under the assumptions that Ω is a convex domain. In this setting, the PDE (1) has a strong solution $u \in H^2(\Omega)$. See [24] for a contraction mapping approach under an assumption that Ω has the C^2 smooth boundary. This C^2 requirement was later removed in [28]. Indeed, this is possible as commented in Chapter 3 in [15]. In addition, in [28], the researchers introduced a new bilinear form $A(u, v)$ and showed that the PDE in (1) has a weak solution in $H^2(\Omega) \cap H_0^1(\Omega)$ by using the well-known Lax-Milgram theorem. More

precisely, they first showed that $\|u\|_\Delta = (\int_\Omega |\Delta u|^2 dxdy)^{1/2}$ is a norm, where Δ is the standard Laplace operator and then define a new non-symmetric bilinear form

$$A(u,v) = \int_\Omega \gamma \mathcal{L}(u) \Delta v dxdy, \tag{2}$$

where $\gamma = (\sum_{i,j=1}^2 |a_{ij}|^2)/(a_{11}+a_{22})^2$ and

$$\mathcal{L}(u) = \sum_{i,j=1}^2 a_{ij} \partial_{ij}^2(u).$$

The bilinear form is associated with the following PDE which is equivalent to the one in (1):

$$\gamma \mathcal{L}(u) = \gamma f, \text{ in } \Omega \tag{3}$$

and $u = 0$ on $\partial\Omega$. The researchers in [28] showed

Lemma 1 *Suppose that Ω is a convex domain. Then there exist two positive constants K_1 and K_2 such that*

$$K_1 |u|_{2,2,\Omega}^2 \leq A(u,u) \leq K_2 |u|_{2,2,\Omega}^2 \tag{4}$$

for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Using the well-known Lax-Milgram theorem, we can establish the existence of a unique weak solution u satisfying $A(u,v) = \int_\Omega \gamma f \Delta v$ for all $v \in H_0^1(\Omega) \cap H^2(\Omega)$. Furthermore, the researchers in [28] concluded that the weak solution u is in fact a strong solution in $H^2(\Omega)$.

In this paper we consider to use bivariate spline functions for numerical solution of (1). Let us introduce the bivariate spline spaces. Let $\triangle$ be a triangulation of Ω and

$$S_d^r(\triangle) = \{s \in C^r(\Omega), s|_t \in \mathcal{P}_d, \forall t \in \triangle\} \tag{5}$$

be the spline space of degree d and smoothness $r \geq -1$ with $d > r$, where $\mathcal{P}_d$ is the space of polynomials of total degree $\leq d$ and $t \in \triangle$ stands for a triangle and $C^r(\Omega)$ is the space of all $r \geq 0$ times continuous functions over the closure of Ω. When $r = -1$, the spline space $S_d^r(\triangle)$ is the discontinuous finite element space. When $r = 0$, bivariate splines are simply standard finite elements. When $r \geq 1$, there are many bivariate spline spaces available for various degree $d \geq 2$. See [20] for detail. These spline functions have been used to solve linear and many other nonlinear partial differential equation(PDE). See, e.g. [2], [23], [17], [1], [16], [27] and etc.. We can use these spline functions, in particular, splines in $S_d^1(\triangle)$ with $d \geq 5$ to numerically solve (1) as the solution in $H^2(\Omega)$.

Indeed, similar to the discussion in [28], the Lax-Milgram theorem implies that there exists a unique spline solution $S_u \in \mathcal{S}_d(\triangle) = S_d^1(\triangle) \cap H_0^1(\Omega) \subset H^2(\Omega) \cap H_0^1(\Omega)$ satisfying $A(S_u,v) = \int_\Omega \gamma f \Delta v$ for all $v \in \mathcal{S}_d(\triangle)$, where $d \geq 5$. Thus, we have

$$A(u - S_u, v) = 0, \quad \forall v \in \mathcal{S}_d(\triangle). \tag{6}$$

Then it follows from (4) that letting $Q_u \in \mathcal{S}_d(\triangle)$ be a good approximation of u in $H^2(\Omega)$, we use (6) to have

$$K_1 |u - S_u|_{2,2,\Omega} \leq A(u - S_u, u - S_u) = A(u - S_u, u - Q_u) \leq K_2 |u - S_u|_{2,2,\Omega} |u - Q_u|_{2,2,\Omega}.$$

That is, we have

$$|u - S_u|_{2,2,\Omega} \leq \frac{K_2}{K_1} |u - Q_u|_{2,2,\Omega}. \tag{7}$$

Furthermore, since

$$\|\nabla(u - S_u)\|^2_{L^2(\Omega)} = |\int_\Omega (u - S_u)\Delta(u - S_u)| \le |u - S_u|_{2,2,\Omega}\|u - S_u\|_{L^2(\Omega)},$$

we use the well-known Poincaré inequality to conclude that

$$\|\nabla(u - S_u)\|_{L^2(\Omega)} \le K_3|u - S_u|_{2,2,\Omega} \le \frac{K_3 K_2}{K_1}|u - Q_u|_{2,2,\Omega} \tag{8}$$

and

$$\|u - S_u\|_{L^2(\Omega)} \le \frac{K_3^2 K_2}{K_1}|u - Q_u|_{2,2,\Omega}. \tag{9}$$

Thus, using our spline space to numerically solve (1) has a much simpler error analysis than the numerical methods based on discontinuous Galerkin finite elements in [28] and the weak Galerkin method in [29]. See our numerical results in §5 based on bivariate spline functions which provide more accurate solutions when using large degree d of spline functions even the testing function is only C^1 and in $H^2(\Omega)$. See [22] for root mean squared errors (RMSE) of spline functions for this type of PDE. These demonstrate that our spline function methods are efficient and effective.

The purpose of this paper is to present a maximum norm estimate: $|u - S_u|_{2,\infty,\Omega}$ if $u \in H^2(\Omega)$ and $|u - S_u|_{1,\infty,\Omega}$ and $|u - S_u|_{\infty,\Omega}$ if u is more smooth, say $u \in H^{m+1}(\Omega)$ for $m \ge 2$. For convenience, we shall write $P_E(u) = S_u$, the projection of u in $\mathcal{S}_d(\triangle)$. To state our main result, we need more notation. For each triangulation $\triangle$, let $\beta_\triangle$ be the shape parameter of $\triangle$ defined by

$$\beta_\triangle := \frac{|\triangle|}{\rho_\triangle}, \tag{10}$$

where $|\triangle|$ is the length of the longest edge of $\triangle$ and $\rho_\triangle$ is the smallest of the radius of the in-circle of triangle $t \in \triangle$. Next a triangulation $\triangle$ is said to be β-quasi-uniform if $\beta_\triangle \le \beta < \infty$. In finite element literature, such a triangulation is called shape-regular (cf. [5]) with regularity parameter β. We shall establish the following

Theorem 1 *Suppose that $\Omega \subset \mathbb{R}^2$ is a convex domain. Suppose that $\triangle$ is a β-quasi-uniform triangulation of Ω. Then there exists a constant $D_8 > 0$ such that for every $u \in H^2(\Omega) \cap H_0^1(\Omega)$*

$$|P_E(u)|_{2,\infty,\Omega} \le D_8|u|_{2,\infty,\Omega}. \tag{11}$$

With the result above we will be able to show that for $u \in W_\infty^{m+1}(\Omega)$ with $m \ge 2$,

$$|u - S_u|_{k,\infty,\Omega} \le C|\triangle|^{m-1}|u|_{m+1,\infty,\Omega}, \quad k = 0, 1, 2 \tag{12}$$

for a positive constant C. See details in Theorem 3 in a later section.

Our major effort in this paper is to establish (11) for the bivariate spline solution to (1). Our proof is based on the ideas in [14], [13] and [21] for the L_∞ norm estimation of the projections in constructing scattered data fitting and interpolation using a spline space which possesses a stable local basis. It is known many spline spaces possess a stable local basis (cf. [20]). For example, for the space of continuous linear finite elements over a triangulation $\triangle$, the standard hat functions at each vertex form a stable local basis. For another example, C^1 quintic Argyris elements form a stable local basis in a superspline subspace in $S_d^1(\triangle)$. For convenience, the concept of a stable local basis will be reviewed in the next section.

