ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Convergence of discrete and penalized least squares spherical splines

Victoria Baramidze[a,*], Ming-Jun Lai[b]

[a] Department of Mathematics, Western Illinois University, 1 University Circle, Macomb, IL 61455, USA
[b] Department of Mathematics, The University of Georgia, Athens, GA 30602, USA

Received 20 September 2010; received in revised form 18 February 2011; accepted 9 March 2011

Communicated by Martin Buhmann

Abstract

We study the convergence of discrete and penalized least squares spherical splines in spaces with stable local bases. We derive a bound for error in the approximation of a sufficiently smooth function by the discrete and penalized least squares splines. The error bound for the discrete least squares splines is explicitly dependent on the mesh size of the underlying triangulation. The error bound for the penalized least squares splines additionally depends on the penalty parameter.

Keywords: Spherical splines; Least squares splines

1. Introduction

Suppose we are given a set of locations on the unit sphere $\mathbb{S}^2$ along with real values associated with these locations. We seek a smooth function defined on $\mathbb{S}^2$ approximating these values.

The problem has applications in atmospheric sciences, geodesy, geometric surface design, etc. For example, meteorological models require initial data for the time evolution equations. Many of the current methods use data associated with the nodes on a uniform grid. Since these types of data sets are often not available, field measurements over non-uniform or scattered locations are first fit by a spline. Then the spline values at the nodes of a uniform grid can be calculated and used in models. In geometric surface design, spherical splines can be constructed

* Corresponding author.
E-mail addresses: V-Baramidze@wiu.edu (V. Baramidze), mjlai@math.uga.edu (M.-J. Lai).

0021-9045/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jat.2011.03.001

to fit point clouds. In geodesy, spherical splines can approximate the geo-potential from satellite measurements around Earth [9].

In many cases spherical splines are proved to be the most convenient tools for interpolation and approximation of scattered data. Spherical splines were introduced in [2] and studied in [3,4,10]. A comprehensive summary of properties of spherical splines can be found in [8]. For comparison of spherical splines with spherical radial basis functions, see [7]. In [6], an efficient iterative computational algorithm is combined with the global fitting methods outlined in [3] to reduce the size of linear systems involved and thus to decrease computational costs. Convergence of the minimal energy interpolating spherical splines was studied in [5]. The results extend the bivariate minimal energy spline interpolation studied in [11] to the spherical setting. Convergence results for the discrete and penalized least squares methods in the bivariate setting were studied in [12–14]. However, to the best of the authors' knowledge, the convergence of the discrete least squares and penalized least squares splines on the sphere have not been explored in the literature.

The least squares methods are the methods of choice for data sets that are very large and contain random noise. Thus it is important to understand the approximating properties of the least squares spherical splines, and determine their convergence order. This is the goal of the paper.

Let us introduce the discrete least squares (DLS) and the penalized least squares (PLS) splines. Suppose $\mathcal{V} = \{v_i, i = 1, \ldots, n\}$ is a given set of locations on the unit sphere $\mathbb{S}^2$, and Δ is a triangulation of $\mathbb{S}^2$. Fix two integers $d > r \geq 0$, and let $S^r_d(\Delta)$ be the space of spherical splines of degree d and smoothness r over the triangulation Δ (to be defined more precisely in Section 2.1). Suppose we are given the set of discrete values $\{f(v), v \in \mathcal{V}\}$ of a function f. We fix a parameter $\lambda \geq 0$ and seek a spline function $s_{\lambda,f} \in S^r_d(\Delta)$, called the penalized least squares spline, satisfying

$$\mathcal{P}_\lambda(s_{\lambda,f}) = \min\{\mathcal{P}_\lambda(s) : s \in S^r_d(\Delta)\}. \tag{1}$$

Here $\mathcal{P}_\lambda(s) := \mathcal{L}(s - f) + \lambda\mathcal{E}(s)$ is called the penalized least squares functional, $\mathcal{L}$ is the discrete least squares functional, and $\mathcal{E}$ is the energy functional, respectively defined by

$$\mathcal{L}(s - f) := \sum_{v \in \mathcal{V}} |s(v) - f(v)|^2, \tag{2}$$

and

$$\mathcal{E}(s) := \int_{\mathbb{S}^2} \sum_{|\alpha|=2} |D^\alpha s|^2.$$

Here $D^\alpha s$, $|\alpha| = 2$ stands for the second order partial derivatives of the spline s. More precisely, s is homogeneously extended to $\mathbb{R}^3$, differentiated, and then restricted back to the sphere for integration. (In our proofs and computational examples we use linear homogeneous extensions, since in this case the energy of a function is equivalent to the square of its second Sobolev semi-norm defined below.)

When $\lambda = 0$, $s_{0,f}$ is called the discrete least squares fit of f. That is,

$$\mathcal{L}(s_{0,f} - f) = \min\{\mathcal{L}(s - f) : s \in S^r_d(\Delta)\}.$$

A triangulation Δ is called β-quasi-uniform when

$$\frac{|\Delta|}{\rho_\Delta} \leq \beta,$$

for some $\beta \geq 0$. Here $\rho_\Delta := \min\{\rho_\tau, \tau \in \Delta\}$ with ρ_τ being the diameter of the largest spherical cap contained in a spherical triangle τ. The mesh size of the triangulation Δ is defined by $|\Delta| := \max\{|\tau|, \tau \in \Delta\}$, with $|\tau|$ being the diameter of the smallest spherical cap containing a triangle τ.

To state our results we need to introduce Sobolev semi-norms. Let

$$|f|_{k,p,\Omega} = \sum_{|\alpha|=k} \|D^\alpha f\|_{p,\Omega}$$

denote the semi-norm of a function f in the Sobolev space $W^{k,p}(\Omega)$, $k \geq 0$, $1 \leq p \leq \infty$, $\Omega \subseteq \mathbb{S}^2$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a triple index with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and D^α stands for partial derivatives, for example $D^{(1,0,1)} = D_x D_z$ (see [1,10,8] for detailed definitions of semi-norms in Sobolev spaces on the unit sphere).

The issue of the existence of DLS splines in $S_d^r(\Delta)$ was addressed in [6]. A sufficient condition for the existence of a unique minimizer of the least squares sum in (2) is that the data sites are evenly distributed over the triangulation Δ with respect to the degree d. This condition requires that for every triangle $\tau \in \Delta$ the matrix

$$[B_{ijk}^d(v_\ell)]_{v_\ell \in \tau \cap \mathcal{V},\ i+j+k=d} \tag{3}$$

is of full rank. Here the spherical homogeneous Bernstein–Bézier basis polynomials B_{ijk}^d are defined with respect to a triangle τ and correspond with the columns of the matrix. They are evaluated at the data sites v_ℓ inside the triangle, with each row of the matrix corresponding to a data site (details are presented in Section 2.1). It is shown in [12] that such sets exist and can be easily constructed.

