

CONVERGENCE ANALYSIS OF A FINITE DIFFERENCE SCHEME FOR THE GRADIENT FLOW ASSOCIATED WITH THE ROF MODEL

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Abstract. We present a convergence analysis of a finite difference scheme for the time dependent partial differential equation called gradient flow associated with the Rudin-Osher-Fatemi model. We devise an iterative algorithm to compute the solution of the finite difference scheme and prove the convergence of the iterative algorithm. Finally computational experiments are shown to demonstrate the convergence of the finite difference scheme. An application for image denoising is given.

1. Introduction. The well-known ROF model may be approximated in the following way

$$\min_{u \in \text{BV}(\Omega)} \int_{\Omega} \sqrt{\epsilon + |\nabla u|^2} dx + \frac{1}{2\lambda} \int_{\Omega} |u - f|^2 dx. \quad (1.1)$$

As $\epsilon > 0$, the above minimizing functional is differentiable. Thus, the Euler-Lagrange equation associated with the above minimization is

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}} \right) - \frac{1}{\lambda} (u - f) = 0. \quad (1.2)$$

Solution of this partial differential equation can be further approximated. Let us consider the time evolution version of the PDE:

$$\begin{cases} \frac{d}{dt} u = \operatorname{div} \left(\frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}} \right) - \frac{1}{\lambda} (u - f) & \in \Omega_T \\ \frac{\partial}{\partial \mathbf{n}} u = 0 & \text{on } \partial\Omega_T \\ u(\cdot, 0) = u_0(\cdot), & \Omega, \end{cases} \quad (1.3)$$

where f is given a noised image, $\Omega_T = [0, T] \times \Omega$, $\frac{\partial}{\partial \mathbf{n}}$ is the outward normal derivative operator. It is called the gradient flow of (1.1). When $\epsilon = 0$, it is called TV flow. Similar partial differential equations also appear in geometry analysis. See references, e.g., [14], [11], [2], [3], [4], and the references therein. The existence, uniqueness, stability of the weak solutions to these time dependent PDE were studied in the literature mentioned above. Numerical solution of the PDE (1.3) using finite elements has been discussed in [9] and [8]. In particular, the researchers showed that the finite element solution exists, is unique, is convergent to the weak solution of the PDE (1.3), the rate of convergence under some sufficient conditions is obtained, and the computation is stable. A fixed point iterative algorithm for the associated system of nonlinear equations was discussed in [17] and its convergence was studied in [6]. Although the finite difference solution of the time dependent PDE (1.3) has been the method of choice for image denoising (e.g. See [16]), no convergence of the finite difference solution to the weak solution of the PDE has been established in the literature so far to the best of the authors' knowledge. See also [7].

The purpose of this paper is to provide a proof of the convergence of the discrete solution obtained from a finite difference scheme for (1.3) to the weak solution. See our Theorem 3.8 in Section 3. Note that the finite difference scheme in (1.5) is slightly different from the traditional ones: forward or backward or central difference scheme. We use the average of forward and backward differences. The advantage of our

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scheme is that the value of the nonlinear term in (1.1) for certain piecewise linear functions is equal to the value of its discretization of the nonlinear term. As the PDE is associated with a convex functional, we use the techniques from convex analysis to help establishing the convergence. In addition, we study how to numerically solve the time dependent PDE (1.3) by using our finite difference scheme. As the finite difference scheme is a system of nonlinear equations, we shall derive an iterative algorithm and show that the iterative solutions are convergent. Again we use our techniques on convex analysis to establish the convergence of the iterative algorithm.

Let us now introduce our finite difference scheme for (1.3). We need some notations. For convenience, let $\Omega = [0, 1] \times [0, 1]$. We let $N > 0$ be a positive integer and divide Ω by equally-spaced points $x_i = ih$ and $y_j = jh$ for $0 \leq i, j \leq N - 1$ where $h = 1/N$. For any $f(x, y)$ defined on Ω , let $f_{i,j}^h = f(x_i, y_j)$ if f is a continuous function on Ω . Otherwise, f^h will be defined as in (2.4). We shall use two different divided differences ∇^+ and ∇^- to approximate the gradient operator. That is,

$$\nabla^+ f_{i,j}^h = \left(\frac{f_{i+1,j}^h - f_{i,j}^h}{h}, \frac{f_{i,j+1}^h - f_{i,j}^h}{h} \right)$$

and

$$\nabla^- f_{i,j}^h = \left(\frac{f_{i,j}^h - f_{i-1,j}^h}{h}, \frac{f_{i,j}^h - f_{i,j-1}^h}{h} \right)$$

for all $0 \leq i, j \leq N - 1$ with $f_{-1,j}^h = f_{0,j}^h, f_{N,j}^h = f_{N-1,j}^h$ for all j and $f_{i,-1}^h = f_{i,0}^h, f_{i,N}^h = f_{i,N-1}^h$ for all i . Furthermore, we define discrete divergence operators div^+ and div^- to approximate the continuous divergence operator, i.e.,

$$\begin{aligned} \text{div}^+(f_{i,j}^h, g_{i,j}^h) &= \begin{cases} f_{0,j}^h/h & i = 0, 0 \leq j \leq N - 1 \\ (f_{i,j}^h - f_{i-1,j}^h)/h & 0 < i < N - 1, 0 \leq j \leq N - 1 \\ -f_{i-2,j}^h/h & i = N - 1, 0 \leq j \leq N - 1 \end{cases} \\ &+ \begin{cases} g_{i,0}^h/h & j = 0, 0 \leq i \leq N - 1 \\ (g_{i,j}^h - g_{i,j-1}^h)/h & 0 < j < N - 1, 0 \leq i \leq N - 1 \\ -g_{i,j-2}^h/h & j = N - 1, 0 \leq i \leq N - 1 \end{cases} \end{aligned}$$

for all $0 \leq i, j \leq N - 1$ and similarly for div^- . By their definitions, we have for every $p \in \mathbb{R}^{N \times N} \times \mathbb{R}^{N \times N}$ and $u \in \mathbb{R}^{N \times N}$

$$\langle -\text{div}^+ p, u \rangle = \langle p, \nabla^+ u \rangle, \quad \langle -\text{div}^- p, u \rangle = \langle p, \nabla^- u \rangle.$$

With these notations, we are able to define a finite difference scheme for numerical solution of the time dependent PDE (1.3).

$$\begin{cases} \frac{d}{dt} u_{i,j} = \frac{1}{2} \text{div}^+ \left(\frac{\nabla^+ u_{i,j}}{\sqrt{\epsilon + |\nabla^+ u_{i,j}|^2}} \right) \\ \quad + \frac{1}{2} \text{div}^- \left(\frac{\nabla^- u_{i,j}}{\sqrt{\epsilon + |\nabla^- u_{i,j}|^2}} \right) - \frac{1}{\lambda} (u_{i,j} - f_{i,j}^h) & 0 \leq i, j \leq N - 1, t \in [0, T] \\ \frac{\partial}{\partial \mathbf{n}} u_{i,j} = 0 & i = 0, N, 0 \leq j \leq N - 1; \\ & j = 0, N, 0 \leq i \leq N - 1, \\ u(x_i, y_j, 0) = u_0^h(x_i, y_j), & 0 \leq i, j \leq N - 1, \end{cases} \quad (1.4)$$

where u_0^h is a discretization of the initial value u_0 according to (2.4). Next we discretize the time domain $[0, T]$ by equally-spaced points $t_k = k\Delta t$, $\Delta t = T/M$. We approximate the $\frac{d}{dt}u_{i,j}$ by $(u_{i,j}^k - u_{i,j}^{k-1})/\Delta t$ to have the fully discrete version of finite difference scheme:

$$\begin{cases} \frac{1}{\Delta t}(u_{i,j}^k - u_{i,j}^{k-1}) = \frac{1}{2} \operatorname{div}^+ \left(\frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) \\ \quad + \frac{1}{2} \operatorname{div}^- \left(\frac{\nabla^- u_{i,j}^k}{\sqrt{\epsilon + |\nabla^- u_{i,j}^k|^2}} \right) - \frac{1}{\lambda}(u_{i,j}^k - f_{i,j}^h) & 0 \leq i, j \leq N-1, 1 \leq k \leq M \\ \frac{\partial}{\partial \mathbf{n}} u_{i,j}^k = 0 & i = 0, N, 0 \leq j \leq N-1; \\ & j = 0, N, 0 \leq i \leq N-1, 0 \leq k \leq M \\ u(x_i, y_j, 0) = u_0^h(x_i, y_j), & 0 \leq i, j \leq N-1. \end{cases} \quad (1.5)$$

We shall first show that the above scheme (1.5) has a uniqueness solution in §2 and we will establish some properties of the solution. Then we show the solution in (1.5) converges to the weak solution of time dependent PDE (1.3) in the sense that the piecewise linear interpolation of the solution vector of (1.5) converges weakly to a function U^* which is the weak solution of the PDE (1.3). These will be done in §3. Next we shall explain how to numerically solve this system of nonlinear equations in §4. We finally report our computational results in §5.

