Tight Wavelet Frames over Bounded Domains

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Abstract. A simple constructive method for a locally supported tight wavelet frame over a bounded domain is introduced in this paper. Examples of B-spline tight wavelet frames and box spline tight wavelet frames over bounded domains are constructed to demonstrate the method.

§1. Introduction

A construction of tight wavelet frames over bounded domains is quite different from construction of tight wavelet frames over unbounded domains. Concrete examples in [3] show that it is impossible to construct tight wavelet frames over bounded domains by modifying tight wavelet frames over unbounded domains. The authors in [3] developed a general theory of non-stationary tight wavelet frame construction over a bounded interval using univariate splines.

In this paper we introduce a simple constructive method for tight wavelet frames over bounded domains. We apply this simple method to construct B-spline tight wavelet frames over interval and box spline tight wavelet frames over bounded domains. One of the advantages of our method is that our method works in the multivariable settings.

To be precise what we study in this paper, let us explain the concept of tight wavelet frames first. A tight wavelet frame over a bounded domain Ω is based on a half infinite sequence of nested subspaces over Ω in $L^2(\Omega)$. That is, a sequence of nested subspaces $\{V_k\}_{k\in\mathbb{Z}_+}$ in $L^2(\Omega)$ satisfies

$$V_1 \subset V_2 \subset \cdots \subset V_k \subset \cdots \to L^2(\Omega)$$
$$\text{and} \quad \overline{\bigcup_{k=1} V_k} = L^2(\Omega).$$

Let $\Phi_k := (\phi_{k,1}, \cdots, \phi_{k,m_k})^T$ be a column vector of locally supported functions in V_k which generate V_k, i.e., $V_k = \text{span}\{\phi_{k,1}, \cdots, \phi_{k,m_k}\}$. Although this sequence of subspaces $\{V_k\}_{k\in\mathbb{Z}_+}$ does not have both translation

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and dilation invariant properties, we say $\{\Phi_k\}_{k\in\mathbb{Z}_+}$ generates a multiresolution analysis(MRA) over bounded domain Ω.

Because V_k is a subspacce of V_{k+1}, the vector Φ_k in V_k can be generated by the column vector Φ_{k+1} which spans V_{k+1}. That is, there exists a matrix P_k of size $m_k \times m_{k+1}$ $(m_k \le m_{k+1})$ such that

$$\Phi_k = P_k \Phi_{k+1}. \tag{1}$$

The matrix P_k is often called a refinement matrix. Let Q_k be a matrix of size $n_k \times m_{k+1}$. Define

$$\Psi_k := Q_k \Phi_{k+1}. \tag{2}$$

Let $\langle f, g\rangle = \int_\Omega fg$ be the standard inner product on $L_2(\Omega)$.

Definition 1. *We say the family of vectors $\{\Psi_k\}_{k\in\mathbb{Z}_+}$ defined in (2) is* a (MRA) tight wavelet frame *associated with $\{\Phi_k\}_{k\in\mathbb{Z}_+}$ in $L^2(\Omega)$ if*

$$\|f\|^2 = \sum_{j=1}^{m_1} |\langle f, \phi_{1,j}\rangle|^2 + \sum_{k=1}^{\infty}\sum_{j=1}^{n_k} |\langle f, \psi_{k,j}\rangle|^2, \qquad \forall f \in L^2(\Omega),$$

where $\Phi_k = (\phi_{k,1}, \cdots, \phi_{k,m_k})^T$ and $\Psi_k = (\psi_{k,1}, \cdots, \psi_{k,n_k})^T$. We call each function $\psi_{k,j}$ for $j = 1, \cdots, n_k$ and $k \in \mathbb{Z}_+$ a tight framelet *(or a tight wavelet frame generator).*

If $\phi_{1,j}, j = 1, \cdots, m_1$ and $\psi_{k,j}, j = 1, \cdots, n_k, k = 1, \cdots$ generate a tight wavelet frame, then

$$f = \sum_{j=1}^{m_1} \langle f, \phi_{1,j}\rangle \phi_{1,j} + \sum_{k=1}^{\infty}\sum_{j=1}^{n_k} \langle f, \psi_{k,j}\rangle \psi_{k,j}$$

for any $f \in L_2(\Omega)$.

The paper is organized as follows. In §2, a tight wavelet frame construction idea is described. In §3, we present a method for tight wavelet frame construction. According to the construction scheme in §3, B-spline tight wavelet frames and box spline tight wavelet frames over bounded domains are constructed in §4 and §5, respectively.

§2. A Constructive Idea

In this section, we assume that we are given refinable vectors Φ_k for V_k with refinable matrix P_k. Let us show that finding a matrix Q_k satisfying

$$I_{m_{k+1}} = P_k^T P_k + Q_k^T Q_k, \tag{3}$$

for a given refinement matrix P_k in (1) is a key step for constructing a tight wavelet frame. Here $I_{m_{k+1}}$ is the standard identity matrix of size m_{k+1}.

