Removal of Gaps among Compound C^1 Bi-Cubic Parametric B-spline Surfaces

Ming-Jun Lai, Jian-ao Lian, and Patrick F. Cassidy

Abstract. Manipulation of control points is a standard procedure in parametric B-spline surface design. Unfortunately, this tool alone is not sufficient to achieve certain goals such as removal of gaps among compound parametric B-spline surfaces. In this paper, we introduce a new approach for removing gaps but maintaining at least the geometrical smoothness condition. If the underlying knot sequences along the connecting boundaries of two parametric B-spline surfaces are proportional, they can be connected in a G^1 fashion without changing the structure of B-spline representation. Otherwise, they can still be connected in a G^1 fashion, but with one requirement, namely, by sacrificing the true B-spline representation along a boundary strip of one of the two B-spline surfaces. Our method involves manipulation of Bézier coefficients along a boundary strip, and we demonstrate the feasibility of this approach by considering the C^1 cubic setting. This procedure is generalized to connecting up to four B-spline surfaces with a common corner.

§0. Prelude

This paper represents a joint work with Charles Chui under the sponsorship of an NSF GOALI grant with matching support by McDonnell Douglas Aerospace (now Boeing Company, St. Louis). The research problem was motivated by the need of connecting surface patches that are designed independently. The basic assumption is that the boundary of each surface patch is supposed to be determined by a given finite set of points. However, certain gaps, though invisible to the human eyes, were usually found to exist when these surface patches are connected by using the CAD tools from Unigraphics. The objective of the paper was, therefore, to develop a design criterion to avoid such gaps, while yielding a reasonably smooth

Wavelets and Splines: Athens 2005
Guanrong Chen and Ming-Jung Lai (eds.), pp. 287–313.

ISBN 0-9728482-6-6

compound surface. We use the G^1 smoothness criterion as the goal in our research. The paper [4], with Charles Chui as a co-author, was completed and submitted in April, 1998, to the Journal of Computer Aided Geometric Design (CAGD) for publication. A revision of the paper was re-submitted in 2003, but was later rejected. To our surprise, however, the same journal, CAGD, has recently published a similar work [12]. Since our paper [4] has not appeared in print, the intention of this writing is to publish its exact content and to point out the similarity with the recently published work [12] in CAGD.

In [12], certain necessary and sufficient conditions for two NURBS patches $R_1(u,v)$ and $R_2(\tilde{u},\tilde{v})$ to join in the G^1 fashion are given under the assumption that $R_1(u,0) = R_2(u,0)$ (cf. (3) in [12, p.289]). This is a very restrictive assumption. Although the G^0 condition implies that $R_1(u,0)$ and $R_2(u,0)$ have a common boundary curve, namely, $R_1(u,0)=R_2(w(u),0)$ with re-parametrization, say $w(u)$, yet the assumption $R_1(u,0) = R_2(u,0)$ rarely happens in practice, and indeed does not even apply to true NURBS. As mentioned above, two adjacent patches are usually determined only by imposing a finite set of common (interpolating) points on the boundaries. That is, it is more practical to assume $R_1(u,0) = R_2(\tilde{u},0)$, only for some finite discrete values of the parameters u and $\tilde{u}$, in the usual design process. In addition, the weaker condition of common interpolating points (as opposed to a common boundary curve) allows us to extend the study to connecting true NURBS patches, as in our later published work [3].

Our study in [4] is based on the practical setting, and an effective algorithm is developed to modify the spline coefficients near the (common) boundary of only one of two spline surface patches to achieve a G^1 compound patch, without gaps. In this regard, we like to mention that although our condition is only sufficient (but not necessary in general), yet it applies to practical applications and can add to the capability of the Unigraphics CAD tools.

In the following, the content of [4] is presented without change. Since this volume is dedicated to Professor Charles K. Chui on the occasion of his 65th birthday, he declines to be a co-author of this paper. We are happy to honor and respect his wish.

§1. Introduction

In most CAD/CAM/CAE systems, a 3-D geometry object, expressed as Bézier, B-spline or NURBS surfaces, often consists of several individually designed surfaces. When these surface patches are put together, the assembled surface, called compound surface, often contains some undesirable discontinuities such as gaps, holes, and overlaps. There are various ways to repair these defects, but the general approach is to carve away portions of boundary strips and fill in the gaps or holes by introducing new surface patches which are relatively much smaller, and usually narrower, than the original ones. For instance, in recent literature [6–9], gap and hole filling algorithms using Coons surface patches and Gregory surface patches were developed for such corrections. These algorithms require a designer to interactively choose a bounding box from the original geometry object before a Coons or Gregory patch is constructed to replace the portion of the original surface patches within the bounding box which contains these discontinuities. The main disadvantage of this general approach in CAD/CAM/CAE applications is that a significant amount of the geometric data set of control points is lost, and the new patches for the repair do not have representation in terms of control points. This paper presents a different approach to correcting these undesirable surface discontinuities. Instead of carving away portions of the boundary and filling in new surface patches, we develop an algorithm for manipulating the control points, and Bézier coefficients when necessary, only near the boundary, to achieve the same goal. Of course this does not work when B-splines with simple knots are considered. Fortunately, in many CAD/CAM/CAE applications, multiple knots are often used. For instance, for cubic B-splines, double interior knots are used, resulting in C^1 continuity instead of C^2 continuity; and for quintic B-splines, usually only C^2 continuity is sufficient in the automobile industry. Even in this multiple knot setting, we will show that manipulating control points is not enough when the knot spacings are arbitrary. Our solution for this situation is to manipulate the Bézier coefficients of the polynomial pieces, but only along the boundary strips. The sacrifice is to slightly reduce the geometric smoothness along these boundary strips. To demonstrate the feasibility and efficiency of our approach, we only consider the C^1 cubic setting, although we believe that our algorithms can be generalized to deal with high order B-spline surfaces. In the following, we first introduce some necessary notations and explain more precisely what we will study in this paper.

Let S and $\widetilde{S}$ be two C^1 bi-cubic parametric B-spline surfaces in the 3-dimensional (or 3-D) space with parametric knot sequences $\{(u_i, v_j) : 0 \le i \le 2m+5,\ 0 \le j \le 2n+5\}$ and $\{(\tilde{u}_i, \tilde{v}_j) : 0 \le i \le 2\tilde{m}+5,\ 0 \le j \le 2\tilde{n}+5\}$, respectively. That is, S and $\widetilde{S}$ can be represented as finite tensor-product cubic B-spline series, $\mathbf{f}$ and $\tilde{\mathbf{f}}$, with double interior knots and four-fold boundary knots. We will assume that S and $\widetilde{S}$ pass through the same set of data points along one of their edges as determined by the knot sequences, or equivalently,

$$\mathbf{f}(u_{2i}, v_2) = \tilde{\mathbf{f}}(\tilde{u}_{2i}, \tilde{v}_2), \quad i = 1, \cdots, \hat{m} := \min(\tilde{m}, m).$$

We mention, however, that the interpolation conditions do not play a significant role when connecting S and $\tilde{S}$ in the G^1 fashion. Observe that although the parametric edges $u_2 \le u \le u_{2\hat{m}}$, $v = v_2$ and $\tilde{u}_2 \le \tilde{u} \le \tilde{u}_{2\hat{m}}, \tilde{v} = \tilde{v}_2$ in 2-D coincide, the two boundary edges, $\mathbf{f}(u, v_2)$ and $\tilde{\mathbf{f}}(\tilde{u}, \tilde{v}_2)$ in the 3-dimension space, usually only meet at the common set of data points described above, and hence, there are $\hat{m} - 1$ possible gaps or overlaps between the two surfaces S and $\widetilde{S}$.

To minimize the modifications of S and $\widetilde{S}$, we first study the possibility of removing these gaps and overlaps simply by manipulating the control points without disturbing the given interpolation data. If this is not feasible, we then modify the boundary strip of S (or of $\widetilde{S}$) to remove these gaps and overlaps. We emphasize that the important features of this approach are that no additional surface patches are used to fill in the gaps, that the modification (if needed) is very minimal, and that the combined surface, without gaps and overlaps, is smooth. The objective of this paper is to establish two mathematical results, derive some useful formulas, and give the corresponding efficient algorithms. First, if one knot sequence, say $\{u_i\}$, is proportional to a subsequence of the other knot sequence $\{\tilde{u}_i\}$, then the two surfaces S and $\tilde{S}$ may be modified to have no gaps and no overlaps, and the combined surface is geometrically continuously differentiable (called G^1). Secondly, if these knot sequences are not proportional, then we present an algorithm to change the Bézier points of the boundary strip of S, say,

$$u_2 \le u \le u_{2\hat{m}} \quad \text{and} \quad v_2 \le v \le v_6,$$

to remove the gaps or overlaps, so that the combined surface is again G^1, and the major portion of S, namely, S minus the boundary strip,

$$\{S(u, v) : u_2 \le u \le u_{2m+4} \quad \text{and} \quad v_6 \le v \le v_{2n+4}\},$$

is still the original C^1 bi-cubic B-spline surface. In other words, C^1 is sacrificed only along the edge of one B-spline surface to yield a global (i.e., combined) G^1 surface which is C^1 outside this boundary strip, and which does not have any gaps and overlaps. This procedure is extended to treat certain multiple (i.e., compound) C^1 bi-cubic B-spline surfaces.

The outline of the paper is as follows. We first introduce the necessary notations and definitions such as Bernstein representation of B-spline surfaces, derive an algorithm for converting a B-spline surface to Bernstein form, and characterize G^1 geometric smoothness conditions. These will be done in §2. The main results of this paper are presented in §3, where we analyze the conditions of the knot sequences for C^0 and G^1 connection of S and $\widetilde{S}$ without sacrificing the B-spline representation; and when the knots are arbitrary, we derive an algorithm for connecting S and $\widetilde{S}$ in a G^1 fashion. Connection of multiple bi-cubic B-spline surfaces is considered in §4. Finally, in §5, we present several numerical examples to demonstrate our results.

§2. Preliminaries

In this section, we introduce some necessary notations and certain elementary properties of B-splines and Bernstein polynomials. We refer the details on these subjects to the references [1, 2, 5, 11].

2.1. C^1 Bi-Cubic B-Spline Surfaces and Bernstein Polynomials

We begin with the definition of C^1 bi-cubic B-spline surfaces. Let $\mathbf{u} = \{u_i\}_{i=0}^{2m+5}$ be a partition of $[0,1]$, defined by

$$\begin{aligned}\mathbf{u} = \{0 = u_0 = \ldots = u_3 < u_4 = u_5 < u_6 = u_7 < \ldots \\ < u_{2m} = u_{2m+1} < u_{2m+2} = \ldots = u_{2m+5} = 1\}.\end{aligned} \tag{2.1}$$

Then $\mathbf{u}$ can be used to define the i^{th} normalized cubic B-splines,

$$\begin{aligned}N_{4,\mathbf{u},i}(x) =& (u_{i+4} - u_i)[u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}](x-u)_+^3, \\ & i = 0, \cdots, 2m+1,\end{aligned} \tag{2.2}$$

where the standard notation of fourth order divided difference with respect to the variable u is used (see [1, 8]). Similarly, let

$$\begin{aligned}\mathbf{v} = \{0 = v_0 = \ldots = v_3 < v_4 = v_5 < v_6 = v_7 < \ldots \\ < v_{2n} = v_{2n+1} < v_{2n+2} = \ldots = v_{2n+5} = 1\}\end{aligned} \tag{2.3}$$

be another knot sequence that governs the normalized cubic B-splines $N_{4,\mathbf{v},j}(x)$, $j = 0, \cdots, 2n+1$.