2. Preliminary Results. We first introduce a weak formulation of PDE (1.3) that is suggested by [9].

DEFINITION 2.1. *We say that $u \in L^1([0, T], \operatorname{BV}(\Omega))$ is a weak solution of (1.3) if u satisfies the initial value and boundary conditions in (1.3) and for any $w \in L^1([0, T], W^{1,1}(\Omega))$ with $\frac{\partial}{\partial \mathbf{n}} w(x, t) = 0$ for all $(t, x) \in [0, T] \times \partial\Omega$,*

$$\int_0^s \int_{\Omega} \frac{d}{dt} u w dx dt + \int_0^s \int_{\Omega} \frac{\nabla u \cdot \nabla w}{\sqrt{\epsilon + |\nabla u|^2}} + \frac{1}{\lambda} \int_0^s \int_{\Omega} (u - f) w dx dt = 0, \quad (2.1)$$

for any $s \in (0, T]$.

It is known (cf. [9]) there exists a unique weak solution U^* satisfying the above weak formulation. U^* is in fact in $L^\infty((0, T], \operatorname{BV}(\Omega))$ if $u^0 \in \operatorname{BV}(\Omega)$ and $f \in L^2(\Omega)$. Following the ideas in [14], the researchers in [9] further showed the weak solution can be characterized by the following inequality.

THEOREM 2.2. *Let u be a weak solution as in Definition 2.1. Then u satisfies the following inequality: for any $s \in (0, T]$,*

$$\begin{aligned} & \int_0^s \int_{\Omega} \frac{d}{dt} v(v - u) dx dt + \int_0^s (J(v) - J(u)) dt \\ & \geq \frac{1}{2} \left[\int_{\Omega} (v(x, s) - u(x, s))^2 dx - \int_{\Omega} (v(x, 0) - u_0(x, 0))^2 dx \right] \end{aligned} \quad (2.2)$$

for all $v \in L^1([0, T], W^{1,1}(\Omega))$ with $\frac{\partial}{\partial \mathbf{n}} v(x, t) = 0$ for all $(t, x) \in [0, T] \times \partial\Omega$, where

$$J(u) = \int_{\Omega} \sqrt{\epsilon + |\nabla u(x, t)|^2} dx + \frac{1}{2\lambda} \int_{\Omega} |f(x, t) - u(x, t)|^2 dx. \quad (2.3)$$

On the other hand, if a function $u \in L^1((0, T], \operatorname{BV}(\Omega))$ satisfies the above inequality (2.2), then u is a weak solution.

Theorem 2.2 is our major tool to establish the convergence of the finite difference solution to the weak solution of the PDE (1.3). We shall use it in the proof of our main result in Theorem 3.8. Next we

introduce some basic notations and prove some basic properties of the solution vector of finite difference scheme (1.5) in the remaining part of this section.

We partition the region $\Omega = [0, 1] \times [0, 1]$ evenly into N by N grids with a grid size of $h = 1/N$, and assume that the pixel value on each grid at index (i, j) is $f_{i,j}^h$,

$$f_{i,j}^h = \frac{1}{h^2} \int_{ih}^{(i+1)h} \int_{jh}^{(j+1)h} f(x) dx, \quad 0 \leq i, j \leq N-1 \quad (2.4)$$

Then the initial data f^h for our numerical scheme is a discretization of the initial data f for PDE (1.3).

$$f^h := \sum_{i,j} f_{i,j}^h \chi_{i,j}(x), \quad (2.5)$$

where $\chi_{i,j}(x)$ is the characteristic function of square $\Omega_{i,j} := [ih, (i+1)h] \times [jh, (j+1)h]$. When there is no ambiguity, we also treat array $\{u^k\}$ as a discrete function (piecewise constant on grids) with $u^k(x) = u_{i,j}^k$ for $x \in \Omega_{i,j}$. In later sections, we will always use superscript (e.g. $u^h(\cdot, t)$ or u^k) to indicate that the function is a discrete function. We also introduce a projecting operator P_h mapping from L^1 to the space of discrete functions

$$P_h f := f^h$$

We define the discrete L^2 norms of f^h in analogue of standard L^2 norms.

$$\|f^h\| := \left\{ \sum_{i,j} (f_{i,j}^h)^2 h^2 \right\}^{1/2}.$$

Furthermore, we define a discretized version of the nonlinear functional (2.3)

$$J^h(v) = \frac{1}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^+ v_{i,j}|^2} h^2 + \frac{1}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^- v_{i,j}|^2} h^2 + \frac{1}{2\lambda} \sum_{i,j} (v_{i,j} - f_{i,j}^h)^2 h^2, \quad (2.6)$$

and the discrete energy functional

$$E^h(v) = J^h(v) + \frac{1}{2\Delta t} \sum_{i,j} (v_{i,j} - u_{i,j}^{k-1})^2 h^2 \quad (2.7)$$

for all arrays $v_{i,j}$, $0 \leq i, j \leq N-1$.

We are now ready to show the following existence and uniqueness results.

THEOREM 2.3. *Fix $N > 0$ and $M > 0$. There exists a unique array $u_{i,j}^k$, $0 \leq i, j \leq N-1$, $0 \leq k \leq M$ satisfying the above system (1.5) of nonlinear equations.*

Proof. Consider the following minimization problem:

$$\min_v E^h(v). \quad (2.8)$$

The Euler-Lagrange equation for its minimizer u^k is

$$\partial E^h(u^k) = \partial J^h(u^k) + \frac{u^k - u^{k-1}}{\Delta t} h^2 = 0.$$

It is straightforward to verify that the subgradient of J^h at u^k is an array with

$$\begin{aligned} & \frac{1}{h^2} \partial J^h(u^k)_{i,j} \\ &= -\frac{1}{2} \operatorname{div}^+ \left(\frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) - \frac{1}{2} \operatorname{div}^- \left(\frac{\nabla^- u_{i,j}^k}{\sqrt{\epsilon + |\nabla^- u_{i,j}^k|^2}} \right) + \frac{1}{\lambda} (u_{i,j}^k - f_{i,j}^h) \end{aligned} \quad (2.9)$$

Then we have

$$\begin{aligned} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} - \frac{1}{2} \operatorname{div}^+ \left(\frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) - \frac{1}{2} \operatorname{div}^- \left(\frac{\nabla^- u_{i,j}^k}{\sqrt{\epsilon + |\nabla^- u_{i,j}^k|^2}} \right) \\ + \frac{1}{\lambda} (u_{i,j}^k - f_{i,j}^h) = 0, \quad 0 \leq i, j \leq N-1, 1 \leq k \leq M \end{aligned} \quad (2.10)$$

which is the equation in (1.5). The existence and uniqueness of $u_{i,j}^k$ follows from the strict convexity of the functional E^h . \square

The following property is a characterization of the discrete solution of (1.5).

