

## IMPROVED ITERATIVELY REWEIGHTED LEAST SQUARES FOR UNCONSTRAINED SMOOTHED $\ell_q$ MINIMIZATION\*

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**Abstract.** In this paper, we first study  $\ell_q$  minimization and its associated iterative reweighted algorithm for recovering sparse vectors. Unlike most existing work, we focus on *unconstrained*  $\ell_q$  minimization, for which we show a few advantages on noisy measurements and/or approximately sparse vectors. Inspired by the results in [Daubechies et al., *Comm. Pure Appl. Math.*, 63 (2010), pp. 1–38] for *constrained*  $\ell_q$  minimization, we start with a preliminary yet novel analysis for *unconstrained*  $\ell_q$  minimization, which includes convergence, error bound, and local convergence behavior. Then, the algorithm and analysis are extended to the recovery of low-rank matrices. The algorithms for both vector and matrix recovery have been compared to some state-of-the-art algorithms and show superior performance on recovering sparse vectors and low-rank matrices.

**Key words.** sparse optimization, sparse vector recovery, compressed sensing, low-rank matrix recovery, matrix completion, iterative reweighted least squares,  $\ell_q$  minimization

**AMS subject classifications.** 49M05, 65B99, 65K10, 90C26, 90C30

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**1. Introduction.** Recovering sparse vectors from linear measurements is one of the central subjects in compressed sensing. Now the study has been extended to recovering low-rank matrices  $X$  from their linear observations  $\mathcal{A}(X)$ , which arises in many applications, for example, system identification [23], model reduction [12], recovering shape and motion from image streams [27, 33], data mining and pattern recognition [11], collaborative prediction [30], and low-dimensional embedding [22]. A particularly interesting case is the matrix completion problem, where  $\mathcal{A}(X)$  is a subset of the entries of  $X$ . It has been shown in [29, 4, 6] that under certain conditions, an  $m$ -by- $n$  matrix  $M$  with rank  $r \leq \min\{m, n\}$  can be exactly recovered from a small number of its entries in  $\Omega \subset [m] \times [n]$  by solving the convex program

$$(1.1) \quad \min_X \|X\|_*, \quad \text{subject to } \mathcal{P}_\Omega(X) = \mathcal{P}_\Omega(M),$$

where  $[m] := \{1, 2, \dots, m\}$ , the nuclear norm  $\|X\|_*$  is the sum of the singular values  $\sigma_i(X)$  of  $X$ , i.e.,  $\|X\|_* = \sum_{i=1}^r \sigma_i(X)$ , and  $\mathcal{P}_\Omega(X) = \mathcal{P}_\Omega(M)$  is short for  $X_{ij} = M_{ij}, (i, j) \in \Omega$ . The work [29] studies the low-rank matrix recovery problem with constraint  $\mathcal{A}(X) = \mathcal{A}(M)$  for general linear operator  $\mathcal{A}: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ .

Various types of algorithms have been proposed for solving problem (1.1), and many of them are extensions or adaptations of their predecessors for sparse vector recovery. They include the singular value thresholding algorithm [3] based on the linearized Bregman algorithm [36, 28], fixed-point continuation code FPCA [24]

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extending FPC [18], the code APGL [32] extending [2], and the code [35] based on the alternating direction method [16]. Algorithms with no vector recovery predecessors include OptSpace [20] and LMaFit [34], which are based on explicit factorizations  $M = USV^*$  and  $M = XY$  and nonlinear least-squares formulations  $\min_{U,S,V} \|\mathcal{P}_\Omega(USV^* - M)\|_F$  and  $\min_{X,Y} \|\mathcal{P}_\Omega(XY - M)\|_F$ , respectively.

Besides the above, the nonconvex  $\ell_q$  quasi-norm  $\|\mathbf{x}\|_q^q = \sum_i |x_i|^q$ ,  $0 < q < 1$ , and its variants have been used to develop algorithms for recovering sparse vectors in [7, 9, 10] and low-rank matrices in [26, 13]. First, compared to  $\ell_1$  norm  $\|\mathbf{x}\|_1$ ,  $\|\mathbf{x}\|_q^q$  for  $0 < q < 1$  makes a closer approximation to the ‘‘counting norm’’  $\|\mathbf{x}\|_0$ , which is the number of nonzero entries of  $\mathbf{x}$ . It is shown in [8] that assuming certain restricted isometry properties (RIPs) of the sensing matrix  $A$ , a sparse vector  $\mathbf{x}^o \in \mathbb{R}^N$  is the  $\ell_q$  minimizer of  $A\mathbf{x} = \mathbf{b}$ , where  $\mathbf{b} := A\mathbf{x}^o$  can have fewer observations than needed by convex  $\ell_1$  minimization. Works [15, 31] derive sufficient conditions in terms of RIP of  $A$  for  $\ell_q$  minimization to recover sparse vectors that are weaker than those known for  $\ell_1$  minimization.

However, the  $\ell_q$  quasi-norm is nonconvex for  $q < 1$ , and  $\ell_q$  minimization is generally NP-hard [17]. Instead of directly minimizing the  $\ell_q$  quasi-norm, which most likely ends up with one of its many local minimizers, algorithms [7, 9, 10] solve a sequence of smoothed subproblems. Specifically, [7] solves reweighted  $\ell_1$  subproblems: given an existing iterate  $\mathbf{x}^{(k)}$ , the algorithm generates a new iterate  $\mathbf{x}^{(k+1)}$  by minimizing  $\sum_i w_i |x_i|$  with weights  $w_i := (\epsilon + |x_i^{(k)}|)^{q-1}$ . To see how it relates to  $\ell_q$  quasi-norm, one can let  $\mathbf{x}^{(k)} = \mathbf{x}$ ,  $\epsilon = 0$ , and  $0/0$  be 0 and then get  $\sum_i w_i |x_i| = \sum_i |x_i|^q = \|\mathbf{x}\|_q^q$ . On the other hand, [9, 10] solves reweighted  $\ell_2$  (more precisely, least-squares) subproblems: at each iteration, they approximate  $\|\mathbf{x}\|_q^q$  by  $\sum_i w_i |x_i|^2$  with weights  $w_i := (\epsilon^2 + |x_i^{(k)}|^2)^{q/2-1}$ .

In the reweighted  $\ell_1/\ell_2$  iterations, a fixed  $\epsilon > 0$  not only avoids division by zero, but it also often causes the limit  $\mathbf{x}^\epsilon = \lim_{k \rightarrow \infty} \mathbf{x}^{(k)}$  to contain very few entries larger than  $O(|\epsilon|)$  in magnitude. In this sense,  $\mathbf{x}^\epsilon$  is often a good approximation of  $\mathbf{x}^o$  up to  $O(|\epsilon|)$ . To recover a sparse vector  $\mathbf{x}$  from  $\mathbf{b} = A\mathbf{x}$ , these algorithms need to vary  $\epsilon$ , starting at a large value and gradually reducing it. In particular, [10] sets  $\epsilon$  in terms of the  $(s+1)$ th largest entry of the latest iterate, where  $s$  is the sparsity guesstimate. Empirical results show that to recover vectors with entries in decaying magnitudes, the reweighted  $\ell_1/\ell_2$  algorithms require significantly fewer measurements than convex  $\ell_1$  minimization, and in compressed sensing this measurement reduction translates to savings in sensing time and cost.

Like other types of algorithms, the above  $\ell_q$  inspired algorithms have been extended to recovering low-rank matrices. In particular, algorithms [26, 13] extend [9, 10] and apply reweighted  $\ell_2$  to approximate  $\|X\|_*$ . Its subproblem will become clear after some notation is introduced.

This paper introduces algorithms for sparse vector and low-rank matrix recoveries based on unconstrained smoothed  $\ell_q$  minimization and reweighted  $\ell_2$  iterations, as well as their convergence results. One of the advantages of an unconstrained model over its constrained counterpart (e.g., [26]) is its suitability for noisy measurements and approximately sparse vector or low-rank matrix recovery. More precisely, we study the following unconstrained smoothed  $\ell_q$  minimization with  $0 < q \leq 1$ :

$$(1.2) \quad \min_{\mathbf{x} \in \mathbb{R}^N} \|\mathbf{x}\|_{q,\epsilon}^q + \frac{1}{2\lambda} \|A\mathbf{x} - \mathbf{b}\|_2^2,$$

where

$$\|\mathbf{x}\|_{q,\epsilon}^q = \sum_{j=1}^N (x_j^2 + \epsilon^2)^{q/2}.$$

This minimization is for sparse vector recovery. For low-rank matrix recovery, we assume the singular values of  $X$  are ordered as  $\sigma_1(X) \geq \sigma_2(X) \geq \dots \geq \sigma_r(X) > \sigma_{r+1}(X) = \dots = \sigma_n(X) = 0$  unless with more specification, where  $r = \text{rank}(X)$ . By writing  $X = U\Sigma V^\top$  in its standard singular value decomposition (SVD), we can extend  $X^\top X = V\Sigma^\top \Sigma V^\top$  to the definition  $(X^\top X)^\alpha := V(\Sigma^\top \Sigma)^\alpha V^\top$ , where  $\alpha$  is a real scalar and  $(\Sigma^\top \Sigma)^\alpha$  is a diagonal matrix with diagonal entries  $(\sigma_i(X))^{2\alpha}$ ,  $i = 1, \dots, n$ . Then, one gets

$$\text{tr}\left((X^\top X)^{q/2}\right) = \text{tr}\left(V(\Sigma^\top \Sigma)^{q/2}V^\top\right) = \text{tr}\left((\Sigma^\top \Sigma)^{q/2}\right) = \sum_{i=1}^n (\sigma_i(X))^q,$$

which is known as the Schatten- $q$  quasi-norm of matrix  $X$ . For  $q = 1, 2$ , this identity reduces to

$$\|X\|_* = \text{tr}\left((X^\top X)^{1/2}\right) = \sum_{i=1}^n \sigma_i(X) \quad \text{and} \quad \|X\|_F^2 = \text{tr}(X^\top X) = \sum_{i=1}^n (\sigma_i(X))^2,$$

respectively, which are the matrix analogues of the vector  $\ell_1$ -norm and squared  $\ell_2$ -norm. We consider minimizing  $\text{tr}\left((X^\top X)^{q/2}\right)$  for  $0 < q \leq 1$ .

Our exposition on low-rank matrix recovery is based on unconstrained smoothed  $\ell_q$  minimization

$$(1.3) \quad \min_X \text{tr}\left((X^\top X + \epsilon^2 I)^{q/2}\right) + \frac{1}{2\lambda} \|\mathcal{A}(X) - \mathbf{b}\|_2^2,$$

where  $I$  is the  $n \times n$  identity matrix,  $\epsilon > 0$  is a smoothing parameter, and  $\mathbf{b}$  is an observation vector with or without noise. By the definition, we have  $(X^\top X + \epsilon^2 I)^{q/2} = V(\Sigma^\top \Sigma + \epsilon^2 I)^{q/2}V^\top$  and

$$\text{tr}\left((X^\top X + \epsilon^2 I)^{q/2}\right) = \sum_{i=1}^n (\sigma_i(X)^2 + \epsilon^2)^{q/2}.$$

Roughly speaking, smoothed  $\ell_q$  minimization is applied to the singular values of  $X$ . Given  $X^{(k)}$  at iteration  $k$ , our algorithm generates  $X^{(k+1)}$  as the unique solution of

$$(1.4) \quad \min_X \frac{q}{2} \text{tr}\left(W^{(k)} X^\top X\right) + \frac{1}{2\lambda} \|\mathcal{A}(X) - \mathbf{b}\|_2^2,$$

where

$$W^{(k)} := \left((X^{(k)})^\top X^{(k)} + \epsilon^2 I\right)^{q/2-1}.$$

The objective function of (1.4) is quadratic in the entries of  $X$ . At  $X^{(k)}$ , the gradient of the objective function in (1.4) is the same as that of the objective function in (1.3).

Our algorithm is different from the ones studied in [26, 13]. The algorithm in [26] is based on the constrained counterpart of (1.3) which minimizes  $\text{tr}\left((X^\top X + \epsilon^2 I)^{q/2}\right)$  for  $0 \leq q \leq 1$  subject to  $\mathcal{A}(X) = \mathbf{b}$ , and it solves the constrained counterpart of (1.4) at each iteration. In addition, the algorithm is analyzed only for  $q = 0, 1$ . In [13], the constrained minimization is restricted to  $q = 1$ , and a different updating rule of  $W^{(k)}$  is used. Under the *strong rank null space property* assumption, their algorithm can be proved to give the unique minimizer.

**1.1. Contributions.** The contributions of this paper include novel algorithms and convergence results for unconstrained  $\ell_q$  minimization in both the vector and matrix settings, relative to the existing papers [21], [10], and [13]. One of the proposed algorithms improves the vector algorithm in [21] by introducing iterative updates of  $\epsilon_k$ ; see Algorithm 2.1. This paper also improves an elementary inequality in [21] and thus gives better estimation of the difference between two consecutive iterates; see the inequalities in Lemma 2.3 and (2.11). In addition, these inequalities are extended from the vector setting to the matrix setting. The iterative update of  $\epsilon_k$  in the proposed algorithms is the same as the one in [10]. The convergence study of the iteratively **weighted** least-squares algorithms for constrained  $\ell_q$  minimization in [10] guides our study of unconstrained  $\ell_q$  minimization. However, while the null space property plays a central role in [10], it is no longer useful for analyzing unconstrained  $\ell_q$  minimization as  $\mathbf{x}^{(k)} - \mathbf{x}^{(0)}$  does not lie in the null space of  $A$  any more. We find a property (cf. Lemma 2.7) to overcome this difficulty for establishing convergence and deriving a local error bound. This study is extended to the setting of low-rank matrix recovery in section 3.

Furthermore, our algorithm for matrix recovery is carefully implemented to exploit problem structures. Starting with an initial rank overestimate  $K$ , our algorithm can automatically decrease it to the true rank by a rank-decreasing strategy. Using the best rank- $K$  approximation of  $(X^{(k)})^\top X^{(k)}$  to update  $W^{(k)}$ , we can efficiently solve (1.4) for matrix completion problems by the *Woodbury matrix identity*. Our algorithm is compared numerically with some state-of-the-art algorithms such as LMaFit [34] and AGPL [32] on matrix completion problems and shows better performance on matrix recovery from noiseless and noisy measurements.

**1.2. Notation.** We let lowercase letters  $p, q, \dots$  denote scalars and bold lowercase letters  $\mathbf{x}, \mathbf{z}, \dots$  denote vectors.  $x_i$  denotes the  $i$ th component of vector  $\mathbf{x}$ . Capital letters  $X, Y, \dots$  are used for matrices and calligraphic letters  $\mathcal{L}, \mathcal{P}, \dots$  for operators or functionals. Greek letters with subscripts such as  $\delta_t$  are reserved for RIP constants. Capital letters such as  $C, C_1, C_2$  are reserved for constants and  $I$  for the identity matrix. Any vector  $\mathbf{x}$  with no more than  $s$  nonzeros is called an  $s$ -sparse vector, and any matrix  $X$  of rank not greater than  $r$  is called an  $r$ -rank matrix. The trace of a square matrix  $X$  is denoted by  $\text{tr}(X)$  and the inner product  $\langle X, Y \rangle := \text{tr}(XY^\top) = \text{tr}(X^\top Y)$  for any  $X, Y \in \mathbb{R}^{m \times n}$ . Other notation is given as it appears.

**1.3. Organization.** The rest of this paper is presented as follows. Section 2 discusses problem (1.2) with  $0 < q \leq 1$ . An iterative reweighted  $\ell_2$  algorithm inspired by [9, 10] is presented for this unconstrained problem with convergence analysis. The algorithm and analysis are extended in section 3 to the low-rank matrix recovery problem (1.3). In section 4, we compare the proposed algorithms for both sparse vector and low-rank matrix recovery to some state-of-the-art algorithms.

**2. Unconstrained  $\ell_q$  minimization with  $0 < q \leq 1$ .** Our algorithm is derived by solving a sequence of subproblems in the form of (1.2), which has the objective

$$L_q(\mathbf{x}, \epsilon, \lambda) = \sum_{j=1}^N (x_j^2 + \epsilon^2)^{q/2} + \frac{1}{2\lambda} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2.$$

For any  $\epsilon > 0$  and  $\lambda > 0$ , the minimization problem (1.2) must have a solution because  $L_q(\mathbf{x}, \epsilon, \lambda)$  is continuous with respect to  $\mathbf{x}$ . Thus it can achieve the minimum over a

bounded set  $\{\mathbf{x} : \|\mathbf{x}\|_2 \leq \Delta\}$ , where  $\Delta$  is a positive constant. In addition,  $L_q(\mathbf{x}, \epsilon, \lambda)$  blows up as  $\|\mathbf{x}\|_2 \rightarrow \infty$ . For convenience, we let  $\mathbf{x}^{\epsilon, \lambda, q}$  denote a critical point of (1.2) and it satisfies the first-order optimality condition

$$(2.1) \quad \left[ \frac{qx_j^{\epsilon, \lambda, q}}{(\epsilon^2 + (x_j^{\epsilon, \lambda, q})^2)^{1-q/2}} \right]_{1 \leq j \leq N} + \frac{1}{\lambda} A^\top (A\mathbf{x}^{\epsilon, \lambda, q} - \mathbf{b}) = 0.$$

Due to the nonlinearity, there is no straightforward method to solve the above system of equations unless for specific instances, such as  $A^\top A$  is a diagonal matrix and  $q = 1$ . We approximately solve the system with a sequence of  $\epsilon$ 's and the method is summarized as Algorithm 2.1.

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ALGORITHM 2.1 (iterative reweighted unconstrained  $\ell_q$  for sparse vector recovery (IRucLq-v)).