In this paper, we will consider C^1 bi-cubic B-spline surfaces of the form

$$S: \ \mathbf{f}(u,v) = \sum_{i=0}^{2m+1} \sum_{j=0}^{2n+1} \mathbf{d}_{i,j} N_{4,\mathbf{u},i}(u) N_{4,\mathbf{v},j}(v), \tag{2.4}$$

where u and v are parameters with $(u,v) \in [0,1]^2$ and $\mathbf{d}_{i,j} \in \mathbb{R}^3$, $i = 0, \ldots, 2m+1, j = 0, \ldots, 2n+1$, are called the *global* control points (or de Boor points) of the surface S.

Next let

$$\phi_{n,i,a,b}(x) = \binom{n}{i} \left(\frac{x-a}{b-a}\right)^i \left(\frac{b-x}{b-a}\right)^{n-i}, \quad 0 \le i \le n, \tag{2.5}$$

be the Bernstein polynomials of degree n relative to the interval $[a,b]$. In the following, we restrict our attention only to the interval $[a,b]$ for the Bernstein polynomials $\phi_{n,i,a,b}(u)$. For this purpose, we introduce the notation

$$B_{n,i,a,b}(u) = \chi_{[a,b]}(u) \phi_{n,i,a,b}(u), \tag{2.6}$$

where the standard notation of the characteristic function $\chi_{[a,b]}$ has been used. Now, since any polynomial may be written in terms of the Bernstein polynomial form, any B-spline can be expressed in terms of $B_{n,i,a,b}(u)$, because of the localness property. For example, when $n = 3$, with the knot sequence $\mathbf{u}$ in (2.1) and by introducing the notation

$$\alpha_i = \frac{u_{2i+4} - u_{2i+2}}{u_{2i+4} - u_{2i}}, \quad \beta_i = \frac{u_{2i+2} - u_{2i}}{u_{2i+4} - u_{2i}} = 1 - \alpha_i, \qquad 0 \le i \le m, \tag{2.7}$$

the B-splines $N_{4,\mathbf{u},i}(x)$ in (2.2), $i = 0, \cdots, 2m+1$, can be represented as

$$\begin{aligned} N_{4,\mathbf{u},2i}(u) &= B_{3,2,u_{2i},u_{2i+2}}(u) + \alpha_i B_{3,3,u_{2i},u_{2i+2}}(u) \\ &\quad + \alpha_i B_{3,0,u_{2i+2},u_{2i+4}}; \\ N_{4,\mathbf{u},2i+1}(u) &= \beta_i B_{3,3,u_{2i},u_{2i+2}}(u) + \beta_i B_{3,0,u_{2i+2},u_{2i+4}}(u) \\ &\quad + B_{3,1,u_{2i+2},u_{2i+4}}(u), \end{aligned} \tag{2.8}$$

where $0 \le i \le m$. Observe from (2.7) that $\alpha_0 = \beta_m = 1$ and $\beta_0 = \alpha_m = 0$. Hence, (2.8) can be rewritten explicitly as

$$\begin{aligned} N_{4,\mathbf{u},0}(u) &= B_{3,0,u_2,u_4}(u), \\ N_{4,\mathbf{u},1}(u) &= B_{3,1,u_2,u_4}(u); \\ N_{4,\mathbf{u},2i}(u) &= B_{3,2,u_{2i},u_{2i+2}}(u) + \alpha_i B_{3,3,u_{2i},u_{2i+2}}(u), \\ &\quad + \alpha_i B_{3,0,u_{2i+2},u_{2i+4}}, \\ N_{4,\mathbf{u},2i+1}(u) &= \beta_i B_{3,3,u_{2i},u_{2i+2}}(u) + \beta_i B_{3,0,u_{2i+2},u_{2i+4}}(u) \\ &\quad + B_{3,1,u_{2i+2},u_{2i+4}}(u), \qquad 1 \le i \le m-1; \\ N_{4,\mathbf{u},2m}(u) &= B_{3,2,u_{2m},u_{2m+2}}(u), \\ N_{4,\mathbf{u},2m+1}(u) &= B_{3,3,u_{2m},u_{2m+2}}(u). \end{aligned}$$

2.2. An Algorithm for Converting Bi-Cubic B-Spline Surfaces in the B-Spline Representation to the Bernstein Form

In view of (2.8), we can write the B-spline surface S in (2.4) in terms of the localized Bernstein polynomials, namely,

$$S: \ \mathbf{f}(u,v) = \sum_{i=0}^{2m+1} \sum_{j=0}^{2n+1} \mathbf{d}_{i,j} N_{4,\mathbf{u},i}(u) N_{4,\mathbf{v},j}(v) \tag{2.9}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=0}^{3} \sum_{\ell=0}^{3} \mathbf{c}_{i,j,k,\ell} B_{3,k,u_{2i},u_{2i+2}}(u) B_{3,\ell,v_{2j},v_{2j+2}}(v), \tag{2.10}$$

where for each *polynomial surface patch*

$$\mathbf{f}(u,v): \quad u \in [u_{2i}, u_{2i+2}], v \in [v_{2j}, v_{2j+2}], i = 1, \ldots, m; \quad j = 1, \ldots, n,$$

the 3-dimensional coefficients $\mathbf{c}_{i,j,k,\ell}: \; 0 \le k, \ell \le 3$, are called the *Bézier coefficients* of the $(i,j)^{\text{th}}$ polynomial surface patch. An algorithm for converting the global control points

$$\{\mathbf{d}_{i,j}: \quad i = 0, \ldots, 2m+1, \; j = 0, \ldots, 2n+1\}$$

to the Bézier coefficients

$$\{\mathbf{c}_{i,j,k,\ell}: \quad 0 \le k, \ell \le 3; \quad i = 1, \ldots, m, \; j = 1, \ldots, n\}$$

is given as follows. Recall the notations of α_i and β_i's in (2.7) and introduce the analogous notation for the knot sequence $\mathbf{v}$. That is, we also set

$$\xi_j = \frac{v_{2j+4} - v_{2j+2}}{v_{2j+4} - v_{2j}}, \quad \eta_j = 1 - \xi_j, \quad 0 \le j \le n. \tag{2.11}$$

Then, by introducing the arrays

$$\{\mathbf{h}_{k,\ell}\}_{0 \le k \le 2m+1, \, 1 \le \ell \le n+1}, \quad \{\mathbf{v}_{k,\ell}\}_{1 \le k \le m+1, \, 0 \le \ell \le 2n+1},$$

and $\{\mathbf{a}_{k,\ell}\}_{1 \le k \le m+1, \, 1 \le \ell \le n+1}$ of sizes $(2m+2) \times (n+1), (m+1) \times (2n+2)$, and $(m+1) \times (n+1)$, respectively, where

$$\begin{aligned}
\mathbf{h}_{2i,j} &:= \xi_{j-1}\mathbf{d}_{2i,2j-2} + \eta_{j-1}\mathbf{d}_{2i,2j-1}, \\
\mathbf{h}_{2i+1,j} &:= \xi_{j-1}\mathbf{d}_{2i+1,2j-2} + \eta_{j-1}\mathbf{d}_{2i+1,2j-1}, \\
&\qquad\qquad i = 0, \cdots, m, \quad j = 1, \cdots, n+1; \\
\mathbf{v}_{i,2j} &:= \alpha_{i-1}\mathbf{d}_{2i-2,2j} + \beta_{i-1}\mathbf{d}_{2i-1,2j}, \\
\mathbf{v}_{i,2j+1} &:= \alpha_{i-1}\mathbf{d}_{2i-2,2j+1} + \beta_{i-1}\mathbf{d}_{2i-1,2j+1}, \\
&\qquad\qquad i = 1, \cdots, m+1, \quad j = 0, \cdots, n; \quad \text{and} \\
\mathbf{a}_{i,j} &:= \alpha_{i-1}\mathbf{h}_{2i-2,j} + \beta_{i-1}\mathbf{h}_{2i-1,j} \\
&= \xi_{j-1}\mathbf{v}_{i,2j-2} + \eta_{j-1}\mathbf{v}_{i,2j-1}, \\
&\qquad\qquad i = 1, \cdots, m+1, \quad j = 1, \cdots, n+1,
\end{aligned}$$

the mn Bézier coefficient matrices of size 4×4 with 3-vector entries

$$[\mathbf{c}_{i,j,k,\ell}]_{0 \le k, \ell \le 3}, \quad i = 1, \cdots, m, \; j = 1, \cdots, n,$$

can be written in the following unified form

$$[\mathbf{c}_{i,j,\ell,3-k}]_{0 \le k,\ell \le 3} = \begin{bmatrix} \mathbf{a}_{i,j+1} & \mathbf{h}_{2i-1,j+1} & \mathbf{h}_{2i,j+1} & \mathbf{a}_{i+1,j+1} \\ \mathbf{v}_{i,2j} & \mathbf{d}_{2i-1,2j} & \mathbf{d}_{2i,2j} & \mathbf{v}_{i+1,2j} \\ \mathbf{v}_{i,2j-1} & \mathbf{d}_{2i-1,2j-1} & \mathbf{d}_{2i,2j-1} & \mathbf{v}_{i+1,2j-1} \\ \mathbf{a}_{i,j} & \mathbf{h}_{2i-1,j} & \mathbf{h}_{2i,j} & \mathbf{a}_{i+1,j} \end{bmatrix}, \tag{2.12}$$

for $i = 1, \cdots, m$, $j = 1, \cdots, n$.

2.3. An Algorithm for Converting Bi-Cubic B-spline Surfaces in the Bernstein Form to the B-Spline Representation

Under certain restrictions on the Bézier coefficients, a surface in its Bernstein representation can be converted to its B-spline representation. In other words, we are interested in converting the Bézier coefficients

$$\{\mathbf{c}_{i,j,k,\ell}: \quad 0 \le k, \ell \le 3; \quad i = 1, \ldots, m,\ j = 1, \ldots, n\}$$

to the global control points

$$\{\mathbf{d}_{i,j}: \quad i = 0, \ldots, 2m+1,\ j = 0, \ldots, 2n+1\}.$$

Suppose that $\{\mathbf{c}_{i,j,k,\ell}: \quad i = 1, \ldots, m,\ j = 1, \ldots, n;\ 0 \le k, \ell \le 3\}$ satisfy

$$\mathbf{c}_{i,j,0,\ell} = \mathbf{c}_{i-1,j,3,\ell} = \alpha_{i-1}\mathbf{c}_{i-1,j,2,\ell} + \beta_{i-1}\mathbf{c}_{i,j,1,\ell}, \quad \ell = 0, \ldots, 3; \quad i = 2, \ldots, m, \quad j = 1, \ldots n \tag{2.13}$$

$$\mathbf{c}_{i,j,k,0} = \mathbf{c}_{i,j-1,k,3} = \xi_{j-1}\mathbf{c}_{i,j-1,k,2} + \eta_{j-1}\mathbf{c}_{i,j,k,1}, \quad \ell = 0, \ldots, 3; \quad i = 1, \ldots, m, \quad j = 2, \ldots, n \tag{2.14}$$

along both the $m-1$ interior vertical grid lines and the $n-1$ interior horizontal grid lines of the parametric domain, respectively. Then it is clear from (2.13) and (2.14) that

$$\alpha_{i-1}\mathbf{c}_{i-1,j,2,0} + \beta_{i-1}\mathbf{c}_{i,j,1,0} = \xi_{j-1}\mathbf{c}_{i,j-1,0,2} + \eta_{j-1}\mathbf{c}_{i,j,0,1}, \quad i = 2, \ldots, m, \quad j = 2, \ldots, n.$$

Hence, under the assumptions (2.13) and (2.14), we have the following conversion formulas.