LEMMA 2.4. *Suppose that array $\{u_{i,j}^k, 0 \leq i, j \leq N-1, 0 \leq k \leq M\}$ is a solution of the finite difference scheme (1.5). Then $u_{i,j}^k$ satisfies the following inequality*

$$\begin{aligned} \sum_{i,j} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} (v_{i,j} - u_{i,j}^k) + \frac{1}{2} \left(\sum_{i,j} \sqrt{\epsilon + |\nabla^+ v_{i,j}|^2} - \sum_{i,j} \sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2} \right) + \\ \frac{1}{2} \left(\sum_{i,j} \sqrt{\epsilon + |\nabla^- v_{i,j}|^2} - \sum_{i,j} \sqrt{\epsilon + |\nabla^- u_{i,j}^k|^2} \right) + \frac{1}{2\lambda} \sum_{i,j} (v_{i,j} - f_{i,j}^h)^2 - \frac{1}{2\lambda} \sum_{i,j} (u_{i,j}^k - f_{i,j}^h)^2 \\ \geq 0 \end{aligned} \quad (2.11)$$

for all arrays $v_{i,j}$ that satisfy the Neumann boundary condition. On the other hand, if an array $\{u_{i,j}^k, 0 \leq i, j \leq N-1, 0 \leq k \leq M\}$ satisfies the above inequality for all $v_{i,j}$ satisfying the discrete Neumann boundary condition in (1.5), then array $\{u_{i,j}^k, 0 \leq i, j \leq N-1\}$ is a solution of (1.5).

Proof. Since u^k is the minimizer of E^h , we have the Euler-Lagrange equation

$$0 = \partial E^h(u^k)$$

i.e.,

$$-\frac{u^k - u^{k-1}}{\Delta t} h^2 = \partial J^h(u^k).$$

By the definition of sub-gradient, for any array $v_{i,j}^h$

$$-\sum_{i,j} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} (v_{i,j}^h - u_{i,j}^k) h^2 \leq J^h(v^h) - J^h(u^k).$$

Rearranging terms in the above inequality and the result follows. \square

The variation of our scheme is also monotone in the following sense.

LEMMA 2.5. *Define discrete function $u^h(t)$ by*

$$u^h(t) := \frac{t - t_{k-1}}{\Delta t} u^k + \frac{t_k - t}{\Delta t} u^{k-1}, \quad t_{k-1} \leq t \leq t_k. \quad (2.12)$$

Then

$$J^h(u^k) \leq J^h(u^h(t)), \quad t_{k-1} \leq t \leq t_k. \quad (2.13)$$

Proof. Since u^k is the minimizer of the following functional

$$E^h(v) = J^h(v) + \frac{1}{2\Delta t} \|u^{k-1} - v\|^2$$

we have

$$J^h(u^k) + \frac{1}{2\Delta t} \|u^{k-1} - u^k\|^2 \leq J^h(u^h(t)) + \frac{1}{2\Delta t} \|u^{k-1} - u^h(t)\|^2. \quad (2.14)$$

For each term in the summation of the L^2 square term on the right-hand side,

$$\begin{aligned} |u^{k-1} - u^h(t)| &= \left| u^{k-1} - \frac{t - t_{k-1}}{\Delta t} u^k + \frac{t_k - t}{\Delta t} u^{k-1} \right| \\ &= \frac{t - t_{k-1}}{\Delta t} |u^k - u^{k-1}| \leq |u^k - u^{k-1}|. \end{aligned}$$

That is

$$\frac{1}{2\Delta t} \|u^{k-1} - u^h(t)\|^2 \leq \frac{1}{2\Delta t} \|u^{k-1} - u^k\|^2.$$

With the above inequality, we conclude the result from (2.14). \square

The following result shows that the computation of finite difference scheme (1.5) is stable.

THEOREM 2.6. *Let $\{u_f^k, 0 \leq k \leq M\}$ be the solution of the system of nonlinear equations (1.5) associated with f^h with initial value u_f^0 . Similarly, let $\{u_g^k, 0 \leq k \leq M\}$ be the corresponding solution of (1.5) associated with g^h with initial value u_g^0 . Then*

$$\|u_f^k - u_g^k\| \leq \max\{\|u_f^0 - u_g^0\|, \|f^h - g^h\|\}, \quad 1 \leq k \leq M. \quad (2.15)$$

Proof. We prove by induction. It is obvious true for $k = 0$. Assume the inequality holds for $k - 1$. Rearrange the L^2 terms in (2.8). We have u_f^k is the minimizer of the following problem.

$$\min_v \frac{h^2}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^+ v_{i,j}|^2} + \frac{h^2}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^- v_{i,j}|^2} + (\mu_1 + \mu_2) \left\| v - (k_1 f^h + k_2 u_f^{k-1}) \right\|^2 \quad (2.16)$$

where $\mu_1 = 1/(2\lambda)$, $\mu_2 = 1/2\Delta t$, and $k_1 = \mu_1/(\mu_1 + \mu_2)$, $k_2 = \mu_2/(\mu_1 + \mu_2)$. By standard stability property of the minimization problem like (2.16) (cf. [18] or Theorem 3.1 in [13])

$$\begin{aligned} \|u_f^k - u_g^k\| &\leq \left\| (k_1 f^h + k_2 u_f^{k-1}) - (k_1 g^h + k_2 u_g^{k-1}) \right\| \\ &\leq k_1 \|f^h - g^h\| + k_2 \left\| u_f^{k-1} - u_g^{k-1} \right\| \\ &\leq \max \left\{ \|f^h - g^h\|, \left\| u_f^{k-1} - u_g^{k-1} \right\| \right\} \\ &\leq \max \left\{ \|f^h - g^h\|, \|u_f^0 - u_g^0\| \right\}. \end{aligned}$$

This completes the proof. \square

REMARK 2.1. *As a direct deduction, if $g^h = u_g^0 = 0$, the solution u_g^k is also zero for all k , then*

$$\|u_f^k\| \leq \max\{\|u_f^0\|, \|f^h\|\}, \quad 1 \leq k \leq M. \quad (2.17)$$

The following lemma discusses the regularity of the discrete solution u^k . In image analysis, the input image usually does not have much regularity. For example, most natural images do not even have weak derivatives. Therefore, to model images, we introduce the notation of Lipschitz space, and treat images as functions in this space.

DEFINITION 2.7. Let $\alpha \in (0, 1]$ be a real number. A function $f \in \text{Lip}(\alpha, L^2(\Omega))$ if $f \in L^2(\Omega)$ and the following quantity

$$|f|_{\text{Lip}(\alpha, L^2(\Omega))} := \sup_{|h| \leq 1} \frac{\|f(\cdot) - f(\cdot + h)\|_{L^2(\Omega_h)}}{|h|^\alpha} \quad (2.18)$$

is finite, where $\Omega_h := \{x \in \Omega, x + th \in \Omega, \forall t \in [0, 1]\}$. We let $\|f\|_{\text{Lip}(\alpha, L^2(\Omega))} = \|f\|_{L^2(\Omega)} + |f|_{\text{Lip}(\alpha, L^2(\Omega))}$.

The parameter α is related to the ‘‘smoothness’’ of functions in the Lipschitz space. Smoother functions belong to Lipschitz spaces with larger α values. For example, a function of bounded variation is a function in $\text{Lip}(1, L^2(\Omega))$.

LEMMA 2.8. Define translation operators $T_{1,0}$ and $T_{0,1}$ by

$$\begin{aligned} (T_{1,0}u^k)_{i,j} &= u_{i+1,j}^k & 0 \leq i, j \leq N-1 \\ (T_{0,1}u^k)_{i,j} &= u_{i,j+1}^k & 0 \leq i, j \leq N-1 \end{aligned}$$

Then if u_0 and f in $\text{Lip}(\alpha, L^2(\Omega))$,

$$\|T_{1,0}u^k - u^k\| \leq (\|u^0\|_{\text{Lip}(\alpha, L^2)} + \|f\|_{\text{Lip}(\alpha, L^2)})h^\alpha$$

and similarly

$$\|T_{0,1}u^k - u^k\| \leq (\|u^0\|_{\text{Lip}(\alpha, L^2)} + \|f\|_{\text{Lip}(\alpha, L^2)})h^\alpha.$$

Proof. We only prove the first inequality. Recall the Euler-Lagrange equation that

$$\frac{u^{k-1} - u^k}{\Delta t} h^2 = \partial J^h(u^k).$$

We write the equation element-wisely as

$$\frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} = \frac{1}{2} \text{div}^+ \left(\frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) + \frac{1}{2} \text{div}^- \left(\frac{\nabla^- u_{i,j}^k}{\sqrt{\epsilon + |\nabla^- u_{i,j}^k|^2}} \right) - \frac{1}{\lambda} (u_{i,j}^k - f_{i,j}^h).$$

