A BIVARIATE SPLINE SOLUTION TO THE HELMHOLTZ EQUATION WITH LARGE WAVE NUMBER

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Abstract. We explain how to use bivariate splines to numerically solve the Helmholtz equation with large wave number, e.g. wave number = 1000 or larger. Although there are many computational methods, e.g. hp finite element methods available, numerical solution of the Helmholtz equation with large wave number still poses a challenge. It is known that there is a so-called pollution error which complicates generation of an accurate solution from the current conventional methods. For example, for linear finite elements, asymptotic error estimates rely on the wave number k and the size h of triangulation satisfying $k^2h \leq C < \infty$. That is, when $k = 1000$, the size of underlying triangulation has to be extremely small which leads to an infeasible numerical computation. In this paper, we use the ideas in [3] for using bivariate splines of arbitrary degree and arbitrary smoothness over arbitrary triangulation to provide a new computational method for the Helmholtz equation. We shall demonstrate numerically that the bivariate spline method enables us to find very accurate solution for large wave number, e.g., $k = 1500$. The accuracy of our numerical experiments does not appear to be affected by the pollution phenomenon. In addition, we shall establish the existence, uniqueness and stability of the weak solution to the Helmholtz equation under the assumption that k^2 is not an eigenvalue of Dirichlet boundary value problem of the Poisson equation. Under this assumption, the standard requirement of strictly star-shaped domain for the well-posedness of the Helmholtz equation is no longer needed.

Key words. Bivariate Spline Functions, Boundary Value Problem, Helmholtz Equation, Large Wave Number

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1. Introduction.

The following partial differential equation, referred to as Helmholtz equation or reduced wave equation is well known:

$$\begin{cases} -\Delta u - k^2 u &= f, \quad \text{in } \Omega \subset \mathbb{R}^2 \\ \mathbf{n} \cdot \nabla u + \mathbf{i} k u &= g \quad \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with Lipschitz boundary, $\mathbf{i} = \sqrt{-1}$ denotes the imaginary unit, $\mathbf{n}$ is the unit normal to $\partial\Omega$, and k is the wave number. This Helmholtz problem arises from many application areas: acoustic scattering, electromagnetic fields, etc.. In particular, the solution to (1.1) provides an approach for numerical solution of Maxwell's equations in a special case. Over many years, the finite element method, discontinuous Galerkin methods, weak-Galerkin methods, and their variants have been used to tackle the numerical solution of the Helmholtz equation (1.1) when wave number k is large. See literature in [23], [11], [12], [10], [27], [6], [7], [8], [32], and etc.. Theoretical study on the existence, uniqueness, stability of the Helmholtz problem (1.1) has been carried out extensively. See existence and uniqueness of the weak solution of (1.1) in [23]. See [5] and [18] for the stability of the weak solution under the assumption of the domain which is strictly star-shaped (see its definition below). However, similar results for other domains have not been established so far. In addition, the numerical computation of solutions to (1.1) remains challenging when the wave number k is large.

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Let us first present a quick review of the study of finite element method, discontinuous Galerkin method, weak Galerkin method and their variations in [11], [12], [10], [27], [6], [7], [8], and [32] for numerical solution to (1.1). In all references mentioned above, the underlying domain Ω has to be a strictly star-shaped domain , which means that there exist a point $\mathbf{x}_0 \in \Omega$ and a positive constant γ_Ω depending only on Ω such that

$$(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n} \geq \gamma_\Omega > 0, \quad \forall \mathbf{x} \in \partial\Omega. \tag{1.2}$$

If $\gamma_\Omega = 0$, Ω is said to be star-shaped domain. Nevertheless, all the computational methods work well for non-convex domains as well as domains which are not strictly star-shaped. Mathematically, it is interesting to have a theory for more general domains.

The convergence analysis of many existing numerical methods has been carried out in the literature. To explain the analysis, we shall use the following norm over a complex-valued Sobolev space $\mathbb{H}^1(\Omega)$ over Ω in the paper:

$$|\!|\!| u |\!|\!|_{1,k,\Omega} := (\|\nabla u\|^2_{L^2(\Omega)} + k^2\|u\|^2_{L^2(\Omega)})^{1/2}. \tag{1.3}$$

This is equivalent to the standard H^1-norm on $\mathbb{H}^1(\Omega)$. In [23], the following result was established.

Theorem 1.1 (Proposition 8.2.7 in [23]). *Let Ω be a bounded star-shaped domain with smooth boundary (or a bounded convex domain). Let $S_h \subset \mathbb{H}^1(\Omega)$ be the finite element space. Then there exists a positive constant C_3 dependent on Ω such that, under the assumption that $(1+k^2h) \leq C$,*

$$|\!|\!| u - u_{FE} |\!|\!|_{1,k,\Omega} \leq C_3 \inf_{s\in S_h} |\!|\!| u - s |\!|\!|_{1,k,\Omega}. \tag{1.4}$$

This result was improved several times in [24, 25] and recently in [6]. That is, letting $S_h \subset \mathbb{H}^1(\Omega)$ be the higher order finite element space of degree p over triangulation $\triangle$ with size $h = |\triangle|$, a subspace of complex-valued Sobolev space $\mathbb{H}^1(\Omega)$, Du and Wu proved the following result in [6]:

Theorem 1.2 (Theorem 5.1 in [6]). *Let u and u_h be the weak solutions satisfying (1.1) and (3.3), respectively. Then there exists a constant C_0 independent of k and h such that if*

$$k(kh)^p \leq C_0 \tag{1.5}$$

then the following estimate holds:

$$|\!|\!| u - u_h |\!|\!|_{1,k,\Omega} \leq (1 + k(kh)^p) \inf_{s\in S_h} |\!|\!| u - s |\!|\!|_{1,k,\Omega}. \tag{1.6}$$

For the internal penalty discontinuous Galerkin (IPDG) method using the spline $S^{-1}_p(\triangle)$ of discontinuous piecewise polynomials of degree p over triangulation $\triangle$ with an internal penalty, Du and Zhu in [7] obtained the following

Theorem 1.3 (Theorem 1 in [7]). *Let u be the weak solutions satisfying (1.1) and u_h be the IPDG solution based on $S^{-1}_p(\triangle)$. Then there exists a constant C_0 independent of k and h such that if*

$$k(kh)^{2p} \leq C_0 \tag{1.7}$$

then the following estimate holds:

$$\|u - u_h\|_E \leq (1 + k(kh)^p) \inf_{s \in S_h} \|u - s\|_E, \tag{1.8}$$

where $S_h = S_p^{-1}(\triangle) \cap H^1(\Omega)$ *and* $\|u - u_h\|_E$ *is the norm in terms of jumps of function values as well as derivative values.*

In addition, the results above improve the study in [12] which has a very complicated proof. Not only does the convergence analysis of numerical methods require $k^{1+q}h \leq C < \infty$ for some $q > 0$, but also numerical experiments show the so-called pollution phenomenon which ruins the accuracy of numerical solution when wave number k is large, even $k = 200$ or $k = 300$.

In this paper, we shall provide a new way to establish the existence, uniqueness and stability of the weak solution to the Helmholtz equation under a new assumption. To replace the assumption that the domain is strictly star-shaped, we assume that for wave number k such that k^2 is not a Dirichlet eigenvalue of the Laplace operator over Ω. Under this new assumption, we are able to establish the coercivity of the sesquilinear form $B(u, v)$ and use the Lax-Milgram theorem to establish the existence and uniqueness of the weak solution to the Helmholtz equation in (1.1). Due to the new assumption, the study leads to the new stability estimate of the weak solution which does not require the classic assumption of strictly star-shaped domains. The new stability estimate enables us to give a new convergence analysis. Although we are not able to find out how the coercivity constant is dependent on k, we identify two cases that desired approximation order will be achieved when $kh \leq C < \infty$. These are much better than the assumptions $(1 + k^2h) < \infty$, (1.5), and (1.7) mentioned above.

In addition to the theoretical analysis of the new stability estimate and convergence analysis, in this paper we shall explain how to use bivariate spline functions to numerically solve (1.1) and demonstrate that our spline method can be more efficient and effective to find the numerical solution of Helmholtz equations with large wave numbers, e.g. $k = 500$— 1500 in Example 5.4 in a later section. We will present a large amount of numerical evidence to demonstrate the convergence of our bivariate spline method. No pollution phenomenon is observed in our computational experiments. Numerical results show that bivariate spline method is much better than the weak-Galerkin(WG) method in [27] and hybridized DG and WG methods in [29], [32] in the sense that we are able to achieve high accuracy and for larger wave numbers. In particular, we present a comparison of degrees of freedom (dof) of our spline method and the weak-Galerkin method to show that the spline method uses fewer dof for a same accuracy than the weak-Galerkin method. More numerical evidence on spline solution to Helmholtz equation over inhomogeneous media and Maxwell equations with time harmonic source term will be reported elsewhere.

The paper is organized as follows. We first explain the existence, uniqueness, and stability of the weak solution in $H^1(\Omega)$ and in a spline space and a convergence analysis of spline weak solution in §2 and §3 and in §4. Next we present our numerical results in §5, where we present some simulation results by using Bessel functions as a known solution and check how accurate our spline solutions are in convex and non-convex settings. Mainly, our spline solutions with various degrees $p = 5, 6, \cdots, 17$ will be used for wave numbers 5— 1500. Finally, we make a few remarks on many unsolved research problems in §6 as the study generates some interesting mathematical problems on the behavior of Dirichlet eigenvalues and coercivity constants L.

2. The Well-Posedness of the Helmholtz BVP. Let $\mathbb{L}^2(\Omega)$ be the space of all complex-valued square integrable functions over Ω and $\mathbb{H}^1(\Omega) \subset \mathbb{L}^2(\Omega)$ be the complex-valued square integrable functions over Ω such that their gradients are in $\mathbb{L}^2(\Omega)$. We introduce the following sesquilinear form:

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla \overline{v} dxdy, \quad u, v \in \mathbb{H}^1(\Omega),$$

where $\overline{u}$ stands for the complex conjugate of the complex-valued function u. Also we need two different inner products:

$$\langle u, v\rangle_\Omega = \int_\Omega u\overline{v} dxdy \quad \forall u, v \in \mathbb{L}^2(\Omega) \text{ and } \langle u, v\rangle_\Gamma = \int_\Gamma u\overline{v} d\Gamma, \quad \forall u, v \in \mathbb{L}^2(\Gamma),$$

be the standard inner product in $\mathbb{L}^2(\Omega)$ and in $\mathbb{L}^2(\Gamma)$, respectively, where $\Gamma = \partial\Omega$. The variational formulation to the Helmholtz problem (1.1) is to find $u \in \mathbb{H}^1(\Omega)$ such that

$$a(u,v) - k^2\langle u, v\rangle_\Omega + \mathbf{i}k\langle u, v\rangle_\Gamma = \langle f, v\rangle_\Omega + \langle g, v\rangle_\Gamma, \quad \forall v \in \mathbb{H}^1(\Omega) \tag{2.1}$$

which is the weak formulation of (1.1). If a function $u \in \mathbb{H}^1(\Omega)$ satisfies the above equation, u is called the weak solution. In this section, we mainly discuss the existence, uniqueness, and stability of the solution to the Helmholtz problem (1.1). We first present a proof of the existence and uniqueness based on the Fredholm alternative theorem and then we use Lax-Milgram theorem to give another proof. Consider the following two second order partial differential equations:

$$\begin{cases} \Delta u + \lambda u & = 0, \text{ in } \Omega \\ \mathbf{n} \cdot \nabla u + \mathbf{i}ku & = 0, \text{ in } \partial\Omega \end{cases} \tag{2.2}$$

and

$$\begin{cases} \Delta u + \lambda u & = f, \text{ in } \Omega \\ \mathbf{n} \cdot \nabla u + \mathbf{i}ku & = 0, \text{ in } \partial\Omega, \end{cases} \tag{2.3}$$

where $\lambda > 0$ is a constant, $f \in L^2(\Omega)$. We have the following well-known result:

THEOREM 2.1 (Fredholm Alternative Theorem). *Fix $\lambda > 0$. Precisely one of the following two statements holds: Either (2.2) has a nonzero weak solution $u \in \mathbb{H}^1(\Omega)$ or there exists a unique weak solution $u_f \in \mathbb{H}^1(\Omega)$ satisfying (2.3).*

We refer to [9] for a proof for the case with Dirichlet boundary condition. A similar argument works for Theorem 2.1 and the detail is left to the interested reader. The following existence and uniqueness is well-known (cf. e.g. [23]). For convenience, we present another proof.