1° For the 4 *corners* of the parametric domain,

$$\begin{aligned} \mathbf{d}_{0,0} &= \mathbf{c}_{1,1,0,0}, \quad \mathbf{d}_{2m+1,0} = \mathbf{c}_{m,1,3,0}, \\ \mathbf{d}_{0,2n+1} &= \mathbf{c}_{1,n,0,3}, \quad \mathbf{d}_{2m+1,2n+1} = \mathbf{c}_{m,n,3,3}. \end{aligned} \tag{2.15}$$

2° Along the horizontal boundaries of the parametric domain,

$$\begin{aligned} \mathbf{d}_{2i-1,0} &= \mathbf{c}_{i,1,1,0}, \quad \mathbf{d}_{2i,0} = \mathbf{c}_{i,1,2,0}, \\ \mathbf{d}_{2i-1,2n+1} &= \mathbf{c}_{i,n,1,3}, \quad \mathbf{d}_{2i,2n+1} = \mathbf{c}_{i,n,2,3}, \quad i = 1, \ldots, m. \end{aligned} \tag{2.16}$$

3° Along the vertical boundaries of the parametric domain,

$$\begin{aligned} \mathbf{d}_{0,2j-1} &= \mathbf{c}_{1,j,0,1}, \quad \mathbf{d}_{0,2j} = \mathbf{c}_{1,j,0,2}, \\ \mathbf{d}_{2m+1,2j-1} &= \mathbf{c}_{m,j,3,1}, \quad \mathbf{d}_{2m+1,2j} = \mathbf{c}_{m,j,3,2}, \quad j = 1, \ldots, n. \end{aligned} \tag{2.17}$$

4° For the *interior* of the parametric domain,

$$\begin{bmatrix} \mathbf{d}_{2i-1,2j} & \mathbf{d}_{2i,2j} \\ \mathbf{d}_{2i-1,2j-1} & \mathbf{d}_{2i,2j-1} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_{i,j,1,2} & \mathbf{c}_{i,j,2,2} \\ \mathbf{c}_{i,j,1,1} & \mathbf{c}_{i,j,2,1} \end{bmatrix}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n. \tag{2.18}$$

2.4. C^0 Continuity Conditions

Let $\mathbf{f}(u,v)$ be a bi-cubic B-spline representation defined on $[0,1]^2$ in (2.13) with knot sequences $\mathbf{u}$ and $\mathbf{v}$ in (2.1) and (2.3), and $\tilde{\mathbf{f}}(u,v)$ be another bi-cubic B-spline representation of a surface $\widetilde{S}$, defined on $[0,1]^2$ and with knot sequences $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$, namely,

$$\begin{aligned} \tilde{\mathbf{u}} = \{0 = \tilde{u}_0 = \ldots = \tilde{u}_3 < \tilde{u}_4 = \tilde{u}_5 < \tilde{u}_6 = \tilde{u}_7 < \ldots \\ < \tilde{u}_{2\tilde{m}} = \tilde{u}_{2\tilde{m}+1} < \tilde{u}_{2\tilde{m}+2} = \ldots = \tilde{u}_{2\tilde{m}+5} = 1\}, \quad (2.19) \\ \tilde{\mathbf{v}} = \{0 = \tilde{v}_0 = \ldots = \tilde{v}_3 < \tilde{v}_4 = \tilde{v}_5 < \tilde{v}_6 = \tilde{v}_7 < \ldots \\ < \tilde{v}_{2\tilde{n}} = \tilde{v}_{2\tilde{n}+1} < \tilde{v}_{2\tilde{n}+2} = \ldots = \tilde{v}_{2\tilde{n}+5} = 1\}. \quad (2.20) \end{aligned}$$

That is, in addition to (2.4), we have

$$\begin{aligned} \widetilde{S}:\ \tilde{\mathbf{f}}(u,v) &= \sum_{i=0}^{2\tilde{m}+1}\sum_{j=0}^{2\tilde{n}+1} \tilde{\mathbf{d}}_{i,j} N_{4,\tilde{\mathbf{u}},i}(u) N_{4,\tilde{\mathbf{v}},j}(v) \\ &= \sum_{i=1}^{\tilde{m}}\sum_{j=1}^{\tilde{n}}\sum_{k=0}^{3}\sum_{\ell=0}^{3} \tilde{\mathbf{c}}_{i,j,k,\ell} B_{3,k,\tilde{u}_{2i},\tilde{u}_{2i+2}}(u) B_{3,\ell,\tilde{v}_{2j},\tilde{v}_{2j+2}}(v). \quad (2.21) \end{aligned}$$

For convenience, similar to (2.7), we also set

$$\widetilde{\alpha}_i = \frac{\widetilde{u}_{2i+4} - \widetilde{u}_{2i+2}}{\widetilde{u}_{2i+4} - \widetilde{u}_{2i}}, \quad \widetilde{\beta}_i = \frac{\widetilde{u}_{2i+2} - \widetilde{u}_{2i}}{\widetilde{u}_{2i+4} - \widetilde{u}_{2i}} = 1 - \widetilde{\alpha}_i, \qquad 0 \le i \le \widetilde{m}. \quad (2.22)$$

We say that the corresponding B-spline surfaces S and $\widetilde{S}$ are C^0 continuously connected (or joined), if for each $i \in \{1, \cdots, \hat{m}\}$, there exists a reparameterization $\tilde{w}_i = \tilde{w}_i(u)$ such that

$$\mathbf{f}(u, v_2) = \tilde{\mathbf{f}}(\tilde{w}_i(u), \tilde{v}_2), \quad u \in [u_{2i}, u_{2i+2}]. \quad (2.23)$$

2.5. G^1 Continuity Condition

Two polynomial surface patches

$$\begin{aligned} & P(u,v), \quad (u,v) \in [u_{2i}, u_{2i+2}] \times [v_{2j}, v_{2j+2}] \\ \text{and} \quad & Q(u,v), \quad (u,v) \in [\tilde{u}_{2i}, \tilde{u}_{2i+2}] \times [\tilde{v}_{2j}, \tilde{v}_{2j+2}], \end{aligned}$$

in the 3-dimensional space are said to be connected in a G^1 fashion, if the two polynomial surface patches are continuously connected in the 3-dimensional space and that for each point on the common boundary edge, there exists a unique plane that is tangent to both polynomial surface patches in the 3-dimensional space. Precisely, there are two conditions (see [5]) given as follows:

(i) there exists a reparameterization $\tilde{w}_j = \tilde{w}_j(v)$ such that $P(u_{2i+2}, v) = Q(\tilde{u}_{2i+2}, \tilde{w}_j(v))$, for all $v \in [v_{2j}, v_{2j+2}]$, and

(ii) there exist three polynomials $\Theta(v), \Phi(v)$, and $\Psi(v)$, such that

$$\Theta(v)\frac{\partial}{\partial u}P(u_{2i+2}, v) = \Phi(v)\frac{\partial}{\partial u}Q(\tilde{u}_{2i+2}, \tilde{w}_j(v)) + \Psi(v)\frac{\partial}{\partial v}Q(\tilde{u}_{2i+2}, \tilde{w}_j(v)) \qquad (2.24)$$

for $v \in [v_{2j}, v_{2j+1}]$.

For a B-spline surface S, if

$$\mathbf{f}(u,v)\Big|_{[u_{2i},u_{2i+2}]\times[v_{2j},v_{2j+2}]} \quad \text{and} \quad \mathbf{f}(u,v)\Big|_{[u_{2i+2},u_{2i+4}]\times[v_{2j},v_{2j+2}]}$$

satisfy the above conditions (i) and (ii), then we say that S is a geometrically differentiable surface. For short, we will say that S satisfies the G^1 condition or S is in a G^1 fashion.

Note that, in (2.24), if $[\tilde{u}_{2i}, \tilde{u}_{2i+2}] = [u_{2i+2}, u_{2i+4}]$ and $[\tilde{v}_{2j}, \tilde{v}_{2j+2}] = [v_{2j}, v_{2j+2}]$ and if $\Theta(v) = 1, \Phi(v) = 1$ and $\Psi(v) = 0$ with $\tilde{w}_j(v) = v$, then P and Q are C^1 continuously connected in the usual sense. That is, in this case, the G^1 condition is the usual C^1 condition.

§3. Main Results

Our first goal is to join S and $\widetilde{S}$ continuously (i.e., without gaps and overlaps) and our second goal is to join S and $\widetilde{S}$ in a G^1 fashion across the common edge of $\widetilde{S}$ and S. However, as we will see later, if the knot sequences $\mathbf{u}$ and $\tilde{\mathbf{u}}$ do not satisfy certain conditions, the second goal cannot be achieved without sacrificing the B-spline representation along a boundary strip of one of the two surfaces. Certainly, without adjusting the control points of either S or $\widetilde{S}$ or both, it is most likely that even the first goal could not be achieved. On the other hand, we do not wish to alter the originally designed surfaces S and $\widetilde{S}$ by too much. So, our plan is to adjust only one strip of S while keeping $\widetilde{S}$ unaltered.

We now introduce the following definition on the knot sequences.

Definition 3.1. *The two knot sequences* $\mathbf{u}$ *and* $\tilde{\mathbf{u}}$ *are proportional if*

$$\alpha_i = \tilde{\alpha}_i, \qquad i = 1, \ldots, \hat{m} - 1, \qquad (3.1)$$

We divide this section into two subsections. In §3.1., we shall discuss the connection of S and $\widetilde{S}$ when their knot sequences are proportional. In §3.2, we will then deal with G^1 connection when the knot sequences are not proportional.