Then subtracting the equation at index $(i+1, j)$ from the same equation at index (i, j) for $0 \leq i \leq N-2$, we obtain

$$\begin{aligned} \frac{u_{i+1,j}^k - u_{i,j}^k}{\Delta t} - \frac{u_{i+1,j}^{k-1} - u_{i,j}^{k-1}}{\Delta t} &= F(\nabla^+ u_{i+1,j}^k, \nabla^+ u_{i,j}^k) + F(\nabla^- u_{i+1,j}^k, \nabla^- u_{i,j}^k) \\ &\quad - \frac{1}{\lambda} (u_{i+1,j}^k - u_{i,j}^k) + \frac{1}{\lambda} (f_{i+1,j}^h - f_{i,j}^h) \end{aligned} \quad (2.19)$$

where $F(\nabla^+ u_{i+1,j}^k, \nabla^+ u_{i,j}^k)$ is defined by

$$F(\nabla^+ u_{i+1,j}^k, \nabla^+ u_{i,j}^k) = \frac{1}{2} \text{div}^+ \left(\frac{\nabla^+ u_{i+1,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i+1,j}^k|^2}} \right) - \frac{1}{2} \text{div}^+ \left(\frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right).$$

Equation (2.19) only holds for $0 \leq i \leq N-2$, $0 \leq j \leq N-1$. Although equation (2.19) is not defined for $i = N-1$, we can set $u_{N+1,j}^k = u_{N,j}^k$ and $f_{N+1,j} = f_{N,j}$, and equation (2.19) still holds. We multiply (2.19) by $u_{i+1,j}^k - u_{i,j}^k$ and add all resulting equations for $0 \leq i, j \leq N-1$ to have

$$\begin{aligned}
& \frac{1}{\Delta t} \sum_{i,j=0}^{N-1} (u_{i+1,j}^k - u_{i,j}^k)^2 \\
&= \frac{1}{\Delta t} \sum_{i,j=0}^{N-1} (u_{i+1,j}^{k-1} - u_{i,j}^{k-1})(u_{i+1,j}^k - u_{i,j}^k) \\
&\quad + \sum_{i,j=0}^{N-1} F(\nabla^+ u_{i+1,j}^k, \nabla^+ u_{i,j}^k)(u_{i+1,j}^k - u_{i,j}^k) + \sum_{i,j=0}^{N-1} F(\nabla^- u_{i+1,j}^k, \nabla^- u_{i,j}^k)(u_{i+1,j}^k - u_{i,j}^k) \\
&\quad - \sum_{i,j=0}^{N-1} \frac{1}{\lambda} (u_{i+1,j}^k - u_{i,j}^k)^2 + \sum_{i,j=0}^{N-1} \frac{1}{\lambda} (f_{i+1,j}^h - f_{i,j}^h)(u_{i+1,j}^k - u_{i,j}^k).
\end{aligned}$$

We show next that the second term is no greater than zero. The third term can be proved to be non-positive similarly. By definition of F ,

$$\begin{aligned}
& \sum_{i,j=0}^{N-1} F(\nabla^+ u_{i+1,j}^k, \nabla^+ u_{i,j}^k)(u_{i+1,j}^k - u_{i,j}^k) \\
&= \sum_{i,j=0}^{N-1} \frac{1}{2} \operatorname{div}^+ \left(\frac{\nabla^+ u_{i+1,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i+1,j}^k|^2}} \right) (u_{i+1,j}^k - u_{i,j}^k) - \sum_{i,j=0}^{N-1} \frac{1}{2} \operatorname{div}^+ \left(\frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) (u_{i+1,j}^k - u_{i,j}^k).
\end{aligned}$$

We use the discrete divergence operators and gradient operators to get

$$\begin{aligned}
& \sum_{i,j=0}^{N-1} F(\nabla^+ u_{i+1,j}^k, \nabla^+ u_{i,j}^k)(u_{i+1,j}^k - u_{i,j}^k) \\
&= \frac{1}{2} \sum_{i,j=0}^{N-1} \left(\operatorname{div}^+ \left(\frac{\nabla^+ u_{i+1,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i+1,j}^k|^2}} \right) - \operatorname{div}^+ \left(\frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) \right) (u_{i+1,j}^k - u_{i,j}^k) \\
&= -\frac{1}{2} \sum_{i,j=0}^{N-1} \left(\left(\frac{\nabla^+ u_{i+1,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i+1,j}^k|^2}} \right) - \left(\frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) \right) \cdot (\nabla^+ u_{i+1,j}^k - \nabla^+ u_{i,j}^k) \\
&\quad - \sum_{j=0}^{N-1} \frac{|\nabla^+ u_{0,j}^k|^2}{\sqrt{\epsilon + |\nabla^+ u_{0,j}^k|^2}}
\end{aligned}$$

Each term in the first sum is non-negative due to the following inequality: for any $x, y \in \mathbf{R}^2$,

$$\left(\frac{x}{\sqrt{\epsilon + |x|^2}} - \frac{y}{\sqrt{\epsilon + |y|^2}} \right) \cdot (x - y) \geq 0.$$

By similar arguments, one has

$$\sum_{i,j=0}^{N-1} F(\nabla^- u_{i+1,j}^k, \nabla^- u_{i,j}^k)(u_{i+1,j}^k - u_{i,j}^k) \leq 0$$

It follows

$$\begin{aligned} \frac{1}{\Delta t} \sum_{i,j=0}^{N-1} (u_{i+1,j}^k - u_{i,j}^k)^2 &\leq \frac{1}{\Delta t} \sum_{i,j=0}^{N-1} (u_{i+1,j}^{k-1} - u_{i,j}^{k-1})(u_{i+1,j}^k - u_{i,j}^k) \\ &\quad - \sum_{i,j=0}^{N-1} \frac{1}{\lambda} (u_{i+1,j}^k - u_{i,j}^k)^2 + \sum_{i,j=0}^{N-1} \frac{1}{\lambda} (f_{i+1,j}^h - f_{i,j}^h)(u_{i+1,j}^k - u_{i,j}^k). \end{aligned}$$

We rewrite the sums in form of discrete integrals and discrete inner products, and apply the arithmetic-geometric inequality

$$\begin{aligned} \frac{1}{\Delta t} \|T_{1,0}u^k - u^k\|^2 &\leq \frac{1}{\Delta t} \langle T_{1,0}u^{k-1} - u^{k-1}, T_{1,0}u^k - u^k \rangle \\ &\quad - \frac{1}{\lambda} \|T_{1,0}u^k - u^k\|^2 + \frac{1}{\lambda} \langle T_{1,0}f - f, T_{1,0}u^k - u^k \rangle \\ &\leq \frac{1}{2\Delta t} \|T_{1,0}u^{k-1} - u^{k-1}\|^2 + \frac{1}{2\Delta t} \|T_{1,0}u^k - u^k\|^2 \\ &\quad - \frac{1}{2\lambda} \|T_{1,0}u^k - u^k\|^2 + \frac{1}{2\lambda} \|T_{1,0}f - f\|^2. \end{aligned}$$

Rearrange and combine similar terms to have

$$\left(\frac{1}{\Delta t} + \frac{1}{\lambda}\right) \|T_{1,0}u^k - u^k\|^2 \leq \frac{1}{\Delta t} \|T_{1,0}u^{k-1} - u^{k-1}\|^2 + \frac{1}{\lambda} \|T_{1,0}f - f\|^2. \quad (2.20)$$

We now prove the following inequality by induction

$$\|T_{1,0}u^k - u^k\|^2 \leq \max\{\|T_{1,0}u^0 - u^0\|^2, \|T_{1,0}f - f\|^2\}. \quad (2.21)$$

It is obvious true for $k = 0$. Assuming the inequality holds for $k - 1$, one can easily see that it also holds for k by (2.20). Therefore, one has

$$\|T_{1,0}u^k - u^k\| \leq \|T_{1,0}u^0 - u^0\| + \|T_{1,0}f - f\| \leq (\|u^0\|_{\text{Lip}(\alpha, L^2)} + \|f\|_{\text{Lip}(\alpha, L^2)})h^\alpha.$$

This completes the proof. \square

3. Main Result and Its Proof. In this section, we shall show that the piecewise linear interpolation of the solution vector of the finite difference scheme (1.5) converges weakly to the solution of the gradient flow (1.3). We assume that the array $\{u_{i,j}^k, 0 \leq i, j \leq N - 1, 0 \leq k \leq M\}$ is the solution vector of (1.5).