Next we shall remark that our proof can be extended to deal with some general second order elliptic PDE in divergence form as well as some PDE of higher order, e.g. a standard biharmonic

equation. Mainly we are able to obtain the maximum norm estimates for bivariate spline solutions to these elliptic PDEs. See details in the Remark Section §4. Finally, we present numerical results in the maximum norm estimate for a PDE with non-differentiable coefficients which were used for testing in [28] and [29].

The rest of the paper is devoted to establish the proof of Theorem 1. As mentioned above, the proofs are motivated by the work in [14], [13], and [21] for scattered data fitting. To present the ideas in the setting of numerical solution of PDE, we include all basic properties and steps. We shall review the concept of stable local basis in the next section. Then we prove the main results in §3, present some remarks in §4 and numerical results in §5.

2 Preliminary on Stable Local Basis of Bivariate Splines

Recall bivariate splines over triangulation $\triangle$ are represented by using Bernstein-Bézier representation. That is, for $s \in S_d^{-1}(\triangle)$, the space of all discontinuous spline functions over $\triangle$,

$$s = \sum_{T \in \triangle} \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^T \tag{13}$$

where $B_{ijk}^T = \dfrac{d!}{i!j!k!} b_1^i b_2^j b_3^k$ with b_1, b_2, b_3 being barycentric coordinates of (x, y) with respect to T (cf. [20]).

We now describe a stable local bases for spline space $\mathcal{S} \subset S_d^r(\triangle)$. Let

$$\mathcal{D}_{d,\triangle} := \cup_{T \in \triangle} \{\xi_{ijk}^T, i + j + k = d\}, \tag{14}$$

with $\xi_{ijk}^T := \dfrac{i\mathbf{u}+j\mathbf{v}+k\mathbf{w}}{d}$ for $T = \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$ be the set of domain points associated with $\triangle$ and d. It is well known that each spline in $S_d^{-1}(\triangle)$ is uniquely determined by the Bézier coefficient c_{ij}^T associated with each domain point ξ_{ijk}^T. A subset $\mathcal{M} \subset \mathcal{D}_{d,\triangle}$ is called a *minimal determining set (MDS)* for $S_d^r(\triangle)$ for $r \geq 0$ if the values of the coefficients of $s \in S_d^r(\triangle)$ associated with domain points in $\mathcal{M}$ uniquely determine all remaining coefficients of s. In the following we shall use clusters of triangles in $\triangle$. For each $T \in \triangle$, let $\text{star}(T) = \{t \in \triangle : T \cap t \neq \emptyset\}$ and for $k \geq 2$,

$$\text{star}^k(T) = \{t \in \triangle : t \cap \text{star}^{k-1}(T) \neq \emptyset\} \tag{15}$$

with $\text{star}^1(T) = \text{star}(T)$. It is known that the number of triangles in $\text{star}^k(T)$ is bounded by

$$\#\{t \in \text{star}^k(T)\} \leq \frac{\pi((k+1)|\triangle|)^2}{\pi \rho_\triangle^2} = \frac{|\triangle|^2}{\rho_\triangle^2}(k+1)^2 \leq \beta^2(k+1)^2. \tag{16}$$

Definition 1 *A basis $\{B_\xi\}_{\xi \in \mathcal{M}}$ for a space $\mathcal{S} \subset S_d^r(\triangle)$ on a triangulation $\triangle$ is a stable local basis, if there exists an integer $\ell \geq 1$ and constants $0 < C_1 < C_2 < \infty$ depending only on d and the smallest angle $\theta_\triangle$ of triangulation $\triangle$ such that*

1) for each $\xi \in \mathcal{M}$, $\text{supp}(B_\xi) \subseteq \text{star}^\ell(T_\xi)$ for some triangle $T_\xi \in \triangle$,

2) for all $\{c_\xi\}_{\xi \in \mathcal{M}}$,

$$C_1 \max_{\xi \in \mathcal{M}} |c_\xi| \leq \| \sum_{\xi \in \mathcal{M}} c_\xi B_\xi \|_{\infty,\Omega} \leq C_2 \max_{\xi \in \mathcal{M}} |c_\xi|, \tag{17}$$

where $\| \cdot \|_{\infty,\Omega} = \| \cdot \|_{0,\infty,\Omega}$ just for convenience.

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For $r = 0$ and any $d \geq 1$, we may choose $\ell = 1$. Also for $r = 1$ and $d \geq 5$, we choose $\ell = 1$. For $r \geq 2$ and $d \geq 3r + 2$, we can choose $\ell = 3$ as explained in [20]. A construction of a stable local basis using the Bernstein-Bézier representation of splines in $S_d^r(\triangle)$ when $d \geq 3r + 2$ is outlined in [20]. For $d < 3r + 2$ over special triangulation, see many constructions in [20]. Next we also need

Lemma 2 (Theorem 5.22 in [20]) *Suppose $d \geq 1$ for $r = 0$ or $d \geq 3r + 2$ for $r \geq 1$. Let $\triangle$ be a quasi-uniform triangulation of Ω. Then $S_d^r(\triangle)$ has a a stable local basis $\{B_\xi\}_{\xi \in \mathcal{M}}$. Furthermore the $\{B_\xi\}_{\xi \in \mathcal{M}}$ is a Riesz basis (with respect to the L_2-norm). That is, there exist constants C_3, C_4 depending on d, β such that*

$$C_3 \sum_{\xi \in \mathcal{M}} |c_\xi|^2 \leq |\triangle|^{-2} \| \sum_{\xi \in \mathcal{M}} c_\xi B_\xi \|_{0,2,\Omega}^2 \leq C_4 \sum_{\xi \in \mathcal{M}} |c_\xi|^2 \tag{18}$$

for all $\{c_\xi\}_{\xi \in \mathcal{M}}$.

3 Main Results and Proofs

In the following we shall assume that our spline space has a stable local support basis $\{B_\xi, \xi \in \mathcal{M}\}$ with support size $\ell \geq 1$. We shall begin with several preparatory lemmas. The first one is well-known in the literature as the Markov Inequality.

Lemma 3 (Markov inequality [20, Theorem 2.32]) *Let T be a triangle. Let $p \in [1, \infty)$ and $d \in \mathbb{N}$ be fixed. There exists a constant D_1 depending only on d and p such that for all nonnegative integers α and β with $0 \leq \alpha + \beta \leq d$, we have*

$$\|D_x^\alpha D_y^\beta s\|_{L^p(T)} \leq \frac{D_1}{\rho_T^{\alpha+\beta}} \|s\|_{L^p(T)}, \quad \forall s \in \mathcal{P}_d, \tag{19}$$

where ρ_T is the in-radius of the triangle T.

This inequality is called the inverse inequality in the finite element literature. However, Markov brothers studied this inequality in the univariate setting in 1889 and 1916. Kellogg extended (cf. [18]) this inequality to the spherical setting in 1928. Hille, Szegö and Tamarkin generalized the inequality in 1937 further. More historical notes can be found in [20] and [30].

Next we need the following lemma whose proof can be found in [3].

Lemma 4 *If a sequence $\{a_i\}_{i=1}^\infty$ satisfies $|a_m| \geq \gamma \sum_{j \geq m} |a_j|$ for all $m \geq 0$ and some $\gamma \in (0, 1)$, then*

$$|a_m| \leq a_0 \frac{(1-\gamma)^m}{\gamma}.$$

We are now ready to discuss some basic properties of bivariate splines.

Lemma 5 *Let $\triangle$ be a β-quasi-uniform triangulation. Let $\mathcal{S}_d(\triangle)$ be the subspace defined by*

$$\mathcal{S}_d(\triangle) := \{s \in S_d^r(\triangle), \text{ such that } s|_{\partial \Omega} = 0\}, \tag{20}$$

for $r \geq 1$ and $d \geq 3r + 2$ or $r = 0$ for all $d \geq 1$. Let $\{B_\xi, \xi \in \mathcal{M}\}$ be a stable local basis for $\mathcal{S}_d(\triangle)$. Then there exist constants $0 < D_2 \leq D_3 < \infty$ depending only on β and d such that

$$D_2 \sum_{\xi \in \mathcal{M}} c_\xi^2 \leq | \sum_{\xi \in \mathcal{M}} c_\xi B_\xi |_{1,2,\Omega}^2 \leq D_3 \sum_{\xi \in \mathcal{M}} c_\xi^2, \tag{21}$$

for all $u \in \mathcal{S}_d(\triangle)$, where D_2 and D_3 are two positive constants dependent on d and the shape parameter β of $\triangle$. In fact, $D_3 = C_4 D_1$.

