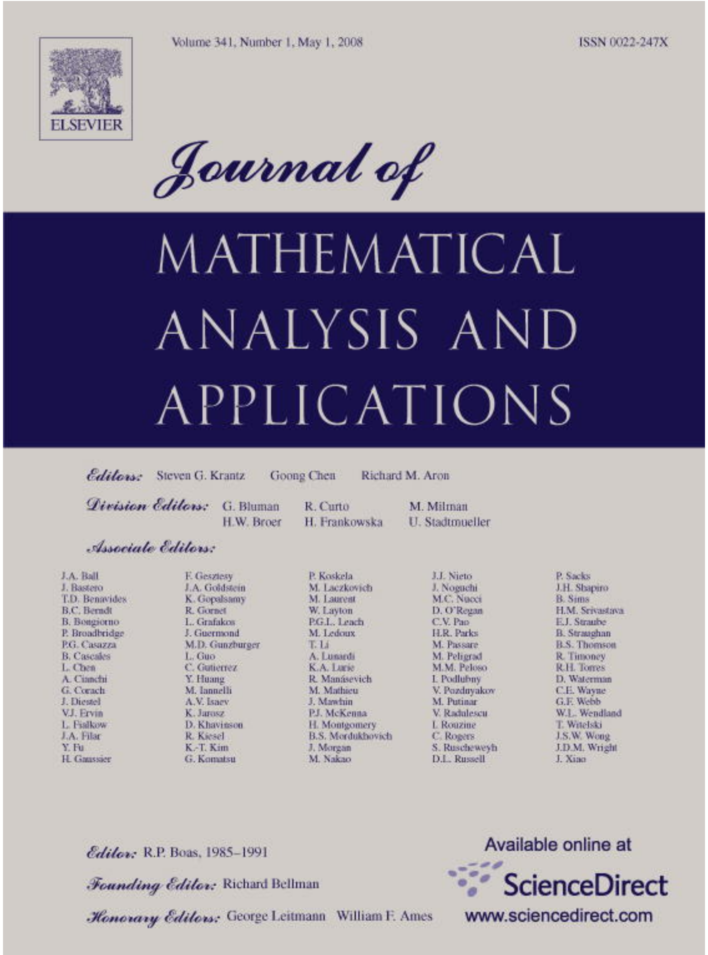
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Convergence of local variational spline interpolation

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Abstract

In this paper we first revisit a classical problem of computing variational splines. We propose to compute local variational splines in the sense that they are interpolatory splines which minimize the energy norm over a subinterval. We shall show that the error between local and global variational spline interpolants decays exponentially over a fixed subinterval as the support of the local variational spline increases. By piecing together these locally defined splines, one can obtain a very good C^0 approximation of the global variational spline. Finally we generalize this idea to approximate global tensor product B-spline interpolatory surfaces.

Keywords: Splines; Interpolation; Approximation

1. Introduction

In the classical problem of variational spline approximation one chooses, for a given function f, a C^2 curve s_f that extremizes the problem

$$\text{minimize}\left\{\int_{[a,b]} \left|s''(t)\right|^2 \mathrm{d}t\colon\ s\in S,\ s(t_i)=f(t_i),\ i=1{:}n\right\},$$

where $\tau := \{a = t_1 < t_2 < \cdots < t_n = b\}$ is a partition of interval $[a,b]$, and S is a space of C^1 functions on $[a,b]$ whose second derivative is square integrable. It is well known that the solutions to the above problem are piecewise polynomial splines that approximate the thin-beam splines of mechanics. It is also well known that to compute these spline, one solves a (banded) linear system of the order of the number of data points within $[a,b]$ that are being interpolated. This is our 'global' solution. The goal of this paper is to compare the error between this global solution and certain 'local' solutions computed by minimizing over small subintervals of $[a,b]$ that contain only a few data points. In particular, we show that the error decreases exponentially as the number of data points is increased. The motivation is that these local solutions require far fewer computations than the global solution. Therefore, if one wants

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to change only a few data points, or if the solution to the problem is required over only a small subinterval of $[a,b]$, then it may be wise to approximate the global spline by these local splines. Moreover, by piecing together these local splines, one obtains a good C^0 approximation to the global spline. Although these approximations are only C^0, we show that the derivatives of these local solutions also approximate well in the sense that their error also decreases exponentially as the number of data points is increased. Hence, our main results show that the error between local and global splines, and their derivatives, both decay exponentially as the number of data points increase.

As stated above, solutions to the above problem are piecewise polynomial. To be more precise, they are piecewise cubic in $C^2[a,b]$, and so it is enough to assume that S is a space of piecewise polynomials (cf. [1–3] and [8]). Without loss of generality we let $S := S^1_{3,\tau}$ be the linear space of C^1 piecewise cubic polynomials on $[a,b]$ with breakpoints t_i. (See the next section for justifications.) In a B-spline basis we are considering all spline functions of order 4 with a double knot at each breakpoint. Now let $\tau_l := [t_l, t_{l+1}]$ be a subinterval of $[a,b]$ for a fixed l. In this paper we consider a local minimization problem, whereby the minimization is carried out over a subinterval $\tau_l^k := [t_{l-k}, t_{l+1+k}]$, where we have assumed that $t_{-i} := t_1$ for all nonnegative integer i and $t_i = t_n$ for all $i > n$. That is, for a given continuous function f on $[a,b]$, let s_f and $s_{f,l,k}$ be solutions to the following problems:

$$\text{minimize } \{E(s) := E_{[a,b]}(s)\colon\ s \in \Lambda_\tau(f)\} \tag{1}$$

and

$$\text{minimize } \{E_{\tau_l^k}(s)\colon\ s \in \Lambda_\tau(f)\}, \tag{2}$$

respectively, with

$$E_I : s \to \int_I |s''(t)|^2 \, dt, \tag{3}$$

where I is a closed subinterval of $[a,b]$ and

$$\Lambda_\tau(f) := \{s \in S\colon\ s(t_i) = f(t_i),\ i = 1{:}n\}. \tag{4}$$

The solution to (1) is typically unique, whereas the solution to (2) is not unique away from the interval τ_l^k where the functional $E_{\tau_l^k}$ has no influence. Hence, we have some freedom in choosing how to extend $s_{f,l,k}$ from τ_l^k to the entire interval $[a,b]$, however, the results in this paper are independent of any extension.

One of the main results in this paper is to show that $\|(s_f - s_{f,l,k})|_{\tau_l}\|_\infty$ decays exponentially to zero as k increases to ∞ and $k < n$, while if $k \geqslant n$ the error is clearly identically zero. Thus, s_f can be approximated by $s_{f,l,k}|_{\tau_l}$ for all $l = 1, \ldots, n-1$. But we point out at the outset that this convergence may not be monotonic, and later give an example to illustrate this point. Since 2D tensor product B-spline interpolatory surfaces play a significant role in applications, we shall generalize the result mentioned above to the 2D setting.

The paper is organized as follows. We begin with a simple fact regarding our choice of spline spaces. Then we establish some stability properties of the spline space in Section 3. In Section 4 we prove our main result in the paper. We then generalize the result for tensor product of B-spline surfaces in Section 5. We shall present some numerical experiments in Section 6 to demonstrate the effectiveness of our local spline scheme. Finally we give several remarks in Section 7.

2. A simple fact

In general we could consider a spline space $S = S^r_{d,\tau}$ with $d \geqslant 3$, and $r = 1$ or $r = 2$. But, by the result in this section, we only need to consider $S^1_{3,\tau}$. The result and proof in this section are well known over a space of piecewise C^2 functions on $[a,b]$, and in particular over C^2 piecewise polynomials. The importance here is that minimizing over larger C^1 spaces of piecewise polynomials, i.e., $S^r_{d,\tau}$ for $d \geqslant 3$ and $r = 1$ or 2, also produces C^2 natural splines.

Define

$$\langle u, v\rangle := \int_a^b u''(t)v''(t)\, dt$$

a semi-inner product, and semi-norm

$$\|s\| := \sqrt{\langle s, s\rangle} = \sqrt{E(s)}.$$

Theorem 1. *Let $f \in S^r_{d,\tau}$ with $d \geqslant 3$ and $r = 1$ or 2. Then a minimizer s_f to (1) is a C^2 piecewise natural cubic spline interpolant to the data $\{(t_i, f(t_i))\}$, $i = 1{:}n$. In particular, $\|s_f\| \leqslant \|f\|$.*

Proof. Since s_f solves (1), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0} \langle s_f + \alpha g, s_f + \alpha g\rangle = 2\langle s_f + \alpha g, g\rangle|_{\alpha=0} = 2\langle s_f, g\rangle = 0$$

for all $g \in S$ such that $g(t_i) = 0$ for $i = 1{:}n$. And so,

$$\begin{aligned}
0 = \langle s_f, g\rangle &= \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} s_f''(t) g''(t)\,\mathrm{d}t \\
&= \sum_{i=1}^{n-1} \left(s_f'' g' - s_f''' g\Big|_{t_i}^{t_{i+1}} + \int_{t_i}^{t_{i+1}} s_f^{(iv)}(t) g(t)\,\mathrm{d}t \right) \\
&= -\big(s_f''\big(t_1^+\big) g'(t_1) + s_f'''\big(t_1^+\big) g(t_1)\big) + \big(s_f''\big(t_n^-\big) g'(t_n) + s_f'''\big(t_n^-\big) g(t_n)\big) \\
&\quad + \sum_{i=2}^{n-1} \big(s_f''\big(t_i^-\big) - s_f''\big(t_i^+\big)\big) g'(t_i) + \big(s_f'''\big(t_i^-\big) - s_f'''\big(t_i^+\big)\big) g(t_i) + \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} s_f^{(iv)}(t) g(t)\,\mathrm{d}t \\
&= -s_f''\big(t_1^+\big) g'(t_1) + s_f''\big(t_n^-\big) g'(t_n) + \sum_{i=2}^{n-1} \big(s_f''\big(t_i^-\big) - s_f''\big(t_i^+\big)\big) g'(t_i) \\
&\quad + \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} s_f^{(iv)}(t) g(t)\,\mathrm{d}t.
\end{aligned}$$