To connect the discrete solution $\{u_{i,j}^k\}$ of (1.5) and the ‘‘continuous’’ weak solution of (1.3), we first construct a function $U_{N,M}(\cdot, t)$ in $W^{1,1}(\Omega)$ for each $t \in [0, T]$ in the form of a linear interpolation of u^k .

Let Δ_N be the following type of triangulation of $\Omega = [0, 1] \times [0, 1]$ with vertices $((i + 1/2)h, (j + 1/2)h), 0 \leq i, j \leq N - 1, h = 1/N$. Suppose the base functions of the continuous linear finite element space $S_1^0(\Delta_N)$ are $\{\phi_{i,j}(x), (i, j) \in \mathbb{Z}^2\}$, where $\phi_{i,j}$ is a scaled and translated standard continuous linear box spline function $\phi(x)$ based on three directions $e_1 = (1, 0), e_2(0, 1)$ and $e_3 = (-1, 1)$, i.e. $\phi_{i,j}(x) := \phi(x/h - (i + 1/2, j + 1/2))$ for any $(i, j) \in \mathbb{Z}^2$.

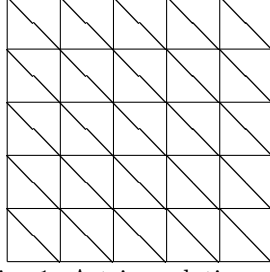


Fig. 1. A triangulation

For any k , we define piecewise linear interpolation $U_{N,M}(x, t_k)$ of u^k on Ω by

$$U_{N,M}(x, t_k) := \sum_{i,j=0}^{N-1} u_{i,j}^k \phi_{i,j}(x). \quad (3.1)$$

Having defined $U_{N,M}(\cdot, t_k)$ for $k = 0, \dots, M$ on Ω , we further define $U_{N,M}(\cdot, t)$ for $t_{k-1} \leq t \leq t_k$ by linear interpolating $U_{N,M}(\cdot, t_{k-1})$ and $U_{N,M}(\cdot, t_k)$ on interval $[t_{k-1}, t_k]$.

$$U_{N,M}(\cdot, t) = \frac{t - t_{k-1}}{\Delta t} U_{N,M}(\cdot, t_k) + \frac{t_k - t}{\Delta t} U_{N,M}(\cdot, t_{k-1}).$$

By the definition of $u^h(t)$ given in (2.12), we can also write $U_{N,M}(\cdot, t)$ as

$$U_{N,M}(\cdot, t) = \sum_{i,j=0}^{N-1} u^h(t) \phi_{i,j}$$

We next prove a sequence of lemmas to explain the properties of $U_{N,M}(\cdot, t)$.

LEMMA 3.1. *Suppose $u_0 \in W^{1,1}(\Omega)$, $f \in L^2(\Omega)$. For any $t \in [0, T]$, $\|\frac{d}{dt} U_{N,M}(\cdot, t)\|_{L^2(\Omega_T)} < C$ for a positive constant C only depending on u_0 and f .*

Proof. Let us write the Euler-Lagrange equation (2.10) in a concise format:

$$\frac{u^{k-1} - u^k}{\Delta t} h^2 = \partial J^h(u^k).$$

The equation above holds element-wise at each index (i, j) . For the equation at each index (i, j) , we multiply both sides by $u_{i,j}^{k-1} - u_{i,j}^k$ and then add the equations for all (i, j) . In terms of the standard inner product notation, we write the result in the following form:

$$\left\langle \frac{u^{k-1} - u^k}{\Delta t}, u^{k-1} - u^k \right\rangle = \langle \partial J^h(u^k), u^{k-1} - u^k \rangle$$

By the definition of sub-differential $\partial J^h(u^k)$

$$\left\langle \frac{u^{k-1} - u^k}{\Delta t}, u^{k-1} - u^k \right\rangle = \langle \partial J^h(u^k), u^{k-1} - u^k \rangle \leq J^h(u^{k-1}) - J^h(u^k).$$

We have

$$\frac{1}{\Delta t} \|u^{k-1} - u^k\|^2 \leq J^h(u^{k-1}) - J^h(u^k), \quad 1 \leq k \leq M.$$

Add the above inequalities for $k = 1, \dots, M$,

$$\sum_{k=1}^M \frac{1}{\Delta t} \|u^{k-1} - u^k\|^2 \leq J^h(u^0) - J^h(u^M). \quad (3.2)$$

Note that

$$\frac{dU_{N,M}(\cdot, t)}{dt} = \sum_{i,j} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} \phi_{i,j}, \quad t^{k-1} < t < t_k.$$

Then applying Cauchy-Schwarz inequality with $|\phi_{i,j}(x)| \leq 1$, we have

$$\begin{aligned} \left\| \frac{dU_{N,M}}{dt} \right\|_{L^2(\Omega_T)}^2 &= \sum_{k=1}^M \int_{\Omega} \left| \sum_{i,j} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} \phi_{i,j} \right|^2 dx \Delta t \\ &\leq 9 \sum_{k=1}^M \left\| \frac{u^k - u^{k-1}}{\Delta t} \right\|^2 \Delta t \leq 9(J^h(u^0) - J^h(u^M)). \end{aligned}$$

Here $u^0 = P_h u_0$. Note that $J^h(u^0)$ is bounded by a positive constant independent of h when $u_0 \in W^{1,1}(\Omega)$. This completes the proof. \square

LEMMA 3.2. *Suppose $u^0, f \in L^2(\Omega)$. Then $\|U_{N,M}\|_{L^2(\Omega_T)} \leq C$ for a constant C only dependent on f and u^0 . Furthermore, $\|U_{N,M}(\cdot, t)\|_{L^2(\Omega)} \leq C$ for a positive constant C for any $t \in [0, T]$.*

Proof. We use (2.17) to bound $\|U_{N,M}\|_{L^2(\Omega_T)}$ and $\|U_{N,M}(\cdot, t)\|_{L^2(\Omega)}$. Recall $u_f^0 = u^0$. It is easy to see for $t = t_k$,

$$\|U_{N,M}(\cdot, t_k)\|_{L^2(\Omega)}^2 \leq \|u_f^k\|^2 \leq \max\{\|u_f^0\|, \|f^h\|\}^2.$$

(cf. [18] or Lemma 2.4 in [13] for the first inequality and Remark 2.1 or (2.17) for the second inequality). Then we have

$$\begin{aligned} \|U_{N,M}\|_{L^2(\Omega_T)}^2 &= \int_0^T \|U_{N,M}(\cdot, t)\|_{L^2(\Omega)}^2 dt \\ &= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \left\| \frac{(t - t_{k-1})U_{N,M}(\cdot, t_k) + (t_k - t)U_{N,M}(\cdot, t_{k-1})}{\Delta t} \right\|_{L^2(\Omega)}^2 dt \\ &\leq \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \|U_{N,M}(\cdot, t_k)\|_{L^2(\Omega)}^2 + \|U_{N,M}(\cdot, t_{k-1})\|_{L^2(\Omega)}^2 dt \\ &\leq \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \|u^k\|^2 + \|u^{k-1}\|^2 dt \leq 2TC^2. \end{aligned}$$

As discussed above, for each $t \in [0, T]$, the integrand is $\|U_{N,M}(\cdot, t)\|_{L^2(\Omega)}^2$ which is less than or equal to $2C^2$ by (2.17). These complete the proof. \square

The above two lemmas ensure that there exists a convergent subsequence from $\{U_{N,M}, N, M \rightarrow \infty\}$ and a function $U^* \in L^2(0, T, L^2(\Omega))$ such that $U_{N,M}$ and $\frac{d}{dt}U_{N,M}$ weakly converge to U^* and $\frac{d}{dt}U^*$ in $L^2(\Omega_T)$.