THEOREM 2.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. Then there exists a unique solution $u \in \mathbb{H}^1(\Omega)$ to (1.1) satisfying (2.1).*

Proof. By Fredholm Alternative Theorem 2.1, let us show that k^2 is not an eigenvalue of (2.2). Otherwise, if there exists a nonzero eigenfunction $u_{k^2} \in \mathbb{H}^1(\Omega)$ satisfying (2.2) with $\lambda = k^2$, then the weak formulation of (2.2), i.e.

$$a(u_{k^2}, v) - k^2\langle u_{k^2}, v\rangle_\Omega + \mathbf{i}k\langle u_{k^2}, v\rangle_\Gamma = 0, \forall v \in \mathbb{H}^1(\Omega),$$

shows that $u_{k^2} = 0$ on Γ by using $v = u_{k^2}$. It follows that $\mathbf{n}\cdot\nabla u_{k^2} = 0$ on Γ by using the boundary condition of (2.2). That is, u_{k^2} is also an eigenfunction of Laplacian operator over Ω associated with Neumann boundary condition. By using the following Lemma 2.3, $u_{k^2} \equiv 0$ as $u_{k^2} \in H_0^1(\Omega)$. This is a contradiction and hence, k^2 is not an eigenvalue of (2.2). Fredholm Alternative theorem implies that (2.3) has a unique solution. ∎

In the proof above, we have used the result of Lemma 2.3. Let us introduce some notation. We first recall that the standard eigenvalue problem associated with Laplacian operator Δ:

$$\begin{cases} \Delta u + \lambda u &= 0, \text{ in } \Omega \\ \quad u &= 0, \text{ in } \partial\Omega. \end{cases} \tag{2.4}$$

If (2.4) has a nonzero solution, λ is called an eigenvalue (or Dirichlet eigenvalue) of the Laplace operator Δ over the underlying domain Ω. It is known that all eigenvalues are positive and there is an infinitely many eigenvalues which increase to infinity. Let us write $\lambda_i, i = 1, \cdots, \infty$ for the eigenvalues and ϕ_i for a normalized eigenfunction associated with λ_i. Similarly, let $v_\nu \in H^1(\Omega)$ be an eigenfunction associated with Neumann eigenvalue ν, i.e. v_ν satisfies the following

$$\begin{cases} -\Delta u - \nu u &= 0, \text{ in } \Omega \\ \quad \mathbf{n}\cdot\nabla u &= 0, \text{ on } \partial\Omega. \end{cases} \tag{2.5}$$

It is known that the Neumann eigenvalues are unbounded, nonnegative, and countably infinite. We are now ready to prove the following

Lemma 2.3 (Filonov, 2004[13]). *For each Neumann eigenvalue $\nu > 0$,*

$$H_0^1(\Omega) \cap ker(-\Delta - \nu I) = \{0\},$$

where I is the identity operator.

Proof. The proof is short and we include it here for convenience. Let $v_\nu \in H^1(\Omega)$ be an eigenfunction associated with Neumann eigenvalue ν, i.e. $v_\nu \in \ker(-\Delta - \nu I)$. If $v_\nu \in H_0^1(\Omega)$, we extend v_ν by zero outside Ω and call it w. Then $w \in H_0^1(\mathbb{R}^2)$ and we have

$$\int_{\mathbb{R}^2} \nabla w \nabla u = \int_\Omega \nabla v_\nu \nabla u = -\nu \int_\Omega v_\nu u = -\nu \int_{\mathbb{R}^2} wu$$

for all $u \in H_0^1(\mathbb{R}^2)$. That is, w is an eigenfunction of the Laplacian operator over $\mathbb{R}^2$ and hence, $w \equiv 0$. ∎

Next let us use the well-known Lax-Milgram theorem to establish the existence, uniqueness, and stability of the weak solution u. To this end, we need some preparatory results. For convenience, let define a sesquilinear form:

$$B(u, v) = a(u, v) - k^2\langle u, v\rangle + \mathbf{i}k\langle u, v\rangle_\Gamma. \tag{2.6}$$

Also, we define

$$|||u|||_{1,k,\Omega} := \left(\|\nabla u\|^2_{L^2(\Omega)} + k^2\|u\|^2_{L^2(\Omega)}\right)^{1/2}.$$

It is easy to see $\|\!|\cdot|\!\|_{1,k,\Omega}$ is a norm on $\mathbb{H}^1(\Omega)$. Associated with this norm, we let $A(u,v) = a(u,v) + k^2\langle u, v\rangle$ be the inner product on $\mathbb{H}^1(\Omega)$ in the rest of the paper.

The following continuity condition of the sesquilinear form $B(u,v)$ is known (cf. [23]).

LEMMA 2.4. *Suppose that $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$. Then*

$$|B(u,v)| \le C_B \|\!|u|\!\|_{1,k,\Omega} \, \|\!|v|\!\|_{1,k,\Omega}, \tag{2.7}$$

where C_B is a positive constant dependent on Ω only.

Recall ϕ_i is a normalized eigenfunction associated with Dirichlet eigenvalue λ_i, $i = 1, \cdots, \infty$ counting the multiplicity. To explain the coercivity of $B(u,v)$, however, we must point out a basic fact that $B(u,v)$ is not coercive when k^2 is a Dirichlet eigenvalue. Indeed, let $u = \phi_i$ be an eigenfunction associated with Dirichlet eigenvalue λ_i. If $k^2 = \lambda_i$, we will have $B(u,v) = 0$ for all $v \in \mathbb{H}^1(\Omega)$ while $u \neq 0$. Thus, in the rest of the paper, we shall often make an assumption that k^2 is not a Dirichlet eigenvalue. Write $Y_i = \text{span}\{\phi_1, \cdots, \phi_i\} \subset H^1_0(\Omega)$. Using Rayleigh-Ritz approximation, it is known (cf. [9]) that

$$\lambda_{i+1} = \min\{\frac{\|\nabla u\|^2}{\|u\|^2} : u \in Y_i^{\perp}\}, \tag{2.8}$$

where $Y_i^{\perp}$ is the orthogonal complement of Y_i in $H^1_0(\Omega)$. Furthermore, let $X_i^{\perp}$ be the orthogonal complement of Y_i in $\mathbb{H}^1(\Omega)$. We shall point out another fact that $\mathbb{H}^1_0(\Omega)$ is not dense in $\mathbb{H}^1(\Omega)$. Otherwise, the testing space in (2.1) could be replaced by $\mathbb{H}^1_0(\Omega)$. Then when $k^2 = \lambda_i$, we could have more than one solution to (2.1) as we know $a(\phi_i, v) - k^2(\phi_i, v) + \mathbf{i}\langle \phi_i, v\rangle_\Gamma = 0$ which can be added into (2.1) for another solution. This would violate Theorem 2.2. Therefore, $X_i^{\perp} \neq Y_i^{\perp}$. Also for convenience, we shall use $\lambda_0 = 0$ in the rest of the paper although λ_0 is not a Dirichlet eigenvalue.

THEOREM 2.5. *Let Ω be a domain with Lipschitz boundary. Let λ_{i+1} be the first eigenvalue of the Laplacian operator over Ω such that $k^2 < \lambda_{i+1}$. Then there exists a lower bound $L > 0$ such that*

$$|B(u,u)| \ge L \|\!|u|\!\|^2_{1,k,\Omega}, \quad \forall u \in X_i^{\perp}. \tag{2.9}$$

Proof. If (2.9) is not true, then there exists a sequence $u_n \in X_i^{\perp}$ such that $\|\!|u_n|\!\|^2_{1,k,\Omega} = 1$ and $|B(u_n,u_n)| \le 1/n$ for $n \ge 1$. The boundedness of u_n in $X_i^{\perp} \subset \mathbb{H}^1(\Omega)$ implies that there exists a $u^* \in \mathbb{H}^1(\Omega)$ such that a subsequence, say the whole sequence $\{u_n, n \ge 1\}$ converges to u^* in $L^2(\Omega)$ norm and converges to u^* weakly in $H^1(\Omega)$ semi-norm by Rellich-Kondrachov Theorem. It follows that

$$a(u_n, u^*) - k^2\langle u_n, u^*\rangle \longrightarrow a(u^*, u^*) - k^2\langle u^*, u^*\rangle,$$

$\|\nabla u_n\| \to \|\nabla u^*\|$, and $\langle u_n, u^*\rangle_\Gamma \to \langle u^*, u^*\rangle_\Gamma$ by using the Sobolev trace theorem. That is,

$$|B(u^*, u^*)| = 0$$

In other words, the real and imaginary parts of $B(u^*, u^*)$ implies that $\|\nabla u^*\|^2_{L^2(\Omega)} = k^2\|u^*\|^2_{L^2(\Omega)}$ and $\int_\Gamma |u^*|^2 d\Gamma = 0$. Thus, $u^* \in \mathbb{H}^1_0(\Omega)$. Furthermore, since u_n is orthogonal to Y_i, so is u^*. It follows that $u^* \in Y_i^{\perp}$. If $u^* \neq 0$, the inequality in (2.8) implies $\lambda_{i+1} \le \dfrac{\|\nabla u^*\|^2}{\|u^*\|^2} = k^2 < \lambda_{i+1}$ which is a contradiction. Thus, we have $u^* \equiv 0$.

On the other hand, $\|u^*\|_{1,k,\Omega} = 1$ because of $\|u_n\|_{1,k,\Omega} = 1$. We get a contradiction again. Therefore, there exists a positive number $L > 0$ satisfying (2.9). ◻

We are now ready to establish the following existence and uniqueness result.

THEOREM 2.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. Then there exists a unique weak solution $u \in H^1(\Omega)$ to (1.1) satisfying (2.1).*

Proof. We decompose $\mathbb{H}^1(\Omega) = X_i^\perp \oplus Y_i$, where $X_i^\perp$ is the orthogonal complement of Y_i in $\mathbb{H}^1(\Omega)$ for each $i \geq 0$ with $Y_0 = \mathbb{H}_0^1(\Omega)$ and $X_0 = \mathbb{H}^1(\Omega)$. Suppose that for an integer i, $\lambda_i < k^2 < \lambda_{i+1}$, where $\lambda_0 = 0$ although it is not an eigenvalue. We shall discuss the case $\lambda_i = k^2$ shortly. We first project the solution onto Y_i which can be done as follows. We compute the projection of f onto Y_i, i.e.

$$f_i = \sum_{j=0}^{i} \langle f, \phi_j \rangle \phi_j. \tag{2.10}$$

Then we can choose $u_i \in \mathbb{H}_0^1(\Omega)$ by

$$u_i = \sum_{j=1}^{i} \frac{1}{-\lambda_j + k^2} \langle f, \phi_j \rangle \phi_j. \tag{2.11}$$

Then it is easy to see that u_i satisfies $\Delta u_i + k^2 u_i = f_i$.

Next we consider $v \in X_i^\perp$ to be the solution

$$\begin{cases} \Delta v + k^2 v &= f - f_i, \quad \text{in } \Omega \subset \mathbb{R}^2 \\ \mathbf{n} \cdot \nabla v + \mathbf{i} k v &= g - \mathbf{n} \cdot \nabla u_i \quad \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Consider its weak formulation and it is easy to see that the right-hand side of the weak formulation is a continuous linear functional. The continuity of $B(u, v)$ and the coercivity (2.9) proved above enable us to use the Lax-Milgram theorem and conclude the existence and uniqueness of the weak solution v of (2.12). Now we can easily check $u = v + u_i$ is the solution of (1.1) satisfying (2.1). The argument of the proof of Theorem 2.2 can be used to establish the uniqueness of this solution by using Lemma 2.3.

Finally, we discuss the case when $k^2 = \lambda_i$. In this case, we shall find a least square solution u_i in Y_i satisfying

$$\min_{v \in Y_i} \max_{\substack{w \in Y_i \\ \|w\|_{1,k} = 1}} |B(v, w) - \langle f_i, w \rangle|. \tag{2.13}$$

Then we solve (2.12) as above and use the uniqueness argument to establish the solution $u = v + u_i$. ◻

Furthermore, the weak solution is stable. Indeed,

THEOREM 2.7. *Suppose that Ω has a $C^{1,1}$ smooth boundary or Ω is convex. Suppose that k^2 is not a Dirichlet eigenvalue of the Laplacian operator over Ω. Let us say $\lambda_i < k^2 < \lambda_{i+1}$ for some $i \geq 0$. Let $u \in \mathbb{H}^1(\Omega)$ be the unique weak solution to (1.1) as explained above. Then there exists a constant $C > 0$ independent of f, g such that*

$$\|u\|_{1,k,\Omega} \leq C(\|f\| + \|g\|_\Gamma) \tag{2.14}$$

for $k \geq 1$, where C is dependent on $\dfrac{1}{1-\lambda_i/k^2}$ and the constant L which is the low bound in (2.9). Furthermore, suppose Ω is convex and $g \in \mathbb{H}^{3/2}(\Gamma)$. Then

$$|u|_{2,2,\Omega} \leq C(1+k)\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)}\right) + \|\nabla_T g\|_{L^2(\Gamma)}. \tag{2.15}$$

where ∇_T stands for the tangential derivative on Γ.

Proof. By using the proof of Theorem 2.6, we use the orthonormality of ϕ_i to have

$$\|\nabla u_i\|^2_{L^2(\Omega)} = \sum_{j=1}^{i}\left(\frac{\lambda_j}{k^2-\lambda_j}\right)^2 |\langle f, \phi_j\rangle|^2 \text{ and } k^2\|u_i\|^2_{L^2(\Omega)} = \sum_{j=1}^{i}\left(\frac{k}{k^2-\lambda_j}\right)^2 |\langle f, \phi_j\rangle|^2.$$

Hence, we have

$$|\!|\!|u_i|\!|\!|_{1,k,\Omega} \leq C_1\|f\|, \tag{2.16}$$

where $C_1 > 0$ is a constant dependent on

$$\max\{\frac{k+\lambda_j}{k^2-\lambda_j}, j=1,\cdots,i\} \leq \frac{k+k^2}{k^2(1-\lambda_i/k^2)} \leq \frac{2}{1-\lambda_i/k^2}$$

as $k \geq 1$ and ϕ_j are orthogonal to each other, and we have used the Bessel inequality $\sum_{j=1}^{i}|\langle f, \phi_i\rangle|^2 = \|f_i\|^2 \leq \|f\|^2$. For convenience, let $C_1 = \dfrac{2}{1-\lambda_i/k^2}$ which will be referred a few times later.