Hence, a necessary condition to solve this for all admissible variations g is that

$$\begin{aligned}
&s_f''\big(t_1^+\big) = 0, \qquad s_f''\big(t_n^-\big) = 0, \\
&s_f''\big(t_i^-\big) = s_f''\big(t_i^+\big) \quad \text{for } i = 2{:}n-1, \\
&s_f^{(iv)}\big|_{(t_i, t_{i+1})} = 0 \quad \text{for } i = 2{:}n-1.
\end{aligned}$$

That is, s_f is a C^2 natural cubic spline. □

3. Stability properties

In this section we derive various stability conditions and inequalities that will be used for the main results of this paper. For B-splines we have the following estimate, specialized here to 2-norms, and modified (weakened) slightly so that we see the dependence of $h_{\min}$ and $h_{\max}$. Here, $\|g\|_2$ denotes the usual L_2 norm on functions $g(t)$ over $[a, b]$, and $\|c\|_2$ is the standard l_2 norm on sequences.

Lemma 2. *(See [3,4].) Let $s := \sum_i c_i N_i(t)$ be a spline function with respect to the B-spline basis $(N_i(t))$ for S. There exists a constant $D_3 > 0$, depending only on the order of the spline, such that*

$$D_3 h_{\min} \|c\|_2^2 \leqslant \|s\|_2^2 \leqslant h_{\max} \|c\|_2^2.$$

The constant D_3 is given in [3], and in particular is independent of the mesh size (and knot spacings). For cubic splines, $D_3 \approx 1/5.6$. Lemma 2 provides a stability estimate for s. Later we will need a stability estimate for s''. For this we need the following polynomial inequality.

Lemma 3. *Let $p(t)$ be any algebraic polynomial such that $p(a) = p(b) = 0$. Then,*

$$\|p\|_2 \leqslant h^2 \|p''\|_2$$

with $h := b - a$.

Proof. By Rolle's theorem there exists c, $a < c < b$, such that $p'(c) = 0$. Then, with $p(a) = p'(c) = 0$, we can represent $p(t)$ as

$$p(t) = p(a) + p'(c)(t-a) + \int_a^t \int_c^s p''(u)\,\mathrm{d}u\,\mathrm{d}s = \int_a^t \int_c^s p''(u)\,\mathrm{d}u\,\mathrm{d}s.$$

And so,

$$\begin{aligned}
\|p\|_2^2 = \int_a^b \left|p(t)\right|^2 \mathrm{d}t &= \int_a^b \left| \int_a^t \int_c^s p''(u)\,\mathrm{d}u\,\mathrm{d}s \right|^2 \mathrm{d}t \\
&\leqslant \int_a^b \left(\int_a^b \int_a^b \left|p''(u)\right| \mathrm{d}u\,\mathrm{d}s \right)^2 \mathrm{d}t = h^3 \left(\int_a^b \left|p''(u)\right| \mathrm{d}u \right)^2 \\
&\leqslant h^4 \int_a^b \left|p''(u)\right|^2 \mathrm{d}u = h^4 \|p''\|_2^2. \qquad \square
\end{aligned}$$

We will also need an L_2 version of Markov's inequality. For polynomials of degree d on the interval $[-1, 1]$ it has the form

$$\|p'\|_2 \leqslant C_d d^2 \|p\|_2.$$

Clearly, we can choose $C_0 = 0$ when $d = 0$. Following [5, Table II], the optimal value for C_3 is approximately 0.7246. By a change of variable, we have the following Markov estimate for polynomials on $[a, b]$,

$$\|p'\|_2 \leqslant 2C_d \frac{d^2}{h} \|p\|_2.$$

Here, the L_2 norms are defined over $[a, b]$, and $h := b - a$. It follows that

$$\|p''\|_2 \leqslant 2C_d \frac{d^2}{h} \|p'\|_2 \leqslant 4C_d^2 \frac{d^4}{h^2} \|p\|_2.$$

Hence, we have

Lemma 4. *For any algebraic polynomial $p(t)$ of degree d on $[a, b]$,*

$$\|p''\|_2 \leqslant \frac{\sqrt{D_u}}{h^2} \|p\|_2 \tag{5}$$

with $\sqrt{D_u} \approx 4C_d^2 d^4$, a constant depending only on the degree d. For $d = 3$, $\sqrt{D_u} \approx 4(0.7246)^2 d^4 \approx 2.1 d^4 \approx 170.11$.

We can now derive a desired stability estimate for the spline space S.

Lemma 5. *Let $s := \sum_i c_i N_i(t)$ be a spline function with respect to the B-spline basis $(N_i(t))$ for S. Assume that $s \in H := \Lambda_\tau(0)$. Then,*

$$D_3 \frac{h_{\min}}{h_{\max}^4} \|c\|_2^2 \leqslant \frac{1}{h_{\max}^4} \|s(t)\|_2^2 \leqslant \|s''(t)\|_2^2 \leqslant D_u \frac{1}{h_{\min}^4} \|s(t)\|_2^2 \leqslant D_u \frac{h_{\max}}{h_{\min}^4} \|c\|_2^2,$$

for constants D_3 and D_u independent of the mesh spacing.

Proof. Let $p_j := s(t)|_{\tau_j}$ for $j = 1{:}n-1$. Each p_j is a polynomial of degree d restricted to τ_j. Since $s \in H$, $s(t_i) = 0$ for $i = 1{:}n$, and so $p_j(t_j) = 0 = p_j(t_{j+1})$ for $j = 1{:}n-1$. Hence, for each j, we can apply Lemma 3 to the polynomial p_j on the interval τ_j. And so we have

$$\begin{aligned}\|s(t)\|_2^2 &= \sum_j \int_{\tau_j} |p_j(t)|^2\,\mathrm{d}t \leqslant \sum_j h_j^4 \int_{\tau_j} |p_j''(t)|^2\,\mathrm{d}t \\ &\leqslant h_{\max}^4 \sum_j \int_{\tau_j} |p_j''(t)|^2\,\mathrm{d}t = h_{\max}^4 \|s''(t)\|_2^2.\end{aligned}$$

Likewise,

$$\begin{aligned}\|s''(t)\|_2^2 &= \sum_j \int_{\tau_j} |p_j''(t)|^2\,\mathrm{d}t \leqslant \sum_j D_u \frac{1}{h_j^4} \int_{\tau_j} |p_j(t)|^2\,\mathrm{d}t \\ &\leqslant D_u \frac{1}{h_{\min}^4} \sum_j \int_{\tau_j} |p_j(t)|^2\,\mathrm{d}t = D_u \frac{1}{h_{\min}^4} \int_a^b |s(t)|^2\,\mathrm{d}t \\ &= D_u \frac{1}{h_{\min}^4} \|s(t)\|_2^2.\end{aligned}$$

Then, by Lemma 2, we have

$$D_3 \frac{h_{\min}}{h_{\max}^4} \|c\|_2^2 \leqslant \frac{1}{h_{\max}^4} \|s(t)\|_2^2 \leqslant \|s''(t)\|_2^2 \leqslant D_u \frac{1}{h_{\min}^4} \|s(t)\|_2^2 \leqslant D_u \frac{h_{\max}}{h_{\min}^4} \|c\|_2^2. \qquad \square$$

Lemma 6. *Let γ and $a_0, \ldots, a_m$ be nonnegative real numbers. Suppose that $\gamma(a_0 + \cdots + a_{k-1}) \leqslant a_k$ for $k = 1, \ldots, m$. Then,*

$$\gamma(1+\gamma)^{k-1} a_0 \leqslant a_k.$$

Proof. We immediately establish that $\gamma a_0 \leqslant a_1$ when $k = 1$. Then, by strong induction,

$$\begin{aligned} a_{k+1} &\geqslant \gamma(a_0 + a_1 + a_2 + \cdots + a_k) \\ &\geqslant \gamma\big(a_0 + \gamma a_0 + \gamma(1+\gamma)a_0 + \cdots + \gamma(1+\gamma)^{k-1} a_0\big) \\ &= \gamma\left((1+\gamma) + \gamma(1+\gamma)\frac{(1+\gamma)^{k-1} - 1}{(1+\gamma) - 1}\right)a_0 \\ &= \gamma\big((1+\gamma) + (1+\gamma)\big((1+\gamma)^{k-1} - 1\big)\big)a_0 \\ &= \gamma(1+\gamma)^k a_0. \qquad \square \end{aligned}$$

Lemma 7. $(\alpha_1 + \cdots + \alpha_m)^2 \leqslant m\alpha_1^2 + \cdots + m\alpha_m^2$.