Indeed, let $\Psi_k = Q_k\Phi_{k+1}$ be a vector of functions. Clearly, $\Psi_k \subset V_{k+1}$. We want to have

$$\langle f, \Phi_{k+1}\rangle^T \Phi_{k+1} = \langle f, \Phi_k\rangle^T \Phi_k + \langle f, \Psi_k\rangle^T \Psi_k, \quad \forall f \in L^2(\Omega). \qquad (4)$$

Let $c_{k,i} := \langle f, \phi_{k,i}\rangle$ for all $i = 1, \cdots, m_k$ and $C_k := (c_{k,1}, \cdots, c_{k,m_k})^T$ be a column vector of size $m_k \times 1$ for any $k \in \mathbb{Z}_+$. In the same way, let $d_{k,j} := \langle f, \psi_{k,j}\rangle$ for all $j = 1, \cdots, n_k$ and $D_k := (d_{k,1}, \cdots, d_{k,n_k})^T$. Then we know

$$\begin{aligned} C_k &= \langle f, \Phi_k\rangle = \langle f, P_k\Phi_{k+1}\rangle = P_k C_{k+1}, \\ D_k &= \langle f, \Psi_k\rangle = \langle f, Q_k\Phi_{k+1}\rangle = Q_k C_{k+1}. \end{aligned} \qquad (5)$$

Thus condition in (4) can be expressed in the following form according to our notations,

$$C_{k+1}^T\Phi_{k+1} = C_k^T P_k\Phi_{k+1} + D_k^T Q_k\Phi_{k+1}.$$

That is, $C_{k+1}^T = C_k^T P_k + D_k^T Q_k$. By using (5), we get

$$C_{k+1}^T C_{k+1} = C_{k+1}^T(P_k^T P_k + Q_k^T Q_k)C_{k+1}.$$

This implies that Q_k must satisfy (3) for all $k \geq 1$. On the other hand, if we find Q_k satisfying (3) for all $k \geq 1$, then we have the above equation and hence, by using (5),

$$C_{k+1}^T C_{k+1} = C_k^T C_k + D_k^T D_k.$$

It follows for any $\ell \in \mathbb{Z}_+$ with $\ell < k$,

$$C_{k+1}^T C_{k+1} = C_\ell^T C_\ell + \sum_{j=\ell}^{k} D_j^T D_j \qquad (6)$$

The condition (3) implies $C_{k+1}^T = C_{k+1}^T(P_k^T P_k + Q_k^T Q_k) = C_k^T P_k + D_k^T Q_k$ and hence,

$$C_{k+1}^T\Phi_{k+1} = C_k^T\Phi_k + Q_k^T\Psi_k = ... = C_\ell^T\Phi_\ell + \sum_{j=\ell}^{k} D_j^T\Psi_j.$$

If $C_{k+1}^T\Phi_{k+1}$ converges to f in $L^2(\Omega)$, for any $\ell \in \mathbb{Z}_+$, we have

$$\begin{aligned}\|f\|^2 =&\langle f, \lim_{k\to+\infty} C_{k+1}^T\Phi_{k+1}\rangle \\ =&\lim_{k\to+\infty}\langle f, C_\ell^T\Phi_\ell + \sum_{j=\ell}^{k} D_j^T\Psi_j\rangle \\ =&C_\ell^T C_\ell + \sum_{j=\ell}^{\infty} D_j^T D_j \\ =&\sum_{j=1}^{m_\ell}|\langle f, \phi_{\ell,j}\rangle|^2 + \sum_{k=\ell}^{\infty}\sum_{j=1}^{n_k}|\langle f, \psi_{k,j}\rangle|^2\end{aligned} \tag{7}$$

If we apply (7) for a fixed f and for all g in $L^2(\Omega)$, then

$$\begin{aligned}\|f+g\|^2 &= \sum_{j=1}^{m_\ell}|\langle f+g, \phi_{\ell,j}\rangle|^2 + \sum_{k=\ell}^{\infty}\sum_{j=1}^{n_k}|\langle f+g, \psi_{k,j}\rangle|^2, \\ \|f-g\|^2 &= \sum_{j=1}^{m_\ell}|\langle f-g, \phi_{\ell,j}\rangle|^2 + \sum_{k=\ell}^{\infty}\sum_{j=1}^{n_k}|\langle f-g, \psi_{k,j}\rangle|^2.\end{aligned}$$