Proof. The second inequality of (21) can be obtained by using the well-known Markov inequality, i.e. Lemma 3 together with Lemma 2. To prove the first inequality, we may assume $|\triangle| = 1$ first. Let

$$D_2' := \inf_{s \in \mathcal{S}_d(\triangle)} \{ |s|_{1,2,\Omega}^2 \text{ such that } \int_\Omega s^2 dx dy = 1 \}. \tag{22}$$

There exists a sequence of splines $s_k \in \mathcal{S}_d(\triangle)$ and $s^* \in \mathcal{S}_d(\triangle)$ such that $s_k \to s^*$ in $H_0^1(\Omega)$ norm. Let $D_2' = |s^*|_{1,2,\Omega}^2$ since $\mathcal{S}_d(\triangle)$ is a finite dimensional space. We claim $D_2' > 0$. Otherwise, we have $0 = D_2' = |s^*|_{1,2,\Omega}^2$. Since $s^* \in \mathcal{S}_d(\triangle)$, $D_2 = 0$ implies that s^* is a piecewise constant function. Using $s^*|_{\partial\Omega} = 0$, we conclude that $s^* \equiv 0$ since $s^* \in C^0(\Omega)$ which contradicts to the fact that $\int_\Omega (s^*)^2 dx dy = 1$.

Now for a general triangulation $\triangle$, say it is the nth uniform refinement of an original triangulation $\triangle_0$ with $|\triangle_0| = 1$. Without loss of generality, we may assume that Ω contains the origin $(0,0)$. We use the substitutions $x = \frac{1}{2^n}\tilde{x}$ and $y = \frac{1}{2^n}\tilde{y}$ to see that

$$\int_\Omega u^2(x,y) dx dy = \frac{1}{2^{2n}} \int_{\Omega_n} u^2(\frac{1}{2^n}\tilde{x}, \frac{1}{2^n}\tilde{y}) d\tilde{x} d\tilde{y},$$

where $\Omega_n = \{(2^n x, 2^n y), (x,y) \in \Omega\}$. As the new triangulation $\triangle_n$ whose vertices are $2^n V$, where V is the collection of all vertices of $\triangle$, we have the size $|\triangle_n| = 2^n |\triangle| = |\triangle_0| = 1$ and hence, we use (22) to have

$$D_2' \int_{\Omega_n} u^2(\frac{1}{2^n}\tilde{x}, \frac{1}{2^n}\tilde{y}) d\tilde{x} d\tilde{y} \le \int_{\Omega_n} [(D_{\tilde{x}} u(\frac{1}{2^n}\tilde{x}, \frac{1}{2^n}\tilde{y}))^2 + (D_{\tilde{y}} u(\frac{1}{2^n}\tilde{x}, \frac{1}{2^n}\tilde{y}))^2] d\tilde{x} d\tilde{y}$$

$$= \frac{1}{2^{2n}} \int_{\Omega_n} [(D_x u(\frac{1}{2^n}\tilde{x}, \frac{1}{2^n}\tilde{y}))^2 + (D_y u(\frac{1}{2^n}\tilde{x}, \frac{1}{2^n}\tilde{y}))^2] d\tilde{x} d\tilde{y}$$

$$= \int_\Omega [(D_x u(x,y))^2 + (D_y u(x,y))^2] dx dy = |u|_{1,2,\Omega}^2.$$

Now combining the above results together, we have

$$D_2' \int_\Omega u^2(x,y) dx dy = \frac{D_2'}{2^{2n}} \int_{\Omega_n} u^2(\frac{1}{2^n}\tilde{x}, \frac{1}{2^n}\tilde{y}) d\tilde{x} d\tilde{y} \le \frac{1}{2^{2n}} |u|_{1,2,\Omega}^2 = |\triangle|^2 |u|_{1,2,\Omega}^2.$$

We now use Lemma 2 to obtain

$$C_3 D_2' \sum_{\xi \in \mathcal{M}} c_\xi^2 \le D_2' |\triangle|^{-2} \int_\Omega |\sum_{\xi \in \mathcal{M}} c_\xi B_\xi|^2 dx dy \le |\sum_{\xi \in \mathcal{M}} c_\xi B_\xi|_{1,2,\Omega}^2$$

which is the desired inequality with $D_2 = C_3 D_2'$. This completes the proof. □

Similar to the proof above, we can have the following

Lemma 6 *Let $\triangle$ be a β-quasi-uniform triangulation. Let $\mathcal{S}_d(\triangle) = S_d^r(\triangle) \cap H^2(\Omega) \cap H_0^1(\Omega)$ be the subspace for $r \ge 1$ and $d \ge 3r + 2$. Suppose that $\{B_\xi, \xi \in \mathcal{M}\}$ is a stable local basis for $\mathcal{S}_d(\triangle)$. Then there exist constants $0 < D_4 \le D_5 < \infty$ depending only on β and d such that*

$$D_4 \sum_{\xi \in \mathcal{M}} c_\xi^2 \le |\triangle|^2 |\sum_{\xi \in \mathcal{M}} c_\xi B_\xi|_{2,2,\Omega}^2 \le D_5 \sum_{\xi \in \mathcal{M}} c_\xi^2, \tag{23}$$

for all $u = \sum_{\xi \in \mathcal{M}} c_\xi B_\xi \in \mathcal{S}_d(\triangle) \in \mathcal{S}_d(\triangle)$, where D_4 and D_5 are two positive constants dependent on d and the shape parameter β of $\triangle$. Furthermore, combing the equivalent relations in (4), we have

$$D_4 K_1 \sum_{\xi \in \mathcal{M}} c_\xi^2 \le |\triangle|^2 A(\sum_{\xi \in \mathcal{M}} c_\xi B_\xi, \sum_{\xi \in \mathcal{M}} c_\xi B_\xi) \le D_5 K_2 \sum_{\xi \in \mathcal{M}} c_\xi^2, \tag{24}$$

Proof. The proof is similar to the one of Lemma 5. The detail is left to the interested reader. □

Recall $A(u - S_u, v) = 0$ for all $v \in \mathcal{S}_d(\triangle)$. This is why we use $S_u = P_E(u)$, a projection of u in $\mathcal{S}_d(\triangle)$. It is easy to see $P_E(u)$ has the following projection property:

$$|P_E(u)|_{2,2,\Omega} \leq \frac{K_2}{K_1} |u|_{2,2,\Omega} \tag{25}$$

for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Indeed, we simply use the equivalent relations in (4), (6), and Cauchy-Schwarz inequality to obtain (25). That is,

$$K_1 |P_E(u)|_{2,2,\Omega}^2 \leq A(P_E(u), P_E(u)) = A(u, P_E(u)) \leq K_2 |u|_{2,2,\Omega} |P_E(u)|_{2,2,\Omega}.$$

Furthermore let us show that the spline projection $P_E(u)$ of function u geometrically decays away from the support of u.

Lemma 7 *There exist constants $0 \leq \sigma < 1$ and D_7, depending only on D_5/D_4, K_2/K_1, such that for any vertex $v \in \triangle$ and any function $u \in H^2(\Omega) \cap H_0^1(\Omega)$ with $supp(u) \subseteq star(v)$*

$$|P_E(u)|_{2,2,\tau} \leq D_7 \sigma^k |u|_{2,2,\Omega}, \tag{26}$$

whenever $\tau \notin star^{k\ell}(v)$ for some $k \geq 1$, where $\ell > 0$ is the size of support of the stable local basis of $\mathcal{S}_d(\triangle)$.