Proof. $m\alpha_1^2 + \cdots + m\alpha_m^2 - (\alpha_1 + \cdots + \alpha_m)^2 = \sum_i \sum_{j \neq i} (\alpha_i - \alpha_j)^2 \geqslant 0$. $\square$

4. Main results

In this section we present our main results in Lemma 8 and Theorem 9. The key idea in the lemma involve an orthogonality condition on the difference between the local and global interpolants, and is similar to that used in [7] and other papers (from which the basic idea originates). However, unlike that paper, the proof is done here in such a way that no 'natural extension' of the spline $s_{f,l,k}$ outside the interval τ_l^k is needed.

Let $H = \Lambda_\tau(0)$ be the linear subspace of S with inner product $\langle f, g\rangle_H := \langle f'', g''\rangle_{L_2,[a,b]}$ and norm $\|\cdot\|_H := \sqrt{\langle\cdot,\cdot\rangle_H}$, and let $\langle f, g\rangle_{H,\tau_l^k} := \langle f'', g''\rangle_{L_2,\tau_l^k}$ with norm $\|\cdot\|_{H,\tau_l^k} := \sqrt{\langle\cdot,\cdot\rangle_{H,\tau_l^k}}$. Hence, $E(s) = \langle s, s\rangle_H$ and $E_{\tau_l^k}(s) = \langle s, s\rangle_{H,\tau_l^k}$. Since s_f solves (1), it follows that, for all $g \in H$,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0}\langle s_f + \alpha g, s_f + \alpha g\rangle_H = 2\langle s_f + \alpha g, g\rangle_H|_{\alpha=0} = 2\langle s_f, g\rangle_H = 0.$$

Likewise, a necessary condition for $s_{f,l,k}$ to solve (2) is that $\langle s_{f,l,k}, g\rangle_{H,\tau_l^k} = 0$ for all $g \in H$. Let $G_k := \{g \in H\colon \operatorname{supp}(g) \subseteq \tau_l^k\}$. Then, $\langle s_{f,l,k}, g\rangle_{H,\tau_l^k} = \langle s_{f,l,k}, g\rangle_H$ for all $g \in G_k$, and so it follows that $\langle s_f - s_{f,l,k}, g\rangle_H = 0$ for all $g \in G_k$. That is, $s_f - s_{f,l,k} \in G_k^\perp$.

Lemma 8. *The error between the local and global spline interpolants on the interval τ_l satisfies*

$$\|s_f - s_{f,l,k}\|_{H,\tau_l} \leqslant C_1\sigma^q \|s_f - s_{f,l,k}\|_{H,\tau_l^{q+2}-\tau_l^{q-1}}$$

for $1 < q \leqslant k$, with $\sigma = \sqrt{\frac{D_u\rho^5}{D_3 + D_u\rho^5}}$, $C_1 = \frac{D_u\rho^5}{D_3\sigma}$, and $\rho := h_{\max}/h_{\min}$.

Note that the constants σ and C_1 depend on the mesh ratio ρ, but not the separate mesh sizes $h_{\min}$ or $h_{\max}$. Note also that $\sigma < 1$.

Proof. Let $s := s_f - s_{f,l,k}$. Then $s \in H$, and it has a representation $s = \sum_i c_i N_i$ for some coefficients c_i. For $q \geqslant 1$, let

$$w_q := \sum_{i\in I_q} c_i N_i \quad \text{and} \quad u_q := s - w_q$$

with respect to the index set

$$I_q := \{i\colon \operatorname{supp}(N_i) \subseteq \tau_l^q\}.$$

Let

$$a_q := \sum_{i\in R_q} c_i^2$$

with $R_q := \{i\colon \operatorname{supp}(N_i) \cap (\tau_l^{q+1} - \tau_l^q) \neq \emptyset\}$. Note that $\langle s, w_q\rangle_H = 0$ when $q \leqslant k$ since $w_q \in G_k$, and that $\operatorname{supp}(u_q) \cap \operatorname{supp}(w_q) \subseteq \tau_l^{q+1} - \tau_l^q$. Then, for $q \leqslant k$,

$$\begin{aligned}
\|w_q\|_H^2 &= \langle w_q, w_q\rangle_H \\
&= \langle s - u_q, w_q\rangle_H \\
&= \langle -u_q, w_q\rangle_H \\
&= \langle -u_q, w_q\rangle_{H,\tau_l^q-\tau_l^{q-1}} \\
&\leqslant \|u_q\|_{H,\tau_l^q-\tau_l^{q-1}} \|w_q\|_{H,\tau_l^q-\tau_l^{q-1}} \\
&\leqslant \bigg\|\sum_{i\in R_{q-1}} c_i N_i\bigg\|_H \|w_q\|_H.
\end{aligned}$$

Hence, $\|w_q\|_H \leqslant \|\sum_{i\in R_{q-1}} c_i N_i\|_H$ when $q \leqslant k$, and so, by Lemma 5,

$$K_1 \sum_{i=0}^{q-2} a_i \leqslant \|w_q\|_H^2 \leqslant \left\| \sum_{i \in R_{q-1}} c_i N_i \right\|_H^2 \leqslant K_2 a_{q-1} \tag{6}$$

for $1 < q \leqslant k$, with $K_1 := D_3 h_{\min}/h_{\max}^4$ and $K_2 := D_u h_{\max}/h_{\min}^4$. And so, for $1 < q \leqslant k$, we have the estimate

$$\gamma \sum_{i=0}^{q-2} a_i \leqslant a_{q-1}$$

with $\gamma := K_1/K_2$. By Lemma 6,

$$a_0 \leqslant \frac{1}{\gamma} \sigma^{q-1} a_q$$

with

$$\sigma := \frac{1}{1+\gamma} = \frac{K_2}{K_1 + K_2}.$$

By Lemma 5

$$a_q = \sum_{i \in R_q} c_i^2 \leqslant \frac{1}{K_1} \|s\|^2_{H, \tau_l^{q+2} - \tau_l^{q-1}},$$

and so

$$a_0 \leqslant \frac{1}{\gamma K_1} \sigma^{q-1} \|s\|^2_{H, \tau_l^{q+2} - \tau_l^{q-1}} = \frac{K_2}{K_1^2} \sigma^{q-1} \|s\|^2_{H, \tau_l^{q+2} - \tau_l^{q-1}}$$

for $1 < q \leqslant k$. Therefore,

$$\begin{aligned} \|s\|^2_{H,\tau_l} &= \left\| \sum_{N_i|_{\tau_l} \neq 0} c_i N_i \right\|^2_{H,\tau_l} \\ &\leqslant K_2 a_0 \quad \text{(by Lemma 5)} \\ &= \frac{K_2^2}{K_1^2} \sigma^{q-1} \|s\|^2_{H, \tau_l^{q+2} - \tau_l^{q-1}} \quad \text{(from above)} \end{aligned}$$

and hence,

$$\|s\|_{H,\tau_l} \leqslant C_1 \sqrt{\sigma^{q-1}} \|s\|_{H, \tau_l^{q+2} - \tau_l^{q-1}}$$

for $1 < q \leqslant k$, with

$$C_1 := \frac{K_2}{K_1} = \frac{D_u \frac{h_{\max}}{h_{\min}^4}}{D_3 \frac{h_{\min}}{h_{\max}^4}} = \frac{D_u \rho^5}{D_3}$$

and

$$\sigma = \frac{K_2}{K_1 + K_2} = \frac{D_u \frac{h_{\max}}{h_{\min}^4}}{D_3 \frac{h_{\min}}{h_{\max}^4} + D_u \frac{h_{\max}}{h_{\min}^4}} = \frac{D_u \rho^5}{D_3 + D_u \rho^5}. \qquad \square$$

Theorem 9. *Let* $f \in L_2^2[a,b] \cap C[a,b]$. *Let* $\rho := h_{\max}/h_{\min}$. *For* $k > 2$ *there exists* $\sigma \in (0,1)$ *such that*

$$\|s_f - s_{f,l,k}\|_{H,\tau_l} \leqslant C_3 \sigma^k \|f\|_H$$

and, if $f \in C^2[a,b]$, *then*

$$\|s_f - s_{f,l,k}\|_{\infty,\tau_l} \leqslant C_4 \sigma^k h_{\max}^{3/2} \|f''\|_\infty$$

with

$$C_3 := \frac{2D_u\rho^5}{D_3\sigma^3}, \qquad C_4 := \frac{4D_u\rho^{11/2}\sqrt{b-a}}{D_3^{3/2}\sigma^3}, \qquad \sigma = \sqrt{\frac{D_u\rho^5}{D_3 + D_u\rho^5}}.$$

In particular, C_3 and σ depend only on the mesh ratio ρ, and $\sigma < 1$.