The condition on the data sites above ensures that a semi-definite inner product $\langle f, g\rangle = \sum_{v\in\mathcal{V}} f(v)g(v)$ defined on a space of bounded functions on the unit sphere becomes definite when restricted to the space of splines $S_d^r(\Delta)$. More specifically, under this condition, if a spline satisfies $\sum_{v_\ell\in\tau\cap\mathcal{V}} s(v_\ell)^2 = 0$ on every τ, we have

$$[s(v_\ell)]_{v_\ell\in\tau\cap\mathcal{V}} = [B_{ijk}^d(v_\ell)]_{v_\ell\in\tau\cap\mathcal{V},\ i+j+k=d}[c_{ijk}]_{i+j+k=d} = \mathbf{0},$$

where $[s(v_\ell)]$ is a vector of the spline values at the data sites inside τ, $[c_{ijk}]$ is a vector of the Bernstein–Bézier coefficients of the spline s restricted to τ, and $\mathbf{0}$ is a vector of zeros of the same length as $[s(v_\ell)]$. If the matrix $[B_{ijk}^d(v_\ell)]$ is of full rank, the coefficients of the spline vanish, and $s|_\tau \equiv 0$ on every triangle (see Section 2.1 for the definition of spherical Bernstein–Bézier polynomials and splines).

It is stated in [6] that, if the vertices of the triangulation Δ form a subset of $\mathcal{V}$, the PLS spline exists and is unique. The derivation of an error bound for the PLS splines involves the estimates for the DLS error. We therefore additionally require that the data sites are evenly distributed over Δ with respect to d.

The constants in the main results stated below depend on the properties of the spline spaces such as degree and quasi-uniformity parameter β of the underlying triangulation. They also depend on the norms of the inverses of the matrices above, which in turn depend on locations of data inside a triangle relative to its vertices. Finally, they depend on the number of data locations in a triangle. To simplify the statements of the theorems we say that the constants depend on the spline space and distribution of the data locations. Later on, as we work through components of the proofs we explain this dependence in detail.

Please cite this article in press as: V. Baramidze, M.-J. Lai, Convergence of discrete and penalized least squares spherical splines, Journal of Approximation Theory (2011), doi:10.1016/j.jat.2011.03.001

Theorem 1. *Suppose the data sites in $\mathcal{V}$ are evenly distributed over the β-quasi-uniform triangulation Δ with respect to the degree d of the spline space $S^r_d(\Delta)$, where $d \geq 3r+2$. Suppose that the mesh size of the triangulation $|\Delta|$ is bounded by 1. Then there exists a constant C depending on the spline space and distribution of the data locations such that for every function f in the Sobolev space $W^{m+1,\infty}(\mathbb{S}^2)$*

$$\|f - s_{0,f}\|_{\infty,\mathbb{S}^2} \leq C|\Delta|^{m+1}|f|_{m+1,\infty,\mathbb{S}^2}.$$

Here m is an integer between 0 and d with $(d-m) \bmod 2 = 0$.

Theorem 2. *Let Δ be a β-quasi-uniform triangulation of the sphere $\mathbb{S}^2$ with $|\Delta| \leq 1$ and let N denote the number of triangles in Δ. Suppose the triangulation Δ satisfies two conditions:*

(1) *the data sites in $\mathcal{V}$ are evenly distributed over Δ with respect to the degree d of the spline space $S^r_d(\Delta)$, where $d \geq 3r+2$;*
(2) *the vertices of the triangulation Δ form a subset of $\mathcal{V}$.*

Let $s_{\lambda,f}$ be the spline minimizing $\mathcal{P}_\lambda$ in (1). Then there exist positive constants C, C' and C'' such that

$$\|f - s_{\lambda,f}\|_{\infty,\mathbb{S}^2} \leq C|\Delta|^{m+1}|f|_{m+1,\infty,\mathbb{S}^2} + \lambda N(C'|\Delta|^{m-1}|f|_{m+1,\infty,\mathbb{S}^2} + C''|f|_{2,\infty,\mathbb{S}^2})$$

for every function f in the Sobolev space $W^{m+1,\infty}(\mathbb{S}^2)$. Here m is an integer between 1 and d with $(d-m) \bmod 2 = 0$. The constants C, C' and C'' depend on the spline space and distribution of the data locations.

Note that following the ideas in [15] one can choose the penalty parameter $\lambda \sim O(|\Delta|^{m+1}/N)$ leading to the order of convergence $O(|\Delta|^{m+1})$ for the penalized least squares approximation.

The proof of Theorem 1 is similar to the analysis of the bivariate discrete least squares fit in [12], and the proof of Theorem 2 is similar to the one given in [14]. However, we introduce a simplified argument in the proof of Theorem 2 based on the derivative estimate

$$\|D^\alpha(f - s_{0,f})\|_{\infty,\mathbb{S}^2} \leq C'''|\Delta|^{m+1-|\alpha|}|f|_{m+1,\infty,\mathbb{S}^2}.$$

The paper is organized as follows. In the preliminary section we introduce spherical polynomials in the Bernstein–Bézier (BB-) form. Several properties of spherical BB-polynomials used in the proof of our main results are briefly reviewed for convenience. We review the existence and properties of stable local bases for spline spaces as well. In Section 3 we study the convergence of the DLS fit. In Section 4 we describe the PLS method and study convergence of the resulting splines. Finally, in Section 5 we present several computational examples to illustrate convergence of the DLS and the PLS spline finding algorithms.

2. Preliminaries

2.1. Spherical splines

Let v_1, v_2, v_3 be three points on the unit sphere $\mathbb{S}^2$, which do not lie on the same great circle. Let $\tau = \langle v_1, v_2, v_3 \rangle$ be a spherical triangle, that is, τ is the smallest domain bounded by the great arcs $\widehat{v_1v_2}$, $\widehat{v_2v_3}$ and $\widehat{v_3v_1}$. The spherical barycentric coordinates of a point v on $\mathbb{S}^2$ relative to τ are the unique real numbers b_1, b_2, b_3 such that

$$v = b_1v_1 + b_2v_2 + b_3v_3.$$

Given an integer $d \geq 0$, the spherical homogeneous Bernstein–Bézier basis polynomials of degree d relative to τ are defined as

$$B_{ijk}^{d,\tau}(v) := \frac{d!}{i!j!k!} b_1^i(v) b_2^j(v) b_3^k(v), \quad i+j+k=d,$$

and

$$P(v) := \sum_{i+j+k=d} c_{ijk} B_{ijk}^{d,\tau}(v)$$

is called a spherical homogeneous Bernstein–Bézier (SBB) polynomial of degree d. $\mathcal{H}_d(\tau)$ denotes the space of all such polynomials.

To prove our results we will need the following case of Lemma 4.4 in [10].

Lemma 3. *Let τ be a spherical triangle with $|\tau| \leq 1$. There exists a positive constant K_1 dependent on d and the smallest angle Θ_τ of τ such that for any $P \in \mathcal{H}_d(\tau)$*

$$A_\tau^{-1/2} \|P\|_{2,\tau} \leq \|P\|_{\infty,\tau} \leq K_1 A_\tau^{-1/2} \|P\|_{2,\tau}. \tag{4}$$

Here A_τ is the area of the spherical triangle τ and $\|\cdot\|_{2,\tau}$, $\|\cdot\|_{\infty,\tau}$ denote the standard L_2 and L_∞ norms on τ.

We will also need a spherical analogue of Markov inequality. See [10,5] or [8] for a proof.

Lemma 4. *Let P be a spherical polynomial of degree d defined on a triangle τ with $|\tau| \leq 1$. There exists a constant K_2 dependent on d, p, k and the smallest angle of τ, such that*

$$|P|_{k,p,\tau} \leq \frac{K_2}{\rho_\tau^k} \|P\|_{p,\tau},$$

for all $1 \leq p \leq \infty$.