3.1. G^1 Connection of S and $\widetilde{S}$ in the B-Spline Representation

To join S and $\widetilde{S}$ continuously across their edges, we will modify S defined by $\mathbf{f}(u, v_2)$, to $S^\star$ defined by $\mathbf{f}^\star(u, v_2)$, where $u \in [u_{2i}, u_{2i+2}]$, so that $S^\star$ and $\widetilde{S}$, defined by $\tilde{\mathbf{f}}(u, \tilde{v}_2)$, $u \in [\tilde{u}_{2i}, \tilde{u}_{2i+2}]$, $i = 1, \ldots, \hat{m}$, connect together continuously. Here, analogous to S and $\widetilde{S}$, we set

$$S^\star : \mathbf{f}^\star(u,v) = \sum_{i=0}^{2m+1} \sum_{j=0}^{2n+1} \mathbf{d}^\star_{i,j} N_{4,\mathbf{u},i}(u) N_{4,\mathbf{v},j}(v) \tag{3.2}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=0}^{3} \sum_{\ell=0}^{3} \mathbf{c}^\star_{i,j,k,\ell} B_{3,k,u_{2i},u_{2i+2}}(u) B_{3,\ell,v_{2j},v_{2j+2}}(v). \tag{3.3}$$

By the C^0 joining condition (2.23) and the algorithm for converting bi-cubic B-spline surfaces to the Bernstein form, we have to choose

$$\begin{aligned} \mathbf{d}^\star_{i,0} &:= \tilde{\mathbf{d}}_{i,0}, \quad i = 0, \ldots, 2\hat{m}, \quad \text{and} \\ \mathbf{d}^\star_{2\hat{m}+1,0} &:= \frac{1}{\beta_{\hat{m}}}(\tilde{\mathbf{d}}_{2\hat{m}+1,0} - \alpha_{\hat{m}} \tilde{\mathbf{d}}_{2\hat{m},0}), \quad \text{when} \quad m \geq \tilde{m}, \quad \text{or} \\ \mathbf{d}^\star_{2\hat{m}+1,0} &:= \tilde{\alpha}_{\hat{m}} \tilde{\mathbf{d}}_{2\hat{m},0} + \tilde{\beta}_{\hat{m}} \tilde{\mathbf{d}}_{2\hat{m}+1,0}, \quad \text{when} \quad m < \tilde{m} \end{aligned} \tag{3.4}$$

and we have to set

$$\alpha_i \mathbf{d}^\star_{2i,0} + \beta_i \mathbf{d}^\star_{2i+1,0} = \tilde{\alpha}_i \tilde{\mathbf{d}}_{2i,0} + \tilde{\beta} \tilde{\mathbf{d}}_{2i+1,0} \tag{3.5}$$

for all $i = 1, \cdots, \hat{m} - 1$. Using (3.4), (2.7), and (2.22), and noting that $\tilde{\mathbf{d}}_{2i,0} \neq \tilde{\mathbf{d}}_{2i+1,0}$ (otherwise, the surface $\widetilde{S}$ has a cusp), the requirement (3.5) implies that $\alpha_i = \tilde{\alpha}_i$ for all i's. That is, if the two knot sequences are proportional, then after choosing $\mathbf{d}^\star_{i,0}$ as in (3.4), but keeping $\mathbf{d}^\star_{i,j} = \mathbf{d}_{i,j}, j \geq 1$ for all i's, $S^\star$ is continuously connected to $\widetilde{S}$.

As we see from the above, the proportionality is a necessary to ensure that $S^\star$ is still a C^1 surface. This proportionality condition on knot sequences turns out to be exactly what is needed for S and $\widetilde{S}$ to be joined in a G^1 fashion without sacrificing the B-spline representations of S and $\widetilde{S}$, as follows.

Theorem 1. *Let S and $\widetilde{S}$ be two bi-cubic B-spline surfaces with control points $\{\mathbf{d}_{i,j}, 0 \leq i \leq 2m+1, 0 \leq j \leq 2n+1\}$ and $\{\tilde{\mathbf{d}}_{i,j}, 0 \leq i \leq 2\tilde{m}+1, 0 \leq j \leq 2\tilde{n}+1\}$, respectively. Assume that the knot sequences $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are proportional as in* (3.1).

1° *If $m = \tilde{m}$, set*

$$\mathbf{d}^\star_{i,0} := \tilde{\mathbf{d}}_{i,0}, \quad i = 0, \ldots, 2\hat{m}+1, \tag{3.7}$$

$$\mathbf{d}^\star_{i,1} := \left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right) \tilde{\mathbf{d}}_{i,0} - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2} \tilde{\mathbf{d}}_{i,1}, \quad i = 0, \ldots, 2\hat{m}+1. \tag{3.8}$$

2° *If $m > \tilde{m}$, set*

$$\begin{cases} \mathbf{d}^\star_{i,0} := \tilde{\mathbf{d}}_{i,0}, \quad i = 0, \ldots, 2\hat{m}, \\ \mathbf{d}^\star_{2\hat{m}+1,0} := \dfrac{1}{\beta_{\hat{m}}}(\tilde{\mathbf{d}}_{2\hat{m}+1,0} - \alpha_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},0}), \end{cases} \tag{3.9}$$

and

$$\begin{cases} \mathbf{d}^\star_{i,1} := \left(1 + \dfrac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)\tilde{\mathbf{d}}_{i,0} - \dfrac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\tilde{\mathbf{d}}_{i,1}, \quad i = 0, \ldots, 2\hat{m}, \\ \mathbf{d}^\star_{2\hat{m}+1,1} := \dfrac{1}{\beta_{\hat{m}}}\Bigg[\left(1 + \dfrac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)(\tilde{\mathbf{d}}_{2\hat{m}+1,0} - \alpha_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},0}) \\ \qquad\qquad - \dfrac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}(\tilde{\mathbf{d}}_{2\hat{m}+1,1} - \alpha_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},1})\Bigg]. \end{cases} \tag{3.10}$$

3° *If $m < \tilde{m}$, set*

$$\begin{cases} \mathbf{d}^\star_{i,0} := \tilde{\mathbf{d}}_{i,0}, \quad i = 0, \ldots, 2\hat{m}, \\ \mathbf{d}^\star_{2\hat{m}+1,0} := \tilde{\alpha}_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},0} + \tilde{\beta}_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m}+1,0}, \end{cases} \tag{3.11}$$

and

$$\begin{cases} \mathbf{d}^\star_{i,1} := \left(1 + \dfrac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)\tilde{\mathbf{d}}_{i,0} - \dfrac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\tilde{\mathbf{d}}_{i,1}, \quad i = 0, \ldots, 2\hat{m}, \\ \mathbf{d}^\star_{2\hat{m}+1,1} := \left(1 + \dfrac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)(\tilde{\alpha}_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},0} + \tilde{\beta}_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m}+1,0}) \\ \qquad\qquad - \dfrac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}(\tilde{\alpha}_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},1} + \tilde{\beta}_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m}+1,1}). \end{cases} \tag{3.12}$$

In addition, keep $\mathbf{d}^\star_{i,j} := \mathbf{d}_{i,j}$ for all other indices i, j. Then the modified surface $S^\star$ is still a C^1 bi-cubic B-spline surface and the surfaces $S^\star$ and $\widetilde{S}$ are connected in a G^1 fashion.

Proof: We will only prove 2°, since the proofs of both 1° and 3° are similar. It is clear, by using the algorithm described in §2.4, that the condition (3.9) implies that $S^\star$ and $\widetilde{S}$ are connected continuously without gaps.

We now show that $S^\star$ and $\widetilde{S}$ are connected in a G^1 fashion with further choice of the control points of $S^\star$ in (3.10). Indeed, we may even choose $\Theta(v) = \Phi(v) = 1$ and $\Psi(v) = 0$ in (2.24), and we have to ensure that

$$\frac{\mathbf{c}^\star_{i,1,k,1} - \mathbf{c}^\star_{i,1,k,0}}{v_4 - v_2} = -\frac{\tilde{\mathbf{c}}_{i,1,k,1} - \tilde{\mathbf{c}}_{i,1,k,0}}{\tilde{v}_4 - \tilde{v}_2}, \quad k = 0, \cdots, 3;\ i = 1, \cdots, \hat{m}, \tag{3.13}$$

in order for $S^\star$ and $\widetilde{S}$ to be joined in the G^1 fashion.

The first $2\hat{m}$ equations for $\mathbf{d}^\star_{i,1}$, $i = 1, \ldots, 2\hat{m}$, in (3.10), imply that the relations in (3.13) hold for $k = 1, 2$ and $i = 1, \ldots, \hat{m}$. That $\widetilde{S}$ is the original bi-cubic C^1 B-spline surface yields

$$\tilde{\mathbf{c}}_{i,1,0,0} = \tilde{\alpha}_{i-1}\tilde{\mathbf{d}}_{2i-2,0} + \tilde{\beta}_{i-1}\tilde{\mathbf{d}}_{2i-1,0},$$

$$\text{and}\quad \tilde{\mathbf{c}}_{i,1,0,1} = \tilde{\alpha}_{i-1}\tilde{\mathbf{d}}_{2i-2,1} + \tilde{\beta}_{i-1}\tilde{\mathbf{d}}_{2i-1,1}, \quad i = 1, \ldots, \hat{m}.$$

Since $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are proportional, we have

$$\mathbf{c}^\star_{i,1,0,0} = \alpha_{i-1}\tilde{\mathbf{d}}_{2i-2,0} + \beta_{i-1}\tilde{\mathbf{d}}_{2i-1,0} = \tilde{\mathbf{c}}_{i,1,0,0}, \quad i = 1, \ldots, \hat{m}.$$

Thus, it follows from (3.10) that

$$\begin{aligned}
\mathbf{c}^\star_{i,1,0,1} &= \alpha_{i-1}\mathbf{d}^\star_{2i-2,1} + \beta_{i-1}\mathbf{d}^\star_{2i-1,1} \\
&= \left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)\alpha_{i-1}\tilde{\mathbf{d}}_{2i-2,0} - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\alpha_{i-1}\tilde{\mathbf{d}}_{2i-2,1} \\
&\quad + \left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)\beta_{i-1}\tilde{\mathbf{d}}_{2i-1,0} - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\beta_{i-1}\tilde{\mathbf{d}}_{2i-1,1} \\
&= \left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)\left(\tilde{\alpha}_{i-1}\tilde{\mathbf{d}}_{2i-2,0} + \tilde{\beta}_{i-1}\tilde{\mathbf{d}}_{2i-1,0}\right) \\
&\quad - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\left(\tilde{\alpha}_{i-1}\tilde{\mathbf{d}}_{2i-2,1} + \tilde{\beta}_{i-1}\tilde{\mathbf{d}}_{2i-1,1}\right) \\
&= \left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)\tilde{\mathbf{c}}_{i,1,0,0} - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\tilde{\mathbf{c}}_{i,1,0,1}, \quad i = 1, \ldots, \hat{m},
\end{aligned}$$

so that

$$\frac{\mathbf{c}^\star_{i,1,0,1}}{v_4 - v_2} = \frac{\tilde{\mathbf{c}}_{i,1,0,0}}{v_4 - v_2} - \frac{\tilde{\mathbf{c}}_{i,1,0,1} - \tilde{\mathbf{c}}_{i,1,0,0}}{\tilde{v}_4 - \tilde{v}_2},$$

which implies that the relations (3.10) for $k = 0$, $i = 1, \ldots, \hat{m}$, hold. Since

$$\mathbf{c}^\star_{i,1,3,1} = \mathbf{c}^\star_{i+1,1,0,1},$$

$$\text{and}\quad \mathbf{c}^\star_{i,1,3,0} = \mathbf{c}^\star_{i+1,1,0,0}, \quad i = 1, \ldots, \hat{m} - 1,$$

we can similarly conclude (3.13) with $k = 3$ and $i = 1, \cdots, \hat{m} - 1$. Finally, for $k = 3$ and $i = \hat{m}$, we have

$$\mathbf{c}^\star_{\hat{m}+1,1,0,0} = \alpha_{\hat{m}}\mathbf{d}^\star_{2\hat{m},0} + \beta_{\hat{m}}\mathbf{d}^\star_{2\hat{m}+1,0} = \tilde{\mathbf{d}}_{2\hat{m}+1,0} = \tilde{\mathbf{c}}_{\hat{m},1,3,0},$$

$$\begin{aligned}
\text{and}\quad \mathbf{c}^\star_{\hat{m},1,3,1} &= \mathbf{c}^\star_{\hat{m}+1,1,0,1} = \alpha_{\hat{m}}\mathbf{d}^\star_{2\hat{m},1} + \beta_{\hat{m}}\mathbf{d}^\star_{2\hat{m}+1,1} \\
&= \left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)\alpha_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},0} - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\alpha_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},1} \\
&\quad + \left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)(\tilde{\mathbf{d}}_{2\hat{m}+1,0} - \alpha_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},0}) \\
&\quad - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}(\tilde{\mathbf{d}}_{2\hat{m}+1,1} - \alpha_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},1}) \\
&= \left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)\tilde{\mathbf{d}}_{2\hat{m}+1,0} - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\tilde{\mathbf{d}}_{2\hat{m}+1,1}.
\end{aligned}$$

Thus, we have

$$\frac{\mathbf{c}^\star_{\hat{m},1,3,1}}{v_4 - v_2} = \frac{\tilde{\mathbf{c}}^\star_{\hat{m},1,3,0}}{v_4 - v_2} - \frac{\tilde{\mathbf{c}}_{\hat{m},1,3,1} - \tilde{\mathbf{c}}_{\hat{m},1,3,0}}{\tilde{v}_4 - \tilde{v}_2},$$

which yields (3.13) with $k = 3$ and $i = \hat{m}$.