Proof. To establish the first inequality we have

$$\begin{aligned}
\|s_f - s_{f,l,k}\|_{H,\tau_l} &\leqslant C_1\sigma^{k-2}\|s_f - s_{f,l,k}\|_{H,\tau_l^k-\tau_l^{k-3}} \quad \text{(by Lemma 8)} \\
&\leqslant C_1\sigma^{k-2}\big(\|s_f\|_{H,\tau_l^k} + \|s_{f,l,k}\|_{H,\tau_l^k}\big) \\
&\leqslant C_1\sigma^{k-2}\big(\|s_f\|_{H} + \|s_{f,l,k}\|_{H,\tau_l^k}\big) \\
&\leqslant C_1\sigma^{k-2}\big(\|f\|_{H} + \|f\|_{H,\tau_l^k}\big) \\
&\leqslant 2C_1\sigma^{k-2}\|f\|_H \\
&= \frac{2C_1}{\sigma^2}\sigma^k\|f\|_H
\end{aligned}$$

by using the minimum property of C^2 natural cubic splines, with $C_1 = \frac{D_u\rho^5}{D_3\sigma}$. For the second estimate, first note that $s_f - s_{f,l,k} = \sum_i \alpha_i N_i$ for some α_i. And so,

$$\begin{aligned}
\|s_f - s_{f,l,k}\|_{\infty,\tau_l}^2 &= \Big\|\sum_{\operatorname{supp}(N_i)\cap\tau_l\neq\emptyset} \alpha_i N_i\Big\|_{\infty,\tau_l}^2 \\
&\leqslant \Big(\sum_{\operatorname{supp}(N_i)\cap\tau_l\neq\emptyset} |\alpha_i|\Big)^2 \quad \big(\text{due to } \big|N_i(t)\big| \leqslant 1\big) \\
&\leqslant 4\sum_{\operatorname{supp}(N_i)\cap\tau_l\neq\emptyset} |\alpha_i|^2 \quad (\text{Lemma 7 and the number of } N_i \text{ supported in } \tau_l) \\
&\leqslant C_2^2\|s_f - s_{f,l,k}\|_{H,\tau_l^1}^2 \quad \text{(by Lemma 5)}
\end{aligned}$$

with $C_2 := 2\sqrt{\frac{h_{\max}^4}{D_3 h_{\min}}}$. And so, putting it together gives

$$\begin{aligned}
\|s_f - s_{f,l,k}\|_{\infty,\tau_l} &\leqslant C_2\|s_f - s_{f,l,k}\|_{H,\tau_l^1} \\
&\leqslant \frac{2C_1}{\sigma^3}C_2\sigma^k\|f\|_H \quad \text{(similar to above estimate)} \\
&\leqslant 2C_1C_2\sigma^{k-3}\|f''\|_\infty\sqrt{b-a},
\end{aligned}$$

with

$$2C_1C_2\sigma^{-3}\sqrt{b-a} = 2\frac{D_u\rho^5}{D_3\sigma}2\sqrt{\frac{h_{\max}^4}{D_3 h_{\min}}}\sigma^{-3}\sqrt{b-a} = \frac{4D_u\rho^{11/2}\sqrt{b-a}}{D_3^{3/2}\sigma^3}h_{\max}^{3/2}. \qquad \square$$

Corollary 10. *Let $f \in C^2[a,b]$. Let $\rho := h_{\max}/h_{\min}$. For $k > 2$ there exists $\sigma \in (0,1)$ such that*

$$\big\|s_f' - s_{f,l,k}'\big\|_{\infty,\tau_l} \leqslant C_5\sigma^k\sqrt{h_{\max}}\|f''\|_\infty$$

with

$$C_5 := \frac{(8\rho + 2h_{\max}^3)D_u\rho^{11/2}\sqrt{b-a}}{D_3^{3/2}\sigma^3} \quad \textit{and} \quad \sigma = \sqrt{\frac{D_u\rho^5}{D_3 + D_u\rho^5}}.$$

Proof. Let $s := s_f - s_{f,l,k}$ and $h := t - u$ with $t, u \in \tau_l$. By Taylor's theorem

$$s(t) = s(u) + s'(u)h + \frac{s''(c)}{2!}h^2$$

for some c between t and u. And so, by Theorem 9,

$$\|s'\|_{\infty,\tau_l} \leqslant \frac{2}{h}\|s\|_{\infty,\tau_l} + \frac{h^2}{2}\|s''\|_{\infty,\tau_l} \leqslant \frac{2}{h}C_4\sigma^k\big(h_{\max}^{3/2}\big)\|f''\|_\infty + \frac{h^2}{2}\|s''\|_{\infty,\tau_l}.$$

Since s'' is linear, it follows that

$$\|s''\|_{\infty,\tau_l} \leqslant \Bigg(\sum_{\operatorname{supp}(N_i)\cap\tau_l\neq\emptyset} |\alpha_i|\Bigg).$$

And so just as in the proof of Theorem 9 it follows that

$$\|s''\|_{\infty,\tau_l} \leqslant C_4\sigma^k h_{\max}^{3/2}\|f''\|_\infty.$$

Therefore,

$$\|s'\|_{\infty,\tau_l} \leqslant \left(\frac{2}{h} + \frac{h^2}{2}\right)C_4\sigma^k h_{\max}\sqrt{h_{\max}}\|f''\|_\infty,$$

with

$$\begin{aligned}\left(\frac{2}{h} + \frac{h^2}{2}\right)C_4 h_{\max} &\leqslant \left(\frac{2}{h_{\min}} + \frac{h_{\max}^2}{2}\right)\frac{4D_u\rho^{11/2}\sqrt{b-a}}{D_3^{3/2}\sigma^3}h_{\max}\\ &= \frac{(8\rho + 2h_{\max}^3)D_u\rho^{11/2}\sqrt{b-a}}{D_3^{3/2}\sigma^3}. \qquad \square\end{aligned}$$

5. Tensor product B-spline surfaces

In this section we generalize the results in the previous section to the tensor B-spline surfaces. The generalization is straightforward and hence, we just outline the steps. Consider a rectangular domain $\Omega := [a,b] \times [c,d]$. Recall τ is a partition of $[a,b]$ and let $\nu = \{c = r_1 < r_2 < \cdots < r_m = d\}$ be a partition of $[c,d]$. Consider a tensor product B-spline space

$$\mathbf{S} := \Big\{s \in C^{1,1}\big([a,b]\times[c,d]\big),\ s = \sum_{i,j} c_{ij}N_{i,\tau}N_{j,\nu}\Big\},$$

where $N_{i,\tau}$ is a B-spline of order 4 with a double knot at each breakpoint of τ and similar for $N_{j,\nu}$. Let $\Omega_{l,k} = [t_l, t_{l+1}] \times [r_k, r_{k+1}]$ be a sub-rectangle of $[a,b]\times[c,d]$. Furthermore, let $\Omega_{l,k}^q = [t_{l-q}, t_{l+1+q}] \times [r_{k-q}, r_{k+1+q}]$ for $q \geqslant 0$, where we have assumed that $t_{-i} := t_1$ for all nonnegative integer i and $t_i = t_n$ for all $i > n$. Similar for breakpoints r_j. For a continuous function f on $[a,b]\times[c,d]$, let s_f and $s_{f,l,k,q}$ be the solutions of the following minimization problems:

$$\text{minimize}\ \big\{E_\Omega(s)\colon\ s \in \Lambda_{\tau,\nu}(f)\big\} \tag{7}$$

and

$$\text{minimize}\ \big\{E_{\Omega_{l,k}^q}(s)\colon\ s \in \mathbf{S},\ s \in \Lambda_{\tau,\nu}(f)\big\}, \tag{8}$$

respectively, with

$$E_I : s \to \int_I \big|s^{(2,2)}(u,v)\big|^2\,\mathrm{d}u\,\mathrm{d}v, \tag{9}$$

where I is a closed subdomain of Ω and

$$\Lambda_{\tau,\nu}(f) := \{s \in \mathbf{S}\colon\ s(t_i, r_j) = f(t_i, r_j),\ i = 1{:}n,\ j = 1{:}m\}. \tag{10}$$

Here,

$$s^{(2,2)}(u, v) := s_{uuvv}(u, v) = \frac{\partial^4 s}{\partial u^2 \partial v^2}(u, v).$$

Just as the natural cubic spline curve minimizes (1), we have the following well-known characterization of the natural bicubic spline (cf. [1] or [2]).

Theorem 11. *A minimizer s_f to* (7) *over the space* $\mathbf{S}$ *is a $C^{2,2}$ piecewise bicubic spline interpolant satisfying the natural boundary conditions*

$$s^{(2,0)}(a, r_j) = s^{(2,0)}(b, r_j) = s^{(0,2)}(t_i, c) = s^{(0,2)}(t_i, d) = 0$$

along on the boundary, for $i = 1{:}n$ and $j = 1{:}m$, and

$$s^{(2,2)}(a, c) = s^{(2,2)}(a, d) := s^{(2,2)}(b, c) = s^{(2,2)}(b, d) = 0$$

at the corners.

Let $h_{\max} = \max\{t_{i+1} - t_i, r_{j+1} - r_j,\ i = 1{:}n,\ j = 1{:}m\}$ and $h_{\min} = \min\{t_{i+1} - t_i, r_{j+1} - r_j,\ i = 1{:}n,\ j = 1{:}m\}$. Let $\|s\|_2$ denotes the usual L_2 norm on function $g(t, r)$ over Ω. By an application of Lemma 2 to tensor products we have

Lemma 12. *Let $s := \sum_{ij} c_{ij} N_{i,\tau}(t) N_{j,\nu}(r)$ be a spline function in $\mathbb{S}$. Then*

$$D_3^2 h_{\min}^2 \|c\|_2^2 \leqslant \|s\|_2^2 \leqslant h_{\max}^2 \|c\|_2^2,$$

where $\|c\|_2^2 = \sum_{ij} |c_{ij}|^2$.

The next result is a generalization of Lemma 3.2. However, the proof is not a straightforward generalization. We use the ideas in the proof of Lemma 5.2 in [6].

Lemma 13. *Let $p(u, v)$ be a bivariate tensor product polynomial of coordinate degrees* $(3, 3)$ *on $I = [a, b] \times [c, d]$ that vanishes at the corner points. Then,*

$$C_0 \|p\|_2 \leqslant h^4 \|p^{(2,2)}\|_2$$

with $h := \max\{b - a, d - c\}$, for some absolute constant $C_0 > 0$.