Let Δ be a triangulation of $\mathbb{S}^2$, i.e. Δ is a collection of spherical triangles such that the union of all triangles in Δ covers $\mathbb{S}^2$, and any two triangles in Δ either do not intersect each other or share an edge or a vertex. Define $S_d^r(\Delta)$ to be the space of piecewise spherical polynomials of degree d and smoothness r on Δ, i.e.

$$S_d^r(\Delta) := \{s : s|_\tau \in \mathcal{H}_d(\tau),\ \forall \tau \in \Delta\} \cap C^r(\mathbb{S}^2).$$

Here $s \in C^r(\mathbb{S}^2)$ if and only if

$$D^\alpha s_d|_{\mathbb{S}^2}$$

is continuous for all α such that $|\alpha| \leq r$, where s_d is the homogeneous extension of s of degree d, i.e.,

$$s_d(v) = |v|^d s\left(\frac{v}{|v|}\right), \quad \forall v \in \mathbb{R}^3 \setminus \{0\}.$$

Define $\text{star}^1(v)$ to be the union of all triangles in Δ that share the vertex v and

$$\text{star}^k(v) := \bigcup \{\text{star}^1(w) : w \text{ is a vertex of } \text{star}^{k-1}(v)\},$$

for $k > 1$. Similarly, for a spherical triangle τ, let $\text{star}^0(\tau) = \tau$ and for $k \geq 1$

$$\text{star}^k(\tau) := \bigcup \{\text{star}^1(w) : w \text{ is a vertex of } \text{star}^{k-1}(\tau)\}.$$

Please cite this article in press as: V. Baramidze, M.-J. Lai, Convergence of discrete and penalized least squares spherical splines, Journal of Approximation Theory (2011), doi:10.1016/j.jat.2011.03.001

We will need a bound on the number n_k of triangles in the kth star around τ which can be found in [5].

Lemma 5. *Suppose Δ is a β-quasi-uniform triangulation with $|\Delta| \le 1$. For a triangle $\tau \in \Delta$ and $k \ge 0$ the number n_k of triangles in* $\mathrm{star}^k(\tau)$ *satisfies*

$$\frac{2}{\pi\beta^2}(2k+1)^2 \le n_k \le \frac{5\beta^2}{4}(2k+1)^2.$$

Note also that for a spherical triangle $|\tau| \le 1$ in Δ, its area A_τ is related to the parameters $|\Delta|$ and ρ_Δ as

$$\frac{\pi\rho_\Delta^2}{5} \le A_\tau \le \frac{\pi|\Delta|^2}{4}. \tag{5}$$

Finally, analogous to the planar case, the smallest angle of a triangle in Δ, denoted by Θ_Δ, is closely related to the quasi-uniformity parameter β [5]. There exists a positive constant K_3 such that

$$\Theta_\Delta \ge \frac{K_3}{\beta}.$$

Therefore, in the following sections we state our results in relation to β rather then the smallest angle of Δ.

2.2. Approximation properties of spherical splines

In Section 13.4 of [8], the authors remark that a construction analogous to the bivariate setting can be followed in order to define a stable local basis $\{B_\xi,\ \xi \in \mathcal{M}\}$ associated with a stable local minimal determining set $\mathcal{M}$ in the spline space $S_d^r(\Delta)$ for $d \ge 3r+2$. Detailed construction of minimal determining sets can be found in [4]. The properties of such bases were studied in [10]. The results that we need are summarized below.

Let $\mathcal{S} := S_d^r(\Delta)$ be a spline space of degree d and smoothness r with $d \ge 3r+2$ over the triangulation Δ (defined in Section 2.1). There exists a minimal determining set $\mathcal{M}$ that is local and stable. There exists a basis $\mathcal{B} := \{B_\xi, \xi \in \mathcal{M}\}$ associated with $\mathcal{M}$ that is local and stable. That is, for any spline $s \in \mathcal{S}$

$$s = \sum_{\xi\in\mathcal{M}} c_\xi B_\xi$$

for some coefficients c_ξ. For any $\xi \in \mathcal{M}$ let τ_ξ be a triangle containing ξ with the area denoted by A_{τ_ξ}. Then

(1) $\mathrm{support}(B_\xi) \subseteq \mathrm{star}^3(\tau_\xi)$;

(2) $\|B_\xi\|_{\infty,\mathbb{S}^2} \le K_4$;

(3) $|c_\xi| \le K_5 A_{\tau_\xi}^{-1/p}\|s\|_{p,\tau_\xi}, \quad \text{for } 1 \le p \le \infty.$ (6)

The constants K_4 and K_5 depend on d, p and β.

It was also shown in [10] that with the basis $\mathcal{B}$ one can construct a quasi-interpolation operator $Q : W^{m+1,p}(\mathbb{S}^2) \to \mathcal{S}$ which achieves the optimal approximation property.

Theorem 6. *Let Δ be a β-quasi-uniform spherical triangulation with $|\Delta| \leq 1$. Let $1 \leq p \leq \infty$ and $d \geq 3r + 2$. For any $f \in W^{m+1,p}(\mathbb{S}^2)$ there exists a spline function $Qf \in S_d^r(\Delta)$ such that*

$$|f - Qf|_{k,p,\mathbb{S}^2} \leq K_6 |\Delta|^{m+1-k} |f|_{m+1,p,\mathbb{S}^2},$$

for all $0 \leq k \leq \min\{r+1, m+1\}$. Here K_6 is a constant depending only on d, p and β, and m is to be taken between 0 and d with $(d - m) \bmod 2 = 0$.

The proof of the above result can also be found in [8,5].

3. Convergence of discrete least squares splines

In this section we investigate the L_∞ error bound for the DLS spline approximation on the sphere. Given a data set $\{(v_i, f(v_i)) : v_i \in \mathcal{V}\}_{i=1}^n$ let $\mathcal{S} := S_d^r(\Delta)$ denote the spline space over a β-quasi-uniform triangulation Δ, $|\Delta| \leq 1$.

Let $\mathcal{X} := B(\mathbb{S}^2)$ be the space of all bounded real-valued functions on the sphere equipped with the semi-definite inner product

$$\langle f, g \rangle_{\mathcal{L}} = \sum_{i=1}^{n} f(v_i) g(v_i).$$

The inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ induces the semi-norm

$$\|f\|_{\mathcal{L}} = \langle f, f \rangle_{\mathcal{L}}^{1/2} = \mathcal{L}(f)^{1/2},$$

with the DLS functional $\mathcal{L}$ defined in (2). Recall that the DLS spline $s_{0,f}$ satisfies

$$\mathcal{L}(s_{0,f} - f) = \min\{\mathcal{L}(s - f), s \in \mathcal{S}\}.$$

As mentioned earlier we require that the data sites v_i, $i = 1, \ldots, n$ are evenly distributed over the triangulation Δ with respect to d. Note that the sets $\tau \cap \mathcal{V}$ involved in (3) are not necessarily disjoint since some data may be located at the vertices or on the edges of Δ. To simplify certain proofs we will assume that there is a way to partition the data locations $\mathcal{V}$ into the subsets $\mathcal{V}_\tau \subset \tau$ such that all $\mathcal{V}_\tau$ are disjoint and $\bigcup_{\tau \in \Delta} \mathcal{V}_\tau = \mathcal{V}$, and the matrices

$$[B_{ijk}^d(v_\ell)]_{v_\ell \in \mathcal{V}_\tau}$$

are of full rank.