Proof. Recall that $\ell \geq 1$ is the size of support of $B_\xi \subset \mathcal{S}_d(\triangle)$. For $v \in \triangle$, we let

$$
\begin{aligned}
\mathcal{M}_k^v : &= \{\xi \in \mathcal{M} : \text{supp}(B_\xi) \cap star^{k\ell}(v) \neq \emptyset\}, \forall k \geq 1; \\
\mathcal{N}_1^v : &= \mathcal{M}_1^v, \\
\mathcal{N}_k^v : &= \mathcal{M}_k^v \backslash \mathcal{M}_{k-1}^v, \forall k \geq 2.
\end{aligned}
$$

Note that these clusters of domain points are defined differently from those in [14]. The following augments are a modification of the ones in [13] and [14] with some improvement.

Write $P_E(u) = \sum_{\xi \in \mathcal{M}} c_\xi B_\xi$ as explained in the previous section, and let

$$u_k := \sum_{\xi \in \mathcal{M}_k^v} c_\xi B_\xi, \quad w_k := P_E(u) - u_k, \quad a_k := \sum_{\xi \in \mathcal{N}_k^v} c_\xi^2,$$

for $k \geq 0$. Since $P_E(u) \in \mathcal{S}_d(\triangle)$, we have

$$\sum_{j \geq k+1} a_j = \sum_{\xi \notin \mathcal{M}_k^v} c_\xi^2 \leq \frac{|\triangle|^2}{D_4} |w_k|_{2,2,\Omega}^2 \tag{27}$$

by using Lemma 6. Since $w_k \in \mathcal{S}_d(\triangle)$, by using (6) we have

$$\int_\Omega \gamma \mathcal{L}(u - P_E(u)) \Delta w_k dx dy = 0. \tag{28}$$

Moreover, $\int_\Omega \gamma \mathcal{L} u \Delta w_k dx dy = 0$ since $supp(u) \subseteq star(v)$ and $supp(w_k)$ lies outside $star^{k\ell}(v)$ for $k \geq 1$. It follows that

$$K_1 |w_k|_{2,2,\Omega}^2 \leq A(w_k, w_k) = \int_\Omega \gamma \mathcal{L}(P_E(u) - u_k) \Delta w_k dx dy = \int_\Omega \gamma \mathcal{L}(u - u_k) \cdot \Delta w_k dx dy$$

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$$= -\int_\Omega \gamma \mathcal{L}(u_k) \cdot \Delta w_k \, dx dy$$

$$= -\int_\Omega \gamma \mathcal{L}\Big(\sum_{\xi \in \mathcal{N}_k^v} c_\xi B_\xi\Big) \Delta w_k \, dx dy \le K_2 | \sum_{\xi \in \mathcal{N}_k^v} c_\xi B_\xi|_{2,2,\Omega} \; |w_k|_{2,2,\Omega}, \tag{29}$$

where we have used Lemma 1, (28) and the fact that $\mathrm{supp}(w_k) \cap B_\xi = \emptyset$ for all $\xi \in N_j^v, j = 0, 1, \cdots, k-1$. Based on the above discussion, we further conclude

$$|w_k|_{2,2,\Omega}^2 \le \frac{K_2^2}{K_1^2} | \sum_{\xi \in \mathcal{N}_k^v} c_\xi B_\xi|_{2,2,\Omega}^2 \le \frac{K_2^2}{K_1^2} \frac{D_5}{|\triangle|^2} \sum_{\xi \in \mathcal{N}_k^v} c_\xi^2 = \frac{K_2^2}{K_1^2} \frac{D_5}{|\triangle|^2} a_k$$

by using (23). Hence by (27),

$$\sum_{j \ge k+1} a_j \le \frac{K_2^2}{K_1^2} \frac{D_5}{D_4} a_k =: D_6 a_k, \tag{30}$$

where $D_6 = \dfrac{K_2^2 D_5}{K_1^2 D_4} > 1$. Let $\gamma := \frac{1}{D_6} < 1$. Then we use Lemma 4 to have

$$a_k \le a_0 \frac{(1-\gamma)^k}{\gamma} = \frac{a_0}{\gamma} \sigma^{2k} = D_6 a_0 \sigma^{2k} \tag{31}$$

with $\sigma := \sqrt{1-\gamma}$. Furthermore, by using (23),

$$a_0 \le \sum_{j \ge 0} a_j = \sum_{\xi \in \mathcal{M}} c_\xi^2 \le \frac{|\triangle|^2}{D_4} |P_E(u)|_{2,2,\Omega}^2 \le \frac{|\triangle|^2}{D_4} \frac{K_2^2}{K_1^2} |u|_{2,2,\Omega}^2, \tag{32}$$

where we have used a property (25) of P_E in the last inequality.

For $\tau \not\subseteq \mathrm{star}^{k\ell}(v)$ for some $k \ge 1$, let us say $\tau \in \mathrm{star}^{(k+1)\ell}(v) \backslash \mathrm{star}^{k\ell}(v)$. If $\xi \in \mathcal{M}_{k-1}^v$, then $\mathrm{supp}(B_\xi) \subseteq \mathrm{star}^{k\ell}(v)$, and therefore $\tau \cap \mathrm{supp}(B_\xi) = \emptyset$. If $\xi \in \mathcal{N}_{k+j}^v, j \ge 2$, we also have $\tau \cap \mathrm{supp}(B_\xi) = \emptyset$. Letting χ_τ be the characteristic function of triangle τ, we further have

$$|P_E(u)|_{2,2,\tau}^2 = |P_E(u)\chi_\tau|_{2,2,\Omega}^2 = | \sum_{\xi \in \mathcal{N}_k^v \cup \mathcal{N}_{k+1}^v} c_\xi B_\xi \chi_\tau|_{2,2,\Omega}^2$$

$$\le \frac{D_5}{|\triangle|^2} \sum_{\xi \in \mathcal{N}_k^v \cup \mathcal{N}_{k+1}^T} c_\xi^2 = \frac{D_5}{|\triangle|^2}(a_k + a_{k+1}) \le \frac{D_5}{D_4} D_6 \frac{K_2^2}{K_1^2} \sigma^{2k}(1 + \sigma^2)|u|_{2,2,\Omega}^2$$

by using (31) and (32). These complete the proof with $D_7 = 2\frac{D_5}{D_4} D_6 \frac{K_2^2}{K_1^2}$. $\square$

Finally we need a result on the spline partition of unity.

Lemma 8 *Suppose that $\triangle$ is a β-quasi-uniform triangulation of Ω with vertices $\mathcal{V}$. For $d \ge 5$, there exists a collection of $\phi_v \in S_d^1(\triangle), v \in \mathcal{V}$ which form a partition of unity:*

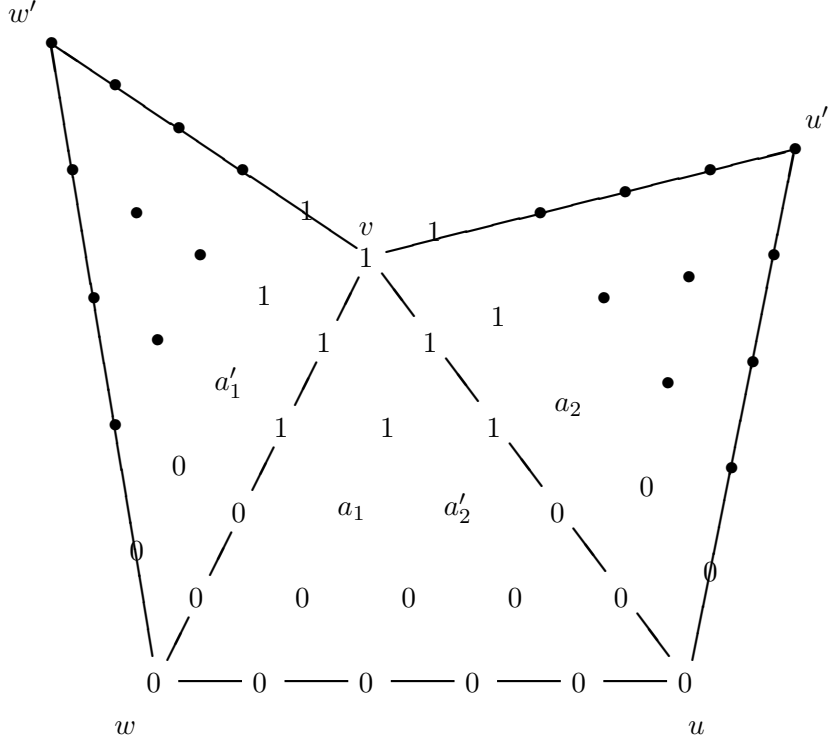$$\sum_{v \in \mathcal{V}} \phi_v \equiv 1. \tag{33}$$

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Figure 1: A Construction of partition of unity

Proof. We only need to construct such a collection in $S_5^1(\triangle)$ as for $d > 5$, the degree raising technique (cf. [20]) can be used to convert such a collection in $S_5^1(\triangle)$ to a collection in $S_d^1(\triangle)$. In addition to the vertices set $\mathcal{V} = \{v_1, v_2, \cdots, v_m\}$, we let $\mathcal{T} = \{T_1, T_2, \cdots, T_n\}$ be the set of all triangles in $\triangle$.