Subtracting the equation (8) from (8), we have

$$4\langle f, g\rangle = 4(\langle \sum_{j=1}^{m_\ell}\langle f, \phi_{\ell,j}\rangle\phi_{\ell,j} + \sum_{k=\ell}^{\infty}\sum_{j=1}^{n_k}\langle f, \psi_{k,j}\rangle\psi_{k,j}\ ,\ g\,\rangle).$$

Thus for all $f \in L^2(\Omega)$ and for all $\ell \in \mathbb{Z}_+$,

$$f = \sum_{j=1}^{m_\ell}\langle f, \phi_{\ell,j}\rangle\phi_{\ell,j} + \sum_{k=\ell}^{\infty}\sum_{j=1}^{n_k}\langle f, \psi_{k,j}\rangle\psi_{k,j}, \quad weakly. \tag{8}$$

Therefore any function in $L^2(\Omega)$ can be analyzed at any level of refinable functions Φ_ℓ and together with tight framelets $\psi_{k,j}$ associated with these refinable functions Φ_k with $k \geq \ell$. Therefore we conclude the following

Theorem 1. *Suppose that Φ_k is a given refinable vector which spans V_k for all $k \geq 1$ with refinable matrix P_k, i.e., $\Phi_k = P_k\Phi_{k+1}$. Suppose Q_k satisfies (3). Let $\Psi_k = Q_k\Phi_k$. Then $\Psi_k, k \in \mathbb{Z}_+$ form a tight wavelet frame. Hence, any $f \in L^2(\Omega)$ can be generated by using Φ_ℓ and Ψ_k with $k \geq \ell$ for any $\ell \geq 1$ as in (8).*

§3. A Constructive Method

According to the constructive idea from the the previous section, we summarize a tight wavelet frame construction over a bounded domain as follows. We begin with a criterion how to compute Q_k satisfying (3).

Theorem 2. *Let $\{V_k\}$ be a MRA generated by a family of functions Φ_k. Denote by P_k the refinable matrix, i.e., $\Phi_k = P_k \Phi_{k+1}$. If $I_{m_k} - P_k P_k^T$ is positive semi-definite for the identity matrix I_{m_k} of size $m_k \times m_k$, then there exists a Q_k satisfying (3) and hence, there exists a tight wavelet frame $\{\Psi_k\}_{k\in\mathbb{Z}_+}$ of $L^2(\Omega)$ defined such a way in (2). Moreover, if each component function $\phi_{k,j}$ of a vector Φ_k is locally supported then each component function $\psi_{k,j}$ of the vector Ψ_k is locally supported.*

Proof: Since the symmetric matrix $I_{m_k} - P_k P_k^T$ is positive semi-definite, there exists a unique lower triangular matrix L_k such that

$$I_{m_k} = P_k P_k^T + L_k L_k^T. \tag{9}$$

Using this lower triangular matrix L_k we let

$$R_k = I_{m_{k+1}+m_k} - \begin{bmatrix} {P_k}^T \\ L_k^T \end{bmatrix} \begin{bmatrix} P_k & L_k \end{bmatrix}. \tag{10}$$

Note that the matrix R_k is symmetric and $R_k^T R_k = R_k$. Writing $R_k = \begin{bmatrix} \widetilde{Q}_k & W_c \end{bmatrix}$ with matrix $\widetilde{Q}_k$ of size $(m_{k+1}+m_k)\times m_{k+1}$ and W_c being the term who cares, we observe

$$\widetilde{Q}_k^T \widetilde{Q}_k = I_{m_{k+1}} - P_k^T P_k.$$

It is clear that the rank of $\widetilde{Q}_k$ is less than or equal to m_{k+1}. Write

$$\widetilde{Q}_k = \begin{bmatrix} J_k \\ \widehat{J}_k \end{bmatrix}$$

with J_k being of size $m_{k+1}\times m_{k+1}$ and $\widehat{J}_k$ of size $m_k \times m_{k+1}$. Then we multiply m_{k+1} Householder transformations $H_{m_k}H_{m_k-1}\cdots H_2H_1$ of size $(m_k+m_{k+1})\times(m_k+m_{k+1})$ on the left side of matrix $\widetilde{Q}_k$. That is,

$$H_{m_k}H_{m_k-1}\cdots H_2H_1\widetilde{Q}_k = \begin{bmatrix} Q_k \\ 0 \end{bmatrix}, \tag{11}$$

where Q_k is a upper triangular matrix of size $m_{k+1}\times m_{k+1}$. Let us denote $U_k := H_{m_k}H_{m_k-1}\cdots H_2H_1$. Then U_k is a unitary matrix and we have

$$Q_k^T Q_k = (U_k\widetilde{Q}_k)^T(U_k\widetilde{Q}_k) = \widetilde{Q}_k^T\widetilde{Q}_k = I_{m_{k+1}} - P_k^T P_k,$$

The matrix Q_k in (11) is the matrix we want to have with full rank m_k. By using Theorem1, we conclude that Ψ_k so defined using the matrix Q_k form a tight wavelet frame.