Under this assumption we conclude the following

(1) the DLS spline exists in $\mathcal{S}$;
(2) $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ is a definite inner product on $\mathcal{S}$, thus $\mathcal{S}$ is a Hilbert space with respect to this inner product;
(3) for any $f \in \mathcal{X}$

$$\|f\|_{\mathcal{L}}^2 = \sum_{v \in \mathcal{V}} f(v)^2 = \sum_{\tau \in \Delta} \sum_{v \in \mathcal{V}_\tau} f(v)^2 = \sum_{\tau \in \Delta} \|f\|_{\mathcal{L},\tau}^2,$$

where $\|f\|_{\mathcal{L},\tau} = (\sum_{v \in \mathcal{V}_\tau} f(v)^2)^{1/2}$ denotes the restriction of $\|f\|_{\mathcal{L}}$ to $\mathcal{V}_\tau$.
(4) for any $f \in \mathcal{X}$ and any $\tau \in \Delta$

$$\|f\|_{\mathcal{L},\tau}^2 = \sum_{v \in \mathcal{V}_\tau} f(v)^2 \leq \sum_{v \in \mathcal{V}_\tau} \|f\|_{\infty,\tau}^2 \leq \max_{T \in \Delta}\{\#\mathcal{V}_T\} \|f\|_{\infty,\tau}^2 = K_7^2 \|f\|_{\infty,\tau}^2, \tag{7}$$

with K_7 independent of τ.

Please cite this article in press as: V. Baramidze, M.-J. Lai, Convergence of discrete and penalized least squares spherical splines, Journal of Approximation Theory (2011), doi:10.1016/j.jat.2011.03.001

Define a projection operator $L : \mathcal{X} \mapsto \mathcal{S}$ by $Lf = s_{0,f}$. Clearly, L is linear. We will prove that L is bounded with respect to $L_\infty(\mathbb{S}^2)$. To this end we use Theorem 3.1 in [12]. The hypotheses of this theorem, (2.1), (3.1) and (3.2), are checked in the following lemmas.

Lemma 7 (*Condition 3.1 in [12]*). *Suppose the data set $\mathcal{V}$ is evenly distributed over the given triangulation Δ with respect to the degree d of the spline space $\mathcal{S}$. Then for every triangle τ in Δ and any spline $s \in \mathcal{S}$*

$$K_8 \|s\|_{\infty,\tau} \le \|s\|_{\mathcal{L},\tau} \tag{8}$$

for some positive constant K_8 depending on $\mathcal{S}$ and the data locations.

Proof. Fix $\tau \in \Delta$. Write $s|_\tau$ in BB-form as

$$s|_\tau = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}.$$

Recall from [2] that B^d_{ijk} are positive on τ and

$$\sum_{i+j+k=d} B^d_{ijk} = (b_1 + b_2 + b_3)^d.$$

The barycentric coordinates b_1, b_2, b_3 on the sphere do not form a partition of unity, however on the triangle τ each is positive and bounded above by 1. Therefore

$$|s|_\tau(v)| \le \sum_{i+j+k=d} |c_{ijk}| B^d_{ijk}(v) \le \max_{i+j+k=d} |c_{ijk}| \sum_{i+j+k=d} B^d_{ijk}(v) \le 3^d \|\mathbf{c}\|_\infty.$$

Since the data locations are evenly distributed over the triangulation Δ, the coefficients $\mathbf{c} = (c_{ijk})_{i+j+k=d}$ of the spline $s|_\tau$ can be found uniquely by solving the system

$$[B^d_{ijk}(v)]_{v\in\mathcal{V}_\tau} \mathbf{c} = M_\tau \mathbf{c} = \mathbf{s},$$

where $\mathbf{s}$ is a vector of values of the spline $s|_\tau$ at the data locations $v \in \mathcal{V}_\tau$. Then

$$\|\mathbf{c}\|_\infty \le \|M_\tau^{-1}\|_\infty \|\mathbf{s}\|_\infty = \|M_\tau^{-1}\|_\infty \max_{v\in\mathcal{V}_\tau} |s(v)| \le \|M_\tau^{-1}\|_\infty \left(\sum_{v\in\mathcal{V}_\tau} |s(v)|^2 \right)^{1/2}$$
$$= \|M_\tau^{-1}\|_\infty \|s\|_{\mathcal{L},\tau}.$$

Therefore

$$|s|_\tau(v)| \le 3^d \|M_\tau^{-1}\|_\infty \|s\|_{\mathcal{L},\tau},$$

thus we have (8) with $K_8 = 1/(3^d \max_{\tau\in\Delta} \|M_\tau^{-1}\|_\infty)$. □

The next condition of Theorem 3.1 (Condition 3.2 in [12]) in [12] is satisfied by (7).

Finally, we check condition (2.1) of [12]. Recall from Section 2 that $\mathcal{M}$ denotes a minimal determining set for $\mathcal{S}$ and $\{B_\xi, \xi \in \mathcal{M}\}$ denotes the basis associated with $\mathcal{M}$.

Lemma 8. *Suppose that $\mathcal{S}$ is a spline space with $d \ge 3r + 2$. There exist constants $0 < K_9$, $K_{10} < \infty$ such that*

$$K_9 \sum_{\xi\in\mathcal{M}} |c_\xi|^2 \le \left\| \sum_{\xi\in\mathcal{M}} c_\xi B_\xi \right\|^2_{\mathcal{L}} \le K_{10} \sum_{\xi\in\mathcal{M}} |c_\xi|^2, \tag{9}$$

for all $(c_\xi)_{\xi\in\mathcal{M}}$.

Please cite this article in press as: V. Baramidze, M.-J. Lai, Convergence of discrete and penalized least squares spherical splines, Journal of Approximation Theory (2011), doi:10.1016/j.jat.2011.03.001

Proof. Since $d \geq 3r+2$, $\mathcal{S}$ possesses a stable local basis $\mathcal{B}$. Let $s = \sum_{\xi \in \mathcal{M}} c_\xi B_\xi$. By (3) in (6) we have

$$\sum_{\xi \in \mathcal{M}} c_\xi^2 \leq K_5^2 \sum_{\xi \in \mathcal{M}} \|s\|_{\infty,\tau_\xi}^2,$$

where τ_ξ is a triangle in Δ containing ξ. Since there is more than one domain point $\xi \in \mathcal{M}$ in every triangle of Δ the sum on the right-hand side of the inequality above has repeating terms. Let $\mathcal{M}_\tau := \mathcal{M} \cap \tau$, $\tau \in \Delta$. Then

$$\sum_{\xi \in \mathcal{M}} \|s\|_{\infty,\tau_\xi}^2 \leq \max_{\tau \in \Delta} \#\mathcal{M}_\tau \sum_{\tau \in \Delta} \|s\|_{\infty,\tau}^2,$$

where $\max_{\tau \in \Delta} \#\mathcal{M}_\tau$ depends on the degree d and smoothness r of $\mathcal{S}$ and can be bounded by a constant depending on d only. Using Lemma 7 to estimate $\|s\|_{\infty,\tau}$ we arrive at

$$\sum_{\xi \in \mathcal{M}} c_\xi^2 \leq K_5^2 K_8^2 \max_{\tau \in \Delta} \#\mathcal{M}_\tau \|s\|_{\mathcal{L}}^2$$

which is equivalent to the left-hand side inequality of (9). Next, to estimate $\|s\|_{\mathcal{L}}$ in the right-hand side inequality of (9) we use (7):

$$\|s\|_{\mathcal{L}}^2 = \sum_{\tau \in \Delta} \|s\|_{\mathcal{L},\tau}^2 \leq \sum_{\tau \in \Delta} K_7^2 \|s\|_{\infty,\tau}^2 = K_7^2 \sum_{\tau \in \Delta} \left\| \sum_{\xi \in \mathcal{M}} c_\xi B_\xi \right\|_{\infty,\tau}^2.$$