Proof. Suppose that the area of the rectangular domain is 1. Let

$$C_0 := \inf\{\|p^{(2,2)}\|_2\colon\ \|p\|_2 = 1,\ p \text{ bicubic that vanishes at the corner points}\}.$$

Let (p_k) be a minimizing sequence with norm $\|p_k\|_2 = 1$ such that $p_k \to p_*$ with $\|p_*\|_2 = 1$ and $C_0 = \|p_*^{(2,2)}\|_2$. Now, if $C_0 = 0$, then necessarily p_* is bilinear. But since p vanishes at the four corner points, it follows that $p_* \equiv 0$, which contradicts the assumption $\|p\|_2 = 1$. Hence, $C_0 > 0$. And so, $C_0 \|p\|_2 \leqslant \|p^{(2,2)}\|_2$.

Now, by a change of variables from the unit square to the rectangle, we have the stated result. □

We can now derive our desired stability estimate for the spline space $\mathbf{S}$.

Lemma 14. *Let $s := \sum_{ij} c_{ij} N_{i,\tau}(t) N_{j,\nu}(r)$ be a spline function in* $\mathbf{S}$. *Assume that $s \in H := \Lambda_{\tau,\nu}(0)$. Then,*

$$D_3^2 \frac{h_{\min}^2}{h_{\max}^8} \|c\|_2^2 \leqslant \frac{1}{h_{\max}^8} \|s\|_2^2 \leqslant E_\Omega(s) \leqslant D_u^2 \frac{1}{h_{\min}^8} \|s\|_2^2 \leqslant D_u^2 \frac{h_{\max}^2}{h_{\min}^8} \|c\|_2^2,$$

for constants $\widetilde{D}_3 := D_3 C_0$ and D_u as in Section 3.

Proof. The inequalities on the right-hand side follow by the Markov-type inequality for p-norms in Lemma 4, and by Lemma 12. The first inequality follows from Lemma 12. For the second inequality, we use Lemma 13. □

The next result and its proof is an extension of Lemma 8 and its proof to 2D.

Lemma 15. *The error between the local and global spline interpolants on the subdomain $\Omega_{l,k}$ satisfies*

$$\|s_f - s_{f,l,k,q}\|_{H,\Omega_{l,k}} \leqslant C_1\sigma^q \|s_f - s_{f,l,k,q}\|_{H,\Omega_{l,k}^{q+2}-\Omega_{l,k}^{q-1}}$$

for $1 < q$, with $\sigma = \sqrt{\frac{D_u\rho^{10}}{\tilde{D}_3+D_u\rho^{10}}}$, $C_1 = \frac{D_u\rho^{10}}{\tilde{D}_3\sigma}$, and $\rho := h_{\max}/h_{\min}$.

Proof. Let $H = \Lambda_{\tau,\nu}(0)$ be the linear subspace of $\mathbf{S}$ with inner product

$$\langle f, g\rangle_H := \langle f^{(2,2)}, g^{(2,2)}\rangle_{L_2(\Omega)}$$

and note that norm $\|s\|_H^2 := E_\Omega(s)$. Next, let

$$\langle f, g\rangle_{H,\Omega_{l,k}^q} := \langle f^{(2,2)}, g^{(2,2)}\rangle_{L_2(\Omega_{l,k}^q)}.$$

Since s_f solves (8), it follows that, for all $g \in H$,

$$\langle s_f, g\rangle_H = 0.$$

Likewise, a necessary condition for $s_{f,l,k,q}$ to solve (9) is that $\langle s_{f,l,k,q}, g\rangle_{H,\Omega_{l,k}^q} = 0$ for all $g \in H$. Let $G_q := \{g \in H:\ \operatorname{supp}(g) \subseteq \Omega_{l,k}^q\}$. Then $\langle s_{f,k}, g\rangle_{H,\Omega_{l,k}^q} = \langle s_{f,l,k,q}, g\rangle_H$ for all $g \in G_q$, and so it follows that $\langle s_f - s_{f,l,k,q}, g\rangle_H = 0$ for all $g \in G_k$.

Let $s = s_f - s_{f,l,k,q}$. Then $s \in H$, and it has a representation $s = \sum_{ij} c_{ij} N_{i,\tau} N_{j,\nu}$ for some coefficients c_{ij}. Let

$$I_r := \{(i, j):\ \operatorname{supp}(N_{i,\tau} N_{j,\nu}) \subseteq \Omega_{i,k}^r\}.$$

For $1 \leqslant r \leqslant q$, let

$$w_r := \sum_{i\in I_r} c_{ij} N_{i,\tau} N_{j,\nu} \quad \text{and} \quad u_r := s - w_r.$$

Let

$$a_r := \sum_{(i,j)\in R_r} c_{ij}^2$$

with $R_r := \{(i, j):\ \operatorname{supp}(N_{i,\tau} N_{j,\nu}) \cap (\Omega_{l,k}^{r+1} - \Omega_{l,k}^r) \neq \emptyset\}$. Note that $\langle s, w_r\rangle_H = 0$ when $r \leqslant q$ since $w_r \in G_q$, and that $\operatorname{supp}(u_r) \cap \operatorname{supp}(w_r) \subseteq \Omega_{l,k}^{r+1} - \Omega_{l,k}^r$. Then, for $r \leqslant q$,

$$\begin{aligned}
\|w_r\|_H^2 &= \langle w_r, w_r\rangle_H \\
&= \langle s - u_r, w_r\rangle_H \\
&= \langle -u_r, w_r\rangle_H \\
&= \langle -u_r, w_r\rangle_{H,\Omega_{l,k}^r - \Omega_{l,k}^{r-1}} \\
&\leqslant \|u_r\|_{H,\Omega_{l,k}^r - \Omega_{l,k}^{r-1}} \|w_r\|_{H,\Omega_{l,k}^r - \Omega_{l,k}^{r-1}} \\
&\leqslant \Big\| \sum_{(i,j)\in R_{r-1}} c_{ij} N_{i,\tau} N_{j,\nu} \Big\|_H \|w_r\|_H.
\end{aligned}$$

Hence, $\|w_r\|_H \leqslant \|\sum_{(i,j)\in R_{r-1}} c_{ij} N_{i,\tau} N_{j,\nu}\|_H$ when $r \leqslant q$, and so, by Lemma 14,

$$K_1 \sum_{\ell=0}^{r-2} a_\ell \leqslant \|w_r\|_H^2 \leqslant \left\| \sum_{(i,j)\in R_{r-1}} c_{ij} N_{i,\tau} N_{j,\nu} \right\|_H^2 \leqslant K_2 a_{r-1} \tag{11}$$

for $1 < r \leqslant q$, with $K_1 := \widetilde{D}_3 h_{\min}^2 / h_{\max}^8$ and $K_2 := D_u h_{\max}^2 / h_{\min}^8$. And so, for $1 < r \leqslant q$, we have the estimate

$$\gamma \sum_{\ell=0}^{r-2} a_\ell \leqslant a_{r-1}$$

with $\gamma := K_1/K_2$. By Lemma 6,

$$a_0 \leqslant \frac{1}{\gamma} \sigma^{r-1} a_r$$

with

$$\sigma := \frac{1}{1+\gamma} = \frac{K_2}{K_1 + K_2}.$$

By Lemma 14,

$$a_r = \sum_{(i,j)\in R_r} c_{ij}^2 \leqslant \frac{1}{K_1} \|s\|^2_{H,\Omega_{l,k}^{r+2} - \Omega_{l,k}^{r-1}},$$

and so

$$a_0 \leqslant \frac{1}{\gamma K_1} \sigma^{r-1} \|s\|^2_{H,\Omega_{l,k}^{r+2} - \Omega_{l,k}^{r-1}} = \frac{K_2}{K_1^2} \sigma^{r-1} \|s\|^2_{H,\Omega_{l,k}^{r+2} - \Omega_{l,k}^{r-1}}$$

for $1 < r \leqslant q$. Therefore,

$$\begin{aligned} \|s\|^2_{H,\Omega_{l,k}} &= \left\| \sum_{N_{i,\tau} N_{j,\nu}|_{\Omega_{l,k}} \neq 0} c_{ij} N_{i,\tau} N_{j,\nu} \right\|^2_{H,\Omega_{l,k}} \\ &\leqslant K_2 a_0 \quad \text{(by Lemma 14)} \\ &= \frac{K_2^2}{K_1^2} \sigma^{r-1} \|s\|^2_{H,\Omega_{l,k}^{r+2} - \Omega_{l,k}^{r-1}} \quad \text{(from above)} \end{aligned}$$

and hence,

$$\|s\|_{H,\Omega_{l,k}} \leqslant C_1 \sqrt{\sigma^{r-1}} \|s\|_{H,\Omega_{l,k}^{r+2} - \tau_{l,k}^{r-1}}$$

for $1 < r \leqslant q$, with

$$C_1 := \frac{K_2}{K_1} = \frac{D_u \frac{h_{\max}^2}{h_{\min}^8}}{\widetilde{D}_3 \frac{h_{\min}^2}{h_{\max}^8}} = \frac{D_u \rho^{10}}{\widetilde{D}_3}$$

and

$$\sigma = \frac{K_2}{K_1 + K_2} = \frac{D_u \frac{h_{\max}^2}{h_{\min}^8}}{\widetilde{D}_3 \frac{h_{\min}^2}{h_{\max}^8} + D_u \frac{h_{\max}^2}{h_{\min}^8}} = \frac{D_u \rho^{10}}{\widetilde{D}_3 + D_u \rho^{10}}. \qquad \square$$

We are now ready to prove the second main theorem in this paper.