Hence, $S^\star$ and $\widetilde{S}$ are indeed joined together in a G^1 fashion along their common boundary edge. The modified surface $S^\star$ is still a bi-cubic B-spline surface with the original control points $\{\mathbf{d}_{i,j}\}_{j\geq 2}$ but with new control points $\{\mathbf{d}^\star_{i,0}, \mathbf{d}^\star_{i,1}\}$ along the boundary strip. So it is still a C^1 surface. We have thus established Theorem 1. □

We remark here that when two knot sequences $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are proportional and $m = \tilde{m}$, we can replace $\tilde{\mathbf{u}}$ by $\mathbf{u}$, so that $\tilde{S}$ can be expressed by using the same knot sequence $\mathbf{u}$ as S. If we use a standard linear transform to convert $\tilde{v}$ from $[0, 1]$ to $[-1, 0]$, then two surfaces S and $\tilde{S}$ can be considered as a single C^1 bi-cubic B-spline surface with parameters $(u, v) \in [0, 1] \times [-1, 1]$.

3.2. G^1 Connection of S and $\widetilde{S}$ in the Bernstein Form

In this section, we allow the underlying knot sequences $\mathbf{u}$ and $\tilde{\mathbf{u}}$ to be arbitrary. In this case, we must sacrifice the bi-cubic B-spline representation in order to make $S^\star$ to be continuously connected to $\widetilde{S}$. Using the Bernstein representation (3.3) for $S^\star$, we can keep $\mathbf{c}^\star_{i,j,k,\ell} = \mathbf{c}_{i,j,k,\ell}$ for all i, j, k, ℓ with $\ell \neq 0$ and $j \neq 1$, but set

$$\mathbf{c}^\star_{i,1,k,0} := \tilde{\mathbf{c}}_{i,1,k,0}, \quad k = 0, \cdots, 3; \quad i = 1, \ldots, \hat{m} - 1.$$

Then the modified surface $S^\star$ of S is connected to $\widetilde{S}$ continuously. Although we have to point out that $S^\star$ is only C^0 along the boundary strip of $S^\star$, we will show that $S^\star$, and hence, S and $\widetilde{S}$ can be connected in the G^1 fashion by further manipulation of certain additional Bézier coefficients. To this end, we need to introduce more notations and, for simplicity, we will only consider $m \geq \tilde{m}$ so that $\hat{m} = \tilde{m}$. Let

$$P(u, v) = \mathbf{f}(u, v)|_{[u_{2i}, u_{2i+2}] \times [v_2, v_4]},$$
$$\text{and} \quad Q(u, v) = \mathbf{f}(u, v)|_{[u_{2i+2}, u_{2i+4}] \times [v_2, v_4]}$$

be two polynomial surface patches of S for a typical i, i.e., $i = 1, \ldots, \hat{m}$. First of all, we adjust the control points of S by letting

$$\mathbf{d}^\star_{i,0} = \tilde{\mathbf{d}}_{i,0}, \quad i = 0, 1, \ldots, 2\hat{m} + 1,$$
$$\mathbf{c}^\star_{i,1,0,0} = \tilde{\mathbf{c}}_{i,1,0,0}, \quad i = 1, \cdots, \hat{m},$$
$$\text{and} \quad \mathbf{c}^\star_{\hat{m}+1,1,0,0} = \tilde{\mathbf{c}}_{\hat{m},1,3,0},$$

in order to connect S and $\widetilde{S}$ continuously. Next we let

$$\mathbf{d}^\star_{i,1} = \left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)\tilde{\mathbf{d}}_{i,0} - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\tilde{\mathbf{d}}_{i,1}, \quad i = 0, \ldots, 2\hat{m},$$

$$\mathbf{d}^\star_{2\tilde{m}+1,1} = \frac{1}{\beta_{\hat{m}}}\Bigg[\left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)(\tilde{\mathbf{d}}_{2\hat{m}+1,0} - \alpha_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},0})$$

$$- \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}(\tilde{\mathbf{d}}_{2\hat{m}+1,1} - \alpha_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},1})\Bigg]; \quad \text{and}$$

$$\mathbf{c}^\star_{i,1,0,1} = \left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)\tilde{\mathbf{c}}_{i,1,0,0} - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\tilde{\mathbf{c}}_{i,1,0,1}, \quad i = 1, \ldots, \hat{m},$$

$$\mathbf{c}^\star_{\hat{m}+1,1,0,1} = \mathbf{c}^\star_{\hat{m},1,3,1} = \left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)\tilde{\mathbf{c}}_{\hat{m},1,3,0} - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\tilde{\mathbf{c}}_{\hat{m},1,3,1}.$$

These ensure that $S^\star$ and $\widetilde{S}$ are connected in a G^1 fashion across their common boundary edge. But it is evident that $S^\star$ is not a C^1 surface, although it is C^0. Thus, we have to adjust the other control points $\mathbf{d}_{i,2}$ and $\mathbf{d}_{i,3}$ of S, $i = 1, \ldots, \hat{m}$, such that $S^\star$ is in a G^1 fashion inside this two-layer boundary strip over $[u_2, u_{2m+4}] \times [v_2, v_6]$. Notice that $\mathbf{f}(u, v), (u, v) \in [u_2, u_{2m+4}] \times [v_6, v_{2n+4}]$ is still a portion of the original C^1 bi-cubic B-spline surfaces. This is indeed possible, as in the following.

Theorem 2. *Let S and $\widetilde{S}$ be two bi-cubic B-spline surfaces with knot sequences $(\mathbf{u}, \mathbf{v})$ and $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ and control points $\{\mathbf{d}_{i,j}, 0 \le i \le 2m+1, 0 \le j \le 2n+1\}$ and $\{\tilde{\mathbf{d}}_{i,j}, 0 \le i \le 2\tilde{m}+1,\ 0 \le j \le 2\tilde{n}+1\}$, respectively. Then while keeping $\mathbf{d}^\star_{i,j} = \mathbf{d}_{i,j}$ for $0 \le i \le 2\hat{m}+1, j \ge 4$, the control points $\mathbf{d}_{i,j}$ for $0 \le i \le 2\hat{m}+1, 0 \le j \le 3$ and Bézier coefficients $\mathbf{c}_{i,0,0,0}$ and $\mathbf{c}_{i,0,0,1}$ can be modified to certain $\mathbf{d}^\star_{i,j}$ and $\mathbf{c}^\star_{i,0,0,0}$ and $\mathbf{c}^\star_{i,0,0,1}$ such that the corrected surface $S^\star$ of S is connected to $\widetilde{S}$ in a G^1 fashion.*

Proof. Note that the Bézier coefficients of $P(u_{2i+2}, v)$ and $Q(u_{2i+2}, v)$ are the same, namely,

$$\begin{cases} \tilde{\alpha}_i\tilde{\mathbf{d}}_{2i,0} + \tilde{\beta}_i\tilde{\mathbf{d}}_{2i+1,0}, \\ \tilde{\alpha}_i\mathbf{d}_{2i,1} + \tilde{\beta}_i\mathbf{d}_{2i+1,1}, \\ \alpha_i\mathbf{d}_{2i,2} + \beta_i\mathbf{d}_{2i+1,2}, \quad \text{and} \\ \xi_1(\alpha_i\mathbf{d}_{2i,2} + \beta_i\mathbf{d}_{2i+1,2}) + \eta_1(\alpha_i\mathbf{d}_{2i,3} + \beta_i\mathbf{d}_{2i+1,3}). \end{cases}$$

Let us compute the first partial derivatives of P and Q at their common edge $u = u_{2i+2}$, $v \in [v_2, v_4]$, as follows:

$$\partial_u P(u, v)|_{u=u_{2i+2}} = \frac{3}{u_{2i+2} - u_{2i}}\sum_{j=0}^{3}(\mathbf{c}_{i,1,3,j} - \mathbf{c}_{i,1,2,j})B_{3,j,v_2,v_4}(v),$$

$$\partial_v P(u,v)|_{u=u_{2i+2}} = \partial_v Q(u,v)|_{u=u_{2i+2}},$$

$$\partial_u Q(u,v)|_{u=u_{2i+2}} = \frac{3}{u_{2i+4}-u_{2i+2}} \sum_{j=0}^{3} (\mathbf{c}_{i+1,1,1,j} - \mathbf{c}_{i+1,1,0,j}) B_{3,j,v_2,v_4}(v),$$

and

$$\partial_v Q(u,v)|_{u=u_{2i+2}} = \frac{3}{v_4 - v_2} \sum_{j=0}^{2} (\mathbf{c}_{i+1,1,0,j+1} - \mathbf{c}_{i+1,1,0,j}) B_{2,j,v_2,v_4}(v).$$

For convenience, we use the notation

$$\begin{aligned} A(t) &= \partial_u Q(u_{2i+2}, v_2 + t(v_4 - v_2)) =: \sum_{j=0}^{3} \mathbf{a}_j B_{3,j}(t), \\ B(t) &= \partial_v Q(u_{2i+2}, v_2 + t(v_4 - v_2)) =: \sum_{j=0}^{2} \mathbf{b}_j B_{2,j}(t), \\ C(t) &= \partial_u P(u_{2i+2}, v_2 + t(v_4 - v_2)) =: \sum_{j=0}^{3} \mathbf{c}_j B_{3,j}(t), \end{aligned} \tag{3.14}$$

where $B_{3,j}(t) := B_{3,j,v_2,v_4}(t)$ and $B_{2,j}(t) := B_{2,j,v_2,v_4}(t), t \in [0,1]$. First observe that

$$\mathbf{a}_0 = \frac{3\tilde{\alpha}_i}{u_{2i+4}-u_{2i+2}} (\tilde{\mathbf{d}}_{2i+1,0} - \tilde{\mathbf{d}}_{2i,0}),$$

$$\mathbf{c}_0 = \frac{3\tilde{\beta}_i}{u_{2i+2}-u_{2i}} (\tilde{\mathbf{d}}_{2i+1,0} - \tilde{\mathbf{d}}_{2i,0}),$$

and

$$\mathbf{a}_1 = \frac{3\tilde{\alpha}_i}{u_{2i+4}-u_{2i+2}} (\mathbf{d}_{2i+1,1} - \mathbf{d}_{2i,1}),$$

$$\mathbf{c}_1 = \frac{3\tilde{\beta}_i}{u_{2i+2}-u_{2i}} (\mathbf{d}_{2i+1,1} - \mathbf{d}_{2i,1}).$$