Moreover, when Φ_k consists of locally supported functions for all $k \geq 1$, each P_k is a banded matrix for $k \in \mathbb{Z}_+$. When P_k is a banded matrix, so is $P_k P_k^T$. It follows that L_k is banded. Thus, it is easy to see from the definition of Q_k that Q_k is banded. Thus, $\psi_{k,j}$ are locally supported for each $k \in \mathbb{Z}_+$ and $j = 1, \cdots, m_{k+1}$. $\square$

§4. B-spline Tight Wavelet Frames

Because of the efficiency and simplicity of computation, B-splines often have been used for constructing wavelet functions. In this section, we apply the constructive method from the proof of Theorem 2 to construct tight wavelet frames over a bounded domain using B-spline functions defined in equally spaced simple knots.

Let us recall the scaling relation of B-spline ϕ^m for $m \geq 2$ (cf.[2]).

$$\phi^m(x) = \sum_{j \in \mathbb{Z}} c_j^m \phi^m(2x - j),$$

where

$$c_j^m = \begin{cases} 2^{-m+1} \begin{pmatrix} m \\ j \end{pmatrix} & \text{for } 0 \leq j \leq m \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Consider B-spline function ϕ^m of order m whose dyadic translations are restricted into domain $[0, b]$, i.e., $\phi^m(2^k \cdot -i)|_{[0,b]}$. Let

$$\phi_{k,j}^m(\cdot) = 2^{k-1} \phi^m(2^{k-1} \cdot -j)\Big|_{[0,b]}$$

and

$$V_k^m := \{\phi_{k,j}^m : 1 \leq j \leq m_k\}.$$

Then the family of nested sequence of subspaces $\{V_k^m : k \in Z_+\}$ is a MRA generated by $\{\phi_{k,1}^m, \cdots, \phi_{k,m_k}^m\}$, where $m_k := 2^{k+1}(m-1)+1$. Thus if we denote

$$\Phi_k^m := (\phi_{k,1}^m, \cdots, \phi_{k,m_k}^m)^T,$$

we can find a refinement matrix P_k^m of size $m_k \times m_{k+1}$ of a vector satisfying $\Phi_k^m = P_k^m \Phi_{k+1}^m$ for each $k \in \mathbb{Z}_+$. First, we check the positive semi-definite property of the matrix $I_{m_k} - P_k^m \cdot P_k^{mT}$ for the identity matrix I_{m_k}.

Lemma 1. *The symmetric matrix $I_{m_k} - P_k^m \cdot P_k^{mT}$ of size $m_k \times m_k$ associated with B-splines of order m is positive semi-definite for each $k \in \mathbb{Z}$ and $m \geq 2$.*

Proof: Let us denote $(p_{i,j}^{m,k}) := P_k^m$. Then for each $i = 1, \cdots, m_k$

$$0 \le \sum_{j=1}^{m_{k+1}} p_{i,j}^{m,k} \le \frac{1}{2} \sum_{j=0}^{m} c_j^m = 1, \tag{13}$$

where c_j^m is in (12) . Let us denote $G_k^m := (g_{i,j}^{m,k}) = P_k^m \cdot P_k^{mT}$. To show that matrix $I_{m_k} - G_k^m$ is positive semi-definite, we use diagonal dominance of matrix $I_{m_k} - G_k^m$. Since matrix G_k^m is symmetry, it is sufficient to check $|1 - g_{i,i}^{m,k}| \ge \sum_{i \ne j} |\, g_{i,j}^{m,k} \,|$ for $i \le \lfloor \frac{m_k}{2} \rfloor + 1$. Notice that

$$g_{i,j}^{m,k} = \sum_{\ell=1}^{m_{k+1}} p_{i,\ell}^{m,k} p_{\ell,j}^{m,k}.$$

Then for each $k \in \mathbb{Z}_+$,

$$\begin{aligned} 1 - |g_{i,i}^{m,k}| - \sum_{j \ne i} |\, g_{i,j}^{m,k} \,| &= 1 - \sum_{j=1}^{m_{k+1}} \sum_{\ell=1}^{m_{k+1}} p_{i,\ell}^{m,k} p_{j,\ell}^{m,k} \\ &= 1 - \left(\sum_{\ell=1}^{m_{k+1}} p_{i,\ell}^{m,k} \right) \left(\sum_{\ell=1}^{m_{k+1}} p_{j,\ell}^{m,k} \right). \end{aligned}$$

Since (13), $1 - |g_{i,i}^{m,k}| \ge \sum_{j \ne i} |\, g_{i,j}^{m,k} \,|$ for all $i = 1, \cdots, m_k$. Therefore the symmetry matrix $I_{m_k} - P_k^m \cdot P_k^{mT}$ is positive semi-definite. □

By the above lemma, we know that for the refinement matrix P_k^m of a vector Φ_k^m whose component B-spline functions generate subspace V_k^m in $L^2([0,b])$ satisfies the sufficient condition in Theorem 2. That is, we can construct B-spline tight wavelet frame over $[0,b]$.