Since v is a weak solution satisfying (2.12) in its weak formulation, we have

$$B(v,v) = \langle f - f_i, v\rangle + \langle g - \mathbf{n}\cdot\nabla u_i, v\rangle_\Gamma.$$

The right-hand side of the above equality can be bounded as follows: letting $\hat{g} = g - \mathbf{n}\cdot\nabla u_i$,

$$\begin{aligned}
|\langle f-f_i, v\rangle| + |\langle \hat{g}, v\rangle| \leq& \|f-f_i\|\|v\| + \|\hat{g}\|_\Gamma\|v\|_\Gamma \\
\leq& \frac{1}{2\epsilon k^2}\|f-f_i\|^2 + \frac{\epsilon}{2}k^2\|v\|^2 + \frac{1}{2\epsilon k}\|\hat{g}\|^2_\Gamma + \frac{\epsilon}{2}k\|v\|^2_\Gamma \\
\leq& \frac{1}{2\epsilon k^2}\|f-f_i\|^2 + \frac{\epsilon}{2}|\!|\!|v|\!|\!|^2_{1,k,\Omega} + \frac{1}{2\epsilon k}\|\hat{g}\|^2_\Gamma + \frac{\epsilon}{2}C_\Omega k\|v\|_{L^2(\Omega)}\cdot\|\nabla v\|_{L^2(\Omega)} \\
\leq& \frac{1}{2\epsilon k^2}\|f-f_i\|^2 + \frac{1}{2\epsilon k}\|\hat{g}\|^2_\Gamma + \epsilon_1|\!|\!|v|\!|\!|^2_{1,k,\Omega}
\end{aligned}$$

for $\epsilon > 0$ with $\epsilon_1 = \epsilon/2 + C_\Omega\epsilon/2$, where we have used the Sobolev trace theorem (cf. Lemma 1.5.1.9 in [16]). Now we use the lower bound in (2.9) to have the inequality in (2.17) by choosing $\epsilon_1 = m/2$ and $\|f_i\| \leq \|f\|$ by the Bessel inequality.

$$|\!|\!|v|\!|\!|_{1,k,\Omega} \leq \frac{C}{k}\|f\| + \frac{C}{\sqrt{k}}\|\hat{g}\|_\Gamma \tag{2.17}$$

for $k \geq 1$.

Next $\|\hat{g}\|^2_\Gamma \leq 2\|g\|^2_\Gamma + 2\|\nabla u_i\|^2_\Gamma$ and although $u_i = 0$ over Γ, we have to estimate ∇u_i over Γ. Let us first use Sobolev trace inequality to have

$$\|\nabla u_i\|^2_\Gamma \leq C_\Omega\|\nabla u_i\|_{L^2(\Omega)}\,|\nabla u_i|_{1,2,\Omega} = C_\Omega\|\nabla u_i\|_{L^2(\Omega)}\,|u_i|_{2,2,\Omega} \tag{2.18}$$

for a positive constant C_Ω dependent on Ω, where $|\cdot|_{\ell,2,\Omega}$ is the ℓth semi-norm for $H^\ell(\Omega)$ for $\ell = 1, 2$. As estimated above, $\|\nabla u_i\|_{L^2(\Omega)} \le \|\!|u_i|\!\|_{1,k,\Omega} \le C_1\|f\|$. So let us concentrate on an estimate for $|u_i|_{2,2,\Omega}$. When Ω has $C^{1,1}$ smooth boundary or Ω is convex, we know that each eigenfunction ϕ_j is in $H^2(\Omega)$ and $|\phi_j|_{2,2,\Omega} \le C_\Omega\|\Delta\phi_j\| = C_\Omega\lambda_j\|\phi_j\|$ for a positive constant C_Ω dependent only on Ω. For simplicity, we write $u_i = \sum_{j=1}^i c_j\phi_j$ to have

(2.19)

$$|u_i|_{2,2,\Omega} \le \sum_{j=1}^i |c_j||\phi_j|_{2,2,\Omega} \le C_\Omega \sum_{j=1}^i |c_j|\|\Delta\phi_j\|_{L^2(\Omega)} \le C_\Omega \sum_{j=1}^i |c_j|\lambda_j\|\phi_j\|_{L^2(\Omega)} = C_\Omega \sum_{j=1}^i |c_j|\lambda_j.$$

As above, $c_j = \dfrac{1}{k^2-\lambda_j}\langle f, \phi_j\rangle$ and thus,

$$\sum_{j=1}^i |c_j|\lambda_j \le \frac{1}{1-\lambda_i/k^2}\sum_{j=1}^i \frac{\lambda_j}{k^2}|\langle f,\phi_j\rangle| \le \frac{1}{1-\lambda_i/k^2}\|f_i\|\left(\sum_{j=1}^i \frac{\lambda_j^2}{k^4}\right)^{1/2}.$$

Let $C_2 = \sqrt{\sum_{j=1}^i \lambda_j^2/k^4}$ which can be estimated by using the so-called Weyl law on the number of Dirichlet eigenvalues over polygonal domain. Indeed, let $N(a)$ be the number of eigenvalues counting the multiplicities less or equal to $a > 0$. The Weyl law says that

$$N(a) = \frac{A_\Omega}{4\pi}a + O(\sqrt{a}) \tag{2.20}$$

(cf. e.g. [4]), where A_Ω stands for the area of Ω. Then $C_2^2 = \dfrac{1}{k^4}\sum_{j=1}^i \lambda_j^2 \le \dfrac{1}{k^4}\lambda_i^2 N(k^2) = B\lambda_i^2/k^2 \le Bk^2$ for another positive constant B. That is, $C_2 \le \sqrt{B}k$. Hence, we have

$$|u_i|_{2,2,\Omega} \le C_1\sqrt{B}k\|f_i\| \le C_1\sqrt{B}k\|f\| \tag{2.21}$$

and together with (2.16), the terms on the right-hand side of (2.18) can be simplified to be

$$\|\nabla u_i\|_\Gamma^2 \le C_\Omega^2 C_1\|f\|C_1\sqrt{B}k\|f_i\| \le C_\Omega^2 C_1^2\sqrt{B}\|f\|^2 k \tag{2.22}$$

and hence from (2.17),

$$\|\!|v|\!\|_{1,k,\Omega} \le \frac{C}{k}\|f\| + \frac{C}{\sqrt{k}}\|g\|_\Gamma + C_1C_\Omega B^{1/4}\|f\|. \tag{2.23}$$

Therefore, we summarize the discussion above to have

$$\begin{aligned}\|\!|u|\!\|_{1,k,\Omega} \le \|\!|v|\!\|_{1,k,\Omega} + \|\!|u_i|\!\|_{1,k,\Omega} &\le \frac{C}{k}\|f\| + \frac{C}{\sqrt{k}}\|g\|_\Gamma + C_\Omega C_1\|f\|B^{1/4} \\ &= C_3(\|f\| + \frac{1}{\sqrt{k}}\|g\|_\Gamma)\end{aligned}$$

for a positive constant C_3 dependent on $2/(1-\lambda_i/k^2)$ and the lower bound L.

Finally, to establish (2.15) we follow the standard approach and apply the formula in Chapter 3, [16] to v. That is, for any $u \in H^2(\Omega)$, we use $\mathbf{v} = \nabla u$ in Theorem 3.1.1.1. in [16] to have

$$\sum_{i,j=1}^{2} \int_{\Omega} (\partial_{ij} u)^2 = \int_{\Omega} (\Delta u)^2 d\mathbf{x} + 2 \int_{\partial\Omega} \nabla_T u \cdot \nabla_T (\nabla u \cdot \mathbf{n}) d\sigma + \int_{\partial\Omega} \left[\mathcal{B}(\nabla_T u, \nabla_T u) + \mathrm{tr}(\mathcal{B})(\nabla u \cdot \mathbf{n})^2 \right] d\sigma, \tag{2.24}$$

where T and $\mathbf{n}$ stand for the tangential and normal direction of Γ, $\mathcal{B}$ is the bilinear form, i.e. the Hessian matrix and tr is the trace operator. Due to the convexity, the last two terms involving the Hessian of the boundary Γ are negative. For our solution v, the first term on the right-hand side above can be estimated as follows: by using the Helmholtz equation,

$$\begin{aligned}\int_{\Omega} |\Delta v|^2 d\mathbf{x} = \int_{\Omega} |f - u_i - k^2 v|^2 d\mathbf{x} &\le 2\|f - u_i\|^2 + 2k^4 \|v\|^2 \\ &\le C(\|f\|^2 + \|u_i\|^2) + 2k^2 \|\!|v|\!\|^2_{1,k,\Omega} \\ &\le C(\|f\|^2 + \|f\|^2/k^2) + 2k^2(\|f\|^2/k^2 + \|g\|^2_\Gamma/k + \sqrt{B}\|f\|^2) \\ &\le Ck^2(\|f\|^2 + \|g\|^2_\Gamma)\end{aligned}$$

for a positive constant C, where we have used (2.16) and (2.17). Next, by using the Robin boundary condition, the second term on the right-hand side of (2.24) is estimated as follows:

$$|\int_{\partial\Omega} \nabla_T v \cdot \nabla_T (\nabla v \cdot \mathbf{n}) d\sigma| \le \|\nabla_T v\|^2_\Gamma + |\int_\Gamma \nabla_T v \nabla_T g d\sigma| \le \frac{3}{2}\|\nabla v\|^2_\Gamma + \frac{1}{2}\|\nabla_T g\|^2_\Gamma.$$

Furthermore, by using Sobolev trace inequality, the first term above on the right-hand side can be estimated by

$$\|\nabla v\|^2_\Gamma \le C_\Omega \|\nabla v\|^2 + \frac{1}{2}|v|^2_{2,2,\Omega} \le C_\Omega \|\!|v|\!\|^2_{1,k,\Omega} + \frac{1}{2}|v|^2_{2,2,\Omega}.$$

Therefore, it follows from (2.24) that

$$\frac{1}{2}|v|^2_{2,2,\Omega} \le Ck^2(\|f\|^2 + \|g\|^2_\Gamma) + \frac{3C_\Omega}{2}\|\!|v|\!\|^2_{1,k,\Omega} + \frac{1}{2}\|\nabla g\|^2_\Gamma$$

Together with (2.21) and (2.23), we have obtained (2.15). □

Note that the stability condition in Theorem 2.7 is for a general bounded domain with $C^{1,1}$ boundary or convex domain as long as k^2 is not a Dirichlet eigenvalue of the Laplacian operator over Ω. It is interesting to know if the lower bound L in (2.9) is dependent on k or not. To this end, we decompose a weak solution u into three parts: $u = u_i + v_i + w$ with $u_i \in Y_i, v_i = Y_i^\perp$ and $w \in (\mathbb{H}^1_0(\Omega))^\perp$. Let us begin with the following

LEMMA 2.8. *There exists a positive constant L_1 such that*

$$|B(w, w)| \ge L_1 \|\!|w|\!\|^2_{1,k,\Omega}, \quad \forall w \in (\mathbb{H}^1_0(\Omega))^\perp. \tag{2.25}$$

Proof. Suppose that we do not have $L_1 > 0$ for (2.25). For each $n > 1$, we have $u_n \in (\mathbb{H}_0^1(\Omega))^\perp$ with $\|\!|u_n|\!\|_{1,k,\Omega} = 1$ such that $|B(u_n, u_n)| \le 1/n$. By Rellich-Kondrachov Theorem, the boundedness of u_k in $\mathbb{H}^1(\Omega)$ implies that there is a convergent subsequence. Without loss of generality, let us say $u_k \to u^*$ in $\mathbb{L}^2(\Omega)$ strongly and in $\mathbb{H}^1(\Omega)$ weakly. It follows that $|B(u^*, u^*)| = 0$. Thus, $\langle u^*, u^*\rangle_\Gamma = 0$, i.e. $u^* \in \mathbb{H}_0^1(\Omega)$. However, $u_n \in (\mathbb{H}_0^1(\Omega))^\perp$ implies that $u^* \in (\mathbb{H}_0^1(\Omega))^\perp$. That is, $u^* \in \mathbb{H}_0^1(\Omega) \cap (\mathbb{H}_0^1(\Omega))^\perp = \{0\}$ which contradicts to the fact $\|\!|u^*|\!\|_{1,k,\Omega} = 1$. Therefore, we have $L_1 > 0$ for (2.25). ◻

Lemma 2.9. *Suppose that k^2 is not a Dirichlet eigenvalue of $-\Delta$ over Ω. Let us say $\lambda_i < k^2 < \lambda_{i+1}$ for some $i \ge 0$. Then there exists a positive constant $L_2 > 0$ such that*

$$|B(u, u)| \ge L_2 \|\!|u|\!\|^2_{1,k,\Omega}, \quad \forall u \in Y_i. \tag{2.26}$$

Proof. To prove (2.26), we assume otherwise. There exists a nonzero $u^* \in Y_i$ such that $B(u^*, u^*) = 0$. It follows that $\|\nabla u^*\|^2 = k^2\|u^*\|^2$. Let us write $u^* = \sum_{j=1}^i c_j\phi_j \in Y_i$. Then we have $\|u^*\|^2 = \sum_{j=1}^i |c_j|^2$ by using the orthonormality of ϕ_j's and similarly, $\|\nabla u^*\|^2 = \sum_{j=1}^i |c_j|^2\lambda_j$. Since $\lambda_j < k^2$ for $j = 1, \cdots, i$, we have $\|\nabla u^*\|^2 < k^2\|u^*\|^2$ which is a contradiction to the face $k^2\|u^*\|^2 = \|\nabla u^*\|^2$.