We simply present the Bernstein-Bézier representation of ϕ_v which is supported on $\mathrm{star}^1(v)$, the union of all triangles in $\triangle$ which has the vertex v. Let $T = \langle v, w, u \rangle$ be a triangle in $\mathrm{star}^1(v)$. For each interior edge e in $\triangle$ which is shared by T_i and T_j. We assign T_i to e if the number i is smaller than the number of j. We use Figure 1 to explain our construction.

The way to choose a_1, a_1' and a_2, a_2' is as follows. As $e = \langle v, w \rangle$ is shared by two triangles $T_i = \langle v, w', w \rangle$ and $T_j = \langle v, w, u \rangle$, if $i < j$, we choose $a_1' = 1/2$ and then use the C^1 smoothness to determine a_1. Otherwise we choose $a_1 = 1/2$ and use the C^1 smoothness condition to determine a_1'. Similar for a_2 and a_2'. Finally, if $e = \langle v, w \rangle$ is a boundary edge, we choose $a_1 = 1/2$.

The Bernstein-Bézier representation of ϕ_v on triangle $\langle v, w', w \rangle$ can be given similarly. For convenience, we simply use $\bullet$ to omit the details. The similar for $\langle v, u, u' \rangle$. Then we can see the Bernstein-Bézier coefficients of the summation of $\phi_v, v \in \mathcal{V}$ over each triangle are all 1 and hence, $\sum_{v \in \mathcal{V}} \phi_v \equiv 1$.
$\square$

We are now ready to prove the main result in this section.

Theorem 2 *Suppose that $\Omega \subset \mathbb{R}^2$ is a polygonal domain. There exists a constant $D_8 > 0$ depending only on d, ℓ, β, K_2/K_1, and D_5/D_4, such that for every $u \in H^2(\Omega) \cap H_0^1(\Omega)$*

$$|P_E(u)|_{2,\infty,\Omega} \le D_8 |u|_{2,\infty,\Omega}. \tag{34}$$

Proof. Let τ be a fixed triangle in $\triangle$. For simplicity, we may assume that $\ell = 1$. We define a new set by

$$R_0^\tau = \tau, R_1^\tau := \mathrm{star}(\tau), R_k^\tau := \mathrm{star}^k(\tau) \backslash \mathrm{star}^{k-2}(\tau), \forall k \ge 2.$$

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Let n_k denote the number of triangles in R_k^τ, $k \geq 0$. As explained in (16), $n_k \leq C(k+1)^2$ for a positive constant dependent on $\beta_\triangle$. Since $\operatorname{star}(v) \subset R_k^\tau$ contains at least one triangle T, the number of such star-shaped sets $\operatorname{star}(v) \subset R_k^\tau$ is bounded by n_k.

Let $\mathcal{V}$ be the collection of all vertices of $\triangle$. We write $u = \sum_{v \in \mathcal{V}} u_v$ with $u_v = u\phi_v \in H^2(\Omega)$ and $\operatorname{supp}(u_v) \subseteq \operatorname{star}(v)$ by using Lemma 8. Since P_E is a linear operator, we use Lemma 7 to have

$$
\begin{aligned}
|P_E(u)|_{2,2,\tau} &\leq \sum_{v \in \mathcal{V}} |P_E(u_v)|_{2,2,\tau} = \sum_{\operatorname{star}(v) \subset R_0^\tau \cup R_1^\tau} |P_E(u_v)|_{2,2,\tau} + \sum_{k \geq 2} \sum_{\operatorname{star}(v) \in R_k^\tau} |P_E(u_v)|_{2,2,\tau} \\
&\leq \sum_{\operatorname{star}(v) \in R_0^\tau \cup R_1^\tau} \frac{K_2}{K_1}|u_v|_{2,2,\Omega} + \sum_{k \geq 2} \sum_{\operatorname{star}(v) \in R_k^\tau} 2\frac{D_5 K_2^2}{D_4 K_1^2}\sigma^k |u_v|_{2,2,\Omega} \\
&\leq \frac{D_5 K_2^2}{D_2 K_1^2}(\max_{T \in \triangle} A_T^{1/2})(n_0 + n_1 + \sum_{k \geq 2} \sigma^k n_k)|u|_{2,\infty,\Omega} \\
&\leq \frac{D_5 K_2^2}{D_2 K_1^2}|\triangle|(n_0 + n_1 + \sum_{k \geq 2} \sigma^k n_k)|u|_{2,\infty,\Omega} = D_7 |\triangle||u|_{2,\infty,\Omega},
\end{aligned}
$$

where we have used the projection property (25) of P_E and A_T stands for the area of triangle T. Since $\sigma < 1$ and $n_k \leq C(k+1)^2$ as in the previous section, we know $D_7 := \frac{D_5 K_2^2}{D_2 K_1^2} \sum_{k \geq 0} \sigma^k n_k < \infty$.

Next since $P_E(u)\chi_\tau$ is a polynomial of degree over τ, we have

$$
|P_E(u_T)|_{1,2,\tau}^2 \geq CA_\tau |P_1(u_T)|_{1,\infty,\tau}^2. \tag{35}
$$

for a positive constant C dependent only on d (cf. Theorem 1.1 in [20]). It follows that

$$
|P_E(u)|_{2,\infty,\tau} \leq \frac{1}{CA_\tau^{1/2}}|P_E(u)|_{2,2,\tau} \leq \frac{D_7|\triangle|}{CA_\tau^{1/2}}|u|_{2,\infty,\Omega}.
$$

Therefore, (34) follows by taking the supremum over all $\tau \in \triangle$ with D_8 dependent on D_7/C and the shape-parameter $\beta_\triangle$. $\qquad\square$

We are now ready to prove the following main results in this section:

Theorem 3 *Suppose that the bilinear form $A(u,v)$ is continuous and satisfies (4). Let u be the weak solution of (1) in $H^2(\Omega) \cap H_0^1(\Omega)$, where Ω is a convex domain. Let $S_u = P_E(u)$ be the weak solution in the spline space $\mathcal{S}_d(\triangle)$ which has a stable local basis. Suppose that $u \in W_\infty^{m+1}(\Omega)$. Then*

$$
|u - S_u|_{2,\infty,\Omega} \leq C|\triangle|^{m-1}|u|_{m+1,\infty,\Omega} \tag{36}
$$

for a positive constant C in (36). If $m \geq 2$, we also have

$$
|u - S_u|_{1,\infty,\Omega} \leq C|\triangle|^{m-1}|u|_{m+1,\infty,\Omega} \text{ and } |u - S_u|_{\infty,\Omega} \leq C|\triangle|^{m-1}|u|_{m+1,\infty,\Omega} \tag{37}
$$

for another positive constants C in (37).

Proof. For $u \in W_\infty^{m+1}(\Omega)$, we let $Q_u \in \mathcal{S}_d(\triangle)$ be the quasi-uniform interpolatory spline satisfying

$$
\|u - Q_u\|_{2,\infty,\Omega} \leq K_0|\triangle|^{m-1}|u|_{m+1,\infty,\Omega}.
$$

(cf. [20]). It is clear that we have $P_E(Q_u) = Q_u$. Then

$$
|u - S_u|_{2,\infty,\Omega} \leq |u - Q_u|_{2,\infty,\Omega} + |P_E(Q_u - u)|_{2,\infty,\Omega} \leq (1 + D_8)|u - Q_u|_{2,\infty,\Omega}.
$$

Combining the above two estimates completes the proof of the estimate in (36).