Theorem 16. *Let $f \in L_2^2([a,b]\times[c,d]) \cap C([a,b]\times[c,d])$. For $q > 2$ there exists $\sigma \in (0,1)$ such that*

$$\|s_f - s_{f,l,k,q}\|_{H,\Omega_{l,k}} \leqslant C_5 \sigma^q \|f\|_H,$$

and, if $f \in C^{2,2}([a,b]\times[c,d])$,

$$\|s_f - s_{f,l,k,q}\|_{\infty,\Omega_{l,k}} \leqslant C_6 \sigma^q \|D^2 f\|_\infty,$$

where

$$\|D^2 f\|_\infty = \sup\{|f^{(2,2)}(u,v)|: u,v \in \Omega\},$$

and with

$$C_5 = 2\frac{D_u \rho^{10}}{\widetilde{D}_3 \sigma^3}, \qquad C_6 := 4C_5 \frac{h_{\max}^4}{\sqrt{\widetilde{D}_3} h_{\min}} \quad \text{and} \quad \sigma = \sqrt{\frac{D_u \rho^{10}}{\widetilde{D}_3 + D_u \rho^{10}}}.$$

Proof. To establish the first inequality we have

$$\begin{aligned}
\|s_f - s_{f,l,k,q}\|_{H,\Omega_{l,k}} &\leqslant C_1 \sigma^{q-2} \|s_f - s_{f,l,k,q}\|_{H,\Omega_{l,k}^q - \Omega_{l,k}^{q-2}} \quad \text{(by Lemma 15)} \\
&\leqslant C_1 \sigma^{q-2} \big(\|s_f\|_H + \|s_{f,l,k,q}\|_{H,\Omega_{l,k}^q}\big) \\
&\leqslant C_1 \sigma^{q-2} \big(\|f\|_H + \|f\|_{H,\Omega_{l,k}^q}\big) \quad \text{(by Theorem 11)} \\
&\leqslant 2C_1 \sigma^{q-2} \|f\|_H
\end{aligned}$$

with $C_1 = \frac{D_u \rho^{10}}{\widetilde{D}_3 \sigma}$. For the second estimate, we first note that $s_f - s_{f,l,k,q} = \sum_{(i,j)} \alpha_{ij} N_{i,\tau} N_{j,\nu}$ for some α_{ij}. And so,

$$\begin{aligned}
\|s_f - s_{f,l,k,q}\|_{\infty,\Omega_{l,k}}^2 &= \Big\| \sum_{\operatorname{supp}(N_{i,\tau} N_{j,\nu}) \cap \Omega_{l,k} \neq \emptyset} \alpha_{ij} N_{i,\tau} N_{j,\nu} \Big\|_{\infty,\Omega_{l,k}}^2 \\
&\leqslant \Big(\sum_{\operatorname{supp}(N_{i,\tau} N_{j,\nu}) \cap \Omega_{l,k} \neq \emptyset} |\alpha_{ij}| \Big)^2 \quad \big(\text{due to } |N_{i,\tau}| \leqslant 1\big) \\
&\leqslant 16 \sum_{\operatorname{supp}(N_{i,\tau} N_{j,\nu}) \cap \Omega_{l,k} \neq \emptyset} |\alpha_{ij}|^2 \\
&\leqslant C_2^2 \|s_f - s_{f,l,k,q}\|_{H,\Omega_{l,k}^1}^2 \quad \text{(by Lemma 14)}
\end{aligned}$$

with $C_2 := 4\sqrt{\frac{h_{\max}^8}{\widetilde{D}_3 h_{\min}^2}} = 4\frac{h_{\max}^4}{\sqrt{\widetilde{D}_3} h_{\min}}$. And so, putting it together gives

$$\begin{aligned}
\|s_f - s_{f,l,k,q}\|_{\infty,\Omega_{l,k}} &\leqslant C_2 \|s_f - s_{f,l,k,q}\|_{H,\Omega_{l,k}^1} \\
&\leqslant 2C_1 C_2 \sigma^{q-3} \|D^2 f\|_\infty \sqrt{(b-a)(d-c)}. \qquad \square
\end{aligned}$$

6. Numerical experiments

In this section we compare some local and global approximations. In Fig. 1, local solutions are plotted for $k = 0$ through $k = 3$ for the function $f(t) = \cos(5t)$ defined over $[-\pi, \pi]$. There are ten breakpoints, $t_1, \ldots, t_{10}$, and the max error $e_k := |s_f - s_{f,l,k}|_{\infty,\tau_l}$ is calculated over the interval $\tau_l := [t_5, t_6] = [-0.34907, 0.34907]$ for each curve. The errors are tabulated for $k = 0$ through $k = 4$ in the table that follows. Note that when $k = 0$, the natural spline $s_{f,j,0}$ is a line segment supported on τ_l, and when $k = 4$ the local and global curves are the same in this example,

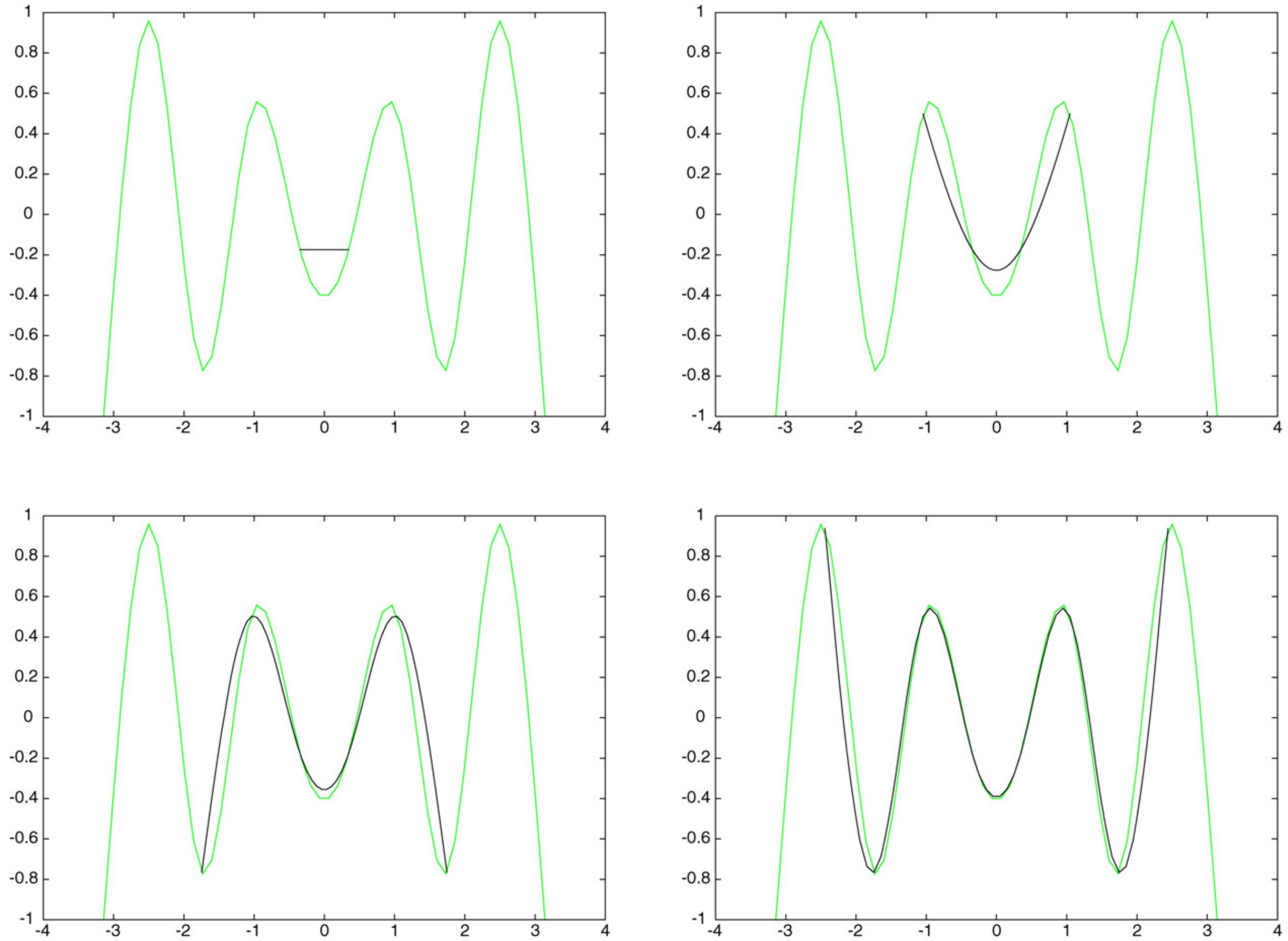

Fig. 1. Plotted are the local solutions for $k = 0, \dots, 3$, respectively, of $f(t) = \cos(5t)$ on $[-\pi, \pi]$ with 10 breakpoints.

hence the error is zero.

Figure 1	$k = 0$	1	2	3	4
$e_k =$	0.2323	0.13166	0.05	0.01295	0
$e_k/e_{k-1} =$		0.56646	0.37978	0.25890	0

In Fig. 2, piecewise local approximations

$$s_{f,k}(t) := \{s_{f,l,k}(t) \colon t \in [t_l, t_{l+1}]\}$$

are plotted for $k = 0, \dots, 3$, together with the global solution. Although these local solutions are not smooth at the breakpoints, one can only see this for the broken line, when $k = 0$. This is not surprising, given that the error between the derivatives of the global and piecewise local interpolants decay exponentially, as governed by Corollary 10.