The size of the support of B-spline tight framelets $\psi_{k,n_1}^m, \cdots, \psi_{k,n_k}^m$ is the same as that of the support of the B-splines $\phi_{k,m_1}^m, \cdots, \phi_{k,m_k}^m$ at each level $k \in \mathbb{Z}_+$ according to our computation below.

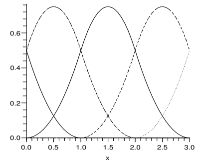
In the following example, we illustrate B-spline tight framelets Ψ_1^m of order $m = 3$ obtained by the matrix Q_1^m for the given matrix P_1^m associated with Φ_1^m. We can compute P_k^m and Q_k^m for any $k \in \mathbb{Z}_+$ and for arbitrary integer order $m \ge 2$ of B-spline functions.

Example 1. *For the quadratic B-spline* ϕ^m *over the interval* $[0,3]$, *where* $m = 3$,*bwe have the column vectors*

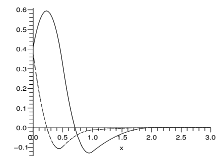
$$\Phi_1^3 = [\ \phi^3(x+2)|_{[0,3]}\ \phi^3(x+1)|_{[0,3]}\ \phi^3(x)|_{[0,3]}\ \phi^3(x-1)|_{[0,3]}\ \phi^3(x-2)|_{[0,3]}\ ]^T$$

$$:= [\phi_{1,1}^3\ \cdots\ \phi_{1,5}^3]^T$$

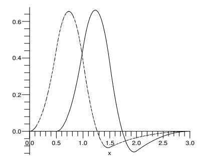
and $\Phi_2^3 = [\ 2\phi^3(2x+4)|_{[0,3]}\ \cdots\ 2\phi^3(2x-4)|_{[0,3]}\ ]^T$. *Then from the relation* $\Phi_1^3 = P_1^3 \cdot \Phi_2^3$, *we have the refinement matrix* P_1^3. *We use the constructive*



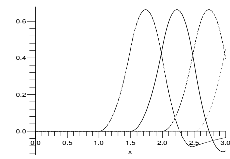
Quadratic B-spline $\{\phi^3_{1,1}, \cdots, \phi^3_{1,5}\}$



Framelets $\psi^3_{1,1}$ and $\psi^3_{1,2}$



Framelets $\psi^3_{1,3}$ and $\psi^3_{1,4}$



Framelets $\psi^3_{1,5}, \psi^3_{1,6}, \psi^3_{1,7}, \psi^3_{1,8}$

Fig. 1. Quadratic B-splines and quadratic B-spline tight framelets of the ground level over a bounded domain $[0, 3]$

method in Theorem 2 to have the matrix Q^3_1. Then we define $\Psi^3_1 := Q^3_1 \cdot \Phi^3_2$. The components $\psi^3_{1,1}, \cdots, \psi^3_{1,8}$ of the column vector Ψ^3_1 are quadratic B-spline tight framelets of the ground level. We illustrate the quadratic B-spine and its tight wavelet framelets of the ground level in Fig 1.

§5. Box Spline Tight Wavelet Frames

Our tight wavelet frame constructive method can be applied in the multivariate setting. In this section we use it to construct tight wavelet frames using box spline functions on three direction mesh.

Let us recall a 3-direction mesh box spline $\phi^{\ell,m,n}(x, y)$ whose Fourier transform is defined as follows for $\ell, n, m \in \mathbb{Z}_+$ (cf.[2]),

$$\widehat{\phi}^{\ell mn}(\xi, \eta) = \left(\frac{1 - e^{-\sqrt{-1}\xi}}{\sqrt{-1}\xi} \right)^{\ell} \left(\frac{1 - e^{-\sqrt{-1}\eta}}{\sqrt{-1}\eta} \right)^{m} \left(\frac{1 - e^{-\sqrt{-1}(\xi+\eta)}}{\sqrt{-1}(\xi + \eta)} \right)^{n}.$$