In fact, L_2 can be found as follows. For any $u = \sum_{j=1}^{i} c_j\phi_j \in Y_i$, we have

$$\begin{aligned}|B(u, u)| = |\|\nabla u\|^2 - k^2\|u\|^2| = \sum_{j=1}^{i} |c_j|^2(k^2 - \lambda_j) \ge \frac{k^2 - \lambda_i}{k^2 + \lambda_i}\sum_{j=1}^{i}(k^2 + \lambda_j)|c_j|^2 \\ = L_2(\|\nabla u\|^2 + k^2\|u\|^2)\end{aligned}$$

with $L_2 = \dfrac{k^2 - \lambda_i}{k^2 + \lambda_i} = \dfrac{1 - \lambda_i/k^2}{1 + \lambda_i/k^2}$. ◻

Finally, we have

Lemma 2.10. *Suppose that k^2 is not a Dirichlet eigenvalue of $-\Delta$ over Ω. Let us say $\lambda_i < k^2 < \lambda_{i+1}$ for some $i \ge 0$. Then there exists a positive constant $L_3 > 0$ such that*

$$|B(u, u)| \ge L_3 \|\!|u|\!\|^2_{1,k,\Omega}, \quad \forall u \in Y_i^\perp. \tag{2.27}$$

Proof. For $w \in Y_i^\perp$, we have

$$\begin{aligned}\|\!|w|\!\|^2_{1,k,\Omega} = B(w, w) + 2k^2\|w\|^2_{L^2(\Omega)} \le |B(w, w)| + 2k^2\|w\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega}\frac{\|w\|_{L^2(\Omega)}}{\|\nabla w\|_{L^2(\Omega)}} \\ \le |B(w, w)| + k\|\!|w|\!\|^2_{1,k,\Omega}\frac{1}{\sqrt{\lambda_{i+1}}}\end{aligned}$$

by using the Cauchy-Schwarz inequality and the Rayleigh-Ritz approximation of the eigenvalues. It follows that

$$(1 - \frac{k}{\sqrt{\lambda_{i+1}}})\|\!|w|\!\|^2_{1,k,\Omega} \le |B(w, w)| \tag{2.28}$$

with $L_3 = 1 - \frac{k}{\sqrt{\lambda_{i+1}}} > 0$. ◻

3. On Spline Weak Solution to Helmholtz Equation. In this section, we mainly explain bivariate spline spaces which will be useful in the study later. We refer to [21] and [3] for detail. Given a polygonal region Ω, a collection $\triangle := \{T_1, ..., T_n\}$ of triangles is an ordinary triangulation of Ω if $\Omega = \cup_{i=1}^n T_i$ and if any two triangles T_i, T_j intersect at most at a common vertex or a common edge. We also assume that triangulation $\triangle$ is quasi-uniform, e.g. $\triangle$ is an nth uniform refinement of a fixed triangulation $\triangle_0$ of Ω. For $r \geq 0$ and $d > r$, let

$$S_p^r(\triangle) = \{s \in C^r(\triangle) : s|_T \in \mathbb{P}_p, \forall T \in \triangle\} \tag{3.1}$$

be the spline space of degree p and smoothness $r \geq 0$ over triangulation $\triangle$.

As solutions to the Helmholtz equation will be a complex solution, let us use a complex spline space in this paper defined by

$$\mathbb{S}_p^r(\triangle) = \{s = s_r + \mathbf{i}s_i, s_i, s_r \in S_p^r(\triangle)\}. \tag{3.2}$$

A spline solution $u_\triangle \in \mathbb{S}_p^r(\triangle)$ is a weak solution of (1.1) if $u_\triangle \in \mathbb{S}_p^r(\triangle)$ satisfies

$$a(u_\triangle, v) - k^2\langle u_\triangle, v\rangle + \mathbf{i}k\langle u_\triangle, v\rangle_{\partial\Omega} = \langle f, v\rangle_\Omega + \langle g, v\rangle_\Gamma, \quad \forall v \in \mathbb{S}_p^r(\triangle) \tag{3.3}$$

which is a standard weak formulation for $r \geq 0$.

The spline space $\mathbb{S}_p^r(\triangle)$ has the similar approximation properties as the standard real-valued spline space $S_p^r(\triangle)$. The following theorem can be established by the same constructional techniques (cf. [20] or [21] for spline space $S_p^r(\triangle)$ for real valued functions):

THEOREM 3.1. *Suppose that $\triangle$ is a γ-quasi-uniform triangulation of polygonal domain Ω. Let $p \geq 3r+2$ be the degree of spline space $\mathbb{S}_p^r(\triangle)$. For every $u \in \mathbb{H}^{m+1}(\Omega)$, there exists a quasi-interpolatory spline function $Q_p(u) \in \mathbb{S}_p^r(\triangle)$ such that*

$$\sum_{T\in\triangle} \|D_x^\alpha D_y^\beta (u - Q_p(u))\|_{2,T}^2 \leq K_5 |\triangle|^{2(m+1-s)} |u|_{2,p+1,\Omega}^2 \tag{3.4}$$

for $\alpha + \beta = s, 0 \leq s \leq m+1$, where $0 \leq m \leq p$, K_5 is a positive constant dependent only on γ, Ω, and p.

We can show the existence and uniqueness of spline weak solution.

THEOREM 3.2. *Let Ω be a polygonal domain and $\triangle$ be a triangulation of Ω. Let $\mathbb{S}_d^1(\triangle)$ be a complex-valued spline space of degree d and smoothness 1 over $\triangle$. Then the spline weak solution to (1.1), i.e. satisfying (3.3) exists and is unique.*

Proof of Theorem 3.2. Let us consider a spline solution $u \in \mathbb{S}_p^1(\triangle) \subset H^1(\Omega)$ which satisfies the weak formulation (3.3) with $r = 1$ for all $v \in \mathbb{S}_p^1(\triangle)$. Then it leads to a system of linear equations due to the finite dimensionality of $\mathbb{S}_p^1(\triangle)$. To see the linear system of equations has a unique solution, we need to show that the solution u has to be zero if the right-hand side is zero, i.e., $f = 0 = g$. That is, we need to show that the solution $u \in \mathbb{S}_p^1(\triangle)$ satisfying the following

$$\int_\Omega \nabla u \cdot \nabla \overline{v} dxdy - k^2 \int_\Omega u\,\overline{v} dxdy + \mathbf{i}k \int_\Gamma u\overline{v} d\Gamma = 0, \quad \forall v \in \mathbb{S}_p^1(\triangle) \tag{3.5}$$

has to be zero. Let $v = u$ in the above equation to have

$$\int_\Omega |\nabla u|^2 dxdy - k^2 \int_\Omega |u|^2 dxdy + \mathbf{i}k \int_\Gamma |u|^2 d\Gamma = 0.$$

We conclude that $\int_\Gamma |u|^2 d\Gamma = 0$ and hence, $u \equiv 0$ on $\Gamma = \partial\Omega$. Hence, it follows from (3.5) that

$$\int_\Omega \nabla u \cdot \nabla \overline{v} dxdy - k^2 \int_\Omega u \, \overline{v} dxdy = 0, \quad \forall v \in \mathbb{S}^1_p(\triangle). \tag{3.6}$$

That is, if $u \neq 0$, u is an eigenfunction in $\mathbb{S}^1_p(\triangle)$ corresponding to eigenvalue k^2.

Furthermore, $\mathbf{n} \cdot \nabla u \equiv 0$ along Γ by the Robin boundary condition. Without loss of generality, we may assume that Ω contains 0. Let $\alpha \in (0,1)$ and $\Omega \subset \Omega_\alpha$ as in Lemma 3.3. In addition, let $\triangle_\alpha$ be a triangulation of Ω_α by adding triangles to the existing $\triangle$. Then the zero boundary conditions of u enable us to extend u outside of Ω by zero and hence, $u \in \mathbb{S}^1_p(\triangle_\alpha)$ because both $u \equiv 0$ and $\mathbf{n} \cdot \nabla u \equiv 0$ along Γ . Hence, u is also an eigenfunction in $\mathbb{S}^1_p(\triangle_\alpha)$ with eigenvalue k^2. By Lemma 3.3, $k^2 = \alpha^2 \lambda_i$ for some $\lambda_i \in \Lambda_1$, the collection of all eigenvalues of Laplacian operator over spline space $\mathbb{S}^1_p(\triangle)$. Since Λ_1 has finitely many eigenvalues; however, $\alpha \in (0,1)$ can be infinitely many and different α implies different λ_i. This leads to a contradiction. Therefore, we conclude that $u \equiv 0$ and hence, there exists a unique solution to the spline weak equation (3.3). ◻

In the proof above, we have used the following

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a domain Lipschitz boundary. Without loss of generality, we may assume $0 \in \Omega$. For each $\alpha \in (0,1)$, we let $\Omega_\alpha = \{(x,y) : (\alpha x, \alpha y) \in \Omega\}$. Let*

$$\Lambda_1 = \{0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots\} \tag{3.7}$$

be the collection of all eigenvalues of Laplace operator $-\Delta$ over Ω. Similarly, let

$$\Lambda_\alpha = \{0 < \lambda_1(\alpha) \leq \lambda_2(\alpha) \leq \cdots \leq \lambda_n(\alpha) \leq \cdots\} \tag{3.8}$$

be the collection of all eigenvalues of $-\Delta$ on Ω_α. Then each eigenvalue $\lambda_i(\alpha) \in \Lambda_\alpha$ is equal to

$$\lambda_i(\alpha) = \alpha^2 \lambda_i, i = 1, 2, \cdots, n, \cdots . \tag{3.9}$$

Proof. For any function $u \in H^1_0(\Omega)$, let $u_\alpha(x,y) = u(\alpha x, \alpha y)$ which is a function in $H^1_0(\Omega_\alpha)$. If u is an eigenfunction of $-\Delta$ over Ω with eigenvalue $\lambda \in \Lambda_1$, we have $-\Delta u = \lambda u$. Thus,

$$-\Delta u_\alpha(x,y) = -\alpha^2 \Delta u(\alpha x, \alpha y) = \alpha^2 \lambda u(\alpha x . \alpha y) = \alpha^2 u_\alpha(x,y).$$

Thus, $\alpha^2 \lambda \in \Lambda_\alpha$ with eigenfunction u_α. Similarly, we can show each eigenvalue $\lambda_\alpha \in \Lambda_\alpha$, $\lambda_\alpha / \alpha^2 \in \Lambda_1$. This completes the proof. ◻

Next we need to show that the spline weak solution $u_\triangle$ are bounded independent of $\triangle$. Following the proof of Theorem 2.5 in the previous section, we have

Theorem 3.4. *Let Ω be a convex domain with Lipschitz boundary satisfying $\lambda_i < k^2 < \lambda_{i+1}$. Letting $u_\triangle \in \mathbb{S}^1_p(\triangle)$ be the spline weak solution satisfying (3.3), suppose that $u_\triangle \in X_i^\perp \cap \mathbb{S}^1_p(\triangle)$. Then there exists a constant M independent of $\triangle$ such that*

$$|\!|\!| u_\triangle |\!|\!|_{1,k,\Omega} \leq M(\|f\| + \|g\|_\Gamma). \tag{3.10}$$

Proof. We simply use the proof of Theorem 2.5 to have

$$L\|u_\triangle\|^2_{1,k,\Omega} \le |B(u_\triangle, u_\triangle)|.$$

Since $B(u_\triangle, u_\triangle) = \langle f, u_\triangle\rangle + \langle g, u_\triangle\rangle_\Gamma$, we use Cauchy-Schwarz inequality to obtain

$$\begin{aligned}|B(u_\triangle, u_\triangle)| &\le \frac{1}{2k\epsilon}\|f\|^2 + \frac{\epsilon}{2}\|u_\triangle\|^2_{1,k,\Omega} + \frac{1}{2k\epsilon}\|g\|^2_\Gamma + \frac{\epsilon}{2}k\|u_\triangle\|^2 \\ &\le \frac{1}{2k\epsilon}\|f\|^2 + \frac{1}{2k\epsilon}\|g\|^2_\Gamma + (\frac{\epsilon}{2} + \frac{C_\Omega\epsilon}{2})\|u_\triangle\|^2_{1,k,\Omega}.\end{aligned}$$

By choosing $\epsilon > 0$ small enough, e.g., $(\frac{\epsilon}{2}+\frac{C_\Omega\epsilon}{2}) \le L/2$, we have (3.10) with $M = 2/L$. ☐

Similarly, we can show that $u_\triangle$ is bounded when $u_\triangle \in \mathbb{S}^1_p(\triangle) \cap Y_i$ by using Lemma 2.9. We leave it to the interested reader. Following the same arguments of Theorem 2.7, we can construct a spline solution $u_\triangle$ based on spline approximations of eigenfunctions ϕ_i's and then due to the C^1 smoothness of the spline solution $u_\triangle$, we can prove the similar result to that of Theorem 2.7. That is, we have

THEOREM 3.5. *Suppose that Ω has a $C^{1,1}$ smooth boundary or Ω is convex. Suppose that k^2 is not a Dirichlet eigenvalue of the Laplacian operator over Ω. Let us say $\lambda_i < k^2 < \lambda_{i+1}$ for some $i \ge 0$. Let $u_\triangle \in \mathbb{S}^1_p(\triangle) \cap \mathbb{H}^1(\Omega)$ be a spline weak solution to (1.1) according to the construction in the proof of Theorem 2.7. Then there exists a constant $C > 0$ independent of f, g such that*

$$\|u_\triangle\|_{1,k,\Omega} \le C(\|f\| + \|g\|_\Gamma) \tag{3.11}$$

for $k \ge 1$, where C is dependent on $\dfrac{1}{1-\lambda_i/k^2}$ and the constant L which is the low bound in (2.9). Furthermore, suppose Ω is convex and $g \in \mathbb{H}^{3/2}(\Gamma)$. Then

$$|u|_{2,2,\Omega} \le C(1+k)\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)}\right) + \|\nabla_T g\|_{L^2(\Gamma)}. \tag{3.12}$$

where ∇_T stands for the tangential derivative on Γ.