Recall the definition of $u^h(t)$ in (2.12) with $u^k = (u_{i,j}^k, 0 \leq i, j \leq N-1)$. That is, $u^h(\cdot, t)$ is a piecewise linear function in t while piecewise constant function in x . However, $U_{N,M}$ is a piecewise linear function in $x \in \Omega$ and piecewise linear function in t . We now further show

LEMMA 3.3. *Suppose $f, u_0 \in \text{Lip}(\alpha, L^2(\Omega))$. Then*

$$\|U_{N,M}(\cdot, t) - u^h(\cdot, t)\|_{L^1([0, T]; L^2(\Omega_T))} \leq CT(\|u^0\|_{\text{Lip}(\alpha, L^2)} + \|f\|_{\text{Lip}(\alpha, L^2)})h^\alpha$$

for a positive constant C dependent only on f and u_0 .

Proof. Let $g(x, t) = U_{N,M}(x, t) - u^h(x, t)$. For any x , $g(x, t)$ is a linear function of t . A direct calculation shows

$$\int_{t_{k-1}}^{t_k} \|g(x, t)\|_{L^2(\Omega)} dt \leq \frac{1}{2} (\|g(x, t_k)\|_{L^2(\Omega)} + \|g(x, t_{k-1})\|_{L^2(\Omega)}) (t_k - t_{k-1}).$$

Adding these inequalities for $k = 1, \dots, M$, we have

$$\int_0^T \|g(x, t)\|_{L^2(\Omega)} dt \leq \Delta t \sum_{k=0}^M \|g(x, t_k)\|_{L^2(\Omega)}. \quad (3.3)$$

Then we only need to bound $\|g(x, t_k)\|$. We note that $g(x, t)$ is a piecewise linear function of x on each sub-grid $\Omega_{i,j} := [ih, (i+1)h] \times [jh, (j+1)h]$, $0 \leq i, j \leq N-1$ for any t . Tedious calculation gives

$$\begin{aligned} \|g(x, t_k)\|_{L^2(\Omega)}^2 &= \sum_{i,j} \int_{\Omega_{i,j}} |U_{N,M}(x, t_k) - u^h(x, t_k)|^2 \\ &\leq \sum_{i,j} Ch^2 \left(|u_{i+1,j}^k - u_{i,j}^k|^2 + |u_{i,j+1}^k - u_{i,j}^k|^2 + |u_{i-1,j}^k - u_{i,j}^k|^2 + |u_{i,j-1}^k - u_{i,j}^k|^2 \right) \\ &\leq C \left(\|T_{1,0}u^k - u^k\|^2 + \|T_{0,1}u^k - u^k\|^2 \right) \\ &\leq 2C(\|f\|_{\text{Lip}(\alpha, L^2)} + \|u_0\|_{\text{Lip}(\alpha, L^2)})^2 h^{2\alpha}. \end{aligned}$$

The last line follows from Lemma 2.8. We substitute the bound for the $\|g(x, t_k)\|_{L^2(\Omega)}$ in inequality (3.3) to complete the proof. \square

LEMMA 3.4. *For all functions v in $L^1([0, T], W^{1,1}(\Omega))$, there is a sequence of functions $\{v_N\}$ in $L^1([0, T], S_1^0(\Delta_N))$ so that*

$$\lim_{N \rightarrow \infty} \|v - v_N\|_{L^1([0, T]; L^2(\Omega))} = 0. \quad (3.4)$$

and

$$\lim_{N \rightarrow \infty} \|v - v_N\|_{L^1([0, T]; W^{1,1}(\Omega))} = 0 \quad (3.5)$$

Proof. For any $0 \leq t \leq T$, define the interpolant $\mathcal{I}^h v$ for $v(\cdot, t)$ in $C(\Omega)$ by

$$\mathcal{I}^h v(x, t) = \sum_{i,j} v((i+1/2)h, (j+1/2)h, t) \phi_{i,j}(x).$$

And for any $t \in [0, T]$, define

$$v_N(x, t) = \mathcal{I}^h v_\epsilon(x, t) \quad (3.6)$$

where v_ϵ is the smoothed v by a symmetric smooth cut-off function ψ_ϵ satisfying (i) $\text{supp} \psi_\epsilon \subset B(0, \epsilon)$ and (ii) $\int_{\mathbb{R}^2} \psi_\epsilon dx = 1$. More precisely,

$$v_\epsilon = \int_{\mathbb{R}^2} v(x-y) \psi_\epsilon(y) dy.$$

Since we need to use the value of v outside Ω in the above integration, we extend v to all of \mathbb{R}^2 by reflecting and translating; Define

$$v(x_1, x_2, t) = v(2 - x_1, x_2, t), \quad \text{for } 1 \leq x_1 \leq 2, \ 0 \leq x_2 \leq 1,$$

and

$$v(x_1, x_2, t) = v(x_1, 2 - x_2, t), \quad \text{for } 0 \leq x_1 \leq 2, \ 1 \leq x_2 \leq 2.$$

Having extended v on 2Ω , we then extend v periodically on all of \mathbb{R}^2 .

It is a classical result(cf. [19]) that for $0 \leq t \leq T$,

$$|v_\epsilon(\cdot, t)|_{W^{1,1}(\Omega)} \leq |v(\cdot, t)|_{W^{1,1}(\Omega)}, \quad (3.7)$$

and

$$\lim_{\epsilon \rightarrow 0} \|v_\epsilon(\cdot, t) - v(\cdot, t)\|_{W^{1,1}(\Omega)} = 0. \quad (3.8)$$

We also know \mathcal{I}^h is a bounded operator from $C^2(\bar{\Omega})$ to $W^{1,1}(\Omega)$, and(cf. [5] or [18])

$$|v_\epsilon(\cdot, t) - \mathcal{I}^h v_\epsilon(\cdot, t)|_{W^{1,1}(\Omega)} \leq Ch|v_\epsilon(\cdot, t)|_{W^{2,1}(\Omega)} \leq C\frac{h}{\epsilon}|v(\cdot, t)|_{W^{1,1}(\Omega)} \quad (3.9)$$

$$\|v_\epsilon(\cdot, t) - \mathcal{I}^h v_\epsilon(\cdot, t)\|_{L^1(\Omega)} \leq Ch|v_\epsilon(\cdot, t) - v(\cdot, t)|_{W^{1,1}(\Omega)} \leq 2Ch|v(\cdot, t)|_{W^{1,1}(\Omega)}. \quad (3.10)$$

Setting $\epsilon = h^{1-\alpha}$, we have

$$\|v_\epsilon(\cdot, t) - \mathcal{I}^h v_\epsilon(\cdot, t)\|_{W^{1,1}(\Omega)} \leq Ch^\alpha|v(\cdot, t)|_{W^{1,1}(\Omega)}, \quad (3.11)$$

and

$$\lim_{h \rightarrow 0} \|v_\epsilon(\cdot, t) - \mathcal{I}^h v_\epsilon(\cdot, t)\|_{W^{1,1}(\Omega)} = 0. \quad (3.12)$$

Finally inequality (3.5) follows from (3.8), (3.12) and Legesuge's Dominated Convergence Theorem. Inequality (3.4) follows from Sobolev embedding theorem(cf. [19], Remark 2.5.2)

$$\|v(\cdot, t) - \mathcal{I}^h v_\epsilon(\cdot, t)\|_{L^2(\Omega)} \leq C \|v(\cdot, t) - \mathcal{I}^h v_\epsilon(\cdot, t)\|_{W^{1,1}(\Omega)}. \quad (3.13)$$

and equation (3.5). \square

We now bound the difference between the two projecting operators: $\mathcal{I}^h v_\epsilon$ and $P_h v_\epsilon$

LEMMA 3.5. For any $v \in W^{1,1}(\Omega)$,

$$\|\mathcal{I}^h v_\epsilon - P_h v_\epsilon\| \leq Ch|v|_{W^{1,1}(\Omega)}. \quad (3.14)$$

Proof.

$$\|\mathcal{I}^h v_\epsilon - P_h v_\epsilon\| \leq \|\mathcal{I}^h v_\epsilon - v_\epsilon\| + \|v_\epsilon - P_h v_\epsilon\|.$$

Now the result follows from (3.10) and Poincaré-Wirtinger inequality(cf. [1])

$$\|v_\epsilon - P_h v_\epsilon\|_{L^2(\Omega)} \leq Ch|v_\epsilon|_{\text{BV}(\Omega)}.$$

\square

We have introduced two notations of total variation, one for functions in $BV(\Omega)$ and the other one for discrete functions. We need to show these two versions of total variation are consistent. We use the following lemma to bound the difference between the continuous variation $J(U_{N,M}(\cdot, t))$ and the discrete variations $J(v_N)$. We bound the difference between $J(v_N(\cdot, t))$ and $J(v_\epsilon^h)$ similarly.