To make our notations simple, let us denote $\phi^\nu := \phi^{\ell,m,n}$. The two-scale relation of 3-direction mesh box splines is

$$\phi^\nu(x, y) = \sum_{i,j \in \mathbb{Z}} c_{i,j} \phi^\nu(2x - i, 2y - j),$$

and its Fourier transformation is

$$\widehat{\phi}^\nu(2\xi, 2\eta) = C(\xi,\eta)\,\widehat{\phi}^\nu(\xi,\eta),$$
$$\text{where} \quad C(\xi,\eta) = \frac{1}{4}\sum_{i,j\in\mathbb{Z}} c_{i,j} e^{\sqrt{-1}(i\xi + j\eta)} \quad \text{and} \quad |\,C(0,0)\,| = 4. \tag{14}$$

Consider a 3-direction mesh box spline ϕ^ν whose dyadic translations are restricted into the domain $[0,a]\times[0,b]$, i.e., $\phi^\nu(2^k x - i, 2^k y - j)|_{[0,a]\times[0,b]}$. Let us denote

$$\phi^\nu_{k,q}(x,y) := 2^{2k}\phi^\nu(2^k x - q_1, 2^k y - q_2)\big|_{[0,a]\times[0,b]}$$

Let m_k be the cardinality of the collection of box splines $\phi^\nu_{k,q}$ which are not zero over $[0,a]\times[0,b]$ and

$$V^\nu_k := \{\phi^\nu_{k,q} : 1 \le q \le m_k\}.$$

Then the family of nested sequence of subspaces $\{V^\nu_k : k \in Z_+\}$ is a MRA generated by $\{\phi^\nu_{k,1}, \cdots, \phi^\nu_{k,m_k}\}$. Thus if we denote

$$\Phi^\nu_k := (\phi^\nu_{k,1}, \cdots, \phi^\nu_{k,m_k})^T,$$

we can find a refinement matrix P^ν_k of size $m_k \times m_{k+1}$ of a vector satisfying $\Phi^\nu_k = P^\nu_k \Phi^\nu_{k+1}$ for each $k \in \mathbb{Z}_+$.

The following lemma says the refinement matrix P^ν_k of a vector Φ^ν_k whose component functions generate subspace V^ν_k in $L^2([0,a]\times[0,b])$ satisfies the sufficient condition in Theorem 2.

Lemma 2. *If P^ν_k is a matrix of size $m_k \times m_{k+1}$ generated by a collection of box spline functions Φ^ν_k over bounded domain, i.e. $\Phi^\nu_k = P^\nu_k \Phi^\nu_{k+1}$, then*

$$I_{m_k} - P^\nu_k \cdot P^{\nu T}_k, \qquad \text{for each } k \in \mathbb{Z}_+$$

is positive semi-definite.

Proof: Let us denote $(p^{\nu,k}_{i,j}) := P^\nu_k$ and $(g^{\nu,k}_{i,j}) := G^\nu_k = P^\nu_k \cdot P^{\nu T}_k$. Because of (14) ,

$$0 \le \sum_{j=0}^{m_{k+1}} p^{\nu,k}_{i,j} \le \frac{1}{4}\sum_{\ell=1}^{m_{k+1}} c_{i,\ell} c_{\ell,j} = 1. \tag{15}$$

To show that matrix $I_{m_k} - G^\nu_k$ is positive semi-definite, we use diagonal dominance of matrix $I_{m_k} - G^\nu_k$. Since the matrix G^ν_k is symmetry, it is sufficient to check $|1 - g^{\nu,k}_{i,i}| \ge \sum_{i\ne j} |\, g^{\nu,k}_{i,j}\,|$ for $i \le \lfloor \frac{m_k}{2} \rfloor + 1$.

$$0 \le g^{\nu,k}_{i,j} = \sum_{\ell=1}^{m_{k+1}} p^{\nu,k}_{i,\ell} p^{\nu,k}_{\ell,j} = 1.$$

Because of (15),

$$1-|g^{\nu,k}_{i,i}|-\sum_{j\neq i}|\,g^{\nu,k}_{i,j}\,| = 1-\sum_{j=1}^{m_{k+1}}\sum_{\ell=1}^{m_{k+1}} p^{\nu,k}_{i,\ell}p^{\nu,k}_{j,\ell}$$
$$= 1-\left(\sum_{\ell=1}^{m_{k+1}} p^{\nu,k}_{i,\ell}\right)\left(\sum_{\ell=1}^{m_{k+1}} p^{\nu,k}_{j,\ell}\right) \geq 0\,.$$

Therefore the symmetry matrix $I_{m_k} - P_k^{\nu}\cdot P_k^{\nu\,T}$ is positive semi-definite. □

Thus we can construct box spline tight wavelet frame over bounded domain $[0,a]\times[0,b]$ by using the constructive scheme in the proof of Theorem 2. The size of support of box spline tight framelets $\psi^{\nu}_{k,n_1},\cdots,$ ψ^{ν}_{k,n_k} is the same as that of the support of the box splines $\phi^{\nu}_{k,m_1},\cdots,$ ϕ^{ν}_{k,m_k} at each level $k\in\mathbb{Z}_+$ according to our computation experience.