Proof. We leave the proof to the interested reader. ☐

4. Convergence of Spline Weak Solutions. In this section, we first use the coercivity in Theorem 2.5 to establish

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Let u be the unique weak solution in $\mathbb{H}^1(\Omega)$ satisfying (2.1) and $u_\triangle \in \mathbb{S}^r_p(\triangle), p \ge 3r+2$ be the spline weak solution to (1.1) satisfying (3.3). Suppose that $u \in (\mathbb{H}^1_0(\Omega))^\perp$ and $u_\triangle \in (\mathbb{H}^1_0(\Omega))^\perp \cap \mathbb{S}^1_p(\triangle)$. Then if $u \in \mathbb{H}^s(\Omega)$ with $1 \le s < p$, there exists $C > 0$ independent of k such that*

$$\|u - u_\triangle\|_{1,k,\Omega} \le C(1 + k|\triangle|)|\triangle|^{s-1}|u|_{s,2,\Omega}, \tag{4.1}$$

where $|u|_{s,2,\Omega}$ is the semi-norm in $\mathbb{H}^s(\Omega)$.

Proof. We use Lemma 2.8 to have

$$L_1\|u - u_\triangle\|^2_{1,k,\Omega} \le |B(u - u_\triangle, u - u_\triangle)|.$$

It follows from (2.1) and (3.3) the orthogonality condition:

$$a(u - u_\triangle, w) - k^2\langle u - u_\triangle, w\rangle + \mathbf{i}\langle u - u_\triangle, w\rangle_{\partial\Omega} = 0, \quad \forall w \in \mathbb{S}^r_p(\triangle). \tag{4.2}$$

That is, $B(u-u_\triangle, w) = 0$ for all $w \in \mathbb{S}^r_p(\triangle)$. By choosing $w = Q_p(v)$, the quasi-interpolatory spline of v as in the previous section, we have

$$|B(u-u_\triangle, u-u_\triangle) = |B(u-u_\triangle, u-Q_p(u)\rangle)| \le C_B \|u-u_\triangle\|_{1,k,\Omega} \|u-Q_p(u)\|_{1,k,\Omega}.$$

In other words,

$$\|u-u_\triangle\|_{1,k,\Omega} \le \frac{C_B}{L_1} \|u-Q_p(u)\|_{1,k,\Omega}. \tag{4.3}$$

Finally, we use the approximation property of spline space $\mathbb{S}^r_p(\triangle)$, i.e. (3.4). For $u \in \mathbb{H}^s(\Omega)$ with $1 \le s \le p$, we use the quasi-interpolatory operator $Q_p(u)$ of u to have

$$\|u-Q_p(u)\|_{1,k,\Omega} \le C(1+k|\triangle|)|\triangle|^{s-1}|u|_{s,2,\Omega}$$

for a constant C dependent on Ω, p and the smallest angle of $\triangle$ only. Therefore, the combination of (4.3) and the estimate above yields (4.1). ∎

Similarly, if $u \in \mathbb{H}^1_0(\Omega)$, we can find spline approximation $u_\triangle$ satisfying (3.3) for $v \in \mathbb{S}^1_p(\triangle) \cap \mathbb{H}^1_0(\Omega)$. Using Lemma 2.10, we have

THEOREM 4.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Suppose that Ω is a domain such that k^2 is not a Dirichlet eigenvalue of the Laplacian over Ω, say $\lambda_i < k^2 < \lambda_{i+1}$ for some $i \ge 0$. Let u be the unique weak solution in $\mathbb{H}^1(\Omega)$ satisfying (2.1) and $u_\triangle \in \mathbb{S}^r_p(\triangle), p \ge 3r+2$ be the spline weak solution to (1.1) satisfying (3.3). Suppose that $u \in Y_i^\perp$ and $u_\triangle \in Y_i^\perp \cap \mathbb{S}^1_p(\triangle)$. Then if $u \in \mathbb{H}^s(\Omega)$ with $1 \le s < p$, there exists $C > 0$ dependent on $1 - k/\sqrt{\lambda_{i+1}}$ such that*

$$\|u-u_\triangle\|_{1,k,\Omega} \le C(1+k|\triangle|)|\triangle|^{s-1}|u|_{s,2,\Omega}, \tag{4.4}$$

where $|u|_{s,2,\Omega}$ is the semi-norm in $\mathbb{H}^s(\Omega)$.

In general, we do not know if the solution u is in $\mathbb{H}^1_0(\Omega)$ or in $(\mathbb{H}^1_0(\Omega))^\perp$. However, we can check if k^2 is an eigenvalue or not.

THEOREM 4.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain or a bounded domain with $C^{1,1}$ boundary. Suppose that Ω is a domain such that k^2 is not a Dirichlet eigenvalue of the Laplacian over Ω. Let u be the unique weak solution in $\mathbb{H}^1(\Omega)$ satisfying (2.1) and $u_\triangle \in \mathbb{S}^r_p(\triangle), p \ge 3r+2$ be the spline weak solution to (1.1) satisfying (3.3). Then if $u \in \mathbb{H}^s(\Omega)$ with $1 \le s < p$, there exists $C > 0$ independent of $|\triangle|$, f and g such that*

$$\|u-u_\triangle\|_{1,k,\Omega} \le C(1+k|\triangle|)|\triangle|^{s-1}(|u|_{s,2,\Omega} + |u_i|_{s,2,\Omega}), \tag{4.5}$$

where $|u|_{s,2,\Omega}$ is the semi-norm in $\mathbb{H}^s(\Omega)$ and u_i is the projection of u in Y_i.

Proof. We simply decompose u to be $v + u_i$, where $u_i \in Y_i$ and $v \in Y_i^\perp$. As the domain Ω is convex or has a $C^{1,1}$ boundary, the regularity theory of Poisson's equation implies that each eigenfunction ϕ_j is very smooth and so is u_i. Thus, v has the same regularity as that of u. For v, we use the coercive condition, i.e. Theorem 2.5 to have

$$L\|v-v_\triangle\|^2_{1,k,\Omega} \le |B(v-v_\triangle, v-v_\triangle)|,$$

where $v_\triangle$ is the spline weak solution to v. Similar to the proof of Theorem 4.1, we have

$$\|v-v_\triangle\|_{1,k,\Omega} \le C(1+k|\triangle|)|\triangle|^{s-1}|v|_{s,2,\Omega} \le C(1+k|\triangle|)|\triangle|^{s-1}(|u|_{s,2,\Omega} + |u_i|_{s,2,\Omega}) \tag{4.6}$$

for another positive constant C dependent on L in (2.9).

Next we discuss the spline approximation $u_{i,\triangle}$ of u_i. The classic theory (cf. [30] and [31]) says that letting $\phi_{j,\triangle} \in S^1_p(\triangle)$ be the spline approximation of eigenfunction ϕ_j using Rayleigh-Ritz approximation method, $\phi_{j,\triangle} \to \phi_j$ very well for each $j = 1, \cdots, i$ in the sense that for $0 \le \ell \le s$,

$$|\phi_j - \phi_{j,\triangle}|_{\ell,2,\Omega} \le C|\triangle|^{s-\ell}|\phi_j|_{s,2,\Omega} \tag{4.7}$$

for a positive constant C independent of $\triangle$, since the spline space $S^1_p(\triangle)$ has the desired approximation power required in the proof of (4.7) (cf. [31]). It follows that $u_{i,\triangle} \to u_i$ and

$$\|\!|u_{i,\triangle} - u_i|\!\|_{1,k,\Omega} \le C|\triangle|^{s-1}(1 + k|\triangle|)\|f\|. \tag{4.8}$$

Indeed, we recall the Weyl law on the number $N(k^2)$ of Dirichlet eigenvalues less or equal to k^2 from [4] and use the formula for u_i in (2.11) to have

$$\begin{aligned}\|u_i - u_{i,\triangle}\| &\le \sum_{j=1}^{i} \frac{1}{k^2-\lambda_j}\|\phi_j - \phi_{j,\triangle}\| \\ &\le \frac{1}{k^2}C_1 N(k^2) C|\triangle|^s \max_{j=1,\cdots,i}|\phi_j|_{s,2,\Omega} \le B_1|\triangle|^s \max_{j=1,\cdots,i}|\phi_j|_{s,2,\Omega}\end{aligned}$$

for a positive constant B_1 dependent on $1 - \lambda_i/k^2$. Similarly, we have

$$\begin{aligned}\|\nabla(u_i - u_{i,\triangle})\| &\le \sum_{j=1}^{i} \frac{1}{k^2-\lambda_j}|\phi_j - \phi_{j,\triangle}|_{1,2,\Omega} \\ &\le \frac{1}{k^2}C_1 N(k^2) C|\triangle|^{s-1} \max_{j=1,\cdots,i}|\phi_j|_{s,2,\Omega} \le B_1|\triangle|^{s-1} \max_{j=1,\cdots,i}|\phi_j|_{s,2,\Omega}\end{aligned}$$

which leads to (4.8). Combining (4.6) and (4.8) completes the proof of Theorem 4.3.□

Let us point out that more detail on computation of eigenvalues and eigenfunctions of $-\Delta$ by using bivariate splines can be found in [19]. Mainly we can show that $\phi_{i,\triangle}$ is a spline weak solution to the eigenfunction equation.

In addition to the lower bound we have established in Theorem 2.5, we can find an estimate for the inf-sup condition. That is, we estimate the following inf-sup condition of $B(u, v)$. The following result was well-known. See, e.g. [10] for the domain which is strictly star-shaped.

THEOREM 4.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded strictly star-shaped domain. Then there exists $C > 0$ (independent of k) such that*

$$\inf_{v\in\mathbb{H}^1(\Omega)} \sup_{u\in\mathbb{H}^1(\Omega)} \frac{Re(B(u,v))}{\|\!|u|\!\|_{1,k,\Omega}\|\!|v|\!\|_{1,k,\Omega}} \ge \frac{C}{1+k}. \tag{4.9}$$

For convenience, we explicitly write down all the detail of a proof based on a standard approach for establishing the inf-sup condition in (4.9). That is, let us first prove the following

LEMMA 4.5. *For each $v \in \mathbb{H}^1(\Omega)$, there exists a $w_v \in \mathbb{H}^1(\Omega)$ such that*

$$Re(B(w_v, v)) \ge \alpha\|\!|v|\!\|^2_{1,k,\Omega} \text{ and } \|\!|w_v|\!\|_{1,k,\Omega} \le \beta\|\!|v|\!\|_{1,k,\Omega} \tag{4.10}$$

for positive constants α and β independent of v, w_v.

Once we have the result in (4.10), we can establish the inf-sup condition (4.9). Indeed,

of Theorem 4.4. It follows from (4.10) we have

$$\mathrm{Re}(B(w_v, v)) \geq \alpha \|v\|_{1,k,\Omega} \|w_v\|_{1,k,\Omega}/\beta$$

or

$$\sup_{u\in\mathbb{H}^1(\Omega)} \frac{\mathrm{Re}(B(u,v))}{\|u\|_{1,k,\Omega}\|v\|_{1,k,\Omega}} \geq \frac{\mathrm{Re}(B(w_v,v))}{\|w_v\|_{1,k,\Omega}\|v\|_{1,k,\Omega}} \geq \frac{\alpha}{\beta}.$$

Taking the inf both sides of the inequality above, we conclude the proof of (4.9). ∎

We now spend some time to prove Lemma 4.5.

Proof of Lemma 4.5. By Theorem 2.2, for each $v \in \mathbb{H}^1(\Omega)$, let $z_v \in \mathbb{H}^1(\Omega)$ be the solution to the Helmholtz equation (1.1) with $f = 2k^2 v$ and $g = 0$ satisfying

$$B(z_v, u) = 2k^2\langle v, u\rangle, \quad \forall u \in \mathbb{H}^1(\Omega).$$

We let $w_v = v + z_v \in \mathbb{H}^1(\Omega)$. To see the first inequality in (4.10), we have

$$\mathrm{Re}(B(w_v, v)) = \mathrm{Re}(B(v, v)) + \mathrm{Re}(B(z_v, v)) = a(v, v) - k^2\langle v, v\rangle + 2k^2\langle v, v\rangle = \|v\|^2_{1,k,\Omega}.$$

That is, the first inequality in (4.10) holds with $\alpha = 1$.

Next by using the stability in [5], i.e. $\|z_v\|_{1,k,\Omega} \leq C2k^2\|v\|$ for a positive constant C independent of k when $k \geq 1$, we have

$$\|w_v\|_{1,k,\Omega} \leq \|v\|_{1,k,\Omega} + \|z_v\|_{1,k,\Omega} \leq \|v\|_{1,k,\Omega} + Ck^2\|v\| \leq C(1 + k)\|v\|_{1,k,\Omega}$$

which is the second inequality in (4.10) with $\beta = C(1 + k)$. ∎

It is interesting to know the estimate for the inf-sup condition when domain Ω is not a strictly star-shaped domain. Using the Dirichlet eigenvalues, we can establish the following

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Suppose that k^2 is not a Dirichlet eigenvalue of $-\Delta$ over Ω. Then there exists $C > 0$ such that*

$$\inf_{v\in\mathbb{H}^1(\Omega)} \sup_{u\in\mathbb{H}^1(\Omega)} \frac{Re(B(u,v))}{\|u\|_{1,k,\Omega}\|v\|_{1,k,\Omega}} \geq L_4. \tag{4.11}$$

Furthermore, L_4 does not go to zero when $k \to \infty$.