LEMMA 3.6. *Let $\{v_N\}$ be the sequence of functions defined as in Lemma 3.4. Then for any $t \in [0, T]$*

$$|J(v_N(\cdot, t)) - J^h(v_\epsilon^h(t))| \leq Ch^\alpha, \quad (3.15)$$

where C depends on v and f . Moreover, for $U_{N,M}(\cdot, t)$ defined in (3.1) we have

$$|J(U_{N,M}(\cdot, t)) - J^h(u^h(t))| \leq Ch^\alpha, \quad (3.16)$$

where C depends on f .

Proof. Note that for any function $v_N(\cdot, t)$ in $S_1^0(\Delta_N)$, the variation term in $J(v_N(\cdot, t))$ is exactly equal to the variation term in $J^h(v_\epsilon^h(t))$. We only need to bound the difference between the second terms in $J(v_N)$ and $J^h(v_\epsilon^h)$.

Let $v_{\epsilon, i, j}^h(t)$ be the value of $v_\epsilon(\cdot, t)$ at point $((i + 1/2)h, (j + 1/2)h)$. Define discrete function $v_\epsilon^h(t)$ by

$$v_\epsilon^h(x, t) := \sum_{i, j} v_{\epsilon, i, j}^h(t) \chi_{i, j}(x), \quad (3.17)$$

and recall f^h is the piecewise constant projection of f , i.e. $f^h = P_h f$.

$$\begin{aligned} |J(v_N(\cdot, t)) - J^h(v_\epsilon^h(t))| &= \left| \frac{1}{2\lambda} \|v_\epsilon^h(\cdot, t) - f^h\|^2 - \frac{1}{2\lambda} \|\mathcal{I}^h v_\epsilon(\cdot, t) - f\|^2 \right| \\ &= \frac{1}{2\lambda} \left| (\|v_\epsilon^h(\cdot, t) - f^h\| - \|\mathcal{I}^h v_\epsilon(\cdot, t) - f\|) (\|v_\epsilon^h(\cdot, t) - f^h\| + \|\mathcal{I}^h v_\epsilon(\cdot, t) - f\|) \right| \\ &\leq \frac{1}{2\lambda} (\|v_\epsilon^h(\cdot, t) - \mathcal{I}^h v_\epsilon(\cdot, t)\| + \|f^h - f\|) C(\|v_\epsilon(\cdot, t)\| + \|f\|) \end{aligned}$$

By standard approximation theory(cf. [19]) and Sobolev inequality

$$\|v_\epsilon^h - \mathcal{I}^h v_\epsilon\| \leq Ch \|Dv_\epsilon\| \leq Ch(|v_\epsilon|_{W^{1,1}} + |v_\epsilon|_{W^{2,1}}) \leq C \frac{h}{\epsilon} |v|_{W^{1,1}},$$

and

$$\|f^h - f\| \leq C |f|_{\text{Lip}(\alpha, L^2)} h^\alpha.$$

Then we proved inequality (3.15) by setting $\epsilon = h^{1-\alpha}$. We can prove (3.16) along the same line of arguments(noting $\|u^h\| \leq 2\|f\|$ and applying Lemma 2.8. We omit the details. \square)

The following proposition is another one of the key ingredients to prove our main results in Theorem 3.8.

PROPOSITION 3.7. *For any test functions v in $L^1([0, T], W^{1,1}(\Omega))$, let $\{v_N\}$ be a sequence defined in Lemma 3.4. Then for $0 < s < T$*

$$\int_0^s \left[\int_\Omega \frac{d}{dt} U_{N,M}(v_N - U_{N,M}) dx + (J(v_N) - J(U_{N,M})) \right] dt \geq -Err_{N,M} \quad (3.18)$$

where $Err_{N,M}$ depends on v and tends to zero as $N, M \rightarrow \infty$ in the following fashion

$$\frac{h^\alpha}{\Delta t} = \frac{M}{TN^\alpha} \rightarrow 0. \quad (3.19)$$

Proof. The idea of the proof is to rewrite the left-hand side of (3.18) as the left-hand side of (2.11) plus some error and bound the error. As the preparation for a long calculation, we first remind the reader that for $t \in (t_{k-1}, t_k)$,

$$U_{N,M}(\cdot, t) = U_{N,M}(\cdot, t_{k-1})(t_k - t)/\Delta t + U_{N,M}(\cdot, t_k)(t - t_{k-1})/\Delta t$$

and

$$\frac{d}{dt}U_{N,M}(\cdot, t) = \frac{U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})}{\Delta t}. \quad (3.20)$$

and $v_N(\cdot, t) = \mathcal{I}^h v_\epsilon(\cdot, t)$ as defined in (3.6).

Without loss of generality, we consider the integration over $[0, T]$ instead of $[0, s]$. We rewrite the first term of the left-hand side of (3.18) as

$$\begin{aligned} & \int_0^T \int_\Omega \frac{d}{dt}U_{N,M}(v_N(\cdot, t) - U_{N,M}(\cdot, t)) dx dt \\ &= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_\Omega \frac{U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})}{\Delta t} (v_N(\cdot, t) - U_{N,M}(\cdot, t)) dx dt \\ &= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_\Omega \frac{U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})}{\Delta t} (v_N(\cdot, t) - U_{N,M}(\cdot, t_k)) dx dt + \text{Err}_1. \end{aligned} \quad (3.21)$$

where

$$\begin{aligned} \text{Err}_1 &= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_\Omega \frac{U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})}{\Delta t} (U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t)) dx dt \\ &= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_\Omega \frac{U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})}{\Delta t} (U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})) \frac{t_k - t}{\Delta t} dx dt. \end{aligned}$$

We bound Err_1 by

$$\begin{aligned} |\text{Err}_1| &\leq \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_\Omega \left| \frac{U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})}{\Delta t} (U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})) \frac{t_k - t}{\Delta t} \right| dx dt. \\ &\leq \sum_{k=1}^M \int_\Omega \left| \frac{U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})}{\Delta t} (U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})) dx \right| \Delta t \\ &= \Delta t \left\| \frac{dU_{N,M}}{dt} \right\|_{L^2(\Omega_T)}^2 \leq C \Delta t, \end{aligned}$$

where the last inequality comes from Lemma 3.1.

To apply the characteristic inequality (2.11), we need to replace all the piecewise linear functions in (3.21) by piecewise constant functions and bound the introduced error. Recall discrete functions $v_\epsilon^h(\cdot, t)$ and $u^h(\cdot, t)$ defined in (3.17) and (2.12) respectively. We replace $v_N(\cdot, t), U_{N,M}(\cdot, t)$ in (3.21) by $v_\epsilon^h(\cdot, t)$, and $u^h(\cdot, t)$ respectively and add an error term. To simplify the presentation, we introduce the following notations to denote the difference between a continuous function and a piecewise constant function;

$$\begin{aligned} \Delta v_N(\cdot, t) &:= v_N(\cdot, t) - v_\epsilon^h(\cdot, t), \\ \Delta U_{N,M}(\cdot, t) &:= U_{N,M}(\cdot, t) - u^h(\cdot, t). \end{aligned}$$

Then

$$\begin{aligned} & \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_{\Omega} \frac{U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})}{\Delta t} (v_N(\cdot, t) - U_{N,M}(\cdot, t_k)) \\ &= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_{\Omega} \frac{u^h(\cdot, t_k) - u^h(\cdot, t_{k-1})}{\Delta t} (v_{\epsilon}^h(\cdot, t) - u^h(\cdot, t_k)) dx + \text{Err}_2, \end{aligned}$$

where Err_2 can be written as

$$\begin{aligned} \text{Err}_2 &= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_{\Omega} \frac{\Delta U_{N,M}(\cdot, t_k) - \Delta U_{N,M}(\cdot, t_{k-1})}{\Delta t} (v_{\epsilon}^h(\cdot, t) - u^h(\cdot, t_k)) \\ &+ \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_{\Omega} \frac{u^h(\cdot, t_k) - u^h(\cdot, t_{k-1})}{\Delta t} (\Delta v_N(\cdot, t) - \Delta U_{N,M}(\cdot, t_k)) \\ &+ \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_{\Omega} \frac{\Delta U_{N,M}(\cdot, t_k) - \Delta U_{N,M}(\cdot, t_{k-1})}{\Delta t} (\Delta v_N(\cdot, t) - \Delta U_{N,M}(\cdot, t_k)). \end{aligned}$$