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**Input:** vector  $\mathbf{b}$ , matrix  $A$  and estimated sparsity level  $s$ ;

**Output:** vector  $\mathbf{x} \in \mathbb{R}^N$ .

Choose *appropriate parameters*  $\lambda > 0, q \in (0, 1]$ .

Initialize  $\mathbf{x}^{(0)}$  such that  $A\mathbf{x}^{(0)} = \mathbf{b}$  and  $\epsilon_0 = 1$ .

**For**  $k = 0, 1, 2, \dots$

Solve *the following linear system for*  $\mathbf{x}^{(k+1)}$ :

$$(2.2) \quad \left[ \frac{qx_j^{(k+1)}}{(\epsilon_k^2 + (x_j^{(k)})^2)^{1-q/2}} \right]_{1 \leq j \leq N} + \frac{1}{\lambda} A^\top (A\mathbf{x}^{(k+1)} - \mathbf{b}) = 0$$

*or equivalently*

$$(2.3) \quad \left( A^\top A + \text{Diag} \left[ \frac{q\lambda}{(\epsilon_k^2 + (x_j^{(k)})^2)^{1-q/2}}, j = 1, \dots, N \right] \right) \mathbf{x}^{(k+1)} = A^\top \mathbf{b}$$

Update  $\epsilon_{k+1} = \min\{\epsilon_k, \alpha \cdot r(\mathbf{x}^{(k+1)})_{s+1}\}$  where  $\alpha \in (0, 1)$  is a constant.

**End For**

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In Algorithm 2.1,  $r(\mathbf{z})$  is the rearrangement of the absolute values of  $\mathbf{z} \in \mathbb{R}^N$  in decreasing order. If  $\epsilon_{k+1} = 0$ , we choose  $\mathbf{x}^{(k+1)}$  to be an approximation of the sparse solution and stop the iteration. Otherwise, we stop the computation within a reasonable time and return the last  $\mathbf{x}^{(k+1)}$ . A similar algorithm is proposed in [21], but it does not have this  $\epsilon$ -update. A large  $\epsilon$  will smooth out small local minimizers, as been explained in [9], so adaptively updating  $\epsilon$  allows one to get close to the global minimizer without getting trapped at a local minimizer.

It is easy to see that the linear system (2.3) is invertible for any  $\mathbf{x}^{(k+1)}$  as long as  $\epsilon_k > 0$ . Once  $\epsilon_{k+1} = 0$ , the iteration is stopped. Thus, Algorithm 2.1 is well defined. It is clear from Algorithm 2.1 that  $\{\epsilon_k\}$  is a nonincreasing sequence which is convergent to some nonnegative number  $\epsilon_*$ . Below we show that the sequence  $\{\mathbf{x}^{(k)}\}$  is bounded and thus has at least a convergent subsequence. We also show that the limit  $\mathbf{x}^*$  of any convergent subsequence is a critical point of (1.2) when  $\epsilon_* > 0$ . When  $\epsilon_* = 0$ , the limit is a sparse vector with sparsity  $\|\mathbf{x}^*\|_0 \leq s$ , where  $\|\mathbf{x}\|_0$  stands for the number of nonzero entries of vector  $\mathbf{x}$ .

DEFINITION 2.1 (RIP). *For integer*  $t = 1, 2, \dots$ , *the restricted isometry constant*  $\delta_t$  *of matrix*  $A$  *is the smallest number such that*

$$(2.4) \quad (1 - \delta_t)\|\mathbf{x}\|_2^2 \leq \|A\mathbf{x}\|_2^2 \leq (1 + \delta_t)\|\mathbf{x}\|_2^2$$

holds for any  $t$ -sparse vector  $\mathbf{x}$ . For simplicity, we say matrix  $A$  satisfies the RIP of order  $t$  with constant  $\delta_t$ .

According to the analysis in [14], the RIP constant satisfies

$$(2.5) \quad \delta_t = \max_{\#(S) \leq t} \|A_S^\top A_S - I\|_2,$$

where  $S \subset [n]$ ,  $\#(S)$  denotes the cardinality of  $S$ ,  $A_S$  is a submatrix of  $A$  obtained by taking all the columns indicated by  $S$ , and  $\|X\|_2$  is the matrix 2-norm which equals the largest singular value  $\sigma_1(X)$ .

Under the RIP assumption, we can ensure that the limit  $\mathbf{x}^*$  is a reasonable approximation of the sparse solution if  $\mathbf{x}^*$  has a very small tail in the sense

$$\sigma_s(\mathbf{x}^*)_p = \inf_{\|\mathbf{y}\|_0 \leq s} \|\mathbf{x}^* - \mathbf{y}\|_p$$

for  $p \geq 1$ , which is the error term of the best  $s$ -term approximation of  $\mathbf{x}^*$  in the  $\ell_p$ -norm.

**2.1. Convergence of Algorithm 2.1.** In the first part of this subsection, we analyze the convergence of Algorithm 2.1 and give an error bound for its limit point  $\mathbf{x}^{\epsilon_*, \lambda}$ . It is always assumed that the true signal  $\mathbf{x}^o$  satisfies  $A\mathbf{x}^o = \mathbf{b}$ . Namely, our results are established on the noiseless scenario. However, all these results can be easily extended to noisy ones. The following theorem summarizes our main result for  $0 < q \leq 1$ . The second part of this subsection shows a stronger convergence result for  $q = 1$ , in which case the problem (1.2) becomes a convex program.

**THEOREM 2.2.** *Suppose that  $\mathbf{x}^o$  is an  $s$ -sparse vector satisfying  $A\mathbf{x}^o = \mathbf{b}$ . Assume that  $A$  satisfies the RIP of order  $2s$  with  $\delta_{2s} < 1$  and the smoothing parameter  $\epsilon_k \rightarrow \epsilon_*$  as  $k \rightarrow \infty$ . Then the sequence  $\{\mathbf{x}^{(k)}\}$  generated by Algorithm 2.1 with  $0 < q \leq 1$  has at least one convergent subsequence. When  $\epsilon_* > 0$ , the limit  $\mathbf{x}^{\epsilon_*, \lambda}$  of any convergent subsequence is a critical point of problem (1.2) with  $\epsilon = \epsilon_*$ , and it satisfies*

$$(2.6) \quad \|\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o\|_2 \leq C_1 \sqrt{\lambda} + C_2 \sigma_s(\mathbf{x}^{\epsilon_*, \lambda})_2.$$

When  $\epsilon_* = 0$ , there must exist a subsequence from  $\{\mathbf{x}^{(k)}\}$  converging to an  $s$ -sparse point  $\mathbf{x}^{0, \lambda}$  which satisfies

$$(2.7) \quad \|\mathbf{x}^{0, \lambda} - \mathbf{x}^o\|_2 \leq C \sqrt{\lambda}.$$

Here,  $C_1$ ,  $C_2$ , and  $C$  are positive constants dependent only on  $\delta_{2s}$ ,  $\|\mathbf{x}^o\|_q$ , and initial point  $\mathbf{x}^{(0)}$ .

*Remark 2.1.* From the results in Theorem 2.2 we can see that when  $\epsilon_* = 0$ , the limit  $\mathbf{x}^{0, \lambda}$  is away from the exact sparse solution by a factor of  $\sqrt{\lambda}$ . If  $\epsilon_* > 0$ , we have to check how small the error  $\sigma_s(\mathbf{x}^{\epsilon_*, \lambda})_2$  is. If the error is very small and  $\lambda$  is very small, then  $\mathbf{x}^{\epsilon_*, \lambda}$  is close to the exact sparse solution. This result provides a reasonable stopping criterion for Algorithm 2.1 when  $\lambda$  is small. Specifically, if the tail of the current iterate  $\mathbf{x}^{(k)}$  is small after the  $s$ th largest term in magnitude, we can terminate the algorithm. Equivalently, if  $\epsilon_k$  is small, we can stop the iteration according to the updating rule of  $\epsilon$ .

*Remark 2.2.* Under the assumption that  $L_q(\mathbf{x}, \epsilon^*, \lambda)$  has finitely many critical points, we can show that the sequence  $\{\mathbf{x}^{(k)}\}$  is convergent when  $\epsilon_* > 0$ . Indeed, let  $\mathbf{y}^i, i = 1, \dots, \ell$  be these critical points and  $c = \min_{i \neq j} \|\mathbf{y}^i - \mathbf{y}^j\|_2$  be the smallest distance among them. For any  $0 < d < c/3$ , there exists an integer  $K_1 > 0$  such that for any  $k \geq K_1$ ,  $\mathbf{x}^{(k)}$  is inside the ball  $B_d(\mathbf{y}^i)$  centered at some  $\mathbf{y}^i$  with radius

d. Otherwise, there would be another new critical point. In addition, (2.11) indicates that there exists another integer  $K_2 > 0$  such that  $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_2 < d$  for any  $k \geq K_2$ . With these two observations, we claim that if  $\mathbf{x}^{(K)} \in B_d(\mathbf{y}^{i_0})$ , then all  $\mathbf{x}^{(k)}$  for  $k \geq K = \max\{K_1, K_2\}$  are inside the same ball, and thus the sequence  $\{\mathbf{x}^{(k)}\}$  converges to  $\mathbf{y}^{i_0}$ .

To prove Theorem 2.2, we begin with the following inequality.

LEMMA 2.3. *Given  $0 < q \leq 1$ , if  $\epsilon_k \geq \epsilon_{k+1} \geq 0$ , then the inequality*

$$(2.8) \quad (\epsilon_k + x^2)^{q/2} - (\epsilon_{k+1} + y^2)^{q/2} - \frac{q(x-y)y}{(\epsilon_k + x^2)^{1-q/2}} \geq \frac{q(x-y)^2}{2(\epsilon_k + x^2)^{1-q/2}}$$

holds for any  $x, y \in \mathbb{R}$ .

*Proof.* We first use the well-known arithmetic-geometric mean inequality (cf. [19, p. 16, (2.5.2)]) to have

$$(2.9) \quad (\epsilon_k + x^2)^{1-q/2}(\epsilon_{k+1} + y^2)^{q/2} \leq \left(1 - \frac{q}{2}\right)(\epsilon_k + x^2) + \frac{q}{2}(\epsilon_{k+1} + y^2).$$

Then we compute the difference

$$\begin{aligned} & (\epsilon_k + x^2)^{q/2} - (\epsilon_{k+1} + y^2)^{q/2} - \frac{q(x-y)y}{(\epsilon_k + x^2)^{1-q/2}} \\ &= \frac{(\epsilon_k + x^2) - (\epsilon_k + x^2)^{1-q/2}(\epsilon_{k+1} + y^2)^{q/2} - q(x-y)y}{(\epsilon_k + x^2)^{1-q/2}} \\ &\geq \frac{(\epsilon_k + x^2) - \left(1 - \frac{q}{2}\right)(\epsilon_k + x^2) - \frac{q}{2}(\epsilon_{k+1} + y^2) - q(x-y)y}{(\epsilon_k + x^2)^{1-q/2}} \\ &= \frac{q\epsilon_k - \epsilon_{k+1} + (x-y)^2}{2(\epsilon_k + x^2)^{1-q/2}} \geq \frac{q(x-y)^2}{2(\epsilon_k + x^2)^{1-q/2}}, \end{aligned}$$

which is the desired inequality.  $\square$

Our next result shows the monotonicity of  $L_q(\mathbf{x}^{(k)}, \epsilon_k, \lambda)$  along the sequence. Inequality (2.10) is similar to that derived in [21], while inequality (2.11) is new.

LEMMA 2.4. *Let  $\mathbf{x}^{(k+1)}$  be the solution of (2.3) for  $k = 0, 1, 2, \dots$ . Then*

$$(2.10) \quad \|\mathbf{Ax}^{(k)} - \mathbf{Ax}^{(k+1)}\|_2^2 \leq 2\lambda \left( L_q(\mathbf{x}^{(k)}, \epsilon_k, \lambda) - L_q(\mathbf{x}^{(k+1)}, \epsilon_{k+1}, \lambda) \right).$$

Furthermore,

$$(2.11) \quad \|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_2^2 \leq C \left( L_q(\mathbf{x}^{(k)}, \epsilon_k, \lambda) - L_q(\mathbf{x}^{(k+1)}, \epsilon_{k+1}, \lambda) \right)$$

for a positive constant  $C$  which is dependent on  $\epsilon_0$  and a bound for  $\mathbf{x}^{(k)}$ ,  $k \geq 0$ .

*Proof.* We first compute

$$\begin{aligned} & L_q(\mathbf{x}^{(k)}, \epsilon_k, \lambda) - L_q(\mathbf{x}^{(k+1)}, \epsilon_{k+1}, \lambda) \\ &= \sum_{j=1}^N (\epsilon_k^2 + |x_j^{(k)}|^2)^{q/2} - \sum_{j=1}^N (\epsilon_{k+1}^2 + |x_j^{(k+1)}|^2)^{q/2} \\ &\quad + \frac{1}{2\lambda} \left( \|\mathbf{Ax}^{(k)} - \mathbf{b}\|_2^2 - \|\mathbf{Ax}^{(k+1)} - \mathbf{b}\|_2^2 \right) \\ &= \sum_{j=1}^N (\epsilon_k^2 + |x_j^{(k)}|^2)^{q/2} - (\epsilon_{k+1}^2 + |x_j^{(k+1)}|^2)^{q/2} + \frac{1}{2\lambda} \|\mathbf{Ax}^{(k)} - \mathbf{Ax}^{(k+1)}\|_2^2 \\ (2.12) \quad & + \frac{1}{\lambda} \left( \mathbf{Ax}^{(k+1)} - \mathbf{b} \right)^T \left( \mathbf{Ax}^{(k)} - \mathbf{Ax}^{(k+1)} \right). \end{aligned}$$

Using (2.2), we have

$$(2.13) \quad \frac{1}{\lambda} \left( A\mathbf{x}^{(k+1)} - \mathbf{b} \right)^T \left( A\mathbf{x}^{(k)} - A\mathbf{x}^{(k+1)} \right) = - \sum_{j=1}^N \frac{qx_j^{(k+1)}(x_j^{(k)} - x_j^{(k+1)})}{(\epsilon_k^2 + |x_j^{(k)}|^2)^{1-q/2}}.$$

Substituting (2.13) into (2.12) and using (2.8) yields

$$(2.14) \quad \begin{aligned} & L_q(\mathbf{x}^{(k)}, \epsilon_k, \lambda) - L_q(\mathbf{x}^{(k+1)}, \epsilon_{k+1}, \lambda) \\ &= \sum_{j=1}^N \left( (\epsilon_k^2 + |x_j^{(k)}|^2)^{q/2} - (\epsilon_{k+1}^2 + |x_j^{(k+1)}|^2)^{q/2} - \frac{qx_j^{(k+1)}(x_j^{(k)} - x_j^{(k+1)})}{(\epsilon_k^2 + |x_j^{(k)}|^2)^{1-q/2}} \right) \\ & \quad + \frac{1}{2\lambda} \|A\mathbf{x}^{(k)} - A\mathbf{x}^{(k+1)}\|_2^2 \\ & \geq \frac{1}{2\lambda} \|A\mathbf{x}^{(k)} - A\mathbf{x}^{(k+1)}\|_2^2 + \sum_{j=1}^N \left( x_j^{(k)} - x_j^{(k+1)} \right)^2 \frac{q}{2(\epsilon_k^2 + |x_j^{(k)}|^2)^{1-q/2}} \end{aligned}$$

from which result (2.10) follows immediately.

For inequality (2.11), we see from (2.10) that  $L_q(\mathbf{x}^{(k)}, \epsilon_k, \lambda)$  is monotonically decreasing. It thus follows that

$$\|\mathbf{x}^{(k)}\|_q^q \leq \|\mathbf{x}^{(k)}\|_{q, \epsilon_k}^q \leq L_q(\mathbf{x}^{(k)}, \epsilon_k, \lambda) \leq L_q(\mathbf{x}^{(0)}, \epsilon_0, \lambda) = \|\mathbf{x}^{(0)}\|_{q, \epsilon_0}^q$$

for all  $k \geq 1$ . That is, there exists a positive number  $\Delta$  such that  $\|\mathbf{x}^{(k)}\|_\infty \leq \Delta$  for all  $k \geq 1$ . Note  $\epsilon_k \leq \epsilon_0$ . Hence,

$$\frac{q}{2(\epsilon_k^2 + |x_j^{(k)}|^2)^{1-q/2}} \geq \frac{q}{2(\epsilon_0^2 + \Delta^2)^{1-q/2}}.$$

Letting  $\frac{1}{C} = \frac{q}{2(\epsilon_0^2 + \Delta^2)^{1-q/2}}$ , we have (2.11) from (2.14).  $\square$

With the above preparations, we are now ready to prove Theorem 2.2.

*Proof of Theorem 2.2.* If  $\epsilon_* > 0$ , the boundedness of  $\{\mathbf{x}^{(k)}\}$  implies that there exists a subsequence  $\{\mathbf{x}^{(k_j)}\}$  converging to some point  $\mathbf{x}^{\epsilon_*, \lambda}$ . Note that (2.11) indicates  $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_2 \rightarrow 0$ . Thus the subsequence  $\{\mathbf{x}^{(k_j+1)}\}$  also converges to  $\mathbf{x}^{\epsilon_*, \lambda}$ . Now, replacing  $\mathbf{x}^{(k)}$ ,  $\mathbf{x}^{(k+1)}$ ,  $\epsilon_k$  with  $\mathbf{x}^{(k_j)}$ ,  $\mathbf{x}^{(k_j+1)}$ ,  $\epsilon_{k_j}$  in (2.2), respectively, and letting  $j \rightarrow \infty$  yields

$$\left[ \frac{qx_j^{\epsilon_*, \lambda}}{(\epsilon_*^2 + (x_j^{\epsilon_*, \lambda})^2)^{1-q/2}} \right]_{1 \leq j \leq N} + \frac{1}{\lambda} A^\top (A\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{b}) = 0.$$

Namely,  $\mathbf{x}^{\epsilon_*, \lambda}$  is a critical point of (1.2) with  $\epsilon = \epsilon_* > 0$ .