Let $\mathcal{M}_\tau$ be a subset of $\mathcal{M}$ containing the points ξ such that the corresponding basis functions B_ξ have support in τ. Then using (2) of (6)

$$\left\| \sum_{\xi \in \mathcal{M}} c_\xi B_\xi \right\|_{\infty,\tau}^2 = \left\| \sum_{\xi \in \mathcal{M}_\tau} c_\xi B_\xi \right\|_{\infty,\tau}^2 \leq \left(\sum_{\xi \in \mathcal{M}_\tau} |c_\xi| \, \|B_\xi\|_{\infty,\tau} \right)^2$$
$$\leq K_4^2 \left(\sum_{\xi \in \mathcal{M}_\tau} |c_\xi| \right)^2 \leq C K_4^2 \sum_{\xi \in \mathcal{M}_\tau} |c_\xi|^2,$$

and thus

$$\|s\|_{\mathcal{L}}^2 \leq C K_4^2 K_7^2 \sum_{\tau \in \Delta} \sum_{\xi \in \mathcal{M}_\tau} |c_\xi|^2.$$

The constant C above depends on the cardinality of $\mathcal{M}_\tau$. We claim that the cardinality of $\mathcal{M}_\tau$ depends on d and β only. Note that $\text{support}(B_\xi) \subseteq \text{star}^3(\tau_\xi)$ by (1) in (6), and therefore $\mathcal{M}_\tau \subseteq \text{star}^3(\tau) \cap \mathcal{M}$. The number of triangles in $\text{star}^3(\tau)$ is n_3. It is bounded by Lemma 5 and the bound depends only on β.

In the last sum ξ is repeated at most $\max \#\{\tau \in \Delta : \xi \in \mathcal{M}_\tau\}$ times. Since $\mathcal{M}_\tau \subseteq \text{star}^3(\tau) \cap \mathcal{M}$, $\max \#\{\tau : \xi \in \mathcal{M}_\tau\} \leq n_3$. Then

$$\sum_{\tau \in \Delta} \sum_{\xi \in \mathcal{M}_\tau} |c_\xi|^2 \leq n_3 \sum_{\xi \in \mathcal{M}} |c_\xi|^2.$$

It follows that

$$\|s\|_{\mathcal{L}}^2 \leq C K_4^2 K_7^2 n_3 \sum_{\xi \in \mathcal{M}} |c_\xi|^2,$$

thus we have proven the second and final inequality in (9). □

The results proved in Lemmas 7 and 8 together with (7) lead to the conclusion that the projection operator L defined earlier as $Lf = s_{0,f}$ is bounded with respect to $L_\infty(\mathbb{S}^2)$.

Theorem 9. *Suppose the given data sites in $\mathcal{V}$ are evenly distributed over the β-quasi-uniform triangulation Δ, $|\Delta| \leq 1$, with respect to the degree d of the spline space $\mathcal{S}$ where $d \geq 3r + 2$. Then there exists a constant $K_{11} > 0$ such that*

$$\|L\|_\infty \leq K_{11}.$$

The constant K_{11} depends on how the data is distributed over the triangulation, how many data sites are located in a triangle, the degree d of the spline space, and the quasi-uniformity parameter β.

Proof. Theorem 3.1 in [12]. □

We are now positioned to prove one of the main results in this paper.

Proof of Theorem 1. Let Qf be the quasi-interpolant defined in Section 2. Since Qf is a spherical spline in $\mathcal{S}$, $LQf = Qf$ and hence

$$\|f - Lf\|_{\infty,\mathbb{S}^2} \leq \|f - Qf\|_{\infty,\mathbb{S}^2} + \|LQf - Lf\|_{\infty,\mathbb{S}^2}.$$

Since L is linear, we have

$$\|f - Lf\|_{\infty,\mathbb{S}^2} \leq \|f - Qf\|_{\infty,\mathbb{S}^2} + \|L(Qf - f)\|_{\infty,\mathbb{S}^2} \leq (1 + \|L\|_\infty)\|f - Qf\|_{\infty,\mathbb{S}^2}.$$

By Theorems 6 and 9

$$\|f - Lf\|_{\infty,\mathbb{S}^2} \leq C|\Delta|^{m+1}|f|_{m+1,\infty,\mathbb{S}^2}. \quad \square$$

In the next section we investigate the behavior of the PLS splines. We will need an estimate involving the derivatives of the DLS splines. Consider the following.

For any $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with nonnegative integers $\alpha_1, \alpha_2, \alpha_3$ satisfying $|\alpha| = \alpha_1+\alpha_2+\alpha_3 \leq r + 1$ and a sufficiently smooth function f, we use Theorem 6 and Lemma 4 to conclude

$$\begin{aligned}\|D^\alpha(f - Lf)\|_{\infty,\mathbb{S}^2} &\leq \|D^\alpha(f - Qf)\|_{\infty,\mathbb{S}^2} + \|D^\alpha(LQf - Lf)\|_{\infty,\mathbb{S}^2} \\ &\leq K_6|\Delta|^{m+1-|\alpha|}|f|_{m+1,\infty,\mathbb{S}^2} + \frac{K_2}{\rho_\Delta^{|\alpha|}}\|LQf - Lf\|_{\infty,\mathbb{S}^2}\end{aligned}$$

for a positive constant K_2 dependent on d, $|\alpha|$ and β. Since, according to Theorem 6

$$\|LQf - Lf\|_{\infty,\mathbb{S}^2} \leq \|L\|_\infty\|Qf - f\|_{\infty,\mathbb{S}^2} \leq K_{11}K_6|\Delta|^{m+1}|f|_{m+1,\infty,\mathbb{S}^2},$$

we have proven the following theorem.

Theorem 10. *Under the conditions of Theorem 1 there exists a constant K_{12} depending on d, β, $|\alpha|$, and $\|L\|_\infty$ such that for every function f in $W^{m+1,\infty}(\mathbb{S}^2)$ for some integer m between 0 and d with $(d - m) \bmod 2 = 0$, we have*

$$\|D^\alpha(f - Lf)\|_{\infty,\mathbb{S}^2} \leq K_{12}|\Delta|^{m+1-|\alpha|}|f|_{m+1,\infty,\mathbb{S}^2}$$

for any $0 \leq |\alpha| \leq \min\{r + 1, m + 1\}$.