When $m \geq 2$, $u \in W_\infty^{m+1}(\Omega)$ implies that u is continuously differentiable by Sobolev imbedding theorem. For any $(x, y) \in \Omega$, the horizontal line passing (x, y) will intersect the $\partial\Omega$ at two points due to the boundedness of Ω. Since $(u - S_u)|_{\partial\Omega} = 0$, we use Roll's theorem to see that $\frac{\partial}{\partial x}(u - S_u) = 0$ at some point (x_0, y) on the line segment within Ω. Then

$$\frac{\partial}{\partial x}(u - S_u)(x, y) = \int_{x_0}^{x} \frac{\partial^2}{\partial x^2}(u - S_u)(s, y)ds$$

and hence, $|\frac{\partial}{\partial x}(u - S_u)(x, y)| \leq C|u - S_u|_{2,\infty,\Omega}$, where $C = |\Omega|$, the size of Ω. Similar for $|\frac{\partial}{\partial y}(u - S_u)(x, y)| \leq C|u - S_u|_{2,\infty,\Omega}$. It follows that

$$|\nabla(u - S_u)|_{\infty,\Omega} \leq C|u - S_u|_{2,\infty,\Omega}.$$

That is, we have the first inequality in (37). The second inequality in (37) follows by using Poincaré's inequality. $\qquad\square$

We remark that we are not able to obtain the maximum norm estimate for $\nabla(u - S_u)$ when $m = 2$ as we can not apply Poincaré's inequality as in the setting of second order elliptic PDE in non-divergence form.

4   Remarks

We have the following three remarks in order.

- 1) The proof in the previous section can be modified to establish the maximum norm estimate for the Poisson equation. That is, let $\mathcal{S}_d(\triangle) = S_d^r(\triangle) \cap H_0^1(\Omega)$ for $r = 0$ and $d \geq 1$ or $r \geq 1$ and $d \geq 3r + 2$. Let $u \in H_0^1(\Omega)$ be the weak solution of the Poisson equation satisfying

  $$a(u, v) = \langle f, v \rangle, \quad v \in H_0^1(\Omega)$$

  with $a(u, v) = \int_\Omega \nabla u \nabla v dx dy$ and $\langle f, v \rangle = \int_\Omega f v dx dy$. Let $P_1(u) \in \mathcal{S}_d(\triangle)$ be the spline solution satisfying $a(P_1(u), v) = \langle f, v \rangle$ for all $v \in \mathcal{S}_d(\triangle)$. Then we have

  Theorem 4 *There exists a constant $D_9 > 0$ depending only on d, ℓ, β and D_3/D_2, such that for every $u \in H_0^1(\Omega)$*

  $$|P_1(u)|_{1,\infty,\Omega} \leq D_9|u|_{1,\infty,\Omega}. \tag{38}$$

  as well as the following maximum norm estimate:

  Corollary 1 *Suppose that the weak solution $u \in W_\infty^{m+1}(\overline{\Omega})$ with $1 \leq m \leq d$. Then*

  $$|u - S_u|_{1,\infty,\Omega} \leq Ch^m|u|_{m+1,\infty,\Omega}, \tag{39}$$

  Furthermore,

  $$\|u - S_u\|_{\infty,\Omega} \leq Ch^m \tag{40}$$

  for a positive constant C dependent only on the shape parameter $\beta_\triangle$.

  Proof. We first recall a spline approximation result from [20].

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Theorem 5 (Theorem 10.10 of [20]) *Let $p \in [1, \infty]$ and $d \in \mathbb{N}$ be given. Suppose that $\triangle$ is a quasi-uniform triangulation of Ω and $d \geq 3r + 2$ for $r \geq 1$. Then for every $u \in W_p^{d+1}(\Omega)$, there exists a spline function $S_u \in \mathcal{S}_d^r(\triangle)$ such that*

$$\|D_1^\alpha D_2^\beta (u - S_u)\|_{L^P(\Omega)} \leq K_0 |\triangle|^{d+1-\alpha-\beta} |u|_{d+1,p,\Omega} \quad \forall 0 \leq \alpha + \beta \leq d, \tag{41}$$

where K_0 is a positive constant depends only on d and the smallest angle of $\triangle$,

Note that when $u = 0$ on $\partial\Omega$, so is S_u based on the constructive proof in [20]. We use (41) with $p = \infty$ to have

$$\begin{aligned}
|u - P_1(u)|_{1,\infty,\Omega} &\leq |u - S_u|_{1,\infty,\Omega} + |S_u - P_1(u)|_{1,\infty,\Omega} \\
&= |u - S_u|_{1,\infty,\Omega} + |P_1(S_u) - P_1(u)|_{1,\infty,\Omega} \\
&\leq (1 + D_9)|u - S_u|_{1,\infty,\Omega} \\
&\leq (1 + D_9)K_0 |\triangle|^d |u|_{d+1,\infty,\Omega}
\end{aligned}$$

if $u \in W_\infty^{d+1}(\Omega)$. When $u \in W_\infty^{m+1}(\Omega)$ with $1 \leq m < d$, we simply use the theory of K-functional (cf. [26] for the univariate setting which can be easily extended to the bivariate setting) to conclude the proof of (39).

Because $u - S_u$ is zero on the boundary $\partial\Omega$ of Ω, we have $\|u - S_u\|_{\infty,\Omega} \leq C|u - u_h|_{1,\infty,\Omega}$ by Poincaré inequality. Thus, we complete the proof of Corollary 1. $\qquad\square$

Note that the above result on the maximum norm estimate for the Poisson equation is a classic result which can be found in books on finite elements, e.g. [6] and [5]. The proof is based on weighted norm estimates and "index engineering" (according to [5]). It is one of the most nontrivial and complicated proofs in the literature of finite elements. Nevertheless, the method of proof has been extended to establish the maximum norm estimates for Stokes equation in [11], [8], and even nonlinear PDE such as Navier-Stokes equations in [12]. See next remark for using our method to establish the maximum norm estimate of the solution of biharmonic equation.

- 2) We can also use the arguments in the previous section to establish the boundedness of the spline solution to the biharmonic equation in the maximum norm:

$$\begin{cases}
\Delta^2 u &= f, \quad \mathbf{x} \in \Omega \\
u &= g, \quad \mathbf{x} \in \partial\Omega \\
\mathbf{n} \cdot \nabla u &= h, \quad \mathbf{x} \in \partial\Omega
\end{cases} \tag{42}$$

for any given continuous functions g and h, where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the standard Laplace operator and $\mathbf{n}$ stands for the outward normal vector along $\partial\Omega$. The existence and uniqueness of the solution of (42) as well as regularity of the solution can be found in [15]. Bivariate splines have been used to solve (42) numerically. See [2]. For simplicity, we consider $g = 0$ and $h = 0$. Let $H_0^2(\Omega) = \{u \in H^2(\Omega) : u|_{\partial\Omega} = 0, \mathbf{n} \cdot \nabla u|_{\partial\Omega} = 0\}$ be another standard Sobolev space. The exact solution $u \in H_0^2$ satisfies

$$\int_\Omega \Delta u \Delta v \, dx dy = \int_\Omega f v \, dx dy, v \in H_0^2(\Omega). \tag{43}$$

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Let $\mathcal{S}_d(\triangle) = S_d^r(\triangle) \cap H_0^2(\Omega)$ with $d \geq 3r + 2$ with $r \geq 1$ (cf. [4] and [20]). Bivariate spline solution to (42) is the spline $S_u \in \mathcal{S}_d(\triangle)$ which satisfies

$$\int_\Omega \Delta S_u \Delta v dx dy = \int_\Omega f v dx dy, v \in \mathcal{S}_d(\Omega). \tag{44}$$

It follows that

$$\int_\Omega \Delta(u - S_u) \Delta v dx dy = 0, \quad v \in \mathcal{S}_d(\Omega). \tag{45}$$

That is, S_u is the unique minimizer of the following minimization problem:

$$S_u := \arg \min_{s \in \mathcal{S}_d(\triangle)} \|\Delta(u - s)\|_{L_2(\Omega)}. \tag{46}$$

In this sense, we can denote it by

$$P_2(u) := u_h. \tag{47}$$

Similarly, we can establish the following

Theorem 6 *Suppose that the shape parameter of the underlying triangulation $\triangle$ satisfies (10). Then*

$$\|P_2\| := \max_{u \in W_\infty^2(\Omega)} \{|P_2(u)|_{2,\infty,\Omega}, |u|_{2,\infty,\Omega} = 1\} \leq D_{10} < \infty \tag{48}$$

for a positive constant D_{10} dependent only on d and β.