We use Lemma 2.4 to get

$$L_q(\mathbf{x}^{\epsilon_*, \lambda}, \epsilon_*, \lambda) \leq L_q(\mathbf{x}^{(k_j)}, \epsilon_{k_j}, \lambda) \leq L_q(\mathbf{x}^{(0)}, \epsilon_0, \lambda) \leq \|\mathbf{x}^{(0)}\|_q^q + N(\epsilon_0)^q = \|\mathbf{x}^{(0)}\|_q^q + N,$$

which implies  $\|A\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{b}\|_2 \leq \sqrt{2\lambda L_q(\mathbf{x}^{\epsilon_*, \lambda}, \epsilon_*, \lambda)} \leq \sqrt{2\lambda(\|\mathbf{x}^{(0)}\|_q^q + N)}$ .



Let  $S$  be the index set of nonzero entries of  $\mathbf{x}^o$  and let  $S^*$  be the index set of  $s$  largest entries in absolute value of  $\mathbf{x}^{\epsilon_*, \lambda}$ . Since  $\|\mathbf{x}^o\|_0 \leq s$ , we have

$$\begin{aligned} \|\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o\|_2 &\leq \|(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_{S \cup S^*}\|_2 + \|(\mathbf{x}^{\epsilon_*, \lambda})_{(S \cup S^*)^c}\|_2 \\ &\leq \frac{1}{\sqrt{1 - \delta_{2s}}} \|A(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_{S \cup S^*}\|_2 + \|(\mathbf{x}^{\epsilon_*, \lambda})_{(S \cup S^*)^c}\|_2 \\ &\leq \frac{1}{\sqrt{1 - \delta_{2s}}} \|A\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{b}\|_2 + \left( \frac{1}{\sqrt{1 - \delta_{2s}}} \|A\|_2 + 1 \right) \|(\mathbf{x}^{\epsilon_*, \lambda})_{(S \cup S^*)^c}\|_2 \\ &\leq \frac{1}{\sqrt{1 - \delta_{2s}}} \sqrt{2\lambda(\|\mathbf{x}^{(0)}\|_q^q + N)} + \left( \frac{1}{\sqrt{1 - \delta_{2s}}} \|A\|_2 + 1 \right) \sigma_s(\mathbf{x}^{\epsilon_*, \lambda})_2. \end{aligned}$$

This completes the proof of (2.6).

If  $\epsilon_* = 0$ , then  $\epsilon_k = 0$  for some  $k$  or  $\epsilon_k = \alpha \cdot r(\mathbf{x}^{(m_k)})_{s+1}$  holds for sufficiently large  $k$  and some integer  $m_k \leq k$ . In the former case,  $\mathbf{x}^{(k)}$  is an  $s$ -sparse vector, and we can let  $\mathbf{x}^{0, \lambda} = \mathbf{x}^{(k)}$ . In the latter case, let  $\mathbf{x}^{0, \lambda}$  be a limit point of  $\{\mathbf{x}^{(m_k)}\}$  since  $\{\mathbf{x}^{(m_k)}\}$  is bounded. Without loss of generality, we assume  $\mathbf{x}^{0, \lambda} = \lim_{k \rightarrow \infty} \mathbf{x}^{(m_k)}$ . Then  $r(\mathbf{x}^{0, \lambda})_{s+1} = \lim_{k \rightarrow \infty} r(\mathbf{x}^{(m_k)})_{s+1} = \lim_{k \rightarrow \infty} \frac{\epsilon_k}{\alpha} = 0$ . That is,  $\mathbf{x}^{0, \lambda}$  is an  $s$ -sparse vector. Therefore, in both cases, we have an  $s$ -sparse limit point  $\mathbf{x}^{0, \lambda}$ . Without loss of generality, we assume  $\mathbf{x}^{(k)} \rightarrow \mathbf{x}^{0, \lambda}$ . Using the RIP of  $A$ , we have

$$\begin{aligned} \|\mathbf{x}^{0, \lambda} - \mathbf{x}^o\|_2 &\leq \frac{1}{\sqrt{1 - \delta_{2s}}} \|A(\mathbf{x}^{0, \lambda} - \mathbf{x}^o)\|_2 = \frac{1}{\sqrt{1 - \delta_{2s}}} \lim_{k \rightarrow \infty} \|A\mathbf{x}^{(k)} - \mathbf{b}\|_2 \\ &\leq \frac{1}{\sqrt{1 - \delta_{2s}}} \lim_{k \rightarrow \infty} \left( 2\lambda L_q(\mathbf{x}^{(k)}, \epsilon_k, \lambda) \right)^{1/2} \\ &\leq \frac{1}{\sqrt{1 - \delta_{2s}}} \left( 2\lambda L_q(\mathbf{x}^{(0)}, \epsilon_0, \lambda) \right)^{1/2} \\ &\leq \frac{1}{\sqrt{1 - \delta_{2s}}} \sqrt{2\lambda \|\mathbf{x}^{(0)}\|_{q, \epsilon_0}^q}, \end{aligned}$$

where we have used Lemma 2.4 in the third inequality. This completes the proof.  $\square$

**Case of  $q = 1$ .** When  $q = 1$ , we can prove a stronger result in the case of  $\epsilon_* > 0$  and estimate the error in the  $\ell_1$  norm. The boundedness of  $\{\mathbf{x}^{(k)}\}$  from Algorithm 2.1 implies that  $\{\mathbf{x}^{(k)}\}$  has a convergent subsequence. Suppose  $\mathbf{x}^{(k_j)} \rightarrow \tilde{\mathbf{x}}$ . We know that  $\tilde{\mathbf{x}}$  satisfies the first-order optimality condition of  $\min_{\mathbf{x}} L_1(\mathbf{x}, \epsilon_*, \lambda)$  when  $\epsilon_* > 0$ . Since  $L_1(\mathbf{x}, \epsilon_*, \lambda)$  is strictly convex when  $\epsilon_* > 0$ ,  $\tilde{\mathbf{x}}$  is the unique minimizer. It follows that  $\{\mathbf{x}^{(k)}\}$  converges and the limit is the unique minimizer. As before let it be denoted by  $\mathbf{x}^{\epsilon_*, \lambda}$ .

**THEOREM 2.5.** *Suppose  $\mathbf{x}^o$  is an  $s$ -sparse vector satisfying  $A\mathbf{x}^o = \mathbf{b}$ , and suppose that the RIP constants  $\delta_{2s}$  and  $\delta_{3s}$  of  $A$  satisfy  $\gamma = \delta_{3s}/(1 - \delta_{2s}) < 1$ . If the limit of  $\{\epsilon_k\}$  is  $\epsilon_* > 0$ , then the sequence  $\{\mathbf{x}^{(k)}\}$  generated by Algorithm 2.1 with  $q = 1$  and  $\alpha \leq \frac{1}{N}$  converges to the unique minimizer  $\mathbf{x}^{\epsilon_*, \lambda}$  of  $L_1(\mathbf{x}, \epsilon_*, \lambda)$ , which satisfies*

$$\|\mathbf{x}^o - \mathbf{x}^{\epsilon_*, \lambda}\|_1 \leq C_1 \sigma_t(\mathbf{x}^o)_1 + C_2 \lambda$$

for an integer  $t < s$  small enough such that  $\frac{1+\gamma}{(1-\gamma)(s+1-t)} < 1$ , where  $C_1$  and  $C_2$  are two positive constants.

**Remark 2.3.** That is, when  $\epsilon_* > 0$ , if the solution  $\mathbf{x}^o$  has sparsity  $\|\mathbf{x}^o\|_0 \leq t$ , then  $\sigma_t(\mathbf{x}^o)_1 = 0$  and hence Algorithm 2.1 can recover the sparse solution  $\mathbf{x}^o$  within an error proportional to  $\lambda$ .

To establish this result, we need a series of lemmas. Let

$$\eta = \mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o,$$

and let  $S_0$  contain the index set of the first  $s$  largest entries of  $\mathbf{x}^o$  in absolute value. In addition, we let  $S_1, S_2, \dots$  be the subsets of the complement  $S_0^c$  of  $S$  in  $\{1, 2, \dots, n\}$ , where  $S_1$  is the set of indices of the first  $s$  largest entries of  $\eta_{S_0^c}$  in absolute value and  $S_2$  is the set of indices of next  $s$  largest entries in absolute value and so on.

LEMMA 2.6. *Suppose that  $A$  satisfies the RIP of order  $2s$  with  $\delta_{2s} < 1$ . Then*

$$(2.15) \quad \|\eta_{S_0 \cup S_1}\|_2 \leq \frac{\delta_{3s}}{1 - \delta_{2s}} \sum_{j \geq 2} \|\eta_{S_j}\|_2 + \frac{\sqrt{2s}}{1 - \delta_{2s}} \lambda.$$

*Proof.* Letting  $S = S_0 \cup S_1$ , by (2.1) we have

$$(2.16) \quad \begin{aligned} \|\eta_S\|_2^2 &= \|(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S\|_2^2 = \langle (\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S, \mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o \rangle \\ &= \langle (\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S, (I - A^T A)(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o) \rangle \\ &\quad + \lambda \left\langle (\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S, \left[ \frac{x_j^{\epsilon_*, \lambda}}{(\epsilon_*^2 + (x_j^{\epsilon_*, \lambda})^2)^{1/2}}, j \in S \right] \right\rangle \end{aligned}$$

since  $\mathbf{x}^{\epsilon_*, \lambda}$  satisfies the first-order optimality condition. The first term of (2.16) is

$$(2.17) \quad \begin{aligned} &\langle (I - A^T A)^T (\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S, (\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o) \rangle \\ &\leq \langle (I - A^T A)^T (\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S, (\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S \rangle \\ &\quad + \sum_{j \geq 2} \langle (I - A^T A)^T (\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S, (\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_{S_j} \rangle \\ &\leq \delta_{2s} \|(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S\|_2^2 + \delta_{3s} \|(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S\|_2 \left( \sum_{j \geq 2} \|(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_{S_j}\|_2 \right), \end{aligned}$$

where we have used the property (2.5) of RIP constant to get (2.17). The second term of (2.16) is

$$(2.18) \quad \begin{aligned} &\lambda \left\langle (\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S, \left[ \frac{x_j^{\epsilon_*, \lambda}}{(\epsilon_*^2 + (x_j^{\epsilon_*, \lambda})^2)^{1/2}}, j \in S \right] \right\rangle \\ &\leq \lambda \|(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S\|_2 \left\| \left[ \frac{x_j^{\epsilon_*, \lambda}}{(\epsilon_*^2 + (x_j^{\epsilon_*, \lambda})^2)^{1/2}}, j \in S \right] \right\|_2 \\ &\leq \lambda \|(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S\|_2 \sqrt{2s}. \end{aligned}$$

We summarize the above to have

$$(2.19) \quad \|(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S\|_2 \leq \delta_{2s} \|(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S\|_2 + \delta_{3s} \sum_{j \geq 2} \|(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_{S_j}\|_2 + \lambda \sqrt{2s}$$

or equivalently

$$(2.20) \quad \|(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_S\|_2 \leq \frac{\delta_{3s}}{1 - \delta_{2s}} \sum_{j \geq 2} \|(\mathbf{x}^{\epsilon_*, \lambda} - \mathbf{x}^o)_{S_j}\|_2 + \frac{\sqrt{2s}}{1 - \delta_{2s}} \lambda. \quad \square$$

Furthermore, with the above inequality, we can show the following.

LEMMA 2.7. *Suppose that  $A$  satisfies the RIP of order  $2s$  with  $\delta_{2s} < 1$ . Then*

$$(2.21) \quad \|\eta_{S_0}\|_1 \leq \frac{\delta_{3s}}{1 - \delta_{2s}} \|\eta_{S_0^c}\|_1 + \frac{s\sqrt{2}}{1 - \delta_{2s}} \lambda.$$

*Proof.* We note that  $\|\eta_{S_j}\|_2 \leq \|\eta_{S_{j-1}}\|_1 / \sqrt{s}$  for all  $j \geq 2$  and

$$(2.22) \quad \sum_{j \geq 2} \|\eta_{S_j}\|_2 \leq \frac{1}{\sqrt{s}} \sum_{j \geq 1} \|\eta_{S_j}\|_1 = \frac{1}{\sqrt{s}} \|\eta_{S_0^c}\|_1.$$

Thus, we can use Lemma 2.6 and (2.22) to get

$$\begin{aligned} \|\eta_{S_0}\|_1 &\leq \sqrt{s} \|\eta_{S_0}\|_2 \leq \sqrt{s} \|\eta_S\|_2 \\ &\leq \frac{\delta_{3s}\sqrt{s}}{1 - \delta_{2s}} \sum_{j \geq 2} \|\eta_{S_j}\|_2 + \frac{s\sqrt{2}}{1 - \delta_{2s}} \lambda \leq \frac{\delta_{3s}\sqrt{s}}{1 - \delta_{2s}} \frac{1}{\sqrt{s}} \|\eta_{S_0^c}\|_1 + \frac{s\sqrt{2}}{1 - \delta_{2s}} \lambda \\ &= \frac{\delta_{3s}}{1 - \delta_{2s}} \|\eta_{S_0^c}\|_1 + \frac{s\sqrt{2}}{1 - \delta_{2s}} \lambda. \end{aligned}$$

Then the desired inequality (2.21) follows.  $\square$

In addition, we need the following lemma, which is a variant of Lemma 4.2 of [10]. Although the idea for the proof is very similar, we include it for convenience.

LEMMA 2.8. *Suppose that the RIP constants  $\delta_{2s}$  and  $\delta_{3s}$  of  $A$  satisfy  $\gamma = \delta_{3s}/(1 - \delta_{2s}) < 1$ . Let  $\beta = \sqrt{2}/(1 - \delta_{2s})$ . Then*

$$\|\eta\|_1 \leq \frac{1 + \gamma}{1 - \gamma} (\|\mathbf{x}^{\epsilon^*, \lambda}\|_1 - \|\mathbf{x}^o\|_1 + 2\sigma_s(\mathbf{x}^o)_1) + \frac{2s\beta}{1 - \gamma} \lambda.$$

*Proof.* We use Lemma 2.7 to have

$$(2.23) \quad \|\eta\|_1 = \|\eta_{S_0}\|_1 + \|\eta_{S_0^c}\|_1 \leq (1 + \gamma) \|\eta_{S_0^c}\|_1 + s\beta\lambda.$$

Then

$$\begin{aligned} \|\eta_{S_0^c}\|_1 &\leq \|(\mathbf{x}^{\epsilon^*, \lambda})_{S_0^c}\|_1 + \|(\mathbf{x}^o)_{S_0^c}\|_1 \\ &= \|\mathbf{x}^{\epsilon^*, \lambda}\|_1 - \|(\mathbf{x}^{\epsilon^*, \lambda})_{S_0}\|_1 + 2\|(\mathbf{x}^o)_{S_0^c}\|_1 - \|(\mathbf{x}^o)_{S_0}\|_1 \\ &\leq \|\mathbf{x}^{\epsilon^*, \lambda}\|_1 - \|\mathbf{x}^o\|_1 + \|(\mathbf{x}^o)_{S_0}\|_1 - \|(\mathbf{x}^{\epsilon^*, \lambda})_{S_0}\|_1 + 2\|(\mathbf{x}^o)_{S_0^c}\|_1 \\ &\leq \|\mathbf{x}^{\epsilon^*, \lambda}\|_1 - \|\mathbf{x}^o\|_1 + \|(\mathbf{x}^o - \mathbf{x}^{\epsilon^*, \lambda})_{S_0}\|_1 + 2\|(\mathbf{x}^o)_{S_0^c}\|_1 \\ &\leq \|\mathbf{x}^{\epsilon^*, \lambda}\|_1 - \|\mathbf{x}^o\|_1 + \gamma \|\eta_{S_0^c}\|_1 + s\beta\lambda + 2\|(\mathbf{x}^o)_{S_0^c}\|_1, \end{aligned}$$

where we have used Lemma 2.7 again in the last inequality. After rearranging the terms, we get

$$(2.24) \quad \|\eta_{S_0^c}\|_1 \leq \frac{1}{1 - \gamma} \left( \|\mathbf{x}^{\epsilon^*, \lambda}\|_1 - \|\mathbf{x}^o\|_1 + s\beta\lambda + 2\|(\mathbf{x}^o)_{S_0^c}\|_1 \right).$$

Combining the inequalities in (2.23) and (2.24), we conclude the result in this lemma.  $\square$

With the above three lemmas, we are ready to prove Theorem 2.5.

*Proof of Theorem 2.5.* The convergence of  $\{\mathbf{x}^{(k)}\}$  to the unique minimizer  $\mathbf{x}^{\epsilon_*, \lambda}$  has been shown by the discussion before Theorem 2.5 or is implied by Remark 2.2 since  $L_1(\mathbf{x}, \epsilon_*, \lambda)$  is strictly convex with respect to  $\mathbf{x}$  when  $\epsilon_* > 0$ .