It follows that $\mathbf{a}_0$ is parallel to $\mathbf{c}_0$ and $\mathbf{a}_1$ is parallel to $\mathbf{c}_1$. Note also that

$$\begin{aligned} \mathbf{a}_2 &= \frac{3\alpha_i}{u_{2i+4}-u_{2i+2}} (\mathbf{d}_{2i+1,2} - \mathbf{d}_{2i,2}) \\ &= \frac{3\beta_i}{u_{2i+2}-u_{2i}} (\mathbf{d}_{2i+1,2} - \mathbf{d}_{2i,2}) = \mathbf{c}_2, \end{aligned}$$

and analogously, $\mathbf{a}_3 = \mathbf{c}_3$. Thus, we have

$$C(t) = A(t) + \hat{\mathbf{c}}_0 B_{3,0}(t) + \hat{\mathbf{c}}_1 B_{3,1}(t),$$

where $\hat{\mathbf{c}}_0 := \mathbf{c}_0 - \mathbf{a}_0$ and $\hat{\mathbf{c}}_1 := \mathbf{c}_1 - \mathbf{a}_1$. If both $\hat{\mathbf{c}}_0 = 0$ and $\hat{\mathbf{c}}_1 = 0$, then $C(t) = A(t)$, so that $\mathbf{f}(u, v)$ is C^1 across the boundary curve $\{(u_{2i+2}, v) : v \in [v_2, v_4]\}$, and we do not need to do anything further. Hence, we consider only the cases where $\hat{\mathbf{c}}_0 \neq 0$ or $\hat{\mathbf{c}}_1 \neq 0$ or both being nonzero. Note that $\tilde{\mathbf{d}}_{2i+1,0} \neq \tilde{\mathbf{d}}_{2i,0}$ and $\mathbf{d}_{2i+1,1} \neq \mathbf{d}_{2i,1}$ (otherwise, $\widetilde{S}$ has a cusp). Meanwhile, $\hat{\mathbf{c}}_0 = 0$ implies $\hat{\mathbf{c}}_1 = 0$ and *vice versa*. Therefore, we only need to consider the case that both $\hat{\mathbf{c}}_0 \neq 0$ and $\hat{\mathbf{c}}_1 \neq 0$.

By the G^1 condition (2.35) in §2.6, there exist three polynomials $\Theta(t), \Psi(t)$ and $\Phi(t)$, such that

$$\Theta(t)A(t) = \Phi(t)C(t) + \Psi(t)B(t). \tag{3.15}$$

Certainly, we would like to choose polynomials Θ, Φ, and Ψ of lowest degrees. We have tried linear polynomials for Φ, Ψ, and Θ, but they are not flexible enough to satisfy (3.15) for those $A(t), B(t)$, and $C(t)$ in (3.14).

We therefore consider quadratic polynomials for $\Theta(t)$ and $\Phi(t)$ and cubic polynomial for $\Psi(t)$. By letting $\mu = \dfrac{\tilde{\beta}_i}{\tilde{\alpha}_i}\dfrac{\alpha_i}{\beta_i}$, we choose

$$\begin{aligned}
&\Psi(t) \equiv 0,\\
&\Theta(t) = \mu\xi_0 B_{2,0}(t) + \mu\xi_2 B_{2,2}(t),\\
&\Phi(t) = \xi_0 B_{2,0}(t) + \xi_2 B_{2,2}(t),
\end{aligned}$$

with $\xi_0 \neq 0$ and $\xi_2 \neq 0$. It is straightforward to verify that (3.15) is satisfied. Hence, the patch

$$\{\mathbf{f}(u, v) : \ (u, v) \in [u_{2i}, u_{2i+4}] \times [v_2, v_4]\}$$

satisfies the G^1 continuity condition.

By comparing the coefficients in both sides of (3.15), we have

$$\begin{aligned}
\mathbf{d}_{2i+1,2} - \mathbf{d}_{2i,2} &= \frac{u_{2i+4} - u_{2i}}{3}\frac{\xi_2}{3(\mu - 1)\xi_0}\hat{\mathbf{c}}_0\\
&= \frac{\xi_2}{3\xi_0(\mu - 1)}\left(\frac{\tilde{\beta}_i}{\beta_i} - \frac{\tilde{\alpha}_i}{\alpha_i}\right)\left(\tilde{\mathbf{d}}_{2i+1,0} - \tilde{\mathbf{d}}_{2i,0}\right)\\
&= \frac{\xi_2}{3\xi_0}\frac{\tilde{\alpha}_i}{\alpha_i}\left(\tilde{\mathbf{d}}_{2i+1,0} - \tilde{\mathbf{d}}_{2i,0}\right).
\end{aligned}$$

Similarly, we also have

$$\mathbf{d}_{2i+1,3} - \mathbf{d}_{2i,3} = 3\frac{\xi_2}{\xi_0}\frac{\tilde{\alpha}_i}{\alpha_i}\left(\mathbf{d}_{2i+1,1} - \mathbf{d}_{2i,1}\right).$$

This completes the proof of Theorem 2. □

The above proof leads to the following algorithm.

3.3. An Algorithm of Connecting S and $\widetilde{S}$ in a G^1 Fashion

We describe, with $m \geq \tilde{m}$, the following six steps for modifying S into $S^\star$:

1° In order to connect $S^\star$ and $\widetilde{S}$ in a C^0 fashion, we let

$$\mathbf{d}^\star_{i,0} := \tilde{\mathbf{d}}_{i,0}, \quad i = 0, \ldots, 2\hat{m}, \quad \text{and}$$
$$\mathbf{d}^\star_{2\hat{m}+1,0} := \frac{1}{\beta_{\hat{m}}}(\tilde{\mathbf{d}}_{2\hat{m}+1,0} - \alpha_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},0}).$$

To connect $S^\star$ and $\widetilde{S}$ in the G^1 fashion across the common boundary edge, we further let

$$\mathbf{d}^\star_{i,1} = \left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)\tilde{\mathbf{d}}_{i,0} - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\tilde{\mathbf{d}}_{i,1}, \quad i = 0, \ldots, 2\hat{m},$$
$$\mathbf{d}^\star_{2\hat{m}+1,1} = \frac{1}{\beta_{\hat{m}}}\Bigg[\left(1 + \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}\right)(\tilde{\mathbf{d}}_{2\hat{m}+1,0} - \alpha_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},0}) - \frac{v_4 - v_2}{\tilde{v}_4 - \tilde{v}_2}(\tilde{\mathbf{d}}_{2\hat{m}+1,1} - \alpha_{\hat{m}}\tilde{\mathbf{d}}_{2\hat{m},1})\Bigg].$$

2° Compute the wrinkles "$\hat{\mathbf{c}}_0$" and "$\hat{\mathbf{c}}_1$":

$$\hat{\mathbf{c}}_0 = \left(\frac{3\tilde{\beta}_i}{u_{2i+2} - u_{2i}} - \frac{3\tilde{\alpha}_i}{u_{2i+4} - u_{2i+2}}\right)\left(\tilde{\mathbf{d}}_{2i+1,0} - \tilde{\mathbf{d}}_{2i,0}\right), \quad \text{and}$$
$$\hat{\mathbf{c}}_1 = \left(\frac{3\tilde{\beta}_i}{u_{2i+2} - u_{2i}} - \frac{3\tilde{\alpha}_i}{u_{2i+4} - u_{2i+2}}\right)\left(\mathbf{d}^\star_{2i+1,1} - \mathbf{d}^\star_{2i,1}\right),$$

for $i = 1, \ldots, \hat{m}$. If both $\hat{\mathbf{c}}_0 = 0$ and $\hat{\mathbf{c}}_1 = 0$ for some i, i.e., when there are no wrinkles, the surface $S^\star$ across boundary $\{(u_{2i+2}, v), v \in [v_2, v_4]\}$ is already C^1. Thus, we skip the following steps 3° and 4° for those indices i's.

3° If there are wrinkles, i.e., $\hat{\mathbf{c}}_0 \neq 0$ and $\hat{c}_1 \neq 0$, choose a parameter $\ell = \dfrac{\xi_2}{\xi_0} \neq 0$ and modify $\mathbf{d}_{2i,2}$ and $\mathbf{d}_{2i,3}$ to iron off the wrinkles. We fix $\mathbf{d}_{2i+1,2}$ and compute a new value of $\mathbf{d}_{2i,2}$ in the following way,

$$\mathbf{d}^\star_{2i+1,2} = \mathbf{d}_{2i+1,2},$$
$$\mathbf{d}^\star_{2i,2} = \mathbf{d}_{2i+1,2} - \frac{\ell}{3}\frac{\tilde{\alpha}_i}{\alpha_i}\left(\tilde{\mathbf{d}}_{2i+1,0} - \tilde{\mathbf{d}}_{2i,0}\right), \quad i = 1, \cdots, \hat{m}.$$

Similarly, we fix $\mathbf{d}_{2i+1,3}$ and adjust $\mathbf{d}_{2i,3}$, as follows.

$$\mathbf{d}^\star_{2i+1,3} = \mathbf{d}_{2i+1,3},$$
$$\mathbf{d}^\star_{2i,3} = \mathbf{d}_{2i+1,3} - 3\ell\frac{\tilde{\alpha}_i}{\alpha_i}\left(\mathbf{d}_{2i+1,1} - \mathbf{d}_{2i,1}\right), \quad i = 1, \cdots, \hat{m}.$$

4° Let $\mathbf{d}^\star_{i,j} = \mathbf{d}_{i,j}$ for all the remaining indices i and j.

5° Finally, apply the algorithm in §2.3 to convert the strip $\{\mathbf{f}^\star(u,v), u_2 \le u \le u_{2m+4},\ v_2 \le v \le v_4\}$ of the bi-cubic B-spline surface $S^\star$ into Bernstein form. Then we set

$$\begin{aligned}
\mathbf{c}^\star_{i,1,3,0} &= \tilde{\alpha}_i \tilde{\mathbf{d}}_{2i,0} + \tilde{\beta}_i \tilde{\mathbf{d}}_{2i+1,0}, \\
\mathbf{c}^\star_{i+1,1,0,0} &= \mathbf{c}^\star_{i,1,3,0}, \\
\mathbf{c}^\star_{i,1,3,1} &= \tilde{\alpha}_i \mathbf{d}_{2i,1} + \tilde{\beta}_i \mathbf{d}_{2i+1,1}, \quad \text{and} \\
\mathbf{c}^\star_{i+1,1,0,1} &= \mathbf{c}^\star_{i,1,3,1}, \qquad i = 1, \cdots, \hat{m}.
\end{aligned}$$

Then the surface $S^\star$ is in G^1 and is connected to $\widetilde{S}$ in the G^1 fashion. □

We remark here any two C^1 bi-cubic B-spline surfaces without interpolation conditions can be connected in a G^1 fashion by following the same procedure.

§4. Connection of Multiple C^1 Bi-Cubic B-Spline Surfaces

The connection of multiple C^1 bi-cubic B-spline surfaces is not an easy task. In the following, we present two situations when three and four bi-cubic B-spline surface can be connected smoothly without any gaps.