With the bound above, we can prove

Corollary 2 *Suppose that the weak solution $u \in W_\infty^{m+1}(\overline{\Omega})$ with $1 \leq m \leq d$. Let $u_h \in \mathcal{S}_d(\triangle)$ be the weak solution of biharmonic equation (42). Then we have*

$$\|\Delta(u - u_h)\|_{\infty,\Omega} \leq C|\triangle|^{m-1}\|u\|_{m+1,\infty,\Omega}. \tag{49}$$

Using Poincaré inequality, we have

$$\|\nabla(u - u_h)\|_{\infty,\Omega} \leq C|\triangle|^{m-1}\|u\|_{m+1,\infty,\Omega}. \tag{50}$$

and

$$\|u - u_h\|_{\infty,\Omega} \leq C|\triangle|^{m-1}\|u\|_{m+1,\infty,\Omega}. \tag{51}$$

for different positive constants C in different inequalities in (49), (50), and (51).

Proof. We leave the proof to the interested reader as the proof is similar to that of Corollary 1. □

Note that the Stokes equations in the 2D setting can be converted into a biharmonic equation based on the stream function formulation (cf. [23]). The bound of the projection of the Stokes equation was derived with log factor and a quasi-optimal rate of convergence was obtained in [8]. Based on the discussion above, the log factor can be removed without introducing the regularized green function.

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- 3) We can continue to use the arguments in the previous section to study the maximum norm estimate for a general second order elliptic PDE in divergence form which can be described as follows.

$$\begin{cases} \mathcal{D}(u) = f, & \mathbf{x} \in \Omega \subset \mathbb{R}^2 \\ u = g, & \mathbf{x} \in \partial\Omega, \end{cases} \tag{52}$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\mathcal{D}$ is a partial differential operator in divergence form:

$$\mathcal{D}(u) := -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\left(A_{ij}\frac{\partial}{\partial x_i}u\right) + \sum_{k=1}^{2} B_k \frac{\partial}{\partial x_k}u + Cu,$$

with $A_{ij} \in L_\infty(\Omega)$, $B_k \in L_\infty(\Omega)$, $C \in L_\infty(\Omega)$, and f is a function in $L_2(\Omega)$. We shall assume that the coefficient matrix $A = [A_{ij}]_{1\leq i,j\leq 2}$ is symmetric and uniformly positive definite over Ω. In this sense, the PDE in (52) is said to be uniform elliptic. Let

$$a(u,v) = \sum_{i,j=1}^{2} \int_\Omega A_{ij}\frac{\partial}{\partial x_i}u\frac{\partial}{\partial x_j}v + \sum_{k=1}^{2} \int_\Omega [B_k\frac{\partial}{\partial x_k}u]v + \int_\Omega Cuv \tag{53}$$

be the bilinear form associated with (52). If $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = g$ satisfies $a(u,v) = \langle f, v\rangle$ for all $v \in H_0^1(\Omega)$, u is called a weak solution to (52).

Without loss of generality, we may assume that $g = 0$. To numerically solve (52), we use bivariate spline functions in $\mathcal{S}_d(\triangle) = S_d^0(\triangle) \cap H_0^1(\Omega)$ for $d \geq 1$ or $\mathcal{S}_d(\triangle) = S_d^1(\triangle) \cap H_0^1(\Omega)$ for $d \geq 3r + 2$ when $r \geq 1$(cf. [4] and [20]). We mainly solve $a(S_u, v) = \langle f, v\rangle$ for all $v \in \mathcal{S}_d(\triangle)$. When the PDE (52) is elliptic and $C(x,y) \geq C_0 > 0$ is sufficiently large, it is known that there exists a unique weak solution $u \in H_0^1(\Omega)$ satisfying $a(u,v) = \langle f, v\rangle$ for all $v \in H_0^1(\Omega)$ by using the well-known Lax-Milgram theorem. Similarly, there exists a unique spline solution $S_u \in \mathcal{S}_d(\triangle)$ satisfying the weak formulation $a(S_u, v) = \langle f, v\rangle$ for all $v \in \mathcal{S}_d(\triangle)$. We mainly solve this equation. See a numerical implementation and numerical results in [2]. We now estimate the error $u - S_u$. It is easy to see

$$a(u - S_u, v) = 0, \forall v \in \mathcal{S}_d(\triangle). \tag{54}$$

We again let $P_e(u) = S_u$, a projection of u in $\mathcal{S}_d(\triangle)$. Another result can be established is

Theorem 7 *Let $\triangle$ be a β-quasi-uniform triangulation with $\beta_\triangle \leq \beta < \infty$. Then*

$$\|P_e\| := \max_{f\in W_\infty^1(\Omega)} \{|P_e(f)|_E, |u|_{1,\infty,\Omega} = 1\} \leq D_{11} < \infty \tag{55}$$

for a positive D_{11} dependent only on d, β, λ and Λ, the smallest and largest eigenvalue of the elliptic operator $\mathcal{D}$.

With this bound, we can prove the following

Corollary 3 *Suppose that the bilinear form $a(u,v)$ is bounded below. Let u be the weak solution of (52) and $S_u = P_e(u)$ be the weak solution of the general elliptic PDE using a spline space $\mathcal{S}_d(\triangle)$ which has a stable local basis. Suppose that $u \in W_\infty^{m+1}(\Omega)$ for $m \geq 1$. Then*

$$\|\nabla(u - u_h)\|_{\infty,\Omega} \leq Ch^m|u|_{m+1,\infty,\Omega} \tag{56}$$

and

$$\|u - u_h\|_{\infty,\Omega} \leq Ch^m|u|_{m+1,\infty,\Omega}, \tag{57}$$

for two positive constants C in (56) and (57).

Proof. The proof is left to the interested reader. □

- 4) Finally, the analysis in this paper can be extended to the 3D and multi-dimensional setting. Many trivariate spline spaces have a stable local basis (cf. [20]). They can be used to solve 3D version PDE discussed in the previous section as well as Remarks above. We leave these study to the interested reader.

5 Numerical Approximation of the PDE in (1)

In this section, we present some numerical results on the maximum norm estimates of bivariate spline solutions to the PDE in (1). Mainly, we solve $u \in S_d^1(\triangle)$ satisfying

$$\int_\Omega \gamma \mathcal{L}(u) \Delta v \, dx \, dy = \int_\Omega \gamma f \Delta v \, dx \, dy, \quad \forall v \in S_d^1(\triangle), \tag{58}$$

or simply

$$\int_\Omega \gamma \mathcal{L}(u) w \, dx \, dy = \int_\Omega \gamma f w \, dx \, dy, \quad \forall w \in S_{d-2}^{-1}(\triangle), \tag{59}$$

where $\gamma = (\sum_{i,j=1}^2 |a_{ij}|^2)/(a_{11} + a_{22})^2$ and

$$\mathcal{L}(u) = \sum_{i,j=1}^2 a_{ij} \partial_{ij}^2(u)$$

as in a previous section. Let T be a triangle in $\triangle$ and

$$S|_T = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^T(x, y),$$

where B_{ijk}^T are Bernstein-Bézier polynomials of degree d. We use $\mathbf{s} = (c_{ijk}^T, i + j + k = d, T \in \triangle)$ to represent the coefficient vector for spline function $S \in S_d^{-1}(\triangle)$. In order to make $S \in S_d^r(\triangle)$, we construct a smoothness matrix $H = H(r)$ such that $H\mathbf{s} = 0$ ensures that S is a function in $S_d^r(\triangle)$. Such a smoothness matrix H has been known for many years (cf. [10]) and MATLAB implementation is realized as explained in [23] and [2]. Also we simply express the boundary condition by $B\mathbf{s} \approx \mathbf{g}$ as in [2].