Local approximations to three functions are plotted in Fig. 3. In Fig. 3, s_f is in a lighter shade and the $s_{f,l,k}$ are in darker type. In each example, all local approximants are plotted for $k = 0, 1, \dots, 20$, which overlap so closely that they are difficult to distinguish from each other, and from s_f. For (a) there are 100 breakpoints ($n = 100$), and for (b) and (c) $n = 200$.

The L_2 and L_∞ errors are given in Table 1. By looking at consecutive terms, ones sees that we can take $\sigma \approx e_k/e_{k-1}$ between 0.2 and 0.4. This is much better than the value of σ suggested in Lemma 8, which is very close to 1 (due to D_u being very large compared to D_3). Hence, the estimate of σ in this paper may not be sharp. But the theoretical and empirical results both show the exponential decay that we hoped to see. And so, in practice, we can piece these local solutions together, i.e., use $s_{f,l,k}|_{\tau_l}$, $l = 1{:}n-1$, to replace s_f. This new curve is only continuous, however due to the exponential decay in the errors of the derivatives as well as function values at the knots, the break in smoothness will not be visually noticeable for large k.

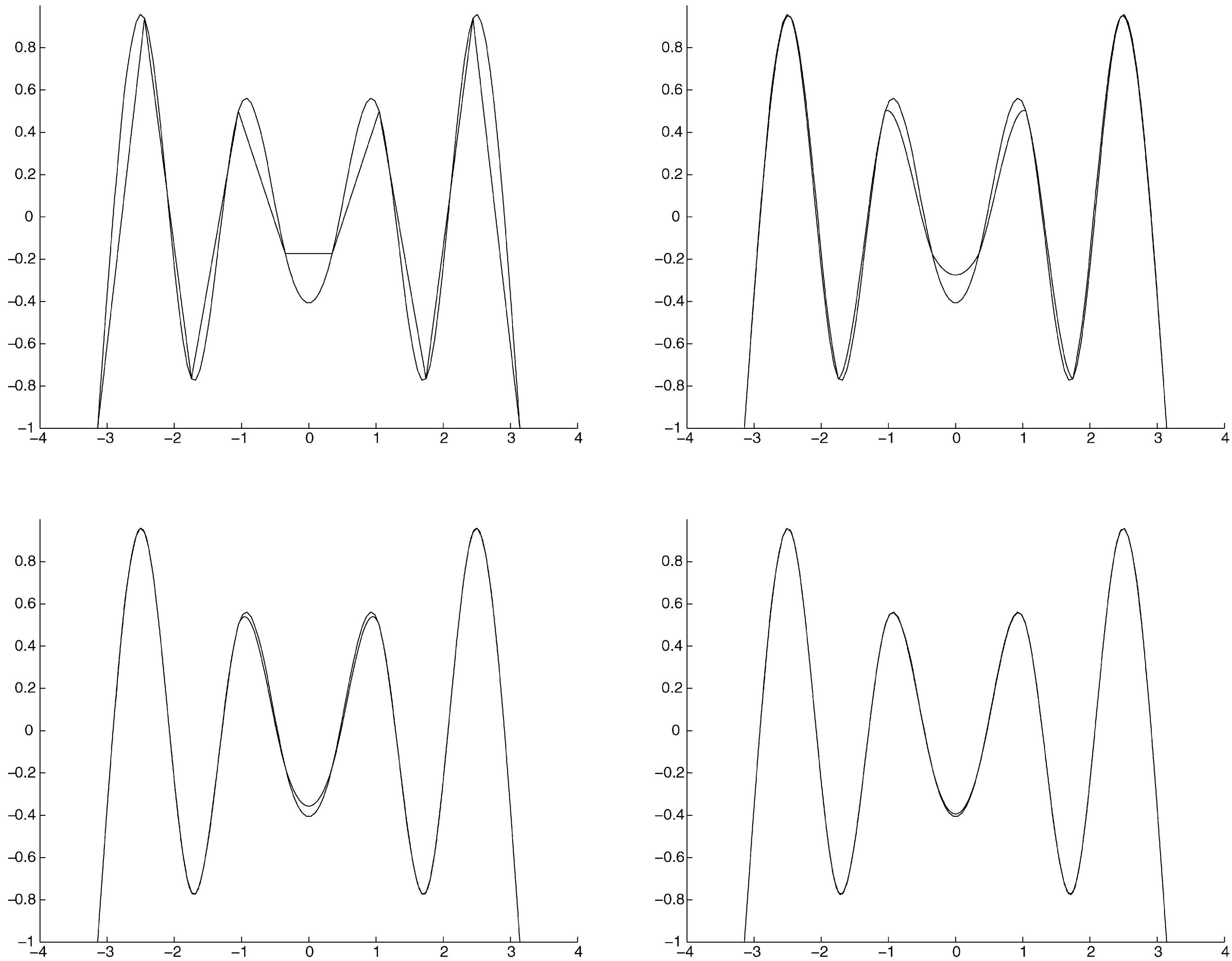

Fig. 2. Piecewise defined local approximations to $f(t) = \cos(5t)$ on $[-\pi, \pi]$ with 10 breakpoints for $k = 0, \ldots, 3$.

Table 1
Errors in L_2 and L_∞ norms for various k

	$k = 0$	1	2	3	4	5	6	7	8	9	10	11
(a)	2.5e–03	4.4e–04	9e–05	1e–05	2e–06	7e–08	7e–08	3e–08	1e–08	3e–09	9e–10	2e–10
(b)	3.1e–03	6.0e–04	1e–04	3e–05	9e–06	2e–06	4e–07	8e–08	1e–08	1e–09	4e–10	3e–10
(c)	2.4e–04	1.9e–05	8e–06	6e–06	7e–06	8.5e–07	2e–07	4e–08	4e–10	3e–09	2e–09	8e–10

The results in this paper show that the errors between local and global variational spline interpolants decrease exponentially as the number of data points increase. However, this convergence may not be monotonic. Suppose we are given uniform knots with data values $(10, 0, -1, 0, 0, 0, 0, -1, 0, 10)$. Then, the L_∞ errors for $s_{f,4,0}$ through $s_{f,4,4}$ are 0.039513, 0.039513, 0.000149, 0.023692 and 0.000000, respectively, as computed over the middle interval. The jump from 0.00149 to 0.023692 is due to the data values -1, which pull the local curve down, away from the global interpolant, which is heavily influenced by the values 10 at the two end points. The last error is identically zero because $s_{f,4,4} = s_f$.

In Fig. 4 the global tensor product solution to the function $s(u, v) = \exp(uv)\sin(20uv)$ on $[-1, 1]^2$ is plotted. Local solutions centered about the rectangle $[0.3939, 0.4141] \times [0.5135, 0.5405]$ were computed for several k. Both the L_2 and L_∞ errors between the local and global solutions over this rectangle are listed in Table 2. As expected, the

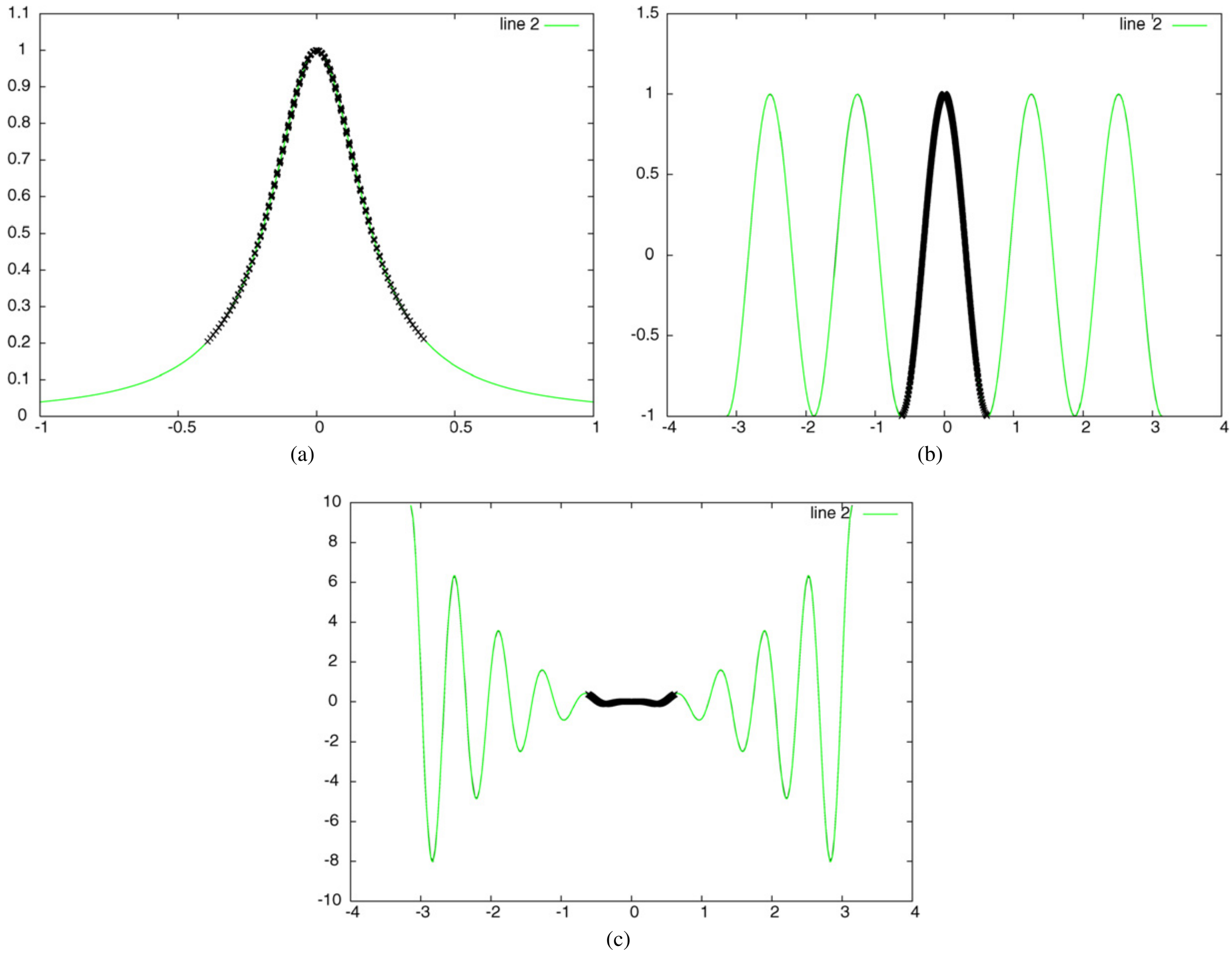

Fig. 3. (a) $f(t) = \frac{1}{25t^2+1}$; (b) $f(t) = \cos(5t)$; (c) $f(t) = t^2\cos(5t)$.