From  $A\mathbf{x}^o = \mathbf{b}$  and  $\epsilon_* \leq \frac{r(\mathbf{x}^{(k)})_{s+1}}{N}$  for all  $k \geq 0$ , it follows that

$$\|\mathbf{x}^{\epsilon_*, \lambda}\|_1 \leq L_1(\mathbf{x}^{\epsilon_*, \lambda}, \epsilon_*, \lambda) \leq L_1(\mathbf{x}^o, \epsilon_*, \lambda) \leq \|\mathbf{x}^o\|_1 + N\epsilon_* \leq \|\mathbf{x}^o\|_1 + r(\mathbf{x}^{\epsilon_*, \lambda})_{s+1}.$$

Using Lemma 2.8 and noting that  $\mathbf{x}^o$  is  $s$ -sparse, we have

$$\begin{aligned} \|\mathbf{x}^o - \mathbf{x}^{\epsilon_*, \lambda}\|_1 &\leq \frac{1+\gamma}{1-\gamma} (\|\mathbf{x}^{\epsilon_*, \lambda}\|_1 - \|\mathbf{x}^o\|_1 + 2\sigma_s(\mathbf{x}^o)_1) + \frac{2s\beta}{1-\gamma}\lambda \\ (2.25) \quad &\leq \frac{1+\gamma}{1-\gamma} r(\mathbf{x}^{\epsilon_*, \lambda})_{s+1} + C\lambda \end{aligned}$$

for  $C = \frac{2s\beta}{1-\gamma}$  and  $\gamma = \frac{\delta_{3s}}{1-\delta_{2s}} < 1$ . Furthermore, based on the rearrangement  $r(\mathbf{x}^{\epsilon_*, \lambda})$  in decreasing order, we can choose  $t < s$  such that

$$(2.26) \quad (s+1-t) \cdot r(\mathbf{x}^{\epsilon_*, \lambda})_{s+1} \leq \sum_{j=t+1}^{s+1} r(\mathbf{x}^{\epsilon_*, \lambda})_j \leq \sigma_t(\mathbf{x}^{\epsilon_*, \lambda})_1 \leq \|\mathbf{x}^o - \mathbf{x}^{\epsilon_*, \lambda}\|_1 + \sigma_t(\mathbf{x}^o)_1.$$

Then, it follows from (2.25) and (2.26) that

$$\begin{aligned} \|\mathbf{x}^o - \mathbf{x}^{\epsilon_*, \lambda}\|_1 &\leq \frac{1+\gamma}{1-\gamma} r(\mathbf{x}^{\epsilon_*, \lambda})_{s+1} + C\lambda \\ &\leq \frac{1+\gamma}{(1-\gamma)(s+1-t)} (\|\mathbf{x}^o - \mathbf{x}^{\epsilon_*, \lambda}\|_1 + \sigma_t(\mathbf{x}^o)_1) + C\lambda. \end{aligned}$$

Therefore, if  $\nu = \frac{1+\gamma}{(1-\gamma)(s+1-t)} < 1$ , then

$$\|\mathbf{x}^o - \mathbf{x}^{\epsilon_*, \lambda}\|_1 \leq \frac{\nu}{1-\nu} \sigma_t(\mathbf{x}^o)_1 + \frac{C}{1-\nu} \lambda,$$

and letting  $C_1 = \frac{\nu}{1-\nu}$  and  $C_2 = \frac{C}{1-\nu}$  completes the proof.  $\square$

**2.2. Local convergence behavior of Algorithm 2.1.** As discussed above, when  $\epsilon_k \rightarrow 0$ , the limit point of the sequence  $\{\mathbf{x}^{(k)}\}$  from Algorithm 2.1 with  $0 < q \leq 1$  is close to the sparse solution within an error proportional to  $\sqrt{\lambda}$ . When  $\epsilon_k \rightarrow \epsilon_* > 0$  and  $q = 1$ ,  $\{\mathbf{x}^{(k)}\}$  converges to the unique minimizer of problem (1.2) with  $\epsilon = \epsilon_*$ . The minimizer is close to a sparse solution within an error proportional to  $\lambda$ . Only the case corresponding to  $q < 1$  and  $\epsilon_* > 0$  needs to be studied further. This subsection gives an analysis of the local convergence behavior of Algorithm 2.1 in this case. The main result is the following theorem.

**THEOREM 2.9.** *Suppose that  $\mathbf{x}^o$  is an  $s$ -sparse vector satisfying  $A\mathbf{x}^o = \mathbf{b}$ . Let  $S_0$  be the index set of the nonzeros of  $\mathbf{x}^o$ . Assume that  $A$  satisfies the RIP of order  $2s$  with  $\delta_{2s} < 1/2$  and  $\gamma = \frac{\delta_{3s}}{1-\delta_{2s}} < 1$ . Let  $\{\mathbf{x}^{(k)}\}$  be the sequence generated by Algorithm 2.1 with  $0 < q \leq 1$  and  $\alpha \leq \frac{1}{N^{1/q}}$  and assume the smoothing parameter  $\epsilon_k \rightarrow \epsilon_* > 0$ . Let  $\eta^{(k)} = \mathbf{x}^{(k)} - \mathbf{x}^o$ . If for some  $k$ ,  $E_k := \|\eta^{(k)}\|_q \leq \rho \min_{i \in S_0} |x_i^o|$  with  $\rho < 1$ , then there **exists** positive constants  $\mu$  and  $C$  such that*

$$(2.27) \quad E_{k+1}^q \leq \mu(E_k)^{q(2-q)} + C\sqrt{\lambda}^q.$$

When  $E_k \rightarrow 0$ , (2.27) holds for all  $k \geq k_0$  for some integer  $k_0 \geq 1$  if  $\lambda = 0$ . If  $\lambda > 0$  is sufficiently small, (2.27) holds for  $k_0 \leq k \leq k_0 + m$  for some integers  $k_0 \geq 1$  and  $m \geq 0$  (to be explained in Remark 2.4).

The proof of Theorem 2.9 will be given after we establish two lemmas. Let us first give some explanation of the results in this theorem.

*Remark 2.4.* For  $q < 1$  and  $\lambda = 0$ ,  $E_k^q$  exhibits a superlinear convergence under the assumption that  $E_k^q$  for some  $k$  is sufficiently small such that  $(E_k^q)^r \mu \leq \delta < 1$  for some  $r \in (0, 1 - q)$ . In case  $\mu < 1$ , we also assume  $E_k \leq 1$ . Then it follows from (2.27) that  $E_{k+1}^q \leq \mu(E_k^q)^r (E_k^q)^{2-q-r} \leq E_k^q$  and hence  $E_{k+1} \leq E_k$ . Thus, we have  $\mu(E_{k+1}^q)^r \leq \mu(E_k^q)^r \leq \delta < 1$  and  $E_{k+1} \leq E_k \leq \rho \min_{i \in S_0} |x_i^o| < 1$ . Hence we can apply (2.27) again. By induction, we see  $\mu(E_{k+m}^q)^r \leq \delta < 1$  and  $E_{k+m} \leq \rho \min_{i \in S_0} |x_i^o|$  for all  $m \geq 1$ . Therefore, by (2.27), we have

$$E_{k+m}^q \leq \delta(E_{k+m-1}^q)^{2-q-r} \leq \delta^{3-q-r}(E_{k+m-2}^q)^{(2-q-r)^2} \dots \leq \delta^\ell (E_k^q)^{(2-q-r)^m},$$

where  $\ell = \sum_{j=0}^{m-1} (2-q-r)^j$ . That is,  $E_k^q$  converges superlinearly. Intuitively, when  $\lambda > 0$  is sufficiently small, the errors  $E_k$  behave similarly. Indeed, under the same assumption on  $E_k$  above, if  $\lambda$  is small such that  $C\sqrt{\lambda^q} \leq \min\{(1-\delta)E_{k+j}^q, j = 1, \dots, m\}$  for another integer  $m$ , then the errors  $E_{k+j}$  for  $0 \leq j \leq m$  also behave in a superlinear fashion.

*Remark 2.5.* Due to the error behavior in Theorem 2.9, if  $E_k^q$  for  $k \geq 1$  are in superlinear fashion, then

$$\left\| \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)} \right\|_q^q \leq E_{k+1}^q + E_k^q,$$

i.e., the difference between two consecutive iterates behaves like that of  $E_k^q$ . If the difference above is in a superlinear fashion, the limit  $\mathbf{x}^{\epsilon_*, \lambda}$  would be close to a sparse solution very likely. This can be a stopping criterion for Algorithm 2.1.

As explained in Theorems 2.2 and 2.5, the limit  $\mathbf{x}^{\epsilon_*, \lambda}$  of any convergent subsequence of  $\{\mathbf{x}^{(k)}\}$  is away from  $\mathbf{x}^o$  by an amount dependent on  $\lambda$  and the residual  $\sigma_s(\mathbf{x}^{\epsilon_*, \lambda})_1$ . Thus, without loss of generality, we next assume that  $\{\mathbf{x}^{(k)}\}$  converges and let  $\eta^{(k)} = \mathbf{x}^{(k)} - \mathbf{x}^o$  for  $k \geq 1$ . Recall that  $S_0$  contains the index set of nonzeros of  $\mathbf{x}^o$  with cardinality  $\#(S_0) = s$ . In addition, we let  $S_1, S_2, \dots$  be the subsets of  $S_0^c$  with  $S_1$  being the set of indices of the first  $s$  largest entries of  $\eta_{S_0^c}^{(k+1)}$  in absolute value and  $S_2$  the set of indices of next  $s$  largest entries and so on.

**LEMMA 2.10.** *Suppose that  $\epsilon_k \rightarrow \epsilon_* > 0$  and suppose that  $A$  satisfies the RIP of order  $2s$  with  $\delta_{2s} < 1$ . Let  $S = S_0 \cup S_1$ . Then*

$$(2.28) \quad \left\| \eta_S^{(k+1)} \right\|_2 \leq \frac{\delta_{3s}}{1 - \delta_{2s}} \sum_{j \geq 2} \left\| \eta_{S_j}^{(k+1)} \right\|_2 + C\lambda$$

for a constant  $C > 0$ .

*Proof.* The proof directly follows that of Lemma 2.6 except for one step which needs to be justified here. The inequality (2.18) is obtained from

$$\left\| \left[ \frac{x_j^{\epsilon_*, \lambda}}{(\epsilon_*^2 + (x_j^{\epsilon_*, \lambda})^2)^{1/2}}, j \in S \right] \right\|_2 \leq \sqrt{2s}.$$

Now the term becomes  $\left\| \left[ \frac{x_j^{(k+1)}}{(\epsilon_k^2 + (x_j^{(k)})^2)^{1/2}}, j \in S \right] \right\|_2$ . Since  $\epsilon_k \geq \epsilon_* > 0$  and  $\{\mathbf{x}^{(k)}\}$  is bounded, then

$$\left\| \left[ \frac{x_j^{(k+1)}}{(\epsilon_k^2 + (x_j^{(k)})^2)^{1-q/2}}, j \in S \right] \right\|_2 \leq \left\| [x_j^{(k+1)} |(\epsilon_*)^{q-2}, j \in S] \right\|_2 \leq \Delta,$$

where  $\Delta$  is a positive constant dependent on  $\epsilon_*$  and the bound of  $\{\mathbf{x}^{(k)}\}$ . Following other steps in the proof of Lemma 2.6 gives the desired result (2.28) with  $C = \frac{\Delta}{1-\delta_{2s}}$ .  $\square$

Furthermore, with the above inequality, we can show the next lemma.

**LEMMA 2.11.** *Suppose that  $A$  satisfies the RIP of order  $2s$  with  $\delta_{2s} < 1$  and  $\gamma = \delta_{3s}/(1-\delta_{2s}) < 1$ . Then for any  $q \in (0, 1]$ ,*

$$(2.29) \quad \left\| \eta_{S_0}^{(k+1)} \right\|_q \leq \frac{\gamma}{1-\gamma} \left\| \eta_{S_0^c}^{(k+1)} \right\|_q + Cs^{1/q-1/2}\lambda$$

for a constant  $C > 0$ .

*Proof.* The idea of the proof is similar to that of Lemma 2.7. We omit the details.  $\square$

We are now ready to prove Theorem 2.9. The proof mainly follows the ideas in [10]. We spell out details in the setting of unconstrained  $\ell_q$  minimization.

*Proof of Theorem 2.9.* By (2.2), we have

$$\sum_{i=1}^N \frac{(x_i^o + \eta_i^{(k+1)})\eta_i^{(k+1)}}{(\epsilon_k^2 + (x_i^{(k)})^2)^{1-q/2}} + \frac{1}{q\lambda} (\eta^{(k+1)})^\top A^\top A \eta^{(k+1)} = 0.$$

Rearranging the terms and noting that  $\mathbf{x}^o$  is supported on  $S_0$ , we have

$$(2.30) \quad \sum_{i=1}^N \frac{(\eta_i^{(k+1)})^2}{(\epsilon_k^2 + (x_i^{(k)})^2)^{1-q/2}} + \frac{1}{q\lambda} (\eta^{(k+1)})^\top A^\top A \eta^{(k+1)} = - \sum_{i \in S_0} \frac{x_i^o \eta_i^{(k+1)}}{(\epsilon_k^2 + (x_i^{(k)})^2)^{1-q/2}}.$$

Using the assumption that  $|\eta_i^{(k)}| \leq E_k \leq \rho \min_{j \in S_0} |x_j^o|$  with  $\rho < 1$ , we get

$$\frac{|x_i^o|}{(\epsilon_k^2 + (x_i^{(k)})^2)^{1-q/2}} \leq \frac{|x_i^o|}{((x_i^o + \eta_i^{(k)})^2)^{1-q/2}} \leq \frac{|x_i^o|^{q-1}}{(1-\rho)^{2-q}}.$$

Letting  $\tilde{C} = \min\{|x_i^o|, x_i^o \neq 0\}$  and  $\bar{C} = (\tilde{C})^{q-1}$ , we have

$$\frac{|x_i^o|}{(\epsilon_k^2 + (x_i^{(k)})^2)^{1-q/2}} \leq \frac{\bar{C}}{(1-\rho)^{2-q}}$$

and hence,

$$- \sum_{i \in S_0} \frac{x_i^o \eta_i^{(k+1)}}{(\epsilon_k^2 + (x_i^{(k)})^2)^{1-q/2}} \leq \sum_{i \in S_0} \frac{\bar{C}}{(1-\rho)^{2-q}} |\eta_i^{(k+1)}| = \frac{\bar{C}}{(1-\rho)^{2-q}} \left\| \eta_{S_0}^{(k+1)} \right\|_1.$$

Thus, by (2.30), we further have

$$(2.31) \quad \begin{aligned} \sum_{i=1}^N \frac{(\eta_i^{(k+1)})^2}{(\epsilon_k^2 + (x_i^{(k)})^2)^{1-q/2}} &\leq \sum_{i=1}^N \frac{(\eta_i^{(k+1)})^2}{(\epsilon_k^2 + (x_i^{(k)})^2)^{1-q/2}} + \frac{1}{q\lambda} \eta^{(k+1)\top} A^\top A \eta^{(k+1)} \\ &\leq \frac{\bar{C}}{(1-\rho)^{2-q}} \left\| \eta_{S_0}^{(k+1)} \right\|_1 \leq \frac{\bar{C}}{(1-\rho)^{2-q}} \left\| \eta_{S_0}^{(k+1)} \right\|_q \\ &\leq \frac{\bar{C}}{(1-\rho)^{2-q}} \left( \frac{\gamma}{1-\gamma} \left\| \eta_{S_0^c}^{(k+1)} \right\|_q + Cs^{1/q-1/2}\lambda \right) \end{aligned}$$

for some positive constant  $C$ , where we have used Lemma 2.11, i.e., the  $\lambda$ -null space property (2.29).

On the other hand, the Cauchy–Schwarz inequality gives

$$\begin{aligned} \left\| \eta_{S_0^c}^{(k+1)} \right\|_q^q &\leq \left( \sum_{i \in S_0^c} \frac{|\eta_i^{(k+1)}|^2}{(\epsilon_k^2 + |x_i^{(k)}|^2)^{1-q/2}} \right)^{q/2} \left( \sum_{i \in S_0^c} (\epsilon_k^2 + |x_i^{(k)}|^2)^{q/2} \right)^{1-q/2} \\ &\leq \left( \sum_{i=1}^N \frac{|\eta_i^{(k+1)}|^2}{(\epsilon_k^2 + |x_i^{(k)}|^2)^{1-q/2}} \right)^{q/2} \left( \sum_{i \in S_0^c} \epsilon_k^q + |x_i^{(k)}|^q \right)^{1-q/2} \\ &\leq \left( \frac{\bar{C}}{(1-\rho)^{2-q}} \left( \frac{\gamma}{1-\gamma} \left\| \eta_{S_0^c}^{(k+1)} \right\|_q + C\lambda \right) \right)^{q/2} \left( \left\| \eta_{S_0^c}^{(k)} \right\|_q^q + N\epsilon_k^q \right)^{1-q/2}, \end{aligned}$$

where we have used (2.31). Squaring both sides of the last inequality yields

$$\left\| \eta_{S_0^c}^{(k+1)} \right\|_q^{2q} \leq \frac{\bar{C}^q}{(1-\rho)^{q(2-q)}} \left( \frac{\gamma^q}{(1-\gamma)^q} \left\| \eta_{S_0^c}^{(k+1)} \right\|_q^q + (C\lambda)^q \right) \left( \left\| \eta_{S_0^c}^{(k)} \right\|_q^q + N\epsilon_k^q \right)^{2-q}.$$

Finally, we have

$$\begin{aligned} N\epsilon_k^q &= (N^{1/q}\epsilon_k)^q \leq (r(\mathbf{x}^{(k)})_{s+1})^q = (r(\mathbf{x}^{(k)})_{s+1})^q - (r(\mathbf{x}^o)_{s+1})^q \\ &\leq |r(\mathbf{x}^{(k)})_{s+1} - r(\mathbf{x}^o)_{s+1}|^q \leq \left\| r(\mathbf{x}^{(k)}) - r(\mathbf{x}^o) \right\|_\infty^q \\ &\leq \|\mathbf{x}^{(k)} - \mathbf{x}^o\|_q^q = \|\eta^{(k)}\|_q^q \end{aligned}$$

by using Lemma 4.1 in [10]. Hence, with  $\bar{\alpha} = \frac{\bar{C}^q}{(1-\rho)^{q(2-q)}} 2^{2-q}$ ,

$$(2.32) \quad \left\| \eta_{S_0^c}^{(k+1)} \right\|_q^{2q} \leq \bar{\alpha} \left( \frac{\gamma^q}{(1-\gamma)^q} \left\| \eta_{S_0^c}^{(k+1)} \right\|_q^q + (C\lambda)^q \right) \|\eta^{(k)}\|_q^{q(2-q)}.$$

If  $\eta_{S_0^c}^{(k+1)} = 0$ , then we can use the  $\lambda$ -null space property (2.29) to see that  $\|\eta^{(k+1)}\|_q \leq C\lambda$  and hence  $E_{k+1} \leq C\lambda \leq C\sqrt{\lambda^q}$ .