In the following example we illustrate some of tight wavelet framelets obtained by setting $\Psi_1^{\nu} = Q_1^{\nu}\Phi_1^{\nu}$ for the given refinement matrix P_1^{ν} associated with the vector Φ_1^{ν} of refinable functions $\phi^{\nu}_{1,1},\cdots,\phi^{\nu}_{1,m_1}$ over a bounded domain, where $\nu=\{1,1,1\}$.

Example 2. *For box spline ϕ^{111} over $[0,2]\times[0,2]$, we set the column vector Φ_1^{111} with all the integer translations of ϕ^{111} over the domain $[0,2]\times[0,2]$ as follows*

$$\begin{aligned}\Phi_1^{111} &= [\ \phi^{111}(x+1,y+1)|_{[0,2]\times[0,2]}\ \cdots\phi^{111}(x-1,y-1)|_{[0,2]\times[0,2]}\ ]^T\\ &:= [\phi^{111}_{1,1}\ \cdots\ \phi^{111}_{1,9}]^T.\end{aligned}$$

Similarly, set the column vector Φ_2^{111} as follows $\Phi_2^{111} = [\ 2\phi^{111}(2x+2,2y+2)|_{[0,4]}\ \cdots\ 2\phi^{111}(2x-2,2y-2)|_{[0,4]}\ ]^T$. Then from the relation $\Phi_1^{111} = P_1^{111}\cdot\Phi_2^{111}$, we have the refinement matrix P_1^{111}. We define $\Psi_1^{111} := Q_1^{111}\Phi_2^{111}$ with the matrix Q_1^{111} obtained by the constructive method. The components of the column vector Ψ_1^{111} are tight framelets for box spline ϕ^{111} of the ground level. We illustrate some of tight frameles in Figures 2, 3, and 4.

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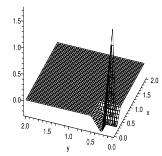