Please cite this article in press as: V. Baramidze, M.-J. Lai, Convergence of discrete and penalized least squares spherical splines, Journal of Approximation Theory (2011), doi:10.1016/j.jat.2011.03.001

4. Convergence of penalized least squares splines

Suppose that a given set of data values $\{v_i, f(v_i)\}_{i=1}^n$ corresponds to a sampling of some function f on the unit sphere. Given a triangulation Δ with $|\Delta| \leq 1$ define

$$\mathcal{X} := \{g \in B(\mathbb{S}^2) : \forall \tau \in \Delta, g|_\tau \in C^1(\tau)\},$$

to be a subspace of bounded functions on the sphere equipped with the semi-definite inner product

$$\langle h, g\rangle_{\mathcal{P}} = \langle h - f, g - f\rangle_{\mathcal{L}} + \lambda\langle h, g\rangle_{\mathcal{E}}$$

with $\lambda > 0$ being a fixed parameter. Here

$$\langle h, g\rangle_{\mathcal{L}} := \sum_{i=1}^{n} h(v_i)g(v_i)$$

as in Section 3. The energy inner product is

$$\langle h, g\rangle_{\mathcal{E}} := \sum_{\tau\in\Delta} \int_\tau \sum_{|\alpha|=2} D^\alpha h D^\alpha g \mathrm{d}\sigma,$$

and the energy semi-norm $\|h\|_{\mathcal{E}}$ is $\sqrt{\langle h, h\rangle_{\mathcal{E}}}$.

Fix a spline space $\mathcal{S} = S_d^r(\Delta)$ with $d \geq 3r+2$ so that $\mathcal{S}$ possesses a stable local basis. Assume that the given data are evenly distributed over the triangulation Δ with respect to the degree d, and the vertices of the triangulation Δ form a subset of $\mathcal{V}$. Then for any function $f \in \mathcal{X}$ there exists the DLS spline $s_{0,f}$ with the approximation properties outlined in Theorems 1 and 10. There exists a unique PLS spline $s_{\lambda,f}$ corresponding to the penalty parameter λ (see [6] for a proof).

The PLS spline corresponding to the penalty parameter λ and approximating the function f is denoted by $s_{\lambda,f}$. It is defined as the spline minimizing the functional

$$\mathcal{P}_\lambda(s) = \mathcal{L}(s - f) + \lambda\mathcal{E}(s)$$

over the space of all spherical splines $\mathcal{S}$. The DLS spline can be characterized by

$$\langle s_{0,f} - f, s\rangle_{\mathcal{L}} = 0, \quad \forall s \in \mathcal{S}.$$

Similarly, the PLS spline $s_{\lambda,f} \in \mathcal{S}$ is characterized by

$$\langle f - s_{\lambda,f}, s\rangle_{\mathcal{L}} = \lambda\langle s_{\lambda,f}, s\rangle_{\mathcal{E}}, \quad \forall s \in \mathcal{S}.$$

The sum of the two equations yields

$$\langle s_{0,f} - s_{\lambda,f}, s\rangle_{\mathcal{L}} = \lambda\langle s_{\lambda,f}, s\rangle_{\mathcal{E}}, \quad \forall s \in \mathcal{S}.$$

In particular, for $s = s_{0,f} - s_{\lambda,f}$ it follows that

$$\|s_{0,f} - s_{\lambda,f}\|_{\mathcal{L}}^2 = \lambda\langle s_{\lambda,f}, s_{0,f}\rangle_{\mathcal{E}} - \lambda\|s_{\lambda,f}\|_{\mathcal{E}}^2 \geq 0. \tag{10}$$

Thus for any $\lambda > 0 \|s_{\lambda,f}\|_{\mathcal{E}}^2 \leq \langle s_{\lambda,f}, s_{0,f}\rangle_{\mathcal{E}}$. By Cauchy–Schwarz's inequality, we have

$$\|s_{\lambda,f}\|_{\mathcal{E}} \leq \|s_{0,f}\|_{\mathcal{E}}. \tag{11}$$

By Cauchy–Schwarz's inequality we get in (10)

$$\|s_{0,f} - s_{\lambda,f}\|_{\mathcal{L}}^2 = \lambda\langle s_{\lambda,f}, s_{0,f} - s_{\lambda,f}\rangle_{\mathcal{E}} \leq \lambda\|s_{\lambda,f}\|_{\mathcal{E}}\|s_{0,f} - s_{\lambda,f}\|_{\mathcal{E}}. \tag{12}$$

Please cite this article in press as: V. Baramidze, M.-J. Lai, Convergence of discrete and penalized least squares spherical splines, Journal of Approximation Theory (2011), doi:10.1016/j.jat.2011.03.001

As in [13], we introduce the following quantity

$$K_{\mathcal{S}} := \sup\left\{\frac{\|s\|_{\mathcal{E}}}{\|s\|_{\mathcal{L}}},\ s \in \mathcal{S},\ s \neq 0\right\}.$$

As we noted in Section 3 since the data are evenly distributed, $\|\cdot\|_{\mathcal{L}}$ is a norm on $\mathcal{S}$ and the constant $K_{\mathcal{S}}$ is well defined. Let us show that $K_{\mathcal{S}}$ is bounded.

It is easy to see that for any $s \in \mathcal{S}$

$$\|s\|_{\mathcal{E}}^2 \le \sum_{\tau\in\Delta} |s|_{2,2,\tau}^2.$$

Since s is a spherical polynomial of degree d on a spherical triangle τ, by Lemma 4

$$|s|_{2,2,\tau}^2 \le \frac{K_2^2}{\rho_\tau^4}\|s\|_{2,\tau}^2$$

for some constant K_2 depending on d and β. By Lemmas 3 and 7

$$\|s\|_{\mathcal{E}}^2 \le \sum_{\tau\in\Delta}\frac{K_2^2}{\rho_\tau^4}\|s\|_{2,\tau}^2 \le \frac{K_2^2}{\rho_\Delta^4}\sum_{\tau\in\Delta} A_\tau\|s\|_{\infty,\tau}^2 \le \frac{K_2^2}{K_8^2}\frac{\max_{\tau\in\Delta} A_\tau}{\rho_\Delta^4}\|s\|_{\mathcal{L}}^2.$$

We therefore get

$$\|s\|_{\mathcal{E}} \le \frac{K_2}{K_8}\frac{\max_{\tau\in\Delta} A_\tau^{1/2}}{\rho_\Delta^2}\|s\|_{\mathcal{L}}.$$

That is,

$$K_{\mathcal{S}} = \sup\left\{\frac{\|s\|_{\mathcal{E}}}{\|s\|_{\mathcal{L}}} : s \in \mathcal{S}, s \neq 0\right\} \le \frac{K_2}{K_8}\frac{\max_{\tau\in\Delta} A_\tau^{1/2}}{\rho_\Delta^2}.$$

Now from (12) and (11) we have

$$\|s_{0,f} - s_{\lambda,f}\|_{\mathcal{L}}^2 \le \lambda K_{\mathcal{S}}\|s_{0,f}\|_{\mathcal{E}}\|s_{0,f} - s_{\lambda,f}\|_{\mathcal{L}}$$

or

$$\|s_{0,f} - s_{\lambda,f}\|_{\mathcal{L}} \le \lambda K_{\mathcal{S}}\|s_{0,f}\|_{\mathcal{E}}.$$

Thus we conclude by Lemma 7 that

$$\|s_{0,f} - s_{\lambda,f}\|_{\infty,\mathbb{S}^2} \le \frac{1}{K_8}\|s_{0,f} - s_{\lambda,f}\|_{\mathcal{L}} \le \frac{\lambda K_{\mathcal{S}}}{K_8}\|s_{0,f}\|_{\mathcal{E}}. \tag{13}$$