Otherwise, we let  $t = \left\| \eta_{S_0^c}^{(k+1)} \right\|_q^q$ ,  $a = \bar{\alpha} \frac{\gamma^q}{(1-\gamma)^q} \left\| \eta^{(k)} \right\|_q^{q(2-q)}$ , which can be bounded independently of  $k$  and  $b = \bar{\alpha}(C\lambda)^q \left\| \eta^{(k)} \right\|_q^{q(2-q)}$ . Then (2.32) becomes  $t^2 - at - b \leq 0$ . It follows that

$$t \leq \frac{1}{2} \left( a + \sqrt{a^2 + 4b} \right) \leq \frac{1}{2} (2a + 2\sqrt{b}).$$

More precisely, we have

$$\left\| \eta_{S_0^c}^{(k+1)} \right\|_q^q \leq \bar{\alpha} \frac{\gamma^q}{(1-\gamma)^q} \left\| \eta^{(k)} \right\|_q^{q(2-q)} + C\sqrt{\lambda^q}$$

for another positive constant  $C$  since  $\|\eta^{(k)}\|_q$  is bounded. By Lemma 2.11, letting  $\theta = \gamma/(1-\gamma)$ ,

$$\left\| \eta^{(k+1)} \right\|_q^q \leq (1 + \theta^q) \left\| \eta_{S_0^c}^{(k+1)} \right\|_q^q + C s^{1-q/2} \lambda^q \leq \frac{\bar{\alpha} \gamma^q (1 + \theta^q)}{(1-\gamma)^q} \left\| \eta^{(k)} \right\|_q^{q(2-q)} + C\sqrt{\lambda^q}$$

with another constant  $C$  in the last inequality which is dependent on  $s$ , where  $\lambda \in (0, 1)$  and  $q \leq 1$ . Letting  $\mu = \frac{\bar{\alpha} \gamma^q (1 + \theta^q)}{(1-\gamma)^q}$ , we finally obtain (2.27) and thus establish the desired inequality.  $\square$

**3. Unconstrained  $\ell_q$  minimization for low-rank matrix recovery.** In this section, we extend the study in the previous section to the low-rank matrix recovery problem (1.3), which has the objective

$$\mathcal{L}_q(X, \epsilon, \lambda) = \text{tr} \left( (X^\top X + \epsilon^2 I)^{q/2} \right) + \frac{1}{2\lambda} \|\mathcal{A}(X) - \mathbf{b}\|_2^2,$$

where  $\mathbf{b}$  is an observation vector with or without noise and  $\mathcal{A}$  is a linear operator from  $\mathbb{R}^{m \times n}$  to  $\mathbb{R}^p$ . Without loss of generality, we assume  $m \geq n$  in this section. The first-order optimality condition of problem (1.3) is  $\nabla_X \mathcal{L}_q(X, \epsilon, \lambda) = 0$ , i.e.,

$$(3.1) \quad qX(X^\top X + \epsilon^2 I)^{q/2-1} + \frac{1}{\lambda} \mathcal{A}^*(\mathcal{A}(X) - \mathbf{b}) = 0,$$

where  $\mathcal{A}^*$  is the adjoint operator of  $\mathcal{A}$ . Similar to the vector case, we approximately solve a sequence of the nonlinear system (3.1) corresponding to a sequence of  $\epsilon$ 's and our algorithm is summarized as follows.

---

ALGORITHM 3.1 (iterative reweighted unconstrained  $\ell_q$  for low-rank matrix recovery (IRucLq-M)).

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**Input:** vector  $\mathbf{b}$ , linear operator  $\mathcal{A}$  and estimated rank  $K$ ;

**Output:** matrix  $X \in \mathbb{R}^{m \times n}$ .

Choose appropriate parameters  $\lambda > 0, q \in (0, 1]$ .

Initialize  $X^{(0)}$  such that  $\mathcal{A}(X^{(0)}) = \mathbf{b}$  and  $\epsilon = \epsilon_0 > 0$ .

**For**  $k = 0, 1, 2, \dots$

Let  $W^{(k)} = ((X^{(k)})^\top X^{(k)} + \epsilon_k^2 I)^{q/2-1}$

Solve the following system for  $X^{(k+1)}$

$$(3.2) \quad \lambda q X^{(k+1)} W^{(k)} + \mathcal{A}^* \left( \mathcal{A}(X^{(k+1)}) \right) = \mathcal{A}^*(\mathbf{b});$$

Update  $\epsilon_{k+1}$  by  $\epsilon_{k+1} = \min \{ \epsilon_k, \alpha \cdot \sigma_{K+1}(X^{(k+1)}) \}$ , where  $\alpha \in (0, 1)$  is a constant.

**End For**

---

In Algorithm 3.1,  $\sigma_{K+1}(X^{(k+1)})$  is the  $(K+1)$ th largest singular value of  $X^{(k+1)}$  and  $W^{(k)}$  is obtained by decomposing  $(X^{(k)})^\top X^{(k)} = V^{(k)}(\Sigma^{(k)})^2(V^{(k)})^\top$  in the SVD format and letting  $W^{(k)} = V^{(k)}((\Sigma^{(k)})^2 + \epsilon_k^2 I)^{q/2-1}(V^{(k)})^\top$ , ~~where  $\Sigma^{(k)}$  is a diagonal matrix with the singular values of  $X^{(k)}$  on its diagonal.~~ If  $\epsilon_{k+1} = 0$ , we return  $X^{(k+1)}$  and terminate the algorithm. Otherwise, we stop the computation in a **reasonable** time and return the last  $X^{(k+1)}$ .

**3.1. Convergence analysis of Algorithm 3.1.** Since the sequence  $\{\epsilon_k\}$  is nonincreasing and lower bounded by zero, it must converge to some  $\epsilon_* \geq 0$ . In this subsection, we show that the sequence  $\{X^{(k)}\}$  is bounded and hence  $\{X^{(k)}\}$  has at least one convergent subsequence. In addition, when  $\epsilon_* > 0$ , the limit of any convergent subsequence is a critical point satisfying the first-order optimality condition (3.1) with  $\epsilon = \epsilon_*$ , and when  $\epsilon_* = 0$ , there exists a convergent subsequence whose limit is a  $K$ -rank matrix. The following theorem summarizes the main convergence result.

**THEOREM 3.1.** *Suppose that the sequence  $\{X^{(k)}\}$  is generated by Algorithm 3.1 with  $0 < q \leq 1$  and  $\epsilon_k \rightarrow \epsilon_*$  as  $k \rightarrow \infty$ . Then  $\{X^{(k)}\}$  has at least one convergent subsequence. If  $\epsilon_* > 0$ , the limit of any convergent subsequence of  $\{X^{(k)}\}$  satisfies the first-order optimality condition (3.1) with  $\epsilon = \epsilon_*$ . If  $\epsilon_* = 0$ , there exists at least one subsequence of  $\{X^{(k)}\}$  converging to a  $K$ -rank matrix  $X^{0,\lambda}$ .*



*Remark 3.1.* Remark 2.2 also applies here, i.e., the assumption of finite critical points implies the convergence of Algorithm 3.1.

Before proving this theorem, let us recall the following reversed version of **the von Neumann** trace inequality.

**LEMMA 3.2.** *Suppose that  $A$  and  $B$  are  $n \times n$  symmetric positive semidefinite matrices. Let  $\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_n(A) \geq 0$  and  $\sigma_1(B) \geq \sigma_2(B) \geq \dots \geq \sigma_n(B) \geq 0$  be singular values of  $A$  and  $B$ , respectively. Then*

$$\operatorname{tr}(AB) \geq \sum_{i=1}^n \sigma_i(A) \sigma_{n-i+1}(B).$$

The proof of this lemma can be found on p. 249 of [25]. Lemma 3.2 will be used a few times in this section. By Lemma 3.2, we can prove the following lemma, which is an extension of Lemma 2.3 in the matrix setting. It is standard in matrix analysis that for a symmetric positive semidefinite matrix  $A$  with eigen-decomposition  $UDU^\top$ , we define  $A^\tau := UD^\tau U^\top$ , where  $\tau \in \mathbb{R}$ .

**LEMMA 3.3.** *Let  $0 \leq c \leq 1/2$ ,  $\epsilon > 0$ , and  $0 < q < 2$ . The inequalities*

$$\begin{aligned} & \operatorname{tr} \left( (\epsilon I + X^\top X)^{\frac{q}{2}} - (\epsilon I + Y^\top Y)^{\frac{q}{2}} - q(X - Y)^\top Y (\epsilon I + X^\top X)^{\frac{q}{2}-1} \right) \\ (3.3) \quad & \geq \operatorname{tr} \left( cq(X - Y)^\top (X - Y) (\epsilon I + X^\top X)^{\frac{q}{2}-1} \right) \end{aligned}$$

$$(3.4) \quad \geq 0$$

hold for any matrices  $X, Y \in \mathbb{R}^{m \times n}$ . Furthermore, we have

- (i) if  $c < 1/2$ , inequalities (3.3) and (3.4) hold with equality if and only if  $X = Y$ ;
- (ii) if  $c = 1/2$ , inequality (3.3) holds with equality if and only if  $X$  and  $Y$  have the same singular value sequence and right singular vectors.

*Proof.* For convenience, let  $[X] := \epsilon I + X^\top X$  and  $[Y] := \epsilon I + Y^\top Y$ . In addition, let  $[x]_i = \epsilon + \sigma_i^2(X)$  and  $[y]_i = \epsilon + \sigma_i^2(Y)$ . Since  $\operatorname{tr} \left( q(X - Y)^\top (X - Y) [X]^{\frac{q}{2}-1} \right) \geq 0$ , if the result is proved for  $c = 1/2$ , it holds for all  $c \leq 1/2$ . Therefore, we let  $c = 1/2$  and derive

$$(3.5)$$

$$\begin{aligned} & \operatorname{tr} \left( [X]^{\frac{q}{2}} - [Y]^{\frac{q}{2}} - q(X - Y)^\top Y [X]^{\frac{q}{2}-1} \right) - \operatorname{tr} \left( \frac{q}{2} (X - Y)^\top (X - Y) [X]^{\frac{q}{2}-1} \right) \\ & = \operatorname{tr} \left( [X]^{\frac{q}{2}} - [Y]^{\frac{q}{2}} - \frac{q}{2} X^\top X [X]^{\frac{q}{2}-1} + \frac{q}{2} Y^\top Y [X]^{\frac{q}{2}-1} \right) \\ & \geq \sum_{i=1}^n \left( [x]_i^{\frac{q}{2}} - [y]_i^{\frac{q}{2}} - \frac{q}{2} \sigma_i(X)^2 [x]_i^{\frac{q}{2}-1} + \frac{q}{2} \sigma_i(Y)^2 [x]_i^{\frac{q}{2}-1} \right) \\ & = \sum_{i=1}^n \left( [x]_i^{\frac{q}{2}} - [y]_i^{\frac{q}{2}} - q(\sigma_i(X) - \sigma_i(Y)) \sigma_i(Y) [x]_i^{\frac{q}{2}-1} - \frac{q}{2} (\sigma_i(X) - \sigma_i(Y))^2 [x]_i^{\frac{q}{2}-1} \right), \end{aligned}$$

where the last inequality follows from

1.  $\operatorname{tr}([X]^{\frac{q}{2}}) = \sum_i \sigma_i([X]^{\frac{q}{2}}) = \sum_i [x]_i^{\frac{q}{2}}$ ,  $\operatorname{tr}([Y]^{\frac{q}{2}}) = \sum_i \sigma_i([Y]^{\frac{q}{2}}) = \sum_i [y]_i^{\frac{q}{2}}$ ,
2.  $\operatorname{tr}(X^\top X [X]^{\frac{q}{2}-1}) = \sum_i \sigma_i(X)^2 [x]_i^{\frac{q}{2}-1}$ ,
3.  $\operatorname{tr}(Y^\top Y [X]^{\frac{q}{2}-1}) \geq \sum_i \sigma_i(Y^\top Y) \sigma_{n-i+1}([X]^{\frac{q}{2}-1}) = \sum_i \sigma_i(Y)^2 [x]_i^{\frac{q}{2}-1}$  by Lemma 3.2.

Thus, the results in (3.3) and (3.4) now follow by using Lemma 2.3 in the previous section for each term in the right-hand side of (3.5), specifically, with  $x = \sigma_i(X)$  and  $y = \sigma_i(Y)$  in (2.8). This inequality holds with equality if and only if  $Y^\top Y$  and  $[X]$  have the same set of eigenvectors.  $\square$

Using Lemma 3.3, we can prove the next lemma.

LEMMA 3.4. *Let the sequence  $\{X^{(k)}\}$  be generated from Algorithm 3.1 with  $0 < q \leq 1$ . Then there exists a constant  $c > 0$  such that for  $k \geq 1$ ,*

$$(3.6) \quad \begin{aligned} & \frac{1}{2\lambda} \left\| \mathcal{A}(X^{(k+1)} - X^{(k)}) \right\|_2^2 + qc \left\| X^{(k+1)} - X^{(k)} \right\|_F^2 \\ & \leq \mathcal{L}_q(X^{(k)}, \epsilon_k, \lambda) - \mathcal{L}_q(X^{(k+1)}, \epsilon_{k+1}, \lambda). \end{aligned}$$

*Proof.* The proof of this lemma is essentially the same as that of Lemma 2.4. However, several steps need to be carefully modified for the matrix setting. Similar to Lemma 2.4, we have

$$\begin{aligned} & \mathcal{L}_q(X^{(k)}, \epsilon_k, \lambda) - \mathcal{L}_q(X^{(k+1)}, \epsilon_{k+1}, \lambda) \\ & = \text{tr} \left( \left( (X^{(k)})^\top X^{(k)} + \epsilon_k^2 I \right)^{\frac{q}{2}} \right) - \text{tr} \left( \left( (X^{(k+1)})^\top X^{(k+1)} + \epsilon_k^2 I \right)^{\frac{q}{2}} \right) \\ & \quad + \frac{1}{2\lambda} \left( \left\| \mathcal{A}(X^{(k)}) - \mathbf{b} \right\|_2^2 - \left\| \mathcal{A}(X^{(k+1)}) - \mathbf{b} \right\|_2^2 \right) \\ & = \text{tr} \left( \left( (X^{(k)})^\top X^{(k)} + \epsilon_k^2 I \right)^{\frac{q}{2}} \right) - \text{tr} \left( \left( (X^{(k+1)})^\top X^{(k+1)} + \epsilon_k^2 I \right)^{\frac{q}{2}} \right) \\ & \quad + \frac{1}{2\lambda} \left\| \mathcal{A}(X^{(k)} - X^{(k+1)}) \right\|_2^2 \\ & \quad - q \text{tr} \left( \left( X^{(k)} - X^{(k+1)} \right)^\top X^{(k+1)} \left( (X^{(k)})^\top X^{(k)} + \epsilon_k^2 I \right)^{\frac{q}{2}-1} \right), \end{aligned}$$

where we have used (3.2) in the last equality.