Box Spline ϕ_{111}

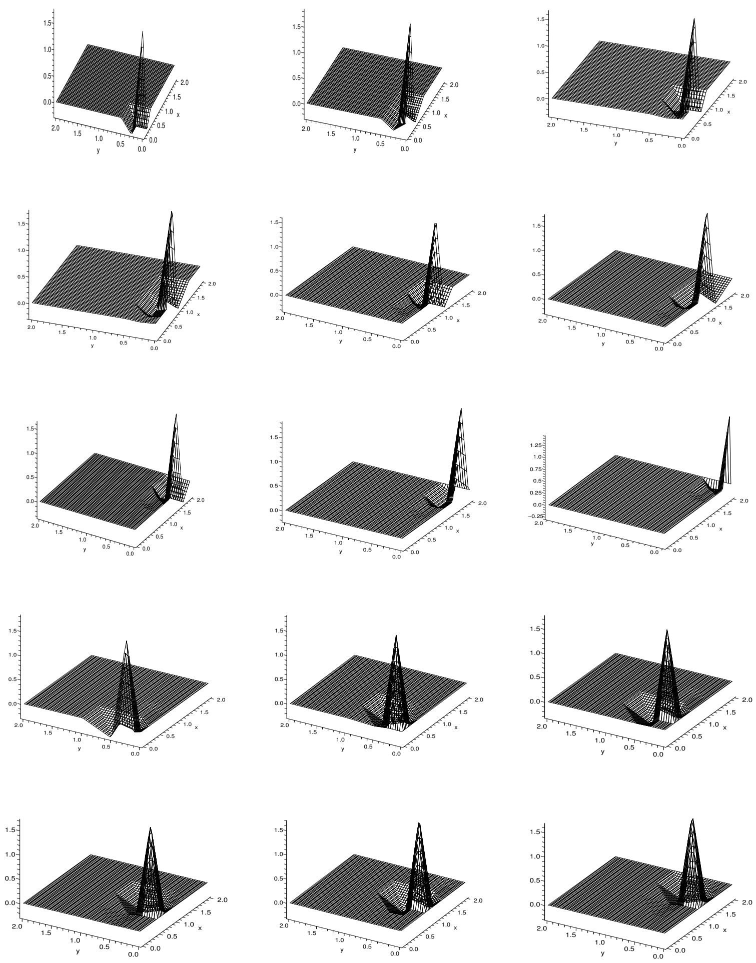

Fig. 2. Box Spline ϕ_{111} and its some of Tight Framelets on a bounded domain

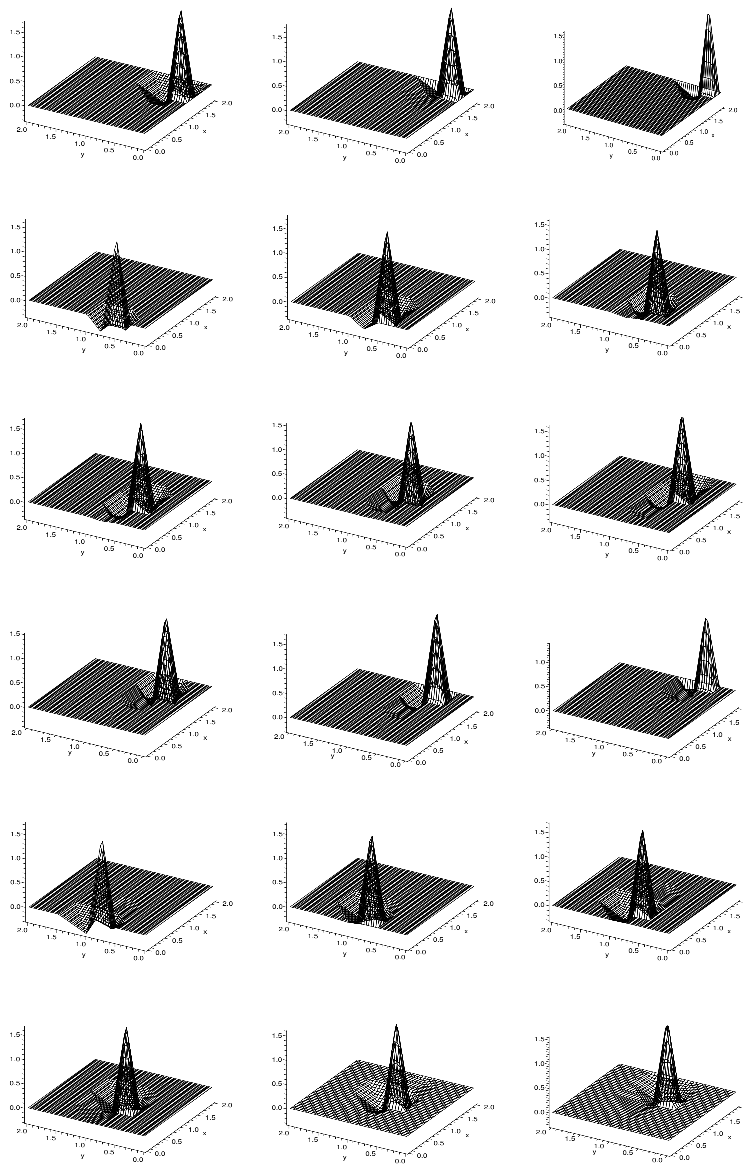

Fig. 3. More Box Spline Tight Framelets located on the bounded domain

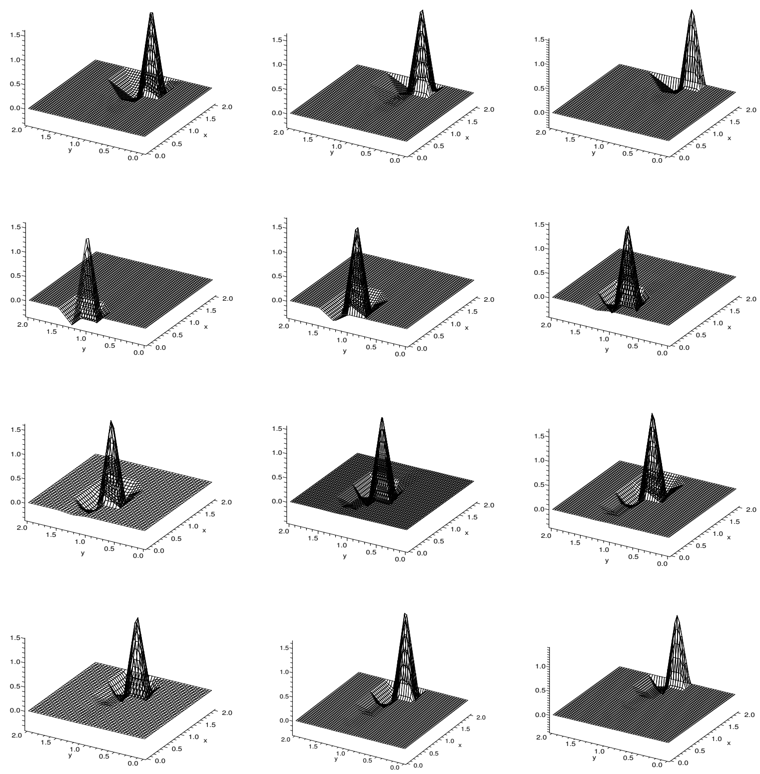

Fig. 4. More Box Spline Tight Framelets located on the bounded domain