Proof. Suppose (4.11) does not hold. Then there exists $v_n \in \mathbb{H}^1(\Omega)$ such that $\|v_n\|_{1,k,\Omega} = 1$ and

$$\sup_{u\in\mathbb{H}^1(\Omega)} \frac{\mathrm{Re}(B(u,v_n))}{\|u\|_{1,k,\Omega}} \leq \frac{1}{n}$$

for $n = 1, \cdots, \infty$. The boundedness of v_n implies that there exists a convergent subsequence by Rellich-Kondrachov Theorem. Without loss of generality we may assume that $v_n \to v^*$ in $\mathbb{L}^2(\Omega)$ norm and in the semi-norm on $\mathbb{H}^1(\Omega)$ with $\|v^*\|_{1,k,\Omega} = 1$. It follows that for each $u \in \mathbb{H}^1(\Omega)$ with $\|u\|_{1,k,\Omega} = 1$, $\mathrm{Re}(B(u, v_n)) \to 0$. Hence, $\mathrm{Re}(B(u, v^*)) = 0$. By using $u = -\mathbf{i}v^*$, we see that $\mathrm{Re}(B(u, v^*)) = \langle v^*, v^*\rangle_\Gamma = 0$. So $v^* = 0$ on Γ. That is, $v^* \in \mathbb{H}^1_0(\Omega)$. It follows that $\mathrm{Re}(B(u, v^*)) = 0$ for all $u \in \mathbb{H}^1_0(\Omega)$. So v^* is an eigenfunction with eigenvalue k^2 which contradicts to the assumption. Hence, we have $L_4 > 0$ in (4.11).

Next let us show that $L_4 \not\to 0$ as $k \to \infty$. As L_4 is dependent on k, let us write L_k for convenience. Since the lower bound $L_k > 0$, we can find v_k with $|||v_k|||_{1,k,\Omega} = 1$ such that

$$\sup_{u \in \mathbb{H}^1(\Omega)} \frac{\mathrm{Re}(B(u, v_k))}{|||u|||_{1,k,\Omega}} \leq 2L_k. \tag{4.12}$$

That is, $\mathrm{Re}(B(v_k, v_k)) \leq 2L_k$. Since $|||v_k|||_{1,k,\Omega} = 1$, we use Rellich-Kondrachov Theorem again to conclude that there exists a $u^* \in \mathbb{H}^1(\Omega)$ such that $v_k \to u^*$ in L^2 norm and $\|\nabla v_k\| \to \|\nabla u^*\|$ without loss of generality. As $k^2\|v_k\|^2 \leq 1$, i.e. $\|v_k\| \leq 1/k$, we have $\|u^*\| \leq 2/k$ for $k > 0$ large enough. It follows that $u^* \equiv 0$. That is, $\nabla u^* \equiv 0$ and hence, $\|\nabla v_k\| \to 0$.

If $L_k \to 0$, we use (4.12) have $|\|\nabla v_k\|^2 - k^2\|v_k\|^2| = |\mathrm{Re}((B(v_k, v_k))| \to 0$. Since $\|\nabla v_k\| \to 0$ mentioned above, it follows that $k^2\|v_k\|^2 \to 0$. However, since $|||v_k|||_{1,k,\Omega} = 1$, we should have $k^2\|v_k\|^2 \to 1$. That is, we got a contradiction. Therefore, L_k does not go to zero when $k \to \infty$. $\square$

Again, it is difficult to determine how the constant L_4 in (4.11) is dependent on k. According to the study in the above, L_4 may be dependent only on $1 - k/\sqrt{\lambda_{i+1}}$ or $1 - \lambda_i/k^2$ instead of $1/(k+1)$ in (4.9). Anyway, another main result in this paper is the following

Theorem 4.7. *Let Ω be a bounded Lipschitz domain. Suppose that k^2 is not a Dirichlet eigenvalue of $-\Delta$ over Ω. Let u be the unique weak solution in $\mathbb{H}^1(\Omega)$ satisfying (2.1) and $u_\triangle \in \mathbb{S}^r_p(\triangle), p \geq 3r + 2$ be the spline weak solution to (1.1) satisfying (3.3). Then if $u \in \mathbb{H}^s(\Omega)$ with $1 \leq s \leq p$, there exists $C > 0$ such that*

$$|||u - u_\triangle|||_{1,k,\Omega} \leq C(1 + k|\triangle|)|\triangle|^{s-1}|u|_{s,2,\Omega}, \tag{4.13}$$

where $|u|_{s,2,\Omega}$ is the semi-norm in $\mathbb{H}^s(\Omega)$ and C is dependent on L_4. If $\Omega \subset \mathbb{R}^2$ is a bounded strictly star-shaped domain, then the approximation constant C in (4.13) can be more precisely written as $C = c(1+k)$ for a positive constant c independent k.

Proof. We simply use the inf-sup condition, i.e. Theorem 4.4 to have

$$|||u - u_\triangle|||^2_{1,k,\Omega} \leq c(1 + k)Re(B(u - u_\triangle, u - u_\triangle)).$$

for a positive constant c independent of k when Ω is a strictly star-shaped domain. It follows from (2.1) and (3.3) the orthogonality condition:

$$a(u - u_\triangle, v) - k^2\langle u - u_\triangle, v\rangle + \mathbf{i}\langle u - u_\triangle, v\rangle_{\partial\Omega} = 0, \quad \forall v \in \mathbb{S}^0_p(\triangle). \tag{4.14}$$

That is, $B(u - u_\triangle, v) = 0$ for all $v \in \mathbb{S}^1_p(\triangle)$. By choosing $v = Q_p(u)$, the quasi-interpolatory spline of u, we have

$$Re(B(u - u_\triangle, u - u_\triangle)) = Re(B(u - u_\triangle, u - Q_p(u))) \leq C_B|||u - u_\triangle|||_{1,k,\Omega}|||u - Q_p(u)|||_{1,k,\Omega}.$$

It follows that

$$|||u - u_\triangle|||_{1,k,\Omega} \leq c(1 + k)|||u - Q_p(u)|||_{1,k,\Omega}. \tag{4.15}$$

Finally, we use the approximation property of spline space $\mathbb{S}^r_p(\triangle)$, i.e. (3.4). For $u \in \mathbb{H}^s(\Omega)$ with $1 \leq s \leq p$, we use the quasi-interpolatory operator $Q_p(u)$ of u to have

$$|||u - Q_p(u)|||_{1,k,\Omega} \leq C(1 + k|\triangle|)|\triangle|^{s-1}|u|_{s,2,\Omega}$$

for a constant C dependent on Ω, p and the smallest angle of $\triangle$ only. With the term above, we can rewrite (4.15) as follows:

$$\|u - u_\triangle\|_{1,k,\Omega} \leq C(1+k)(1+k|\triangle|)|\triangle|^{s-1}|u|_{s,2,\Omega}. \tag{4.16}$$

for another positive constant C. These complete the proof of Theorem 4.7.

If we use Theorem 4.6 in the place of Theorem 4.4 above, we can get the estimate in (4.13). These complete the proof. ∎

Finally, another approach to studying the convergence of the spline method is to use the duality arguments. In this setting, we shall assume that Ω is convex or has a C^2 smooth boundary. For convenience, let $e_\triangle = u - u_\triangle$ be the error function of the spline approximation of u. Then we know that there exists a unique solution $\psi \in \mathbb{H}^1(\Omega)$ of the Helmholtz problem (1.1) with $f = k^2 e_\triangle$ and $g = 0$ satisfying

$$B(\psi, u) = \langle k^2 e_\triangle, u\rangle, \quad u \in \mathbb{H}^1(\Omega). \tag{4.17}$$

Since $u_\triangle \in \mathbb{S}^1_p(\triangle)$, we have $\nabla(e_\triangle) \in \mathbb{H}^1(\Omega)^2$. Hence, $\nabla\psi$ satisfies the following weak formulation of the Helmholtz equation:

$$B(\nabla\psi, u) = \langle k^2 \nabla e_\triangle, u\rangle, \quad u \in \mathbb{H}^1(\Omega). \tag{4.18}$$

Since Ω is either convex or has a smooth $C^{1,1}$ boundary, we see that $\nabla\psi \in \mathbb{H}^2(\Omega)$ by using an argument in the proof of Theorem 2.7, mainly, the estimate in (2.15) and satisfy

$$|\nabla\psi|_{H^2(\Omega)} \leq C_\Omega(1+k)k^2\|\nabla e_\triangle\|_{L^2(\Omega)} \text{ or } |\psi|_{H^3(\Omega)} \leq C_\Omega(1+k)k^2\|\nabla e_\triangle\|_{L^2(\Omega)}, \tag{4.19}$$

for a positive constant C_Ω.

As $\psi \in \mathbb{H}^3(\Omega)$, we can use the quasi-interpolant $Q_p(\psi) \in \mathbb{S}^1_p(\triangle)$ to have

$$\|\psi - Q_p(\psi)\|_{L^2(\Omega)} \leq C|\triangle|^3|\psi|_{H^3(\Omega)} \text{ and } \|\nabla(\psi - Q_p(\psi))\|_{L^2(\Omega)} \leq C|\triangle|^2|\psi|_{H^3(\Omega)} \tag{4.20}$$

by using Theorem 3.1.

By using (4.17), the orthogonality (4.14), the approximation property (4.20), and (4.19), we apply Lemma 2.4 to have

$$\begin{aligned}
k^2\langle e_\triangle, e_\triangle\rangle &= \mathrm{Re}(B(e_\triangle,\psi) = \mathrm{Re}(B(e_\triangle, \psi - Q_p(\psi))) \\
&\leq C_B\|e_\triangle\|_{1,k,\Omega}\,\|\psi - Q_p(\psi))\|_{1,k,\Omega} \\
&\leq C_B\|e_\triangle\|_{1,k,\Omega}\,C|\triangle|^2(1+k|\triangle|)|\psi|_{H^3(\Omega)} \\
&\leq C\|e_\triangle\|_{1,k,\Omega}\,|\triangle|^2(1+k|\triangle|)C_\Omega(1+k)k^2\|\nabla e_\triangle\|_{L^2(\Omega)} \\
&\leq C\|e_\triangle\|^2_{1,k,\Omega}\,|\triangle|^2(1+k|\triangle|)(1+k)k^2
\end{aligned}$$

for positive constants C which may be different in different lines above and all these constants are independent of k and $h = |\triangle|$. If $C|\triangle|^2k^2(1+k) := \alpha < 1/2$, we have

$$\begin{aligned}
\|e_\triangle\|^2_{1,k,\Omega} &= \mathrm{Re}(B(e_\triangle, e_\triangle)) + 2k^2\langle e_\triangle, e_\triangle\rangle \\
&= \mathrm{Re}(B(e_\triangle, u - Q_p(u))) + 2\alpha\|e_\triangle\|^2_{1,k,\Omega} \\
&= C\|e_\triangle\|_{1,k,\Omega}\|u - Q_p(u)\|_{1,k,\Omega} + 2\alpha\|e_\triangle\|^2_{1,k,\Omega}
\end{aligned}$$

for a positive constant C which is independent of k. As $2\alpha < 1$, we have

$$\|e_\triangle\|_{1,k,\Omega} \le \frac{C}{1-2\alpha}\|u - Q_p(u)\|_{1,k,\Omega}. \tag{4.21}$$

By the approximation results, $\|u - Q_p(u)\|_{1,k,\Omega} \le C|\triangle|^{m-1}(1+k|\triangle|)|u|_{m,2,\Omega}$, we are able to finish the proof of the results in Theorem 4.8 by using (4.21).

THEOREM 4.8. *Let $\Omega \subset \mathbb{R}^2$ be a convex domain with $C^{1,1}$ boundary. Suppose that k^2 is not a Dirichlet eigenvalue. Let u be the unique weak solution in $\mathbb{H}^1(\Omega)$ satisfying (2.1) and $u_\triangle \in \mathbb{S}_p^r(\triangle), p \ge 3r+2$ be the spline weak solution to (1.1) satisfying (3.3). Then if $u \in \mathbb{H}^m(\Omega)$ with $1 \le m \le p$, there exists $C > 0$ such that*

$$\|u - u_\triangle\|_{1,k,\Omega} \le \frac{C}{1-\alpha}(1+k|\triangle|)|\triangle|^{m-1}|u|_{m,2,\Omega}, \tag{4.22}$$

with $\alpha = 2C_\Omega|\triangle|^2k^2(1+k) < 1$, where $|u|_{m,2,\Omega}$ is the semi-norm in $\mathbb{H}^m(\Omega)$.

That is, when $h = |\triangle|$ is sufficiently small, the approximation constant is indeed independent of k as we have seen from our numerical experimental results in the next section.