Now we can use Lemma 3.3 to get

$$\begin{aligned} & \mathcal{L}_q(X^{(k)}, \epsilon_k, \lambda) - \mathcal{L}_q(X^{(k+1)}, \epsilon_{k+1}, \lambda) \\ & \geq \frac{1}{2\lambda} \left\| \mathcal{A}(X^{(k)} - X^{(k+1)}) \right\|_2^2 \\ & \quad + \frac{q}{2} \text{tr} \left( \left( X^{(k)} - X^{(k+1)} \right)^\top \left( X^{(k)} - X^{(k+1)} \right) \left( (X^{(k)})^\top X^{(k)} + \epsilon_k^2 I \right)^{\frac{q}{2}-1} \right) \\ & \geq \frac{1}{2\lambda} \left\| \mathcal{A}(X^{(k)} - X^{(k+1)}) \right\|_2^2 \\ & \quad + \frac{q}{2} \sum_{i=1}^n \sigma_i \left( \left( X^{(k)} - X^{(k+1)} \right)^\top \left( X^{(k)} - X^{(k+1)} \right) \right) \left( \sigma_i(X^{(k)})^2 + \epsilon_k^2 \right)^{\frac{q}{2}-1} \\ & \geq \frac{1}{2\lambda} \left\| \mathcal{A}(X^{(k)} - X^{(k+1)}) \right\|_2^2 + qc \sum_{i=1}^n \sigma_i \left( \left( X^{(k)} - X^{(k+1)} \right)^\top \left( X^{(k)} - X^{(k+1)} \right) \right) \\ & = \frac{1}{2\lambda} \left\| \mathcal{A}(X^{(k)} - X^{(k+1)}) \right\|_2^2 + qc \left\| X^{(k+1)} - X^{(k)} \right\|_F^2. \end{aligned}$$

Here, the second inequality follows from Lemma 3.2, and it implies

$$\mathcal{L}_q(X^{(k+1)}, \epsilon_{k+1}, \lambda) \leq \mathcal{L}_q(X^{(k)}, \epsilon_k, \lambda).$$

Hence,

$$(3.7) \quad (\sigma_i(X^{(k)})^2 + \epsilon_k^2)^{\frac{q}{2}} \leq \text{tr} \left( ((X^{(k)})^\top X^{(k)} + \epsilon_k^2 I)^{\frac{q}{2}} \right) \leq \mathcal{L}_q(X^{(k)}, \epsilon_k, \lambda) \leq \dots \leq \mathcal{L}_q(X^{(0)}, \epsilon_0, \lambda)$$

holds for any  $i = 1, \dots, n$ . Namely,  $\sigma_i(X^{(k)})^2 + \epsilon_k^2$  is upper bounded so  $(\sigma_i(X^{(k)})^2 + \epsilon_k^2)^{q/2-1}$  is lower bounded by a positive constant  $c$ , from which the third inequality follows. This completes the proof.  $\square$

Now, we are ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* The inequalities in (3.7) imply that the sequence  $\{X^{(k)}\}$  is bounded. Hence, there is a subsequence  $\{X^{(k_j)}\}$  converging to some point  $\hat{X}$ . If  $\epsilon_* > 0$ , the weighting matrix  $W^{(k_j)}$  is always well defined since  $\epsilon_{k_j} \geq \epsilon_*$ . Note that  $X^{(k_j+1)}$  solves (3.2) with  $W = W^{(k_j)}$ , i.e.,

$$(3.8) \quad \lambda q X^{(k_j+1)} W^{(k_j)} + \mathcal{A}^*(\mathcal{A}(X^{(k_j+1)})) = \mathcal{A}^*(\mathbf{b}).$$

Then  $\{X^{(k_j+1)}\}$  must converge to some  $\bar{X}$ . Now it follows from (3.6) that  $\|X^{(k)} - X^{(k+1)}\|_F \rightarrow 0$ , and thus

$$\|\hat{X} - \bar{X}\|_F = \lim_{j \rightarrow \infty} \|X^{(k_j)} - X^{(k_j+1)}\|_F \rightarrow 0.$$

Therefore,  $\hat{X} = \bar{X}$ . Letting  $j \rightarrow \infty$  in (3.8) gives

$$\lambda q \hat{X} \hat{W} + \mathcal{A}^*(\mathcal{A}(\hat{X})) = \mathcal{A}^*(\mathbf{b})$$

with  $\hat{W} = (\hat{X}^\top \hat{X} + \epsilon_*^2 I)^{q/2-1}$ , which implies that  $\hat{X}$  satisfies (3.1) with  $\epsilon = \epsilon_*$ .

If  $\epsilon_* = 0$ , then from the updating rule of  $\epsilon_k$ , it must hold that  $\epsilon_k = 0$  for some  $k$  or  $\epsilon_k = \alpha \cdot \sigma_{K+1}(X^{(m_k)})$  for some integer  $m_k \leq k$  when  $k$  is sufficiently large. In the first case, we have  $\sigma_{K+1}(X^{(k)}) = 0$ . Thus  $X^{(k)}$  is a  $K$ -rank matrix, and we let  $X^{0,\lambda} = X^{(k)}$ . In the second case, we let  $X^{0,\lambda}$  be the limit of a convergent subsequence of  $\{X^{(m_k)}\}$ . Without causing confusion, we still denote the subsequence as  $\{X^{(m_k)}\}$ . Then we have  $\sigma_{K+1}(X^{0,\lambda}) = \lim_{k \rightarrow \infty} \sigma_{K+1}(X^{(m_k)}) = \lim_{k \rightarrow \infty} \frac{\epsilon_k}{\alpha} = \frac{\epsilon_*}{\alpha} = 0$ . Thus, in both cases,  $X^{0,\lambda}$  is a  $K$ -rank matrix, and this completes the proof.  $\square$

**3.2. Error analysis of Algorithm 3.1.** Under the matrix-RIP assumption, this subsection gives an error analysis of Algorithm 3.1.

**DEFINITION 3.5 (matrix-RIP).** For integer  $r = 1, 2, \dots$ , the matrix-RIP constant  $\delta_r$  of  $\mathcal{A}$  is the smallest number such that

$$(3.9) \quad (1 - \delta_r) \|X\|_F^2 \leq \|\mathcal{A}(X)\|_2^2 \leq (1 + \delta_r) \|X\|_F^2$$

holds for all  $r$ -rank matrices  $X$ . For simplicity, we say that  $\mathcal{A}$  satisfies the matrix-RIP of order  $r$  with constant  $\delta_r$ .

Let  $e_r(X)$  be the error term of the best  $r$ -rank approximation of  $X$  in the Frobenius norm, i.e.,

$$(3.10) \quad e_r(X) = \min_{\text{rank}(Z) \leq r} \|X - Z\|_F.$$

Similarly,  $\rho_r(X)$  is defined as the error term of the best  $r$ -rank approximation of  $X$  in nuclear norm, i.e.,

$$(3.11) \quad \rho_r(X) = \min_{\text{rank}(Z) \leq r} \|X - Z\|_*.$$

In what follows, we assume that the underlying true matrix  $X^o$  satisfies  $\mathcal{A}(X^o) = \mathbf{b}$ . In general, the vector  $\mathbf{b}$  can be contaminated by noise. Therefore,  $X^o$  may not satisfy  $\mathcal{A}(X^o) = \mathbf{b}$  any more. Hence, our analysis always assumes that  $\mathbf{b}$  contains no noise. However, it is not difficult to extend our results to the noisy case. In the remaining part of this section, we analyze the recoverability of Algorithm 3.1. Theorem 3.6 considers the case of  $0 < q \leq 1$  and Theorem 3.7 focuses on the case of  $q = 1$ .

**THEOREM 3.6.** *Suppose that  $X^o$  is a  $K$ -rank matrix satisfying  $\mathcal{A}(X^o) = \mathbf{b}$ . Assume that  $\mathcal{A}$  satisfies the matrix-RIP of order  $2K < \min\{m, n\}$  with  $\delta_{2K} < 1$ . Moreover, assume that  $\epsilon_*$  is the limit of sequence  $\{\epsilon_k\}$ . If  $\epsilon_* > 0$ , then any limit point  $X^*$  of the sequence  $\{X^{(k)}\}$  generated by Algorithm 3.1 with  $0 < q \leq 1$  satisfies*

$$(3.12) \quad \|X^* - X^o\|_F \leq C_1 \sqrt{\lambda} + C_2 e_K(X^*).$$

When  $\epsilon_* = 0$ , there exists at least one subsequence converging to a  $K$ -rank matrix  $X^*$  which satisfies

$$(3.13) \quad \|X^* - X^o\|_F \leq C_1 \sqrt{\lambda},$$

where  $C_1, C_2$  are two positive constants and  $e_K(X^*)$  is defined in (3.10).

*Proof.* Recall that we have proved that any limit point  $X^*$  of  $\{X^{(k)}\}$  is a critical point when  $\epsilon_* > 0$ , and when  $\epsilon_* = 0$ , there exists at least one subsequence  $\{X^{(k_j)}\}$  converging to a  $K$ -rank matrix  $X^*$ . Hence, we only need to prove inequalities (3.12) and (3.13) to complete the proof.

Let us consider the case of  $\epsilon_* = 0$  first. Note  $\text{rank}(X^* - X^o) \leq \text{rank}(X^*) + \text{rank}(X^o) \leq 2K$ . Since  $\mathcal{A}$  satisfies the matrix-RIP of order  $2K$  with  $\delta_{2K} < 1$ , we have

$$\begin{aligned} \|X^* - X^o\|_F &\leq \frac{1}{\sqrt{1 - \delta_{2K}}} \|\mathcal{A}(X^* - X^o)\|_2 \\ &= \frac{1}{\sqrt{1 - \delta_{2K}}} \lim_{j \rightarrow \infty} \left\| \mathcal{A}(X^{(k_j)}) - \mathbf{b} \right\|_2 \\ &\leq \frac{1}{\sqrt{1 - \delta_{2K}}} \lim_{j \rightarrow \infty} (2\lambda \mathcal{L}_q(X^{(k_j)}, \epsilon_{k_j}, \lambda))^{1/2} \\ &\leq \frac{1}{\sqrt{1 - \delta_{2K}}} (2\lambda \mathcal{L}_q(X^{(0)}, \epsilon_0, \lambda))^{1/2}, \end{aligned}$$

where we have used the monotonicity of the sequence  $\{\mathcal{L}_q(X^{(k)}, \epsilon_k, \lambda)\}$  in the last inequality. Note that  $\mathcal{L}_q(X^{(0)}, \epsilon_0, \lambda)$  is independent of  $\lambda$  since  $\mathcal{A}(X^{(0)}) = \mathbf{b}$ . Therefore, in the case of  $\epsilon_* = 0$ , the limit  $X^*$  is close to the low-rank solution  $X^o$  within an error proportional to  $\sqrt{\lambda}$ .

Next we consider the case of  $\epsilon_* > 0$ . Again, by the monotonicity of  $\mathcal{L}_q(X^{(k)}, \epsilon_k, \lambda)$ , we have

$$\mathcal{L}_q(X^*, \epsilon_*, \lambda) = \lim_{j \rightarrow \infty} \mathcal{L}_q(X^{(k_j)}, \epsilon_{k_j}, \lambda) \leq \mathcal{L}_q(X^{(0)}, \epsilon_0, \lambda),$$

from which it follows that

$$(3.14) \quad \|\mathcal{A}(X^*) - \mathbf{b}\|_2 \leq \sqrt{2\lambda \mathcal{L}_q(X^*, \epsilon_*, \lambda)} \leq \sqrt{2\lambda \mathcal{L}_q(X^{(0)}, \epsilon_0, \lambda)}.$$

Write  $X^*$  in full-SVD format:  $X^* = U_K \Sigma_K V_K^\top + \tilde{U}_K \tilde{\Sigma}_K \tilde{V}_K^\top$ . Here  $U_K$  and  $V_K$  consist of the  $K$  left and right singular vectors of  $X^*$  corresponding to the first  $K$

largest singular values of  $X^*$ , respectively, and  $\tilde{U}_K$  and  $\tilde{V}_K$  consist of another  $m - K$  left singular vectors and another  $n - K$  right singular vectors of  $X^*$ , respectively. In addition, write  $X^o$  in economy-SVD format  $X^o = U_{\mathbf{b}} \Sigma_{\mathbf{b}} V_{\mathbf{b}}^\top$ . If necessary, we add more orthogonal columns to make  $U_{\mathbf{b}} \in \mathbb{R}^{m \times K}$  and  $V_{\mathbf{b}} \in \mathbb{R}^{n \times K}$ . Since  $2K < \min\{m, n\}$ , we can always find a column orthogonal matrix  $U \in \mathbb{R}^{m \times (n-2K)}$  such that  $U^\top U_K = 0$  and  $U^\top U_{\mathbf{b}} = 0$ . Let  $\mathcal{P}_{2K} = I - UU^\top$  be a projection and its complementary projection  $\mathcal{P}_{2K}^c = UU^\top$ . Noting that  $\text{rank}(\mathcal{P}_{2K}(X)) \leq 2K$  for any  $X \in \mathbb{R}^{m \times n}$ , we have

$$\begin{aligned}
\|X^* - X^o\|_F &\leq \|\mathcal{P}_{2K}(X^* - X^o)\|_F + \|\mathcal{P}_{2K}^c(X^* - X^o)\|_F \\
&\leq \frac{1}{\sqrt{1 - \delta_{2K}}} \|\mathcal{A}(\mathcal{P}_{2K}(X^* - X^o))\|_2 + \|\mathcal{P}_{2K}^c(X^* - X^o)\|_F \\
&= \frac{1}{\sqrt{1 - \delta_{2K}}} \|\mathcal{A}(X^* - X^o) - \mathcal{A}(\mathcal{P}_{2K}^c(X^* - X^o))\|_2 + \|\mathcal{P}_{2K}^c(X^* - X^o)\|_F \\
&\leq \frac{1}{\sqrt{1 - \delta_{2K}}} \|\mathcal{A}(X^*) - \mathbf{b}\|_2 + \left( \frac{1}{\sqrt{1 - \delta_{2K}}} \|\mathcal{A}\| + 1 \right) \|\mathcal{P}_{2K}^c(X^* - X^o)\|_F \\
&= \frac{1}{\sqrt{1 - \delta_{2K}}} \|\mathcal{A}(X^*) - \mathbf{b}\|_2 + \left( \frac{1}{\sqrt{1 - \delta_{2K}}} \|\mathcal{A}\| + 1 \right) \left\| U^\top \tilde{U}_K \tilde{\Sigma}_K \right\|_F \\
&\leq \frac{1}{\sqrt{1 - \delta_{2K}}} \|\mathcal{A}(X^*) - \mathbf{b}\|_2 + \left( \frac{1}{\sqrt{1 - \delta_{2K}}} \|\mathcal{A}\| + 1 \right) e_K(X^*),
\end{aligned}$$

where we have used the fact that  $\|AB\|_F \leq \|A\|_2 \|B\|_F$  for any matrices  $A, B$  of appropriate size in the last inequality, and  $\|\mathcal{A}\|$  is the norm of operator. Applying (3.14) to the last inequality completes the proof.  $\square$

**Case of  $q = 1$ .** When  $q = 1$  and  $\epsilon_* > 0$ , the function  $\mathcal{L}_1(X, \epsilon_*, \lambda)$  is strictly convex with respect to  $X$ . Hence, the critical point  $X^*$  is the unique minimizer. In this case, we are able to get a stronger result described as follows.

**THEOREM 3.7.** *Suppose that  $X^o$  is  $K$ -rank and  $\mathcal{A}(X^o) = \mathbf{b}$ . Assume that  $\{X^{(k)}\}$  is a sequence generated by Algorithm 3.1 with  $q = 1$  and  $\alpha \leq 1/n$ , and assume that the limit of  $\{\epsilon_k\}$  is  $\epsilon_* > 0$ . Suppose that  $\mathcal{A}$  satisfies the matrix-RIP of order  $3K < \min\{m, n\}$  such that  $\gamma = \frac{\delta_{3K}}{1 - \delta_{2K}} < 1$ . If  $\nu = \frac{1 + \gamma}{(1 - \gamma)(K - t + 1)} < 1$  for some integer  $t < K$ , then  $\{X^{(k)}\}$  converges to the unique minimizer  $X^*$  of (1.3) with  $\epsilon = \epsilon_*$ , which satisfies*

$$(3.15) \quad \|X^o - X^*\|_* \leq C_1 \sqrt{K} \lambda + C_2 \rho_t(X^o),$$

where  $C_1, C_2$  are two positive constants and  $\rho_t(X^o)$  is defined in (3.11).

**Remark 3.2.** According to (3.15), if  $\text{rank}(X^o) \leq t < K$ , then  $X^*$  differs from the low-rank solution  $X^o$  by an amount proportional to  $\lambda$ . In addition, if  $\lambda = 0$ , which corresponds to the constrained problem with constraint  $\mathcal{A}(X) = \mathbf{b}$ , then Algorithm 3.1 will successfully recover the low-rank matrix  $X^o$ .

To prove the theorem, we first show some lemmas. Denote  $Y = X^* - X^o$ . Let  $U_{\mathbf{b}} \Sigma_{\mathbf{b}} V_{\mathbf{b}}^\top$  be the best rank- $K$  approximation of  $X^o$  with  $U_{\mathbf{b}}$  and  $V_{\mathbf{b}}$  containing the left and right  $K$  singular vectors of  $X^o$  corresponding to the first  $K$  largest singular values, respectively, and  $\Sigma_{\mathbf{b}}$  being a diagonal matrix with the first  $K$  largest singular values on its diagonal. Define a projection  $\mathcal{P}_K = U_{\mathbf{b}} U_{\mathbf{b}}^\top$  and its complementary projection  $\mathcal{P}_K^c = I - \mathcal{P}_K$ . Split  $Y$  as  $Y = \mathcal{P}_K(Y) + \mathcal{P}_K^c(Y)$ . Suppose the SVD of  $\mathcal{P}_K^c(Y)$  is  $\mathcal{P}_K^c(Y) = \tilde{U} \tilde{\Sigma} \tilde{V}^\top$ . We further split  $\mathcal{P}_K^c(Y) = \sum_{i \geq 1} Y_i$  with  $Y_i = \hat{U}_i \hat{\Sigma}_i \hat{V}_i^\top$ , where  $\hat{U}_1$  consists of the  $K$  columns of  $\tilde{U}$  corresponding to the first  $K$  largest singular values,  $\hat{U}_2$  the next  $K$  columns of  $\tilde{U}$  corresponding to the next  $K$  largest singular values,

and so on.  $\widehat{V}_1, \widehat{V}_2, \dots$  and  $\widehat{\Sigma}_1, \widehat{\Sigma}_2, \dots$  are obtained accordingly. For convenience, let  $\mathcal{P} = \mathcal{P}_K$ ,  $\mathcal{P}^c = \mathcal{P}_K^c$ , and  $\widehat{Y} = \mathcal{P}(Y) + Y_1$  in the following analysis.