Let us study the right-hand side of the above inequality. Suppose that $f \in W^{m+1,\infty}(\mathbb{S}^2)$, $m \ge 1$, and $r \ge 1$. Then $s_{0,f} - f \in W^{2,\infty}(\mathbb{S}^2)$. By the definition of $\|\cdot\|_{\mathcal{E}}$, Lemma 3, triangle inequality, and Theorem 10 we have

$$\begin{aligned}\|s_{0,f}\|_{\mathcal{E}} &\le |s_{0,f}|_{2,2,\mathbb{S}^2} \le \sum_{\tau\in\Delta}|s_{0,f}|_{2,2,\tau} \le \sum_{\tau\in\Delta} A_\tau^{1/2}|s_{0,f}|_{2,\infty,\tau} \le N\max_{\tau\in\Delta} A_\tau^{1/2}|s_{0,f}|_{2,\infty,\mathbb{S}^2}\\ &\le N\max_{\tau\in\Delta} A_\tau^{1/2}(|s_{0,f} - f|_{2,\infty,\mathbb{S}^2} + |f|_{2,\infty,\mathbb{S}^2})\\ &\le N\max_{\tau\in\Delta} A_\tau^{1/2}(K_{12}|\Delta|^{m-1}|f|_{m+1,\infty,\mathbb{S}^2} + |f|_{2,\infty,\mathbb{S}^2}) =: C_f,\end{aligned}$$

where $N = \#\{\tau : \tau \in \Delta\}$. Thus, it follows by (13) that

$$\|s_{0,f} - s_{\lambda,f}\|_{\infty,\mathbb{S}^2} \leq \frac{\lambda K_{\mathcal{S}}}{K_8} C_f. \tag{14}$$

By (14), (5) and the estimate for $K_{\mathcal{S}}$ we have

$$\|s_{0,f} - s_{\lambda,f}\|_{\infty,\mathbb{S}^2} \leq \lambda N \beta^2 \frac{\pi}{4} \frac{K_2}{K_8^2} (K_{12}|\Delta|^{m-1}|f|_{m+1,\infty,\mathbb{S}^2} + |f|_{2,\infty,\mathbb{S}^2}). \tag{15}$$

Proof of Theorem 2. By the triangle inequality

$$\|f - s_{\lambda,f}\|_{\infty,\mathbb{S}^2} \leq \|s_{0,f} - f\|_{\infty,\mathbb{S}^2} + \|s_{0,f} - s_{\lambda,f}\|_{\infty,\mathbb{S}^2}.$$

Apply Theorem 1 and (15). □

5. Computational examples

In this section we present four computational examples. In Example 1 we demonstrate that for a sufficiently smooth function the error of the DLS fit decreases as $O(|\Delta|^{d+1})$. In Example 2 we show that the error for the PLS fit depends linearly on the parameter λ. In Example 3 we demonstrate how the penalty parameter can be adjusted with the refinement of the triangulation Δ to achieve the convergence order $O(|\Delta|^{d+1})$ for the PLS spline. In Example 4 we show that when the data values contain random noise, the PLS fit with an appropriate choice of parameter λ can perform better than the DLS fit.

Example 1. Let Δ_0 be a triangulation of the unit sphere with 8 triangles and 6 vertices $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. We refine Δ_0 uniformly to get Δ_1 by partitioning each triangle into four using midpoints of the edges. Similarly Δ_{k+1} denotes the uniform refinement of Δ_k for $k = 1, 2, 3$. We use a set of 47,106 data sites almost uniformly scattered over the sphere and a test function

$$f(x, y, z) = 1 + x^8 + e^{2y^3} + e^{2z^2} + 10xyz.$$

The data sites form a subset of domain points corresponding to degree 8 over the last refinement Δ_4, that is

$$\mathcal{V} \subset \left\{ \frac{iv_1 + jv_2 + kv_3}{\|iv_1 + jv_2 + kv_3\|} : v_1, v_2, v_3 \text{ are vertices of } \tau \in \Delta_4, i + j + k = 8 \right\}$$

(some domain points that belong to the edges of triangulation Δ_4 are removed, and all repetitions are eliminated).

We use the iterative algorithm in [6] to find the DLS spline $s_{0,f}$ in spaces $S_d^1(\Delta_k)$, $d = 5, 6, 7$. The maximum values $e_{d,k}$, $d = 5, 6, 7$, $k = 0, \ldots, 4$ of the error $|s_{0,f} - f|$ evaluated over the data locations are listed in Table 1, and the ratios of the error values are recorded in Table 2.

Example 2. Next we consider the PLS fit in the spline spaces $S_d^1(\Delta_3)$, $d = 5, 6$ over the triangulation Δ_3, that consists of 512 triangles. We sample the function $g = x^d$ over 47,106 scattered data locations as in Example 1. We compute the PLS splines $s_{\lambda,g}$ with the parameter $\lambda_k = 1/2^k$, for $k = 2, 3, 4, 5$. The maximum values $e_{d,k}$ of the error $|s_{\lambda,g} - g|$ over the data locations are listed in Table 3. Note that the function x^d in this example has its $(d + 1)$ Sobolev semi-norm vanishing on the sphere, while its second Sobolev semi-norm remains nonzero. This choice of test function allows us to trace the dependence of the error on the parameter λ.

Table 1
Error in the approximation of f by the DLS splines.

	$e_{d,0}$	$e_{d,1}$	$e_{d,2}$	$e_{d,3}$	$e_{d,4}$
$S_5^1(\Delta_k)$	3.2326e−01	1.7535e−02	9.2120e−04	2.2662e−05	4.0062e−07
$S_6^1(\Delta_k)$	4.8938e−02	2.1900e−03	4.9109e−05	1.2004e−06	9.6883e−09
$S_7^1(\Delta_k)$	3.8714e−02	5.5652e−04	1.5188e−05	1.3509e−07	2.8862e−10

Table 2
Ratios of errors in the approximation of f by the DLS splines.

	$e_{d,0}/e_{d,1}$	$e_{d,1}/e_{d,2}$	$e_{d,2}/e_{d,3}$	$e_{d,3}/e_{d,4}$
$S_5^1(\Delta_k)$	18.44	19.03	40.65	56.57
$S_6^1(\Delta_k)$	22.35	44.59	40.91	123.90
$S_7^1(\Delta_k)$	69.56	36.64	112.43	468.05

Table 3
Linear dependence of the error on λ, maximal error values over the data locations.

$e_{d,k}$	$e_{d,2}$	$e_{d,3}$	$e_{d,4}$	$e_{d,5}$
$S_5^1(\Delta_3)$	7.7918e−3	3.9232e−3	1.9680e−3	9.8513e−4
$S_6^1(\Delta_3)$	1.3865e−2	7.0475e−3	3.5482e−3	1.7767e−3

Table 4
Linear dependence of the error on λ, ratios of maximal error values over the data locations.

	$e_{d,2}/e_{d,3}$	$e_{d,3}/e_{d,4}$	$e_{d,4}/e_{d,5}$
$S_5^1(\Delta_3)$	1.9861	1.9935	1.9977
$S_6^1(\Delta_3)$	1.9673	1.9862	1.9970

The ratios of the error values are listed in Table 4. As expected they approach 2 as λ is reduced by a factor of 2. It is clear from Table 4 that the error in approximation of f by the PLS spline depends on the parameter λ linearly.