4.1. G^1 Connection of Multiple C^1 Bi-Cubic B-Spline Surfaces

Let $S, \widetilde{S}, \widehat{S}$ be three C^1 bi-cubic B-spline surfaces as in Fig. 1 with parametric knot sequences $(\mathbf{u}, \mathbf{v})$, $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$, and $(\hat{\mathbf{u}}, \hat{\mathbf{v}}) := \{(\hat{u}_i, \hat{v}_j) : 0 \le i \le 2p+5, 0 \le j \le 2q+5\}$, respectively. Suppose that they satisfy the interpolation conditions (3.1) as well as

$$\widehat{\mathbf{f}}(\hat{u}_2, \hat{v}_{2J-2j}) = \widetilde{\mathbf{f}}(\tilde{u}_2, \tilde{v}_{2j+2}), \quad j = 0, \cdots, \hat{n}_1, \tag{4.1}$$

$$\widehat{\mathbf{f}}(\hat{u}_2, \hat{v}_{2J+2j}) = \mathbf{f}(u_2, v_{2j+2}), \quad j = 0, \cdots, \hat{n}_2, \tag{4.2}$$

for some $J \in \{1, \cdots, q\}$ and $\hat{n}_1 < J,\ \hat{n}_2 < q - J$.

To connect these B-spline surfaces in the G^1 fashion, we first connect S and $\widetilde{S}$ and then connect the resulting combined surface $\check{S}$ to $\widehat{S}$, where $\check{S}$ is the union of $\widetilde{S}$ and the modified $S^\star$ of S.

We first assume that $\mathbf{u}$ is proportional to $\tilde{\mathbf{u}}$. Then by Theorem 1, S can be modified to be $S^\star$, so that $S^\star$ and $\widetilde{S}$ are connected in the G^1 fashion. Next we have to assume that $\mathbf{v}$ and a subsequence of $\hat{\mathbf{v}}$ are proportional. Also we have to assume that $\tilde{\mathbf{v}}$ and another subsequence of

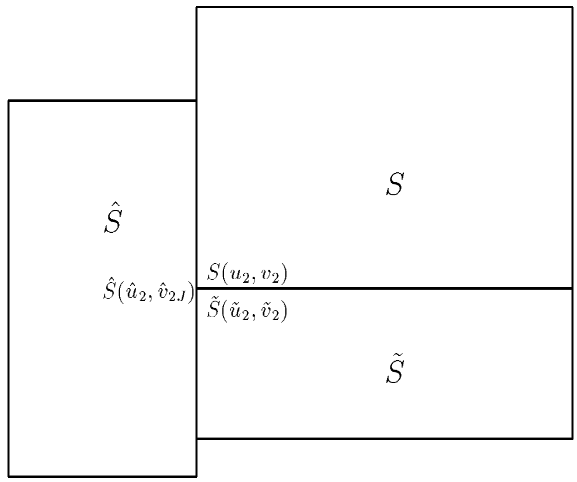

Fig. 1. An illustration for surfaces S, $\widetilde{S}$, and $\widehat{S}$.

$\hat{\mathbf{v}}$ are proportional. That is, letting $\hat{\xi}_j = \dfrac{\hat{v}_{2j+4} - \hat{v}_{2j+2}}{\hat{v}_{2j+4} - \hat{v}_{2j}}$ and $\hat{\eta}_j = 1 - \hat{\xi}_j$, we have to assume that

$$\begin{aligned} \hat{\eta}_{J-j-2} &= \tilde{\xi}_{j+1}, \quad j = 0, \cdots, \hat{n}_1 - 1 \\ \hat{\xi}_{J+j} &= \xi_{j+1}, \quad j = 0, \cdots, \hat{n}_2 - 1. \end{aligned}$$

Then Theorem 1 may be applied to connect the compound B-spline surfaces $\check{S}$ and $\widehat{S}$ in the G^1 fashion, except for the common corner shared by three surfaces. A similar analysis as given in §3.1 shows that it is also necessary and sufficient to have

$$\begin{aligned} &u_{2i+2} - u_{2i} = \tilde{u}_{2i+2} - \tilde{u}_{2i}, \quad i = 1, \cdots, \hat{m} - 1, \qquad \text{and} \\ &\frac{\hat{v}_{2J+2} - \hat{v}_{2J}}{v_4 - v_2} = \frac{\hat{v}_{2J+4} - \hat{v}_{2J+2}}{v_6 - v_4} = \cdots = \frac{\hat{v}_{2J+2\hat{n}_2} - \hat{v}_{2J+2\hat{n}_2-2}}{v_{2\hat{n}_2+2} - v_{2\hat{n}_2}} \\ = &\frac{\hat{v}_{2J} - \hat{v}_{2J-2}}{\tilde{v}_2 - \tilde{v}_4} = \frac{\hat{v}_{2J-2} - \hat{v}_{2J-4}}{\tilde{v}_4 - \tilde{v}_6} = \cdots = \frac{\hat{v}_{2J-2\hat{n}_1+2} - \hat{v}_{2J-2\hat{n}_1}}{\tilde{v}_{2\hat{n}_1} - \tilde{v}_{2\hat{n}_1+2}}. \end{aligned}$$

Furthermore, assume that the parametric knot sequences $\mathbf{u}, \mathbf{v}, \tilde{\mathbf{v}}, \hat{\mathbf{v}}$ do not satisfy the proportionality conditions. To connect these three surfaces together in a G^1 fashion (cf. Fig. 1), we first apply the algorithm in §3.2 to connect S and $\widetilde{S}$ in a G^1 fashion. Let $\check{S}$ be the union of the resulting two surfaces. Note that S and $\widetilde{S}$ are joined in a C^1 fashion across their common boundary edge. We then apply the algorithm in §3.2 to join $\check{S}$ and $\widehat{S}$. We treat $\check{S}$ as if it is in the position of $\widetilde{S}$ in the algorithm, and modify some control points as well as some Bézier coefficients of $\widehat{S}$ to connect $\check{S}$ and $\widehat{S}$ in a G^1 fashion. Then the combined surface, consisting of S, $\widetilde{S}$, and $\widehat{S}$, is a G^1 surface.

Similarly, we can treat the G^1 connection of four bi-cubic B-spline surfaces, if the underlying knot sequences have certain stronger proportionality, e.g., all knot sequences are equally spaced with the same spacing. For another example, supposing that the knot sequences of B-spline surfaces S_1 and S_2 are proportional and the knot sequences of S_3 and S_4 are proportional, then we can first apply Theorem 1 to connect S_1 and S_2 in the G^1 fashion. In this case, S_1 and S_2 can be viewed as one bi-cubic B-spline surface. Similarly, S_3 and S_4 can be viewed as another bi-cubic B-spline surface. Then we apply Theorem 2 or Algorithm in §3.2. Hence, the four B-spline surface can be connected in G^1 fashion.

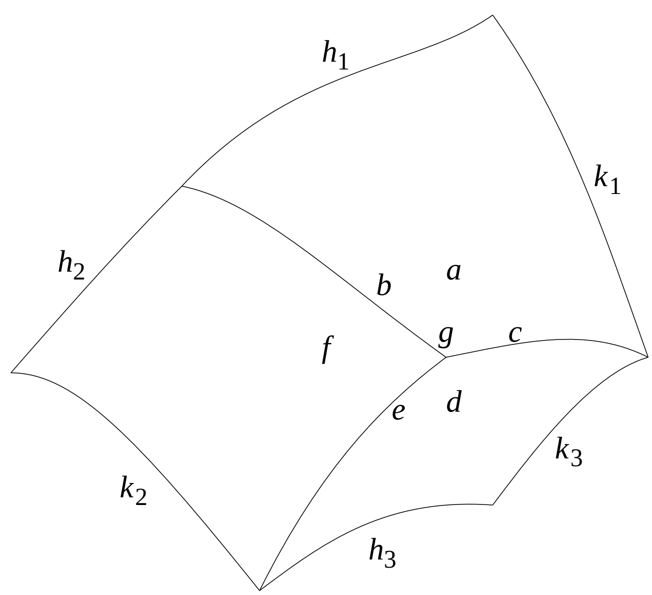

Fig. 2. An illustration for three corner polynomial surfaces.

4.2. G^1 Connection of Three C^1 Bi-Cubic B-Spline Surfaces (continued)

We next show that the methods introduced in §3 do not work for C^1 connection of three C^1 bi-cubic B-spline surfaces $S, \widetilde{S}$, and $\widehat{S}$ as shown in Fig. 2. Let $h_1, k_1, h_2, k_2, h_3, k_3$ be the lengths of the corresponding parametric intervals, as indicated in the diagram, and set

$$\alpha = \frac{h_1}{h_2}, \quad \beta = \frac{k_1}{k_3}, \quad \gamma = \frac{k_2}{h_3}.$$

The C^0 continuity requires that the Bézier coefficients agree along the three boundary curves. So we consider the seven Bézier coefficients $\mathbf{a}$, $\mathbf{b}$, ..., $\mathbf{g}$ as shown in Fig. 2. Then the C^1 continuity at the central point is equivalent to

$$\begin{aligned} \mathbf{b} - \mathbf{a} &= \alpha(\mathbf{f} - \mathbf{b}), \quad & \mathbf{g} - \mathbf{c} &= \alpha(\mathbf{e} - \mathbf{g}), \\ \mathbf{c} - \mathbf{a} &= \beta(\mathbf{d} - \mathbf{c}), \quad & \mathbf{g} - \mathbf{b} &= \beta(\mathbf{e} - \mathbf{g}), \\ \mathbf{g} - \mathbf{b} &= \gamma(\mathbf{c} - \mathbf{g}), \quad & \mathbf{e} - \mathbf{f} &= \gamma(\mathbf{d} - \mathbf{e}). \end{aligned}$$

It is easy to verify, by using the fact that $\alpha, \beta, \gamma > 0$, that these six equalities imply that $\mathbf{a} = \mathbf{b} = \cdots = \mathbf{g}$, which violates the true C^1 meaning at the common corner point shared by the three surfaces. That is, we have shown that C^1 connection of three C^1 bi-cubic B-spline surfaces with a common corner is not possible.

§5. Numerical Examples

In the following we present several examples to demonstrate the algorithms for joining two bi-cubic B-spline surfaces in a G^1 fashion. In fact, as we remarked earlier, the interpolation conditions of the two surfaces are not essential. Our algorithms are able to join any two surfaces in a G^1 fashion.

Example 1. *Consider two C^1 bi-cubic parametric B-spline surfaces S and $\widetilde{S}$ that interpolate certain data sets at the knots. Clearly, S and $\widetilde{S}$ are not joined continuously. See S and $\widetilde{S}$ in Fig. 3. After applying the algorithm in §3.3, we obtain a new surface which satisfies a G^1 continuity condition, as shown in Fig. 4.*

Example 2. *Let S and $\widetilde{S}$ be two bi-cubic B-spline surfaces as shown in Fig. 5. After applying the algorithm for joining S and $\widetilde{S}$ in a G^1 fashion in §3.4, we obtain new surfaces which satisfy the G^1 continuity condition, as shown in Fig. 6, Fig. 7, Fig. 8, and Fig. 9.*

Acknowledgments. This research was supported by the National Science Foundation GOALI program under grant # DMI-96-34833, with matching support by McDonnell Douglas Aerospace (now Boeing Company, St. Louis).

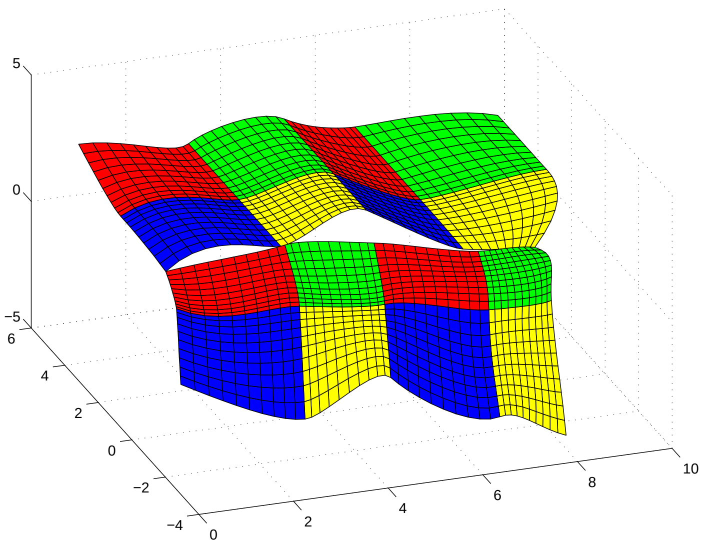

Fig. 3. S and $\widetilde{S}$ only interpolate at some common data points.