5. Numerical Solution to Helmholtz Equations. In this section, we shall present our computational method and then report some numrical results. Our computational algorithm is given as follows. For spline space $\mathbb{S}_p^1(\triangle)$, let $\mathbf{c}$ be the coefficient vector associated with each spline function $s \in \mathbb{S}_p^1(\triangle)$. In the implementation explained in [3], $\mathbf{c}$ is a stack of the polynomial coefficients over each triangle in $\triangle$. Let H be the smoothness matrix such that $s \in \mathbb{S}_p^1(\triangle)$ if and only if $H\mathbf{c} = 0$. Next let $\mathbf{f}$ and $\mathbf{g}$ be the vectors of coefficients for the spline approximations for the source functions f and g, respectively. Let M and K be the mass and stiffness matrices as in [3]. Then the spline solution to (3.3) can be given in terms of these matrices as follows:

$$\overline{\mathbf{c}}^\top K\mathbf{c}_\triangle - k^2\overline{\mathbf{c}}^\top M\mathbf{c}_\triangle + \mathbf{i}\overline{\mathbf{c}}^\top M_\Gamma\mathbf{c}_\triangle = \overline{\mathbf{c}}^\top M\mathbf{f} + \overline{\mathbf{c}}^\top M_\Gamma\mathbf{g}, \quad \forall \mathbf{c} \in \mathbb{R}^N \tag{5.1}$$

for $\mathbf{c}$ and $\mathbf{c}_\triangle$ which satisfies $H\mathbf{c}_\triangle = 0$ and $H\mathbf{c} = 0$, where $\overline{\mathbf{c}}$ is the standard conjugate of $\mathbf{c}$ and $N = (d+1)(d+2)T/2$, where T is the number of triangles in $\triangle$. To solve this constrained system of linear equations, we use the so-called the constrained iterative minimization method described in [3]. That is, we solve the following constrained minimization:

$$\min_{\mathbf{c}} \frac{1}{2}(\overline{\mathbf{c}}^\top K\mathbf{c} - k^2\overline{\mathbf{c}}^\top M\mathbf{c} + \mathbf{i}\overline{\mathbf{c}}^\top M_\Gamma\mathbf{c}) - \overline{\mathbf{c}}^\top M\mathbf{f} - \overline{\mathbf{c}}^\top M_\Gamma\mathbf{g}, \tag{5.2}$$

subject to $H\mathbf{c} = 0$, where M_Γ is the mass matrix over the boundary. The constrained iterative minimization method in [3] provides an efficient way to find the solution of the minimization above. In fact, we only do 3 iterations. This is a difference between our spline method and the standard high order finite element method. It makes spline solutions more accurate for the Helmholtz problem with high wave numbers.

Next we report numerical results based on our bivariate spline functions. Solving the Helmholtz equations with bivariate splines offers advantages over the existing finite element framework including high order FEM, internal penalty and hybridized discontinuous Galerkin methods, and weak Galerkin methods. Our implementation is relativley straightforward and allows us to find accurate spline solutions of arbitrary degree and smoothness for problems involving large wave numbers. More precisely,

(1) we are able to solve the Helmholtz problem with large wave number $1 \leq k \leq 500$ using our splines of degree $p \geq 5$ and $h = 1/64$ using a laptop computer; using a high memory (1000GB) node from the Sapelo 2 cluster at University of Georgia, we are able to find accurate solutions to the Helmholtz equation with wave numbers from 500–1500 by using spline functions of degree 12 and $h = 1/100$. For example, the calculation Example 5.4 for $k = 1500$ used just under 630GB of memory.

(2) our numerical evidence strongly shows that $hk \leq d/2$ will enable us to find very accurate approximation in H^1 norm for various wave numbers $k = 500-1000$ and degrees $d = 5, \cdots, 17$, and no pollution error occurs. See Example 5.3.

(3) we are able to solve the Helmholtz problem with accurate numerical solutions over domains which are not strictly star-shaped. See Example 5.8.

We shall present some simulation results based on an exact solution and see how accurate our bivariate spline method can be. Some comparisons with the weak-Galerkin method will be shown, and spline solutions to the Helmholtz problem (1.1) for large wave numbers k will be demonstrated. In the following examples, we solve the following 2D Helmholtz problem:

$$\begin{cases} -\Delta u - k^2 u & = f, \text{ in } \Omega, \\ \alpha(\nabla u \cdot \mathbf{n}) + \beta u & = g, \text{ on } \Gamma = \partial\Omega \end{cases} \tag{5.3}$$

over various domains Ω and for various $k \geq 1$.

Example 5.1. For Example 1 (5.1,...,5.5), we take Ω to be unit regular hexagon with center $(0,0)$ as seen in [11] and [27]. Here we take $f = \frac{\sin(kr)}{r}$, $\alpha = 1$, $\beta = \mathbf{i}k$, and g is chosen so that the exact solution is given by:

$$u = \frac{\cos(kr)}{k} - \frac{\cos(k) + \mathbf{i}\sin(k)}{k(J_0(k) + \mathbf{i}J_1(k))} J_0(kr)$$

in polar coordinates, where $J_\nu(z)$ are Bessel function of the first kind and $r = \sqrt{x^2+y^2}$.

In Fig. 1 we show plots of the spline solution $u_s \in S_9^1$ (real and imaginary parts) to Eq. (5.3) with wave number $k = 100$. We also use spline functions in S_d^1 degree $d = 5, ..., 17$ to approximate the solution over the domain shown in Fig. 1, left. The relative errors in the L^∞ norm as well as the root mean square error based on 67201 equally-spaced points within Ω are shown in Table 1 ($k = 100$) and Table 2 ($k = 300$).

TABLE 1
The relative L^2 and H^1 root mean square errors and relative maximum errors based on C^1 spline functions of various degrees (wave number =100)

d	h	rel. L2 error	rel. H1 error	ℓ_∞ error	$\lvert u\rvert_{1,\infty}$
5	0.125	1.3487e+00	1.3455e+00	7.1285e-02	4.2410e+00
7	0.125	7.1281e-01	7.2441e-01	7.6888e-02	6.0462e+00
9	0.125	3.0860e-02	3.5515e-02	2.3120e-03	1.5114e-01
11	0.125	9.8578e-04	1.8734e-03	7.0982e-05	1.1142e-02
13	0.125	4.4347e-05	1.3758e-04	2.5585e-06	1.3807e-03
15	0.125	2.5829e-06	4.8840e-06	1.2549e-07	3.5538e-05
17	0.125	1.2643e-07	2.4334e-07	5.8799e-09	1.3866e-06

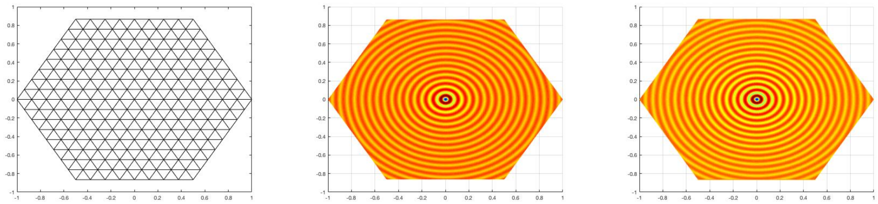

FIG. 1. *Real part of the spline solution $u_s \in S^1_9$ (left) and imaginary part (right)*

TABLE 2
L^2 and H^1 root mean square errors and relative maximum errors based on C^1 spline functions of various degrees for wave number k=200

d	h	rel. L2 error	rel. H1 error	ℓ_∞ error	$\|u\|_{1,\infty}$
5	0.063	1.3054e+00	1.4632e+00	2.9566e-02	1.0032e+01
7	0.063	1.0274e+00	1.0207e+00	5.4012e-02	1.6182e+01
9	0.063	5.0853e-02	5.7129e-02	1.8468e-03	5.6149e-01
11	0.063	1.1197e-03	1.5021e-03	4.3664e-05	1.2068e-02
13	0.063	4.8710e-05	1.0916e-04	1.7632e-06	9.6094e-04
15	0.063	2.2405e-06	4.8859e-06	7.6330e-08	3.4396e-05
17	0.063	1.0917e-07	2.5422e-07	3.8012e-09	2.5664e-06

Example 5.2. I think we should remove table 3, or this example. Table 3's format doesn't match the others We next solve (5.3) again over the unit regular hexagon with center at $(0, 0)$ (as shown in Fig. 1) for large wave number $k = 400$ and $k = 500$. We use uniformly refined triangulations to find spline solutions of (5.3) which accurately approximate the exact solution for $k = 400$ and $k = 500$ as shown in Table 3 and Table 4, respectively.

TABLE 3
Spline Solutions (d=10, r=1) of Helmholtz Equation with wave number $k = 400$

wave no.	h	d	rel. L2 error	rel. H1 error	RMSE$(u - u_\triangle))$	RMSE$(\nabla(u - u_\triangle))$
400	0.250	10	5.768511	9.815614	0.049095	0.820569
400	0.125	10	4.190938	4.797113	0.049276	0.826168
400	0.063	10	1.330066	1.329840	0.060649	1.018848
400	0.031	10	0.011629	0.013092	0.005256	0.093720
400	0.016	10	0.000081	0.000082	0.000439	0.007419

Example 5.3. In this example, we report the relative L^2 and H^1 error results with the size of the triangulation h chosen so that that the wave number and size of mesh satisfy $kh = p/2$ in Table 5. As we use the spline space S^1_8, so that $kh = 4$ for various k. Our numerical results in Table 5 suggest that there is no pollution error phenomenon or the pollution error is well controlled in our spline method.

Example 5.4. In this example, we show the accuracy of spline solutions for high wave numbers. Again, we solve (5.3) over the unit hexagon. We report the relative errors for our spline solutions with $d = 10$, $r = 1$ with high wave numbers in Table 6 and $d = 12$ and $r = 1$ in Table 7.

Example 5.5. To see the degrees of freedom when solving (5.3), let us present two

TABLE 4
Spline Solutions (d=12, r=1) of Helmholtz Equation with wave number $k = 500$

wave no.	h	d	rel. L2 error	rel. H1 error	ℓ_∞ error	$\|u\|_{1,\infty}$ error
500	0.125	12	1.4541e+00	1.2967e+00	1.0588e-02	9.0560e+00
500	0.062	12	1.1921e+00	1.1743e+00	1.4517e-02	7.6721e+00
500	0.031	12	6.3515e-03	8.6685e-03	7.6444e-05	7.8923e-02
500	0.016	12	9.8523e-08	8.7072e-07	1.5062e-09	7.3869e-06

TABLE 5
Accuracy of spline approximation for fixed $kh = 4$

wave no. k	h	kh	rel. L2 error	rel. H1 error	ℓ_∞ error	$\|u\|_{1,\infty}$ error
60	0.067	4	3.3041e-06	1.3658e-05	3.5912e-07	6.4930e-05
120	0.033	4	3.6576e-06	1.1103e-05	1.5460e-07	6.5157e-05
180	0.022	4	1.6801e-06	1.4977e-05	7.8387e-08	5.8730e-05
240	0.017	4	1.6748e-06	1.5134e-05	6.8373e-08	6.9122e-05
300	0.013	4	1.6735e-06	1.5231e-05	6.1516e-08	7.8338e-05
360	0.011	4	1.6738e-06	1.5294e-05	5.6416e-08	8.6716e-05
420	0.010	4	1.6725e-06	1.5342e-05	5.2400e-08	9.4442e-05

tables for our spline method with the weak Galerkin method in [27]. For wave number $k = 1$, we compare the accuracy of spline solutions from the space S^1_5 to piecewise constant weak Galerkin solutions (relative error results from [27]) along with degree of freedom counts. For the piecewise constant WG method, we calculate the degrees of freedom by $dof_{cwg} = \#(E) + \#(T)$. For splines in $S^1_5(\triangle)$, we report an only upper bound on the degrees of freedom for convenience; $dof_{S^1_5} < 2\#(V) + \#(E)(d-1) + \#(E)(d-3)$. We write $\#(V), \#(E)$, and $\#(T)$ to denote the number of vertices, edges, and triangles in a given triangulation. The numerical results are shown in Table 8.

We also compare our spline method with piecewise linear weak Galerkin method using the results in [27]. In Table 9, a comparison of relative errors of the solutions of spline S^1_5 and piecewise linear weak Galerkin solutions from [27] is shown, along with degree of freedom counts. For piecewise linear WG, we calculate $dof_{lwg} = 2\#(E) + 3\#(T)$. The spline method provides a more accurate solution using far fewer degrees of freedom.

Example 5.6. Let us consider the Helmholtz boundary value problem over a non-convex domain, shown left in Fig. 2. For this example, $\alpha = 1$, $\beta = \mathbf{i}k$ and source functions f and g are chosen so that the analytic solution to (5.3) is given by

$$u = J_\xi(kr)\cos(\xi\theta).$$

As above, r and θ are the usual polar coordinates, k is the wavenumber, and J_ξ is a Bessel function of the first kind. This is another standard testing function studied in [27] and [17]. We study three situations where $\xi = 1$, $3/2$, and $2/3$. Plots of the spline solutions from S^1_5 for $k = 4$ and $k = 20$ are shown in Fig. 2– 4. We summarize numerical results for each of these three cases in Tables 10, 12, and 11.

Example 5.7. Certainly, we are interested in exploring numerical solution to a nonconvex domain with larger wave numbers $k = 100, 200, 300$. As referenced in [27], the computation for $\xi = 2/3$ is more challenging than the case where $\xi = 1$ and $\xi = 3/2$, but the spline solution in S^1_{10} is nonetheless highly accurate. Fig. 5 shows

TABLE 6
Accuracy of spline solution in S^1_{10} for various large wave numbers

wavenumber k	h	rel. L2 error	rel. H1 error	ℓ_∞ error	$\lvert u\rvert_{1,\infty}$ error
500	0.016	5.4581e-06	6.7180e-05	1.1751e-07	5.9545e-04
600	0.016	1.3501e-04	4.0429e-04	1.7397e-06	2.6984e-03
700	0.016	2.0630e-03	2.5532e-03	3.3968e-05	1.6688e-02
800	0.008	8.0733e-07	7.6723e-06	1.3186e-08	6.6425e-05
900	0.008	1.9449e-06	2.3920e-05	3.5339e-08	1.8619e-04
1000	0.008	7.3629e-06	6.7027e-05	9.8781e-08	8.5276e-04

TABLE 7
Accuracy of spline solution in S^1_{12} for various large wave numbers

wavenumber k	h	rel. L2 error	rel. H1 error	ℓ_∞ error	$\lvert u\rvert_{1,\infty}$ error
1100	0.0100	2.9026e-05	4.9725e-05	1.8495e-07	4.6325e-04
1200	0.0100	8.5032e-05	1.3023e-04	4.5849e-07	9.1309e-04
1300	0.0100	3.8509e-04	4.3707e-04	5.2119e-06	2.0724e-03
1400	0.0100	1.9326e-03	1.9489e-03	2.3369e-05	1.7926e-02
1500	0.0100	8.3163e-03	8.2431e-03	5.3503e-05	1.2119e-01

graphs of the spline solutions to the BVP with $\xi = 1$, left, and $\xi = 3/2$, right, where $k = 100$. Relative errors are given in the plots; all relative L^2 and H^1 errors are on the order of 10^{-3}.