We let S_u be the spline solution with the coefficient vector $\mathbf{s}$ which is the minimizer of (60) Find $\mathbf{u}$ satisfying

$$\min \frac{h^2}{2}(\|H\mathbf{s}\|^2 + \|B\mathbf{s} - \mathbf{g}\|^2), \quad \text{subject to } K_\gamma \mathbf{u} = M_\gamma \mathbf{f}, \tag{60}$$

where K_γ and M_γ are the matrices associated with the integrals in (59) and report the maximum norm error (RMSE) of $u - S_u$, $|\nabla(u - S_u)| = (|\frac{\partial}{\partial x}(u - S_u)| + |\frac{\partial}{\partial y}(u - S_u)|)/2$ and $|\nabla^2(u - S_u)| = (|\frac{\partial^2}{\partial x^2}(u - S_u)| + 2|\frac{\partial^2}{\partial x \partial y}(u - S_u)| + |\frac{\partial^2}{\partial y^2}(u - S_u)|)/4$ based on their values over 333×333 equally-spaced points over Ω.

Example 1 *In this example, we show the performance of our spline solutions for a PDE with non-differentiable coefficients and nonsmooth exact solution $u = xy(e^{1-|x|} - 1)(e^{1-|y|} - 1)$ which satisfies*

$$2\frac{\partial^2}{\partial x^2}u + 2sign(x)sign(y)\frac{\partial^2}{\partial x \partial y}u + 2\frac{\partial^2}{\partial y^2}u = f(x, y), \quad (x, y) \in \Omega \subset \mathbb{R}^2 \tag{61}$$

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where $u = 0$ on the boundary of $\Omega = [-1, 1] \times [-1, 1]$ as in [28]. Note that the coefficients of the PDE above are not differentiable and hence, the PDE can not be written in a divergence form. Also note that the solution is in $H^2(\Omega)$, but not continuously twice differentiable. We shall use $S_d^1(\triangle_\ell)$ to construct the numerical solution and use $S_{d-2}^{-1}(\triangle_\ell)$ in (59) for the testing spline space for $d \geq 5$ with $\triangle_\ell$ shown in Figure 2.
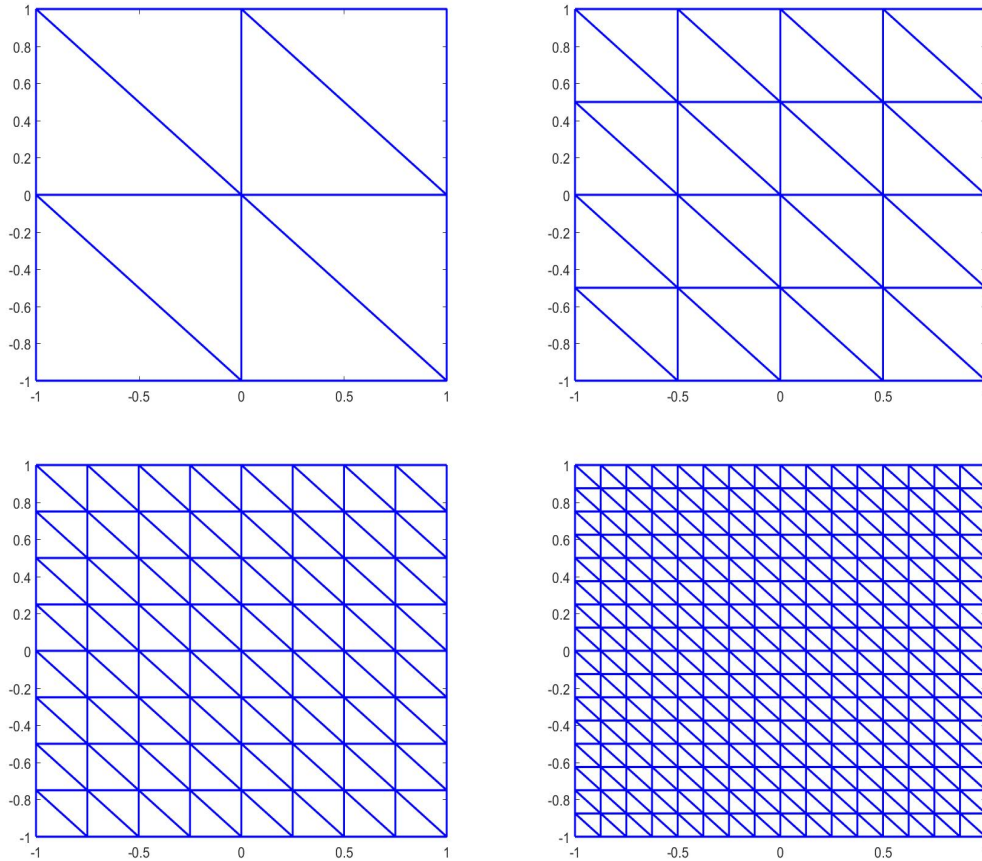

Figure 2: A triangulation (top-left) and its uniform refinements

In all numerical experiments reported above, the smoothness conditions and the boundary conditions of S_u for various degree d are satisfied within $1e - 10$ or less. However, in Tables 3 and 4, we have seen the convergence rates decrease at the last refinement of triangulation. This is probably due to the accuracy of MATLAB. Thatis, the error between two consecutive iterative spline coefficient vectors is within tolerance $1e - 15$ although the maximum norm error $u - S_u$ is still not within $1e - 12$.

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$|\triangle|$	$u - S_u$	rate	$\nabla(u - S_u)$	rate	$\nabla^2(u - S_u)$	rate
0.7071	1.542612e-04	0.00	9.776078e-04	0.00	2.275737e-02	0.00
0.3536	6.760710e-06	4.51	4.308411e-05	4.50	2.111036e-03	3.41
0.1768	3.990941e-07	4.08	1.851706e-06	4.54	1.507160e-04	3.80
0.0884	2.426117e-08	4.04	9.370397e-08	4.30	1.221727e-05	3.62

Table 1: The maximum norm error of spline solutions in $S_5^1(\triangle_\ell)$ by using testing space $S_3^{-1}(\triangle_\ell)$ of PDE (61) based on uniform triangulations in Figure 2

$|\triangle|$	$u - S_u$	rate	$\nabla(u - S_u)$	rate	$\nabla^2(u - S_u)$	rate
0.7071	1.280177e-05	0.00	6.840177e-05	0.00	1.895400e-03	0.00
0.3536	2.180512e-07	5.88	1.629599e-06	5.39	9.776263e-05	4.29
0.1768	3.367514e-09	6.02	3.163704e-08	5.69	3.265764e-06	4.87
0.0884	4.267892e-11	6.30	5.641191e-10	5.81	9.446271e-08	5.09

Table 2: The maximum norm error of spline solutions in $S_6^1(\triangle_\ell)$ by using testing space $S_4^{-1}(\triangle_\ell)$ of PDE (61) based on uniform triangulations in Figure 2

$|\triangle|$	$u - S_u$	rate	$\nabla(u - S_u)$	rate	$\nabla^2(u - S_u)$	rate
0.7071	7.188952e-07	0.00	3.430211e-06	0.00	1.160564e-04	0.00
0.3536	9.438582e-09	6.25	3.744586e-08	6.52	2.497923e-06	5.53
0.1768	1.458943e-10	6.02	4.439200e-10	6.40	4.856558e-08	5.67
0.0884	3.145667e-11	2.21	7.216256e-11	2.62	1.108986e-08	2.34

Table 3: The maximum norm error of spline solutions in $S_7^1(\triangle_\ell)$ by using testing space $S_5^{-1}(\triangle_\ell)$ of PDE (61) based on uniform triangulations in Figure 2

$|\triangle|$	$u - S_u$	rate	$\nabla(u - S_u)$	rate	$\nabla^2(u - S_u)$	rate
0.7071	3.263471e-08	0.00	1.649284e-07	0.00	6.712776e-06	0.00
0.3536	2.494362e-10	7.03	1.333012e-09	6.95	9.971102e-08	6.08
0.1768	2.432667e-11	3.36	1.597217e-10	3.06	3.871430e-08	1.47

Table 4: The maximum norm error of spline solutions in $S_8^1(\triangle_\ell)$ by using testing space $S_6^{-1}(\triangle_\ell)$ of PDE (61) based on uniform triangulations in Figure 2

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