Example 3. In this example we demonstrate how λ can be chosen to achieve the order of approximation $O(|\Delta|^{d+1})$ for PLS splines. In our error estimate, Theorem 2, the last term does not depend on the size of the triangulation $|\Delta|$. It is proportional to λN, where N is the number of triangles. To have $\lambda N \sim O(|\Delta|^{d+1})$ we must have $\lambda \sim O(|\Delta|^{d+1}/N)$. Since with each refinement the number of triangles increases by a factor of 4, while the triangulation size decreases by a factor of 2, we decrease λ by a factor of 2^{d+3}. We sample the function f as in Example 1 at 47,106 almost uniformly scattered locations over the unit sphere. We find the PLS splines in $S_d^r(\Delta)$, $r = 1, d = 5, 6$ over the triangulations Δ_k, $k = 0, 1, 2, 3, 4$ with the penalty parameters $\lambda = 1/2^{k(d+3)+2(d-4)+1}$. The factor $1/2^{2(d-4)+1}$ in λ reflects the fact that for the initial triangulation Δ, λ is chosen increasingly small as the degree of the space rises. The factor is chosen to match the error in the approximation of f by PLS spline with that in the approximation of f by the DLS splines in Table 1. The errors in maximal norm are evaluated over the data locations and are listed in Table 5. The ratios of errors as we refine triangulations

Please cite this article in press as: V. Baramidze, M.-J. Lai, Convergence of discrete and penalized least squares spherical splines, Journal of Approximation Theory (2011), doi:10.1016/j.jat.2011.03.001

Table 5
Error in the approximation of f by the PLS splines.

	$e_{d,0}$	$e_{d,1}$	$e_{d,2}$	$e_{d,3}$	$e_{d,4}$
S_5^1	3.2072e−1	1.7561e−2	9.2116e−4	2.2665e−5	4.0066e−7
S_6^1	5.3473e−2	2.1840e−3	4.9124e−5	1.2004e−6	9.6886e−9
S_7^1	3.9077e−2	5.5688e−4	1.5188e−5	1.3509e−7	2.2828e−10

Table 6
Ratios of errors in the approximation of f by the PLS splines.

	$e_{d,0}/e_{d,1}$	$e_{d,1}/e_{d,2}$	$e_{d,2}/e_{d,3}$	$e_{d,3}/e_{d,4}$
S_5^1	18.26	19.06	40.64	56.57
S_6^1	24.48	44.46	40.92	123.90
S_7^1	70.17	36.67	112.43	591.77

Table 7
Maximum errors for least squares fittings for data with 5% noise.

	Δ_1	Δ_2	Δ_3
DLS	0.0551	0.1349	0.3251
PLS	0.0451	0.0545	0.0522
λ	$1/2^6$	$1/2^4$	$1/2^5$

Table 8
Maximum errors for least squares fittings for data with 2.5% noise.

	Δ_1	Δ_2	Δ_3
DLS	0.0259	0.0674	0.1625
PLS	0.0218	0.0316	0.0322
λ	$1/2^7$	$1/2^5$	$1/2^6$

Table 9
Maximum errors for least squares fittings for data with 1% noise.

	Δ_1	Δ_2	Δ_3
DLS	0.0105	0.0270	0.0650
PLS	0.0097	0.0149	0.0169
λ	$1/2^9$	$1/2^6$	$1/2^7$

are listed in Table 6. Comparison with Tables 1 and 2 allows us to conclude that our choices of λ are reasonable.

Example 4. Finally, we present the case when the data values contain random noise. That is, let $\mathbf{y}$ be the values of f over the 47,106 points as in Example 1. Let $\mathbf{e}$ be a vector of 47,106 uniformly distributed random numbers in $[-1, 1]$. We compute both least squares spherical spline fits to $\widetilde{\mathbf{y}} = \mathbf{y} + \sigma\mathbf{e}\mathbf{y}$ over the spline spaces $S_6^1(\Delta_k)$ and $k = 1, 2, 3$, $\sigma = 0.05, 0.025, 0.01$ (by the product $\mathbf{ey}$ we mean entry by entry multiplication). The maximum values of $|s_{\lambda,f} - f|$ over the 47,106 points are recorded in Tables 7–9. By adjusting the parameter λ, we are able to find PLS

Please cite this article in press as: V. Baramidze, M.-J. Lai, Convergence of discrete and penalized least squares spherical splines, Journal of Approximation Theory (2011), doi:10.1016/j.jat.2011.03.001

splines, which have maximum errors smaller than corresponding DLS splines. The respective values of λ are listed under the error values for PLS splines.

Acknowledgment

The second author was supported by the National Science Foundation under the grant #DMS 0713807.

References

[1] R. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] P. Alfeld, M. Neamtu, L.L. Schumaker, Bernstein–Bézier polynomials on spheres and sphere-like surfaces, Comput. Aided Geom. Design 13 (1996) 333–349.
[3] P. Alfeld, M. Neamtu, L.L. Schumaker, Fitting scattered data on sphere-like surfaces using spherical splines, J. Comput. Appl. Math. 73 (1996) 5–43.
[4] P. Alfeld, M. Neamtu, L.L. Schumaker, Dimension and local bases of homogeneous spline spaces, SIAM J. Math. Anal. 27 (1996) 1482–1501.
[5] V. Baramidze, M.J. Lai, Error bounds for minimal energy interpolatory spherical splines, in: C.K. Chui, M. Neamtu, L.L. Schumaker (Eds.), Approximation Theory XI, Nashboro Press, Brentwood, 2005, pp. 25–50.
[6] V. Baramidze, M.J. Lai, C.K. Shum, Spherical splines for data interpolation and fitting, SIAM J. Sci. Comput. 28 (2006) 241–259.
[7] G. Fasshauer, L.L. Schumaker, in: M. Daehlen, T. Lyche, L.L. Schumaker (Eds.), Scattered Data Fitting on the Spehere, in: Mathematical Methods for Curves and Surfaces II, Vanderbilt Univ. Press, 1998, pp. 117–166.
[8] M.J. Lai, L.L. Schumaker, Spline Functions over Triangulations, Cambridge University Press, Cambridge, 2007.
[9] M.J. Lai, C.K. Shum, V. Baramidze, P. Wenston, Triangulated spherical splines for geopotential reconstruction, J. Geod. 83 (8) (2008) 695–708.
[10] M. Neamtu, L.L. Schumaker, On the approximation order of splines on spherical triangulations, Adv. Comput. Math. 21 (2004) 3–20.
[11] M. von Golitschek, M.J. Lai, L.L. Schumaker, Error bounds for minimal energy interpolatory splines, Numer. Math. 93 (2002) 315–331.
[12] M. von Golitschek, L.L. Schumaker, Bounds on projections onto bivariate polynomial spline spaces with stable local bases, Constr. Approx. 18 (2002) 241–254.
[13] M. von Golitschek, L.L. Schumaker, Penalized least squares fitting, in: Algorithms for Approximation II, Chapman and Hall, London, 1990.
[14] M. von Golitschek, L.L. Schumaker, Penalized least squares fitting, Serdica 18 (2002) 1001–1020.
[15] H. Wendland, C. Rieger, Approximate interpolation with applications to selecting smoothing parameters, Numer. Math. 101/4 (2005) 729–748.

Please cite this article in press as: V. Baramidze, M.-J. Lai, Convergence of discrete and penalized least squares spherical splines, Journal of Approximation Theory (2011), doi:10.1016/j.jat.2011.03.001