In Fig. 6, graphs of the spline solutions in $S^1_{10}(\triangle)$ to the more difficult BVP with $\xi = 2/3$ are shown for wave number $k = 200$, left, and $k = 300$, right. Relative errors are given in the plots; all relative L^2 and H^1 errors are on the order of 10^{-2}.

Example 5.8. [Domain with Hole] Here we reproduce example 2 from [32], solving the Helmholtz equation (5.3) with Dirichlet boundary condition ($\alpha = 0$, $\beta = 1$) and exact solution

$$u(x, y) = e^{-\mathbf{i}kx} \tag{5.4}$$

over an interesting domain with hole. In Fig. 7, we show the real and imaginary spline solutions for wavenumber $k = 20$. The plots are visually indistinguishable from those of the exact solution.

We also solve (5.3) with Robin boundary condition as in the previous examples with $\alpha = 1$, $\beta = \mathbf{i}k$ and $g(x, y)$ is induced by the exact solution (5.4). Table 13 show that the errors for spline solutions from the space $S^1_6(\triangle)$ to the Helmholtz problem with wave number $k = 30$.

6. Future Research Problems.

Remark 6.1. We have assumed Ω is convex or is a bounded domain with $C^{1,1}$ boundary. This requirement can be weakened by using the new condition called domain with positive reach as explained in [14]. Under the positive reach condition, the solution of Poisson equation will be in $H^2(\Omega)$. Similarly, the solution to Helmholtz equation will be in $H^2(\Omega)$. We leave the details for future study.

Remark 6.2. We are working to extend these results to the 3D setting using trivariate splines of arbitrary degree and smoothness. Numerical results will be reported soon. See [26].

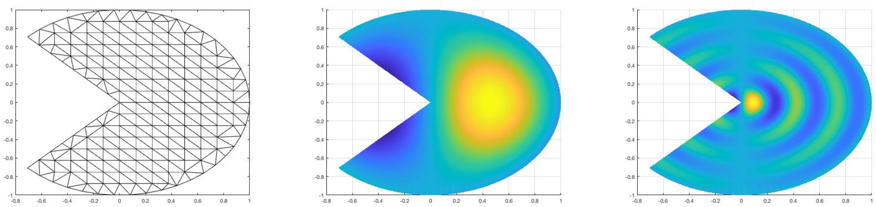

FIG. 2. *Triangulation of the domain and spline solution $u_s \in S_5^1$ to the non-convex Helmholtz problem with exact solution $u = J_\xi(kr)\cos(\xi\theta)$, with $\xi = 1$ and $k = 4$ (left) and $k = 20$ (right)*

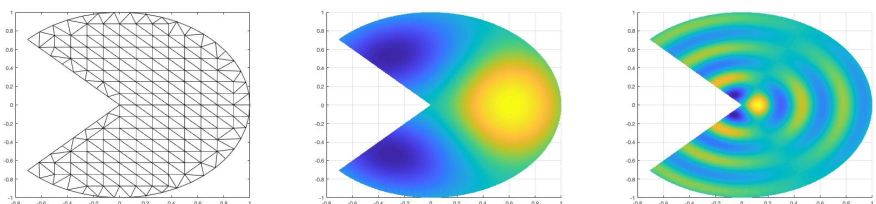

FIG. 3. *Spline solution from S_5^1 to non-convex Helmholtz problem with exact solution $u = J_\xi(kr)\cos(\xi\theta)$, where $\xi = 3/2$ and $k = 4$ (left) and $k = 20$ (right)*

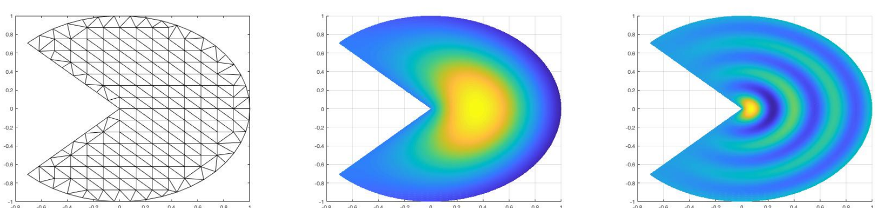

FIG. 4. *Spline solution to non-convex Helmholtz problem with exact solution $u = J_\xi(kr)\cos(\xi\theta)$, with $\xi = 2/3$ and $k = 4$ (left) and $k = 20$ (right).*

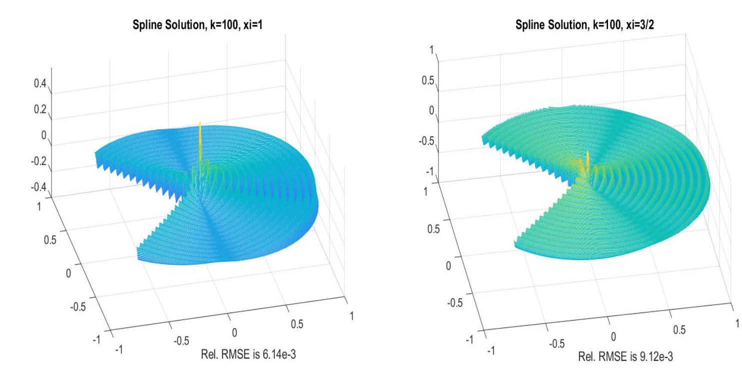

FIG. 5. *Spline solution in $S_{10}^1(\triangle)$ to the Helmholtz problem with exact solution $u = J_\xi(kr)\cos(\xi\theta)$ and wave number $k = 100$ and $\xi = 1$(left), and $\xi = 3/2$ (right)*

Table 8
Comparison of our spline method with piecewise constant weak Galerkin method

	piecewise constant WG			Spline S_5^1		
$\|\triangle\|$	rel. L2 error	rel. H1 error	dof	rel. L2 error	rel. H1 error	dof
1.000	-	-	-	9.285e-07	1.948e-05	98
0.500	4.170e-03	2.490e-02	66	1.672e-08	6.819e-07	332
0.250	1.050e-03	1.110e-02	252	6.635e-10	2.224e-08	1214
0.125	2.630e-04	5.380e-03	984	-	-	-
0.062	6.580e-05	2.670e-03	3888	-	-	-
0.031	1.650e-05	1.330e-03	15456	-	-	-
0.016	4.110e-06	6.650e-04	61632	-	-	-

Table 9
Comparison of our spline method with piecewise constant line Galerkin method

	linear WG			Spline S_5^1		
$\|\triangle\|$	rel. L2 error	rel. H1 error	dof	rel. L2 error	rel. H1 error	dof
1.000	-	-	-	4.287e-03	1.489e-02	98
0.500	-	-	-	1.183e-04	7.197e-04	332
0.250	2.580e-04	9.480e-03	600	2.019e-06	2.546e-05	1214
0.125	3.460e-05	2.310e-03	2352	3.525e-08	8.609e-07	4634
0.062	4.470e-06	5.740e-04	9312	2.411e-09	2.866e-08	18098
0.031	5.640e-07	1.430e-04	37056	-	-	-
0.016	7.060e-08	3.580e-05	147840	-	-	-
0.008	8.790e-09	8.960e-06	590592	-	-	-

Remark 6.3. When extending the study in the 3D setting, a major difficulty is the approximation order of trivariate spline spaces. That is, the similar results to the bivariate spline setting in [20] are not available. We shall leave them to our study on trivariate splines for Helmholtz equation and Maxwell's equations.

Remark 6.4. As pointed out in several places in previous sections, the dependence of constants L and L_4 on wave number k is not clear when the domain Ω is not a strictly star-shaped domain. It is interesting to find out. The authors will continue their study on this issue.

Remark 6.5. Several estimates discussed in previous sections are dependent on whether the number k^2 is a Dirichlet eigenvalue or not. As the theory of the existence and uniqueness to Helmholtz equation (1.1) has no such requirement, it is interesting to remove such a condition. For example, it is also interesting to extend the stability result in Theorem 4.6 when k^2 is a Dirichlet eigenvalue.

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TABLE 10
Numerical results of spline approximation $\in S_5^1$ over nonconvex domain with $\xi = 1$

	wavenumber=4		wavenumber=20	
$\lvert\Delta\rvert$	rel. L2 error	rel. H1 error	rel. L2 error	rel. H1 error
1.0000	1.1242e-03	4.6766e-03	1.3420e+00	1.6892e+00
0.5000	2.0562e-04	8.2798e-04	8.9020e-01	8.7483e-01
0.2500	3.4424e-06	3.0885e-05	1.0677e-01	1.1434e-01
0.1250	8.1231e-08	1.1162e-06	1.6385e-03	4.1769e-03
0.0625	-	-	2.0492e-05	1.3421e-04
0.0312	-	-	6.5958e-06	8.2274e-06

TABLE 11
Numerical results of spline approximation $\in S_5^1$ over nonconvex domain with $\xi = 3/2$

	wavenumber=4		wavenumber=20	
$\lvert\Delta\rvert$	rel. L2 error	rel. H1 error	rel. L2 error	rel. H1 error
1.0000	1.0993e-01	1.4599e-01	1.9511e+00	2.3335e+00
0.5000	2.7023e-03	2.1935e-02	1.0055e+00	1.0333e+00
0.2500	1.0796e-03	5.7067e-03	4.4131e-02	6.5030e-02
0.1250	2.2659e-04	2.0220e-03	6.4977e-03	1.1583e-02
0.0625	4.9490e-05	7.1555e-04	1.3156e-03	3.4129e-03
0.0312	1.0926e-05	2.2876e-04	2.8728e-04	1.0494e-03

TABLE 12
Numerical results of spline approximation $\in S_5^1$ over nonconvex domain with $\xi = 2/3$

	wavenumber=4		wavenumber=20	
$\lvert\Delta\rvert$	rel. L2 error	rel. H1 error	rel. L2 error	rel. H1 error
0.5000	9.1279e-03	5.3100e-02	1.4984e+00	1.5024e+00
0.2500	3.3169e-03	3.0808e-02	9.5122e-01	9.4475e-01
0.1250	1.2753e-03	1.8893e-02	8.1904e-03	2.6594e-02
0.0625	4.9854e-04	1.0827e-02	3.1416e-03	1.4909e-02
0.0312	1.9433e-04	5.6974e-03	1.2276e-03	7.7787e-03

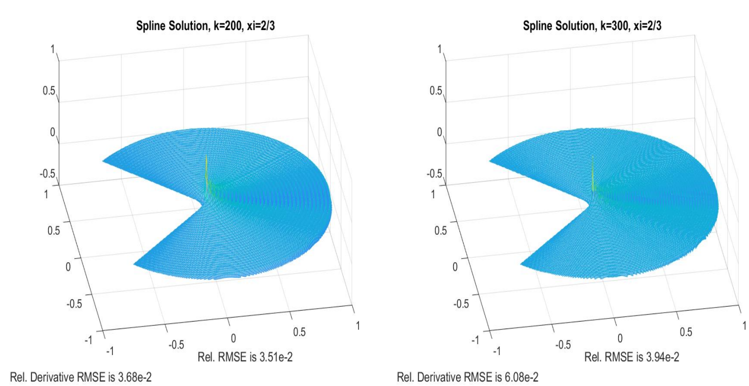

FIG. 6. *Spline solution $\in S_1^1 0$ to the Helmholtz problem with exact solution $u = J_\xi(kr)\cos(\xi\theta)$ and and $\xi = 2/3$. The wave numbers are $k = 200$ (left) and $k = 300$(right).*

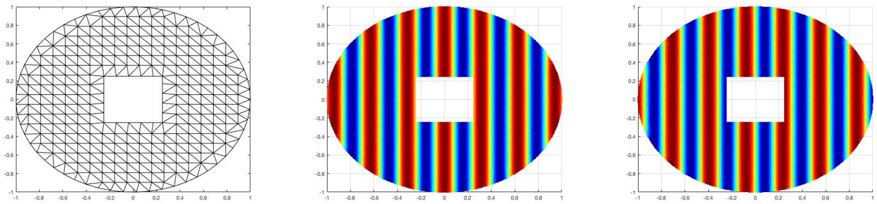

FIG. 7. *Triangulation with $h = 1/8$ (left); real part (center) and imaginary part (right) of $d = 10$, $r = 1$ spline solution to non-convex Helmholtz problem with exact solution $u = e^{-ikx}$, $k = 20$.*

TABLE 13
Error table for spline solutions in S_6^1 to the Helmholtz problem with Robin boundary condition and wave number $k = 30$.

$\|\Delta\|$	rel. L2 error	L2 order	rel. H1 error	H1 order
0.2500	1.105e-01	–	1.305e-01	–
0.1250	4.432e-04	7.962	1.719e-03	6.247
0.0625	3.094e-06	7.162	2.370e-05	6.180
0.0312	1.888e-08	7.356	3.338e-07	6.150

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