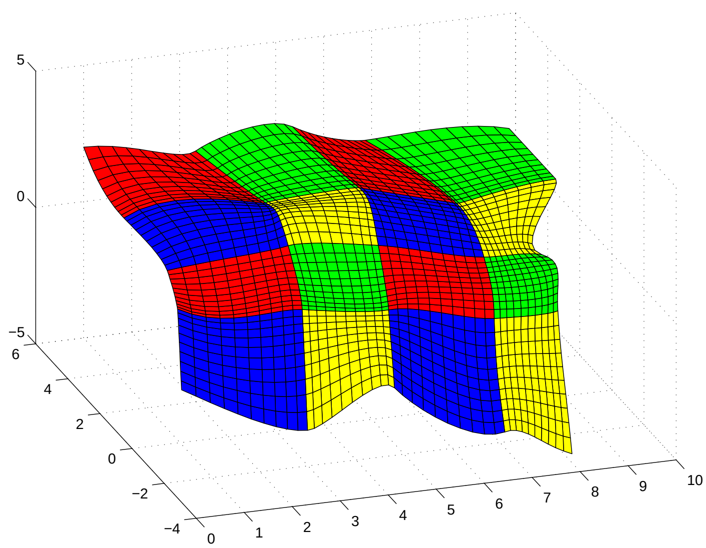

Fig. 4. S and $\widetilde{S}$ are joined in a G^1 fashion.

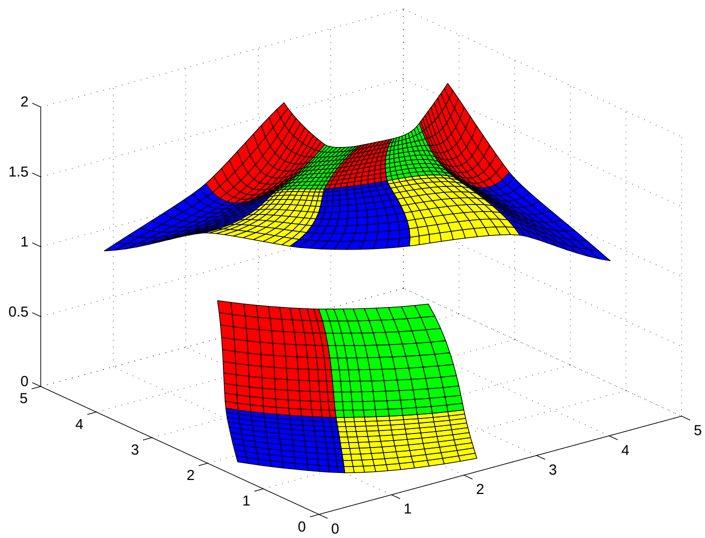

Fig. 5. Two disjoint surfaces S and $\widetilde{S}$.

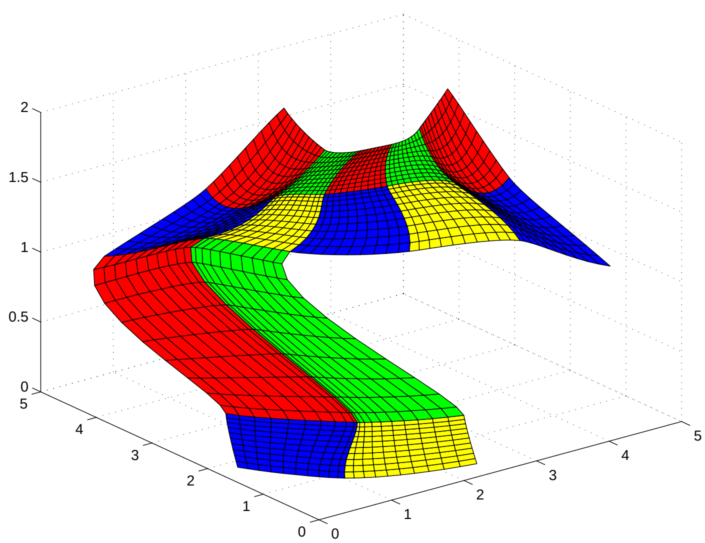

Fig. 6. S and $\widetilde{S}$ are joined in a G^1 fashion.

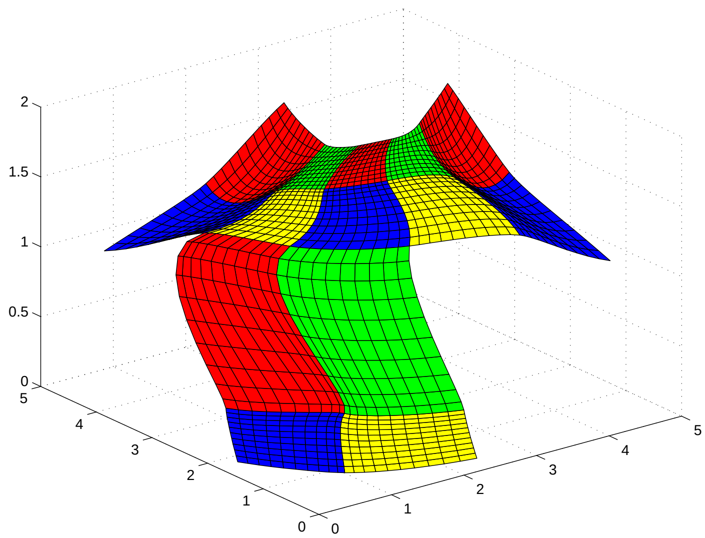

Fig. 7. S and $\widetilde{S}$ are joined in a G^1 fashion.

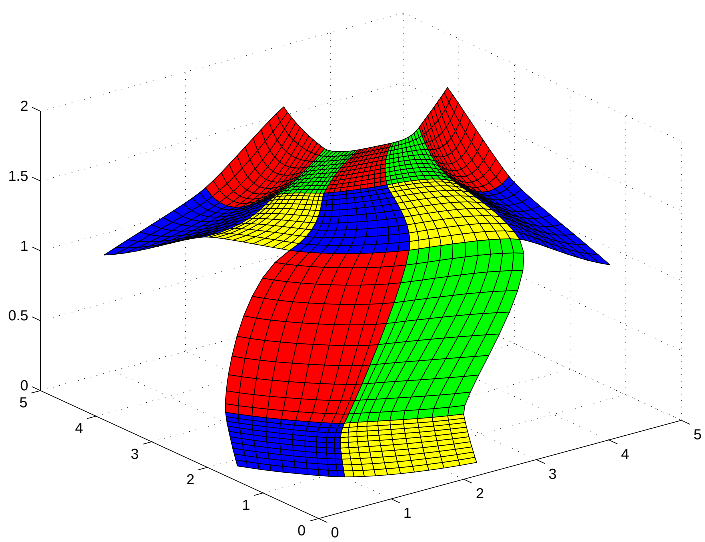

Fig. 8. S and $\widetilde{S}$ are joined in a G^1 fashion.

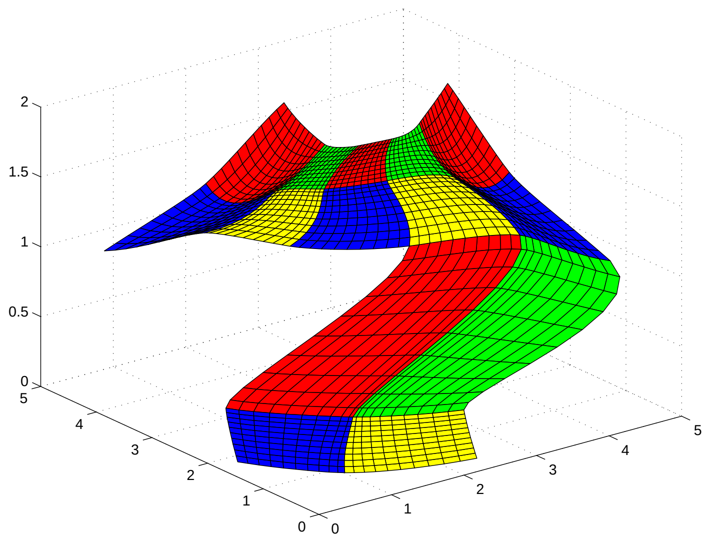

Fig. 9. S and $\widetilde{S}$ are joined in a G^1 fashion.

References

1. de Boor, C., *A Practical Guide to Splines*, Springer Verlag, 1978.
2. Chui, C. K., *Multivariate Splines*, SIAM Publications, 1988.
3. Chui, C. K., M. J. Lai, and J.-A. Lian, Algorithms for G^1 connection of multiple parametric bi-cubic NURBS surfaces, Numerical Algorithms **23** (2000), 285–313.
4. Chui, C. K., M. J. Lai, J.-A. Lian, and P. F. Cassidy, Removal of gaps among compound C^1 bi-cubic parametric B-spline surfaces, in manuscript (presented at the *NSF Design and Manufacturing Grantees Conference*, Jan. 5–8, 1998, Monterrey, Mexico, 141–142.)
5. Farin, G., *Curves and Surfaces for Computer Aided Geometric Design*, Academic Press, San Diego, CA, 1995.
6. Farin, G. and B. Hamann, Current trends in geometric modeling and selected computational applications, J. Comput. Physics **138** (1997), 1–15.
7. Hamann, B. and B. A. Jean, Interactive surface correction based on a local approximation scheme, Comput. Aided Geom. Design **13** (1996), 351–368.
8. Jean, B. A. and B. Hamann, Interactive techniques for correcting CAD/CAM data, in *Numerical Grid Generation in Computational*

Fluid Dynamics and Related Fields, N. P. Weatherill, P. R. Eiseman, J. Häuser and J. F. Thompson, eds., Pineridge Press Ltd., Swansea, U. K. 1994, 317–328.

9. Konno, K. and H. Chiyokura, An approach of designing and controlling free-form surfaces by using NURBS boundary Gregory patches, Comput. Aided Geom. Design **13** (1996), 825–849.
10. Liu, D. and J. Hoschek, GC^1 continuity conditions between adjacent rectangular and triangular Bézier surface patches, Computer-Aided Design **21** (1989), 194–200.
11. Schumaker, L. L., *Spline Functions: Basic Theory*, Wiley, New York, 1981.
12. Che, X. J., X. Z. Liang, and Q. Li, G^1 continuity conditions of adjacent NURBS surfaces, Comput. Aided Geom. Design **22** (2005), 285–298.

Ming-Jun Lai
Department of Mathematics
University of Georgia
Athens, GA 30602-7403
mjlai@math.uga.edu

Jian-ao Lian
Department of Mathematics
Prairie View A&M University
Prairie View, Texas 77446-4189
JiLian@pvamu.edu

&

Patrick F. Cassidy
The Boeing Company
P.O. Box 516
St. Louis, MO 63166
Patrick.Cassidy@MW.Boeing.com