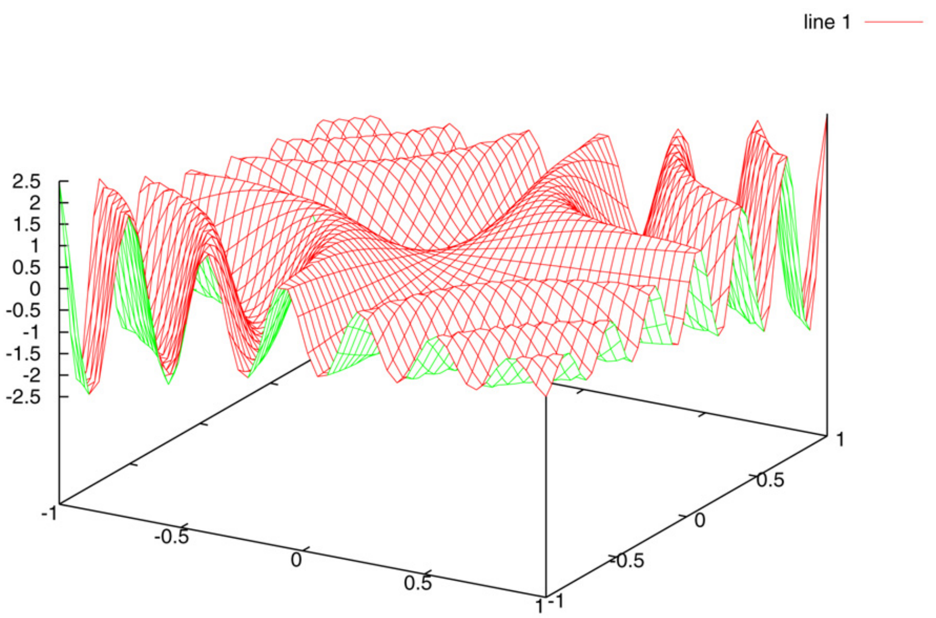

Fig. 4. Tensor product approximation of $s(u, v) = \exp(uv)\sin(20uv)$ on $[-1, 1]^2$, with 100×75 breaks, $\tau = [0.3939, 0.4141] \times [0.5135, 0.5405]$.

Table 2
Errors in L_2 and L_∞ norms for various k

Figure 4	4	5	6	7	8	9	10	11	12
$e_k =$	1.2e–02	2.3e–03	5.7e–04	1.3e–04	2.9e–05	5.8e–06	9.5e–07	1.1e–07	6.3e–08
	2.7e–08	1.0e–08	3.4e–09	1.0e–09	3.1e–10	9.1e–11	2.3e–11	7.0e–12	9.1993e–13
$e_k/e_{k-1} =$		0.19288	0.24315	0.23370	0.21879	0.19701	0.16386	0.11818	0.56185
		0.37311	0.33709	0.31421	0.29553	0.28485	0.26186	0.29410	0.13113

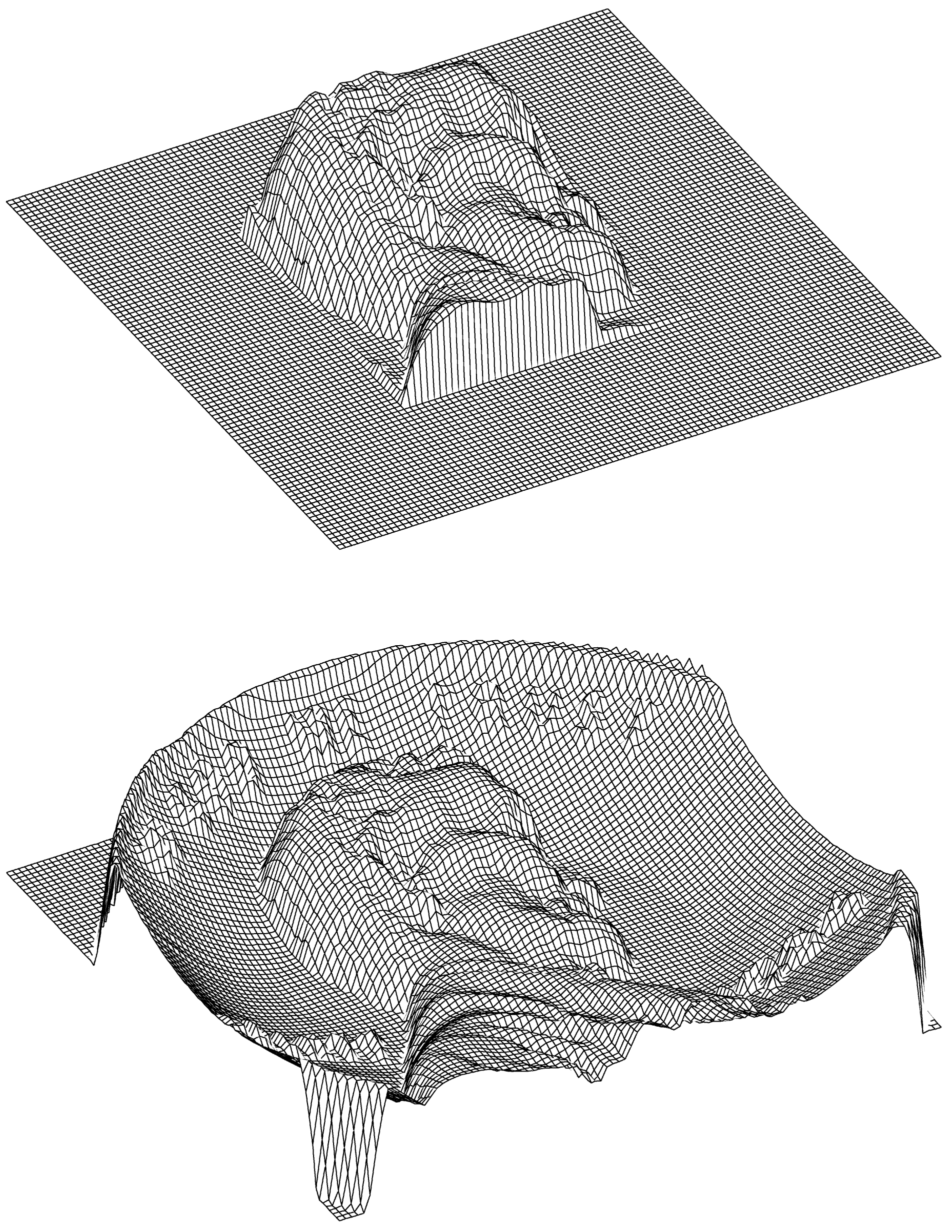

Fig. 5. Local and global interplants to penny data.

decay is exponential as k increases. Similar to the results for curves, the exponential constant is between 0.1 and 0.4 here, and in particular much less than the theoretical value of σ derived in this paper.

As a final comparison, in Fig. 5 we plot local and global solutions to a ‘penny’ data set. Here, one can see that very little detail is lost in the middle part of the local solution, compared to the global solution. That is, Lincoln’s head is well approximated using only local data.

7. Final remarks

We conclude with the following remarks:

Remark 1. In Section 5 we used an energy functional based on a fourth-order derivative. It is possible to use the following energy functional:

$$E_\Omega(s) = \int_\Omega \left(\left(\frac{\partial^2 s}{\partial x^2} \right)^2 + \left(\frac{\partial^2 s}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 s}{\partial y^2} \right)^2 \right) \mathrm{d}x\, \mathrm{d}y$$

which involves only second-order derivatives. This will make the proof more complicated. In fact, under this energy functional, Lai and Schumaker studied the convergence of local and global bivariate piecewise polynomial interpolations over triangulations (cf. [7]).

Remark 2. Clearly, the result in Section 5 can be extended to approximate tensor product of B-spline functions of more than 2D dimensions. Also, we can extend it to parametric B-spline surfaces $(x(u, v), y(u, v), z(u, v))$ with $x(u, v)$, $y(u, v)$ and $z(u, v)$ being tensor product of B-spline functions. Moreover, we expect that these results will lay the groundwork for work on other variational curve and surface problems, and in particular variational subdivision, as has been studied by the first author.

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