LEMMA 3.8. *Suppose that  $\mathcal{A}$  satisfies the matrix-RIP of order  $2K < \min\{m, n\}$  with  $\delta_{2K} < 1$ . Let  $\gamma = \frac{\delta_{3K}}{1-\delta_{2K}}$  and  $C = \frac{\sqrt{n}}{1-\delta_{2K}}$ . Then*

$$(3.16) \quad \left\| \widehat{Y} \right\|_F \leq \gamma \sum_{i \geq 2} \|Y_i\|_F + C\lambda.$$

*Proof.* Note  $\left\| \widehat{Y} \right\|_F^2 = \langle \widehat{Y}, Y \rangle$ . Hence,

$$(3.17) \quad \begin{aligned} \left\| \widehat{Y} \right\|_F^2 &= \left\langle \widehat{Y}, X^* - X^o - \lambda X^* \left( (X^*)^\top X^* + \epsilon_*^2 I \right)^{-1/2} - \mathcal{A}^* \mathcal{A} (X^* - X^o) \right\rangle \\ &= \left\langle \widehat{Y}, (I - \mathcal{A}^* \mathcal{A})(Y) \right\rangle - \lambda \left\langle \widehat{Y}, X^* \left( (X^*)^\top X^* + \epsilon_*^2 I \right)^{-1/2} \right\rangle, \end{aligned}$$

where we have used the fact that  $X^*$  is a critical point of  $\mathcal{L}_1(X, \epsilon_*, \lambda)$  in the first equality.

For the first term in (3.17), we have

$$\begin{aligned} \left\langle \widehat{Y}, (I - \mathcal{A}^* \mathcal{A})(Y) \right\rangle &= \left\langle (I - \mathcal{A}^* \mathcal{A})^*(\widehat{Y}), \widehat{Y} + \sum_{i \geq 2} Y_i \right\rangle \\ &= \left\langle (I - \mathcal{A}^* \mathcal{A})^*(\widehat{Y}), \widehat{Y} \right\rangle + \sum_{i \geq 2} \left\langle (I - \mathcal{A}^* \mathcal{A})^*(\widehat{Y}), Y_i \right\rangle \\ &\leq \delta_{2K} \left\| \widehat{Y} \right\|_F^2 + \sum_{i \geq 2} \delta_{3K} \left\| \widehat{Y} \right\|_F \|Y_i\|_F, \end{aligned}$$

where we have used an alternative definition of matrix-RIP of  $\mathcal{A}$ . This definition is similar to the vector case (2.5) and can be proved equivalent to (3.9) in essentially the same way as in [14] by noting that the operator  $I - \mathcal{A}^* \mathcal{A}$  is self-adjoint. We leave the details to the interested reader.

For the second term in (3.17), note that  $\left\| X^* \left( (X^*)^\top X^* + \epsilon_*^2 I \right)^{-1/2} \right\|_F \leq \sqrt{n}$  by straightforward calculations. Hence,

$$-\lambda \left\langle \widehat{Y}, X^* \left( (X^*)^\top X^* + \epsilon_*^2 I \right)^{-1/2} \right\rangle \leq \sqrt{n} \lambda \left\| \widehat{Y} \right\|_F.$$

Now, we can summarize the above discussions to have

$$\left\| \widehat{Y} \right\|_F^2 \leq \delta_{2K} \left\| \widehat{Y} \right\|_F^2 + \delta_{3K} \left\| \widehat{Y} \right\|_F \sum_{i \geq 2} \|Y_i\|_F + \sqrt{n} \lambda \left\| \widehat{Y} \right\|_F.$$

Dividing both sides in the above inequality by  $\left\| \widehat{Y} \right\|_F$  and rearranging the terms yield

(3.16).  $\square$

LEMMA 3.9. *Under the same assumptions as in Lemma 3.8, we have*

$$(3.18) \quad \left\| \mathcal{P}(Y) \right\|_* \leq \gamma \left\| \mathcal{P}^c(Y) \right\|_* + \sqrt{K} C \lambda,$$

where  $\gamma = \frac{\delta_{3K}}{1-\delta_{2K}}$  and  $C = \frac{\sqrt{n}}{1-\delta_{2K}}$ .

*Proof.* Note  $\|\mathcal{P}(Y)\|_*^2 \leq K \|\mathcal{P}(Y)\|_F^2 \leq K \|\widehat{Y}\|_F^2$  and  $\|Y_i\|_F \leq \|Y_{i-1}\|_* / \sqrt{K}$  for  $i \geq 2$ . Then from Lemma 3.8, we have

$$\|\mathcal{P}(Y)\|_* \leq \sqrt{K} \|\widehat{Y}\|_F \leq \sqrt{K} \left( \gamma \sum_{i \geq 2} \|Y_i\|_F + C\lambda \right) \leq \gamma \sum_{i \geq 1} \|Y_i\|_* + \sqrt{K}C\lambda,$$

which together with  $\sum_{i \geq 1} \|Y_i\|_* = \|\mathcal{P}^c(Y)\|_*$  completes the proof.  $\square$

LEMMA 3.10. *Suppose the assumptions in Lemma 3.8 hold and assume  $\gamma = \frac{\delta_{3K}}{1-\delta_{2K}} < 1$ . Then*

$$(3.19) \quad \|X^* - X^o\|_* \leq \frac{1+\gamma}{1-\gamma} (\|X^*\|_* - \|X^o\|_* + 2\rho_K(X^o)) + \frac{2\sqrt{K}}{1-\gamma}C\lambda,$$

where  $C = \frac{\sqrt{n}}{1-\delta_{2K}}$  and  $\rho_K(X^o)$  is defined in (3.11).

*Proof.* From Lemma 3.9, it follows that

$$(3.20) \quad \|Y\|_* \leq \|\mathcal{P}(Y)\|_* + \|\mathcal{P}^c(Y)\|_* \leq (1+\gamma) \|\mathcal{P}^c(Y)\|_* + \sqrt{K}C\lambda.$$

Moreover, we have

$$\begin{aligned} \|\mathcal{P}^c(Y)\|_* &\leq \|\mathcal{P}^c(X^*)\|_* + \|\mathcal{P}^c(X^o)\|_* \\ &= \|X^*\|_* - \|\mathcal{P}(X^*)\|_* + 2\|\mathcal{P}^c(X^o)\|_* - \|\mathcal{P}^c(X^o)\|_* \\ &\leq \|X^*\|_* - \|X^o\|_* + \|\mathcal{P}(X^o)\|_* - \|\mathcal{P}(X^*)\|_* + 2\|\mathcal{P}^c(X^o)\|_* \\ &\leq \|X^*\|_* - \|X^o\|_* + \|\mathcal{P}(X^o - X^*)\|_* + 2\|\mathcal{P}^c(X^o)\|_* \\ &\leq \|X^*\|_* - \|X^o\|_* + \gamma \|\mathcal{P}^c(Y)\|_* + \sqrt{K}C\lambda + 2\|\mathcal{P}^c(X^o)\|_*, \end{aligned}$$

where we have used (3.18) in the last inequality. Rearranging the terms in the last inequality yields

$$(3.21) \quad \|\mathcal{P}^c(Y)\|_* \leq \frac{1}{1-\gamma} \left( \|X^*\|_* - \|X^o\|_* + \sqrt{K}C\lambda + 2\|\mathcal{P}^c(X^o)\|_* \right).$$

Substituting (3.21) into (3.20) and rearranging the terms, we get the desired result.  $\square$

Now, we are ready to prove Theorem 3.7.

*Proof of Theorem 3.7.* Since  $\mathcal{L}_1(X, \epsilon_*, \lambda)$  is strictly convex with respect to  $X$ , Remark 3.1 implies that the whole sequence  $\{X^{(k)}\}$  converges to the unique minimizer  $X^*$ .

Following the proof of Theorem 2.5 in the beginning, we have

$$(3.22) \quad \|X^*\|_* \leq \mathcal{L}_1(X^*, \epsilon_*, \lambda) \leq \mathcal{L}_1(X^o, \epsilon_*, \lambda) \leq \|X^o\|_* + n\epsilon_* \leq \|X^o\|_* + \sigma_{K+1}(X^*),$$

which together with Lemma 3.10 indicates

$$(3.23) \quad \begin{aligned} \|X^* - X^o\|_* &\leq \frac{1+\gamma}{1-\gamma} (\|X^*\|_* - \|X^o\|_* + 2\rho_K(X^o)) + \frac{2\sqrt{K}}{1-\gamma}C\lambda \\ &\leq \frac{1+\gamma}{1-\gamma} \sigma_{K+1}(X^*) + \frac{2\sqrt{K}}{1-\gamma}C\lambda, \end{aligned}$$

where  $C = \frac{\sqrt{n}}{1-\delta_{2K}}$  and we have used the fact that  $X^o$  is  $K$ -rank. For  $t < K$ , we have

$$(3.24) \quad (K-t+1)\sigma_{K+1}(X^*) \leq \sum_{i=t+1}^{K+1} \sigma_i(X^*) \leq \rho_t(X^*) \leq \|X^* - X^o\|_* + \rho_t(X^o),$$

where we have used the following argument: letting  $X_{[t]}^o = U_{\mathbf{b},t}\Sigma_{\mathbf{b},t}V_{\mathbf{b},t}^\top$  with  $U_{\mathbf{b},t}$  being the first  $t$  singular vectors of  $U_{\mathbf{b}}$ ,  $\Sigma_{\mathbf{b},t}$  is the diagonal matrix with the first  $t$  singular values on its diagonal, and  $V_{\mathbf{b},t}$  the first  $t$  singular vectors of  $V_{\mathbf{b}}$ , then

$$\rho_t(X^*) \leq \|X^* - X_{[t]}^o\|_* \leq \|X^* - X^o\|_* + \|X^o - X_{[t]}^o\|_* = \|X^* - X^o\|_* + \rho_t(X^o).$$

Substituting (3.24) into (3.23) yields

$$\|X^* - X^o\|_* \leq \frac{1+\gamma}{(1-\gamma)(K-t+1)} (\|X^* - X^o\|_* + \rho_t(X^o)) + \frac{2\sqrt{K}}{1-\gamma} C\lambda.$$

Letting  $\nu = \frac{1+\gamma}{(1-\gamma)(K-t+1)}$ ,  $C_1 = \frac{2C}{(1-\gamma)(1-\nu)}$  and  $C_2 = \frac{\nu}{1-\nu}$ , we have

$$\|X^* - X^o\|_* \leq C_1\sqrt{K}\lambda + C_2\rho_t(X^o),$$

which completes the proof.  $\square$

In a similar way, we can extend the discussion of local convergence behavior at the end of the previous section to the matrix recovery setting. We leave it to the interested reader.

**4. Computational results.** In this section, we present several numerical experiments to illustrate the effectiveness of Algorithms 2.1 and 3.1. For Algorithm 2.1, we compare it with several state-of-the-art solvers on recovering sparse vectors. For Algorithm 3.1, we compare it with two matrix completion solvers, i.e., we take  $\mathcal{A} = \mathcal{P}_\Omega$  in (1.3). All our tests were performed on a Lenovo D20 Workstation with 40 GB of RAM and two Intel Xeon E5506 processors, each of them with four cores.

**4.1. Sparse vector recovery.** In Algorithm 2.1, for given  $q$  and  $\lambda$ , the step size  $\alpha$  and the sparsity estimate  $s$  are most important. Due to the nonconvexity of the problem (for  $q < 1$ ), too small  $\alpha$  may cause the algorithm to stagnate at local minima. In our tests, we took  $\alpha = 0.9$ , which worked well for the tested sparse vectors. We did not assume prior information of the true sparsity level and simply took  $s = \lfloor \frac{m}{2} \rfloor$ , where  $m$  is the row number of matrix  $A$ . For all the tests, we used zero vectors as the starting points as we determined that using a zero vector or a pseudoinverse solution of  $A\mathbf{x}^{(0)} = \mathbf{b}$  as an initial point for Algorithm 2.1 leads to a similar performance of sparse vector recovery.

**Choice of  $q$ .** First, we tested Algorithm 2.1 on recovering sparse vectors with  $q$  varying among  $\{0.1, 0.5, 0.7, 1\}$ . In this test,  $A$  was generated by MATLAB's command `randn(64, 256)`. The true vector  $\mathbf{x}^o$  had  $t$  nonzeros with each one entry generated according to the standard Gaussian distribution and  $t$  varying among  $\{8, 10, 12, \dots, 32\}$ . The location of nonzeros was uniformly randomly generated. The parameter  $\lambda$  was set to  $10^{-6}$  for all  $q$ 's. Although the best  $\lambda$  should be dependent on  $q$  in general, we considered the noiseless case, and  $\lambda = 10^{-6}$  is small enough to approximately enforce  $A\mathbf{x} = A\mathbf{x}^o$ . We let the algorithm run to 1,000 iterations. The recovery was regarded successful if  $\frac{\|\mathbf{x}^r - \mathbf{x}^o\|_2}{\|\mathbf{x}^o\|_2} \leq 10^{-3}$ , where  $\mathbf{x}^r$  stands for a recovered



vector. The left picture in Figure 4.1 shows the frequency of successful recovery using Algorithm 2.1 over 100 independent trials for various  $q$ 's and  $t$ 's. From the figure, we can see that Algorithm 2.1 with  $q = 0.1, 0.5$  performed better than  $q = 0.7$  and much better than  $q = 1$ . In addition,  $q = 0.5$  gave slightly higher success frequency than  $q = 0.1$ . We emphasize that our results do not counter the intuition that a smaller  $q$  should recover more sparse vectors. This is because a smaller  $q$  makes the minimizing functional more nonconvex and thus more difficult to solve. We found that if we decreased  $\epsilon$  more slowly, the performance of Algorithm 2.1 with  $q = 0.1$  could be further improved. However, the running time also became much longer.

**Comparison with other solvers.** Second, we compared Algorithm 2.1 (IRucLq-v) with three existing  $\ell_1$  solvers,  $\ell_1$  magic [5], reweighted  $\ell_1$  [7], and the homotopy method [1], and one  $\ell_q$  solver (Lq-FL) [15].  $\ell_1$  magic and reweighted  $\ell_1$  solve the constrained  $\ell_1$  minimization

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ subject to } A\mathbf{x} = A\mathbf{x}^\circ.$$

The homotopy method solves the unconstrained  $\ell_1$  minimization

$$(4.1) \quad \min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{1}{2\tau} \|A\mathbf{x} - A\mathbf{x}^\circ\|_2^2.$$

The  $\ell_q$  method in [15] solves the constrained  $\ell_q$  minimization

$$\min_{\mathbf{x}} \|\mathbf{x}\|_q^q \text{ subject to } A\mathbf{x} = A\mathbf{x}^\circ.$$

Note that the  $\ell_q$  method in [15] uses the output of  $\ell_1$  magic as the initial vector, and a sequence of  $q$ 's are used during the iterations to produce the sparsest solution. In this test,  $A$  had the size of  $50 \times 250$ , and each element was generated according to the Gaussian distribution  $\mathcal{N}(0, \frac{1}{50})$ . This kind of matrix was also tested in [10]. The true vector  $\mathbf{x}^\circ$  was generated in the same way as in the previous test. For Algorithm 2.1, we used  $\lambda = 10^{-6}$  and  $q = 0.5$ , and for the homotopy method, we took  $\tau = 10^{-6}$  in (4.1). We let each algorithm run to the maximum number of iterations  $maxit = 1000$ , which is sufficiently large for all of them. All other settings of the compared algorithms were left to their default ones. If  $\frac{\|\mathbf{x}^r - \mathbf{x}^\circ\|_2}{\|\mathbf{x}^\circ\|_2} \leq 10^{-3}$ , the recovery was regarded as successful. The right picture in Figure 4.1 plots the success frequency of each method over 500 independent trials. From the figure, we can see that our method gives the highest successful rate.

**4.2. Matrix completion.** This section reports some numerical results on solving matrix completion problems using Algorithm 3.1. For all the numerical tests, we used  $X^{(0)} = \mathcal{P}_\Omega(M)$  as the starting point.

**Rank estimation.** In Algorithm 3.1,  $K$  is one of the most important parameters. Since the true rank  $r$  is generally unknown, we deliberately used the overestimate  $K = \lceil 1.5r \rceil$  throughout our tests, unless otherwise specified. During the iterations of our algorithm,  $K$  was updated dynamically as did in [34]. Specifically, suppose that  $X$  was the current iterate and  $\lambda_{K_{\min}} \geq \lambda_{K_{\min}+1} \geq \dots \geq \lambda_{K+1} > 0$  were the  $(K_{\min})$ th through  $(K+1)$ th largest eigenvalues of  $X^T X$ , where  $K_{\min}$  was the user-specified minimum rank estimate. Let  $\bar{\lambda}_i = \lambda_i / \lambda_{i+1}$  for  $i = K_{\min}, \dots, K$ , and suppose

$$\widehat{K} = \operatorname{argmin}_{K_{\min} \leq i \leq K} \bar{\lambda}_i.$$

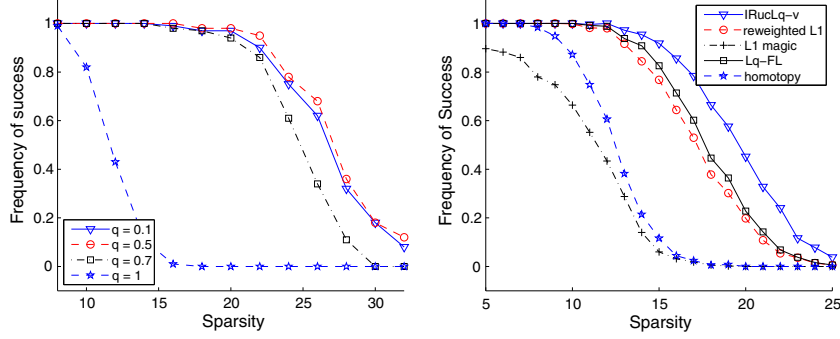


FIG. 4.1. Comparison results of recoverability. Left: Algorithm 2.1 with different  $q$ 's. Right: Algorithm 2.1 with  $q = 0.5$  compares with three  $l_1$  solvers and one  $l_q$  solver on recovering sparse vectors.

If the condition

$$\frac{(K - K_{\min} + 1)\bar{\lambda}_{\hat{K}}}{\sum_{i \neq \hat{K}} \bar{\lambda}_i} > 10,$$

was satisfied, which means there is a “big” jump between  $\lambda_{\hat{K}}$  and  $\lambda_{\hat{K}+1}$ , then we reduced  $K$  to  $\hat{K}$ . We found from our numerical experiments that whenever this adjustment was applied,  $\hat{K}$  became equal to the true rank  $r$ , so there was no need to repeat this adjustment on each problem.

**Choice of  $q$ .** As in sparse vector recovery, we first numerically compared the solutions of Algorithm 3.1 with different values of  $q$  on  $100 \times 100$  matrices to identify a good value of  $q$  for the remaining tests. In this test, each matrix was exactly low-rank and had the form  $M = M_L M_R$ , where  $M_L$  and  $M_R$  were generated by MATLAB's commands `randn(m,r)` and `randn(r,n)`, respectively. The maximum number of iterations was set to 1000, and the parameters  $\lambda$  and  $\alpha$  were set to  $10^{-6}$  and 0.9, respectively. A fixed sampling ratio  $SR = 0.5$  was used in this test, where  $SR \triangleq \#(\Omega)/(mn)$ . We compared four different values of  $q = 0.1, 0.5, 0.7, 1$  with initial rank estimate  $K = \lfloor 1.5r \rfloor$  and minimum rank estimate  $K_{\min} = 5$ . We regarded the recovery **successful** if  $\frac{\|M^r - M\|_F}{\|M\|_F} \leq 10^{-3}$ , where  $M^r$  stands for a recovered solution. The left picture in Figure 4.2 depicts the success frequency over 100 independent trials for each  $q$  and  $r$ . From the figure, we can see that Algorithm 3.1 with  $q = 1$  performed

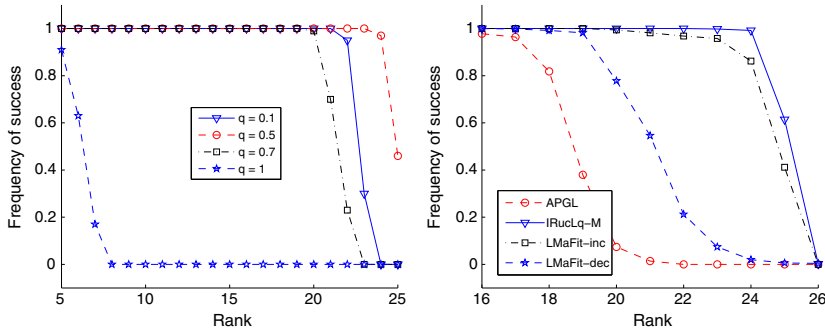


FIG. 4.2. Comparison results of recoverability. Left: Algorithm 3.1 with different  $q$ 's. Right: Algorithm 3.1 with  $q = 0.5$  compares with APGL and LMAFit on recovering low-rank matrices.

much worse than  $q = 0.1, 0.5, 0.7$ . Again,  $q = 0.5$  gives the best performance. The reason for the relatively lower success frequency of  $q = 0.1$  over  $q = 0.5$  is similar to that for the vector case. Thus, in the remaining tests, we used  $q = 0.5$ .

**Comparison on synthetic data.** Second, we compared Algorithm 3.1 (IRucLq-M) with two recent matrix completion solvers APGL [32] and LMaFit [34]. These two algorithms compare favorably with a number of other methods including FPCA [24] and OptSpace [20] on many types of matrices. APGL solves

$$\min_X \|X\|_* + \frac{1}{2\mu} \|\mathcal{P}_\Omega(X - M)\|_F^2,$$

and LMaFit solves

$$\min_{X,Y} \|\mathcal{P}_\Omega(XY - M)\|_F^2 \text{ subject to } X \in \mathbb{R}^{m \times K}, Y \in \mathbb{R}^{K \times n},$$

where  $K$  is an estimated rank and can be fixed or dynamically updated. In this test, each matrix was exactly low-rank and had the form  $M = M_L M_R$ , where  $M_L$  and  $M_R$  were generated by MATLAB's commands `rand(m,r)-0.5` and `rand(r,n)-0.5`, respectively. It is worth mentioning that matrices with uniformly random entries are usually more difficult to recover than those with Gaussian random entries. We let IRucLq-M and APGL run to a maximum number of iterations `maxit` = 1000 and LMaFit to `maxit` = 5000 since LMaFit converges relatively slowly and takes less time per iteration. For IRucLq-M, we set  $\alpha = 0.9$  and  $\lambda = 10^{-6}$ . Initial rank estimate  $K = \lfloor 1.5r \rfloor$  was used and the minimum rank estimate was set to  $K_{\min} = 5$ . For LMaFit, both increasing-rank (LMaFit-inc) and decreasing-rank strategies (LMaFit-dec) were compared corresponding to its parameter `est_rank` set to 2 and 1, respectively. Initial rank estimate  $K = 5$  was used for LMaFit-inc and the value of maximum rank estimate parameter `rank_max` set to  $\lfloor 1.5r \rfloor$ . The increase step parameter `rk_inc` was set to 1. For LMaFit-dec, the initial rank estimate was set to  $K = \lfloor 1.5r \rfloor$  and the value of minimum rank estimate parameter `rank_min` set to 5. For APGL, we set its parameters `truncation` = 1 and `truncation_gap` = 10. The initial value of  $\mu$  was  $\mu_0 = 10^{-2} \|\mathcal{P}_\Omega(M)\|_2$ , where  $\|\mathcal{P}_\Omega(M)\|_2$  equals the largest singular value of  $\mathcal{P}_\Omega(M)$ . It was dynamically updated by the continuation technique  $\mu_k = \max(0.7\mu_{k-1}, \mu_{\min})$ , where  $\mu_{\min} = 10^{-6} \|\mathcal{P}_\Omega(M)\|_2$  was used. All other parameters related to LMaFit and APGL were set to their default values. Similar to the previous test, if  $\frac{\|M^r - M\|_F}{\|M\|_F} \leq 10^{-3}$ ,  $M^r$  is regarded as a successful recovery. The right picture in Figure 4.2 plots the success frequency of each algorithm for different ranks over 500 independent trials. From the figure, we can see that IRucLq-M gave the best recoverability.

**Algorithm acceleration.** We notice that (3.2) with  $\mathcal{A} = \mathcal{P}_\Omega$  is expensive to solve with large-scale data since we need to solve  $m$  linear equations

$$(4.2) \quad \lambda q X^{i \rightarrow} W^{(k-1)} + (\mathcal{P}_\Omega(X))^{i \rightarrow} = (\mathcal{P}_\Omega(M))^{i \rightarrow}, \quad i = 1, \dots, m,$$

where  $X^{i \rightarrow}$  denotes the  $i$ th row of  $X$ . To tackle this difficulty, we follow [13] and keep the best rank- $K$  approximation of  $X^\top X$ , which is formed by the  $K$  largest eigenvalues and their corresponding eigenvectors, while updating the weighting matrix  $W$ . Then, we can exploit the *Woodbury matrix identity* to solve (4.2). More precisely, suppose that  $X$  is the current iterate and  $X^\top X = V \Sigma^2 V^\top$  is the eigen-decomposition of  $X^\top X$ . We approximate  $X^\top X$  by  $V \Sigma_t^2 V^\top$ , where  $\Sigma_t$  is a diagonal matrix with diagonal entries  $(\Sigma_t)_{jj} = \sigma_j(X)$  if  $j \leq K$  and zero otherwise. Now suppose  $\epsilon$  and  $W$  are

updated to  $\epsilon_+ = \min\{\epsilon, \alpha \cdot \sigma_{K+1}(X)\}$  and  $W_+ = V(\Sigma_t^2 + \epsilon_+^2 I)^{q/2-1} V^\top$ , respectively. Let  $V_K \in \mathbb{R}^{n \times K}$  be the matrix consisting of the  $K$  columns of  $V$  corresponding to the first  $K$  largest eigenvalues, and let  $D_K$  be the  $K \times K$  diagonal matrix with diagonal elements  $(D_K)_{jj} = (\sigma_j^2 + \epsilon_+^2)^{q/2-1} - \epsilon_+^{q-2}$  for  $j = 1, \dots, K$ . Then, we can write  $W_+ = V_K D_K V_K^\top + \epsilon_+^{q-2} I$ . Moreover, note that  $(\mathcal{P}_\Omega(X))^{i \rightarrow} = X^{i \rightarrow} E_i$ , where  $E_i \in \mathbb{R}^{n \times n}$  is the diagonal matrix with diagonal elements  $(E_i)_{jj} = 1$  if  $(i, j) \in \Omega$  and zero otherwise. Replacing  $W^{(k-1)}$  in (4.2) by the updated weighting matrix  $W_+$ , we need to solve

$$(4.3) \quad X^{i \rightarrow} (\lambda q V_K D_K V_K^\top + \lambda q \epsilon_+^{q-2} I + E_i) = M^{i \rightarrow} E_i, \quad i = 1, \dots, m.$$

For convenience, letting  $\bar{D}_K = \lambda q D_K$ ,  $\bar{E}_i = \lambda q \epsilon_+^{q-2} I + E_i$  and using the *Woodbury matrix identity*, we find the explicit solutions of (4.3) as

$$X^{i \rightarrow} = M^{i \rightarrow} E_i (\bar{E}_i^{-1} - \bar{E}_i^{-1} V_K (\bar{D}_K^{-1} + V_K^\top \bar{E}_i^{-1} V_K)^{-1} V_K^\top \bar{E}_i^{-1}), \quad i = 1, \dots, m.$$

Since  $\bar{D}_K^{-1} + V_K^\top \bar{E}_i^{-1} V_K$  is  $K \times K$ , its inverse is less expensive to compute. This leads to an accelerated algorithm which is called *truncated-IRucLq-M*, or *t-IRucLq-M*.

Next, we compared IRucLq-M and t-IRucLq-M on recovering low-rank matrices of size  $200 \times 200$ . Each matrix had the form of  $M = M_L M_R$ , where  $M_L$  and  $M_R$  were generated by MATLAB's commands `randn(m, r)` and `randn(r, n)`, respectively. We fixed  $K = r$ , i.e., we used the correct rank estimate, and  $r$  varied among  $\{41, 44, 47, 50, 53\}$ . The parameters  $\alpha$  and  $\lambda$  were set to  $\alpha = 0.9$  and  $\lambda = 10^{-6}$  for both IRucLq-M and t-IRucLq-M. We terminated the algorithms if

$$(4.4) \quad \begin{aligned} & \sigma_{K+1}(X^{(k)}) \leq \text{tol for some } k \\ \text{or} & \frac{|\sigma_{K+1}(X^{(k)}) - \sigma_{K+1}(X^{(k-1)})|}{\max\{1, \sigma_{K+1}(X^{(k-1)})\}} \leq \text{tol for three consecutive } k\text{'s,} \end{aligned}$$

where  $\text{tol} = 10^{-5}$  was used in this test. In addition, we set  $\text{maxit} = 1000$  for both IRucLq-M and t-IRucLq-M. Figure 4.3 plots the success frequency and average running time over 100 independent trials. From the results, we can see that in general t-IRucLq-M is faster than IRucLq-M with no quality loss. For this reason, only t-IRucLq-M is used in the remaining experiments.

**Comparison on a real image.** Finally, we applied Algorithm 3.1 to grayscale image recovery from partial observations. The original image (in Figure 4.4) has a

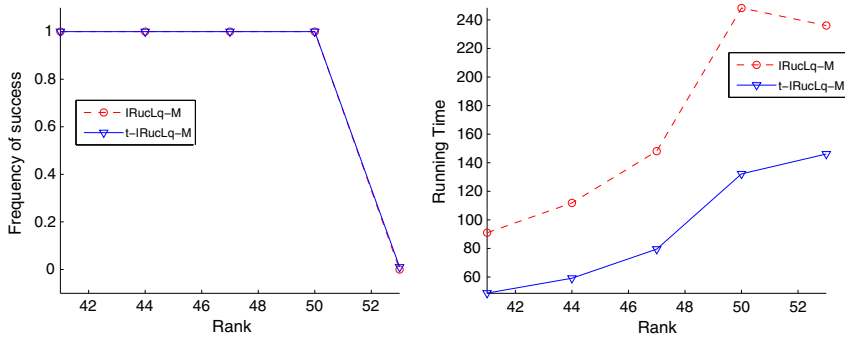


FIG. 4.3. Comparison of IRucLq-M and t-IRucLq-M. Left: success frequency. Right: average running time.

FIG. 4.4. Grayscale image of  $512 \times 512$  pixels

resolution of  $512 \times 512$ . In this test, the underlying matrix  $M$  was not exactly low-rank and had the form of

$$M = M^o + \sigma \frac{\|M^o\|_F \Xi}{\|\Xi\|_F},$$

where  $M^o$  is the matrix representation of the image (not contaminated),  $\sigma$  varied among  $\{0.01, 0.05, 0.10\}$ , and  $\Xi$  was white noise generated by MATLAB command `randn(m,n)`.

For t-IRucLq-M, we used  $\alpha = 0.9$  and heuristically set  $\lambda = 10^{-2}\sigma$ . The rank estimate was fixed to  $K = 40$ . For LMaFit, both fixing-rank (LMaFit-fix) and increasing-rank (LMaFit-inc) strategies were compared corresponding to the parameter `est_rank` set to 0 and 2, respectively. For LMaFit-fix, the rank estimate was fixed to  $K = 40$ . For LMaFit-inc, the initial rank estimate was set to  $K = 5$  and the maximum rank estimate was set to `rank_max=50`. Actually, we also tested both t-IRucLq-M and LMaFit with the decreasing-rank (LMaFit-dec) strategy. We observe that t-IRucLq-M would never decrease the initial rank estimate  $K$ . Hence, it was the same as that with a rank fixing strategy. In addition, t-IRucLq-M made better recovery for larger  $K$ . However, LMaFit-dec gave a worse solution than that by LMaFit-fix or LMaFit-inc. For APGL, the continuation technique  $\mu_k = \max(0.7\mu_{k-1}, \mu_{\min})$  was used with  $\mu_0 = 0.1 \|\mathcal{P}_\Omega(M)\|_2$  and  $\mu_{\min} = 10^{-3} \|\mathcal{P}_\Omega(M)\|_2$ . The stopping tolerance was set to  $10^{-3}$  for both t-IRucLq-M (see (4.4)) and APGL and  $10^{-4}$  for LMaFit since we saw that  $10^{-3}$  was too loose for LMaFit. The maximum number of iterations was set to 2000 for all three algorithms. All other parameters for LMaFit and APGL were set to the same values as in the previous test.

CPU time (sec), peak-signal noise ratio (PSNR), and mean square error (MSE) were employed to measure the performance of the algorithms. Table 4.1 lists the average results of 100 independent trials corresponding to different sampling ratios  $SR = 0.3, 0.4, 0.5$ . From the table, we see that in most cases t-IRucLq-M obtains better solutions than those by APGL with comparable speed. LMaFit is faster than t-IRucLq-M, but it gives worse solutions with both fixing-rank and increasing-rank strategies in all cases. We found that LMaFit could not improve the solution much, even we let it run more iterations, say, to 5000.

TABLE 4.1

Results of image recovery by *t-IRucLq-M*, *APGL*, and *LMaFit*. Best solutions are in boldface.

Problem		APGL				LMaFit-fix			LMaFit-inc			t-IRucLq-M		
SR	$\sigma$	$\mu$	Time	PSNR	MSE	Time	PSNR	MSE	Time	PSNR	MSE	Time	PSNR	MSE
0.3	0.01	8.06e-2	16.9	24.45	3.59e-3	6.84	17.26	1.90e-2	9.99	18.73	1.34e-2	24.7	<b>24.64</b>	<b>3.44e-3</b>
0.3	0.05	8.06e-2	16.6	23.35	4.62e-3	5.32	16.98	2.02e-2	8.76	17.99	1.59e-2	23.9	<b>23.92</b>	<b>4.06e-3</b>
0.3	0.10	8.06e-2	23.6	21.34	7.35e-3	5.60	16.19	2.42e-2	10.3	16.54	2.22e-2	23.6	<b>22.33</b>	<b>5.85e-3</b>
0.4	0.01	1.07e-1	11.7	<b>26.41</b>	<b>2.29e-3</b>	2.77	23.42	4.56e-3	9.36	22.77	5.31e-3	11.4	26.29	2.35e-3
0.4	0.05	1.07e-1	17.9	24.84	3.28e-3	2.68	22.65	5.44e-3	9.51	21.76	6.69e-3	12.9	<b>25.50</b>	<b>2.82e-3</b>
0.4	0.10	1.07e-1	24.9	22.12	6.13e-3	3.03	20.71	8.50e-3	9.91	19.89	1.03e-2	15.9	<b>23.68</b>	<b>4.29e-3</b>
0.5	0.01	1.34e-1	17.8	<b>28.07</b>	<b>1.56e-3</b>	1.20	25.52	2.80e-3	8.06	25.58	2.77e-3	9.06	27.47	1.79e-3
0.5	0.05	1.34e-1	22.8	25.84	2.60e-3	1.41	24.92	3.22e-3	8.36	24.85	3.28e-3	10.9	<b>26.48</b>	<b>2.25e-3</b>
0.5	0.10	1.34e-1	26.3	22.39	5.77e-3	1.37	23.26	4.72e-3	8.41	22.86	5.18e-3	10.3	<b>24.33</b>	<b>3.69e-3</b>

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