The three terms in Err_2 can be bounded in a similar fashion. We only give the details of the bounds for the first and second terms. The third term can be bounded similarly. We first point out the following facts, $\|v_{\epsilon}^h\|, \|u^h\| \leq C$ that can be easily proved with Lemma 2.6. Note that by Lemma 3.3

$$\|\Delta U_{N,M}\|_{L^1([0,T];L^2(\Omega))} \leq CT(\|u^0\|_{\text{Lip}(\alpha,L^2(\Omega))} + \|f\|_{\text{Lip}(\alpha,L^2(\Omega))})h^{\alpha}.$$

By using Cauchy-Schwarz inequality, the first term in Err_2 can be bounded by

$$\frac{2}{\Delta t} \|\Delta U_{N,M}\|_{L^2([0,T];L^2(\Omega))} (\|v_{\epsilon}^h\| + \|u^h\|) \leq CT(\|u^0\|_{\text{Lip}(\alpha,L^2(\Omega))} + \|f\|_{\text{Lip}(\alpha,L^2(\Omega))}) \frac{h^{\alpha}}{\Delta t}.$$

Next we look at the second term in Err_2 .

$$\begin{aligned} \|\Delta v_N(\cdot, t)\|_{L^2(\Omega)} &= \|\mathcal{I}^h v_{\epsilon}(\cdot, t) - \text{P}_h v_{\epsilon}(\cdot, t)\|_{L^2(\Omega)} \\ &\leq \|\mathcal{I}^h v_{\epsilon} - v_{\epsilon}\|_{L^2(\Omega)} + \|v_{\epsilon} - \text{P}_h v_{\epsilon}\|_{L^2(\Omega)} \\ &\leq C\|\mathcal{I}^h v_{\epsilon} - v_{\epsilon}\|_{W^{1,1}(\Omega)} + Ch|v_{\epsilon}|_{W^{2,1}(\Omega)} \\ &\leq C\|\mathcal{I}^h v_{\epsilon} - v_{\epsilon}\|_{W^{1,1}(\Omega)} + C\frac{h}{\epsilon}|v_{\epsilon}|_{W^{1,1}(\Omega)} \leq Ch^{\alpha}\|v(\cdot, t)\|_{W^{1,1}(\Omega)} \end{aligned}$$

by using (3.11)(and recall that $\epsilon = h^{1-\alpha}$).

Then the second term in Err_2 is bounded by

$$\begin{aligned} & \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_{\Omega} \frac{u^h(\cdot, t_k) - u^h(\cdot, t_{k-1})}{\Delta t} (\Delta v_N(\cdot, t) - \Delta U_{N,M}(\cdot, t_k)) dx dt \\ &\leq \sum_{k=1}^M \int_{t_{k-1}}^{t_k} C \left\| \frac{d}{dt} U_{N,M} \right\|_{L^2(\Omega)} \|\Delta v_N(\cdot, t) - \Delta U_{N,M}(\cdot, t_k)\|_{L^2(\Omega)} dt \\ &\leq C (\|\Delta v_N\|_{L^1([0,T];L^2(\Omega))} + \|\Delta U_{N,M}\|_{L^1([0,T];L^2(\Omega))}) \\ &\leq CT(\|u_0\|_{\text{Lip}(\alpha,L^2)} + \|f\|_{\text{Lip}(\alpha,L^2)} + \|v\|_{L^1([0,T];W^{1,1}(\Omega))})h^{\alpha}, \end{aligned}$$

where we have used Lemmas 3.1, 3.3 and 3.5. We also bound the other two terms with the order of h being 1 and $1 + \alpha$ respectively. Consuming all higher orders of h , the left side of (3.18) can be bounded

from below by

$$\sum_{k=1}^M \int_{t_{k-1}}^{t_k} \sum_{i,j} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} (v_{\epsilon,i,j}^h - u_{i,j}^k) h^2 - C(\|u_0\|_{\text{Lip}(\alpha,L^2)} + \|f\|_{\text{Lip}(\alpha,L^2)} + \|v\|_{L^1([0,T];W^{1,1}(\Omega))})Th^\alpha.$$

We sum up our bound on (3.21) as

$$\begin{aligned} & \int_0^T \int_{\Omega} \frac{d}{dt} U_{N,M}(v_N(\cdot, t) - U_{N,M}(\cdot, t)) dx dt \\ & \geq \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \sum_{i,j} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} (v_{\epsilon,i,j}^h - u_{i,j}^k) h^2 \\ & \quad - C(\|u_0\|_{\text{Lip}(\alpha,L^2)} + \|f\|_{\text{Lip}(\alpha,L^2)} + \|v\|_{L^1([0,T];W^{1,1}(\Omega))})T \frac{h^\alpha}{\Delta t} - C\Delta t. \end{aligned} \quad (3.22)$$

We next bound the second term of the left-hand side of (3.18)(the variation term),

$$\begin{aligned} \int_0^T J(v_N(\cdot, t)) - J(U_{N,M}(\cdot, t)) dt &= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} J(v_N(\cdot, t)) - J(U_{N,M}(\cdot, t)) dt \\ &= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} J^h(v_\epsilon^h(t)) - J^h(u^h(t_k)) dt + \text{Err}_3 \end{aligned}$$

with

$$\begin{aligned} \text{Err}_3 &= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} J(v_N(\cdot, t)) - J^h(v_\epsilon^h(t)) dt - \sum_{k=1}^M \int_{t_{k-1}}^{t_k} J^h(u^h(t)) - J^h(u^h(t_k)) dt - \\ & \quad \sum_{k=1}^M \int_{t_{k-1}}^{t_k} J(U_{N,M}(\cdot, t)) - J^h(u^h(t)) dt \end{aligned}$$

By Lemma 3.6, the first and the third term can be bounded by $C_1 h^\alpha T$ and $C_2 h^\alpha T$ respectively. To bound the second term we use the convexity of J^h and the monotonicity of J^h shown in Lemma 2.5,

$$\begin{aligned} & \left| \sum_{k=1}^M \int_{t_{k-1}}^{t_k} J^h(u^h(t)) - J^h(u^h(t_k)) dt \right| \\ & \leq \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \left| \frac{t - t_{k-1}}{\Delta t} J^h(u^h(t_k)) + \frac{t_k - t}{\Delta t} (J^h(u^h(t_{k-1})) - J^h(u^h(t_k))) \right| dt \\ & = \sum_{k=1}^M |J^h(u^h(t_{k-1})) - J^h(u^h(t_k))| \int_{t_{k-1}}^{t_k} \frac{t_k - t}{\Delta t} dt \leq \sum_{k=1}^M 2Ch^\alpha \Delta t = CTh^\alpha, \end{aligned}$$

where we have used Lemma 3.6.

Collecting the results together, we have

$$\int_0^T J(v_N(\cdot, t)) - J(U_{N,M}(\cdot, t)) dt \geq \sum_{k=1}^M \int_{t_{k-1}}^{t_k} J^h(v_\epsilon^h(\cdot, t)) - J^h(u^h(t_k)) dt - Ch^\alpha T. \quad (3.23)$$

Put all the bounds (3.22) and (3.23) together, we have

$$\begin{aligned} & \int_0^T \int_{\Omega} \frac{d}{dt} U_{N,M}(v_N(\cdot, t) - U_{N,M}(\cdot, t)) dxdt + \int_0^T J(v_N(\cdot, t)) - J(U_{N,M}(\cdot, t)) dt \\ & \geq \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \left\{ \sum_{i,j} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} (v_{i,j} - u_{i,j}^k) h^2 + J^h(v_{\epsilon}^h(\cdot, t)) - J^h(u^h(t_k)) dt \right\} \\ & \quad - C(\|u_0\|_{\text{Lip}(\alpha, L^2)} + \|f\|_{\text{Lip}(\alpha, L^2)} + \|v\|_{L^1([0, T]; W^{1,1}(\Omega))}) T \frac{h^\alpha}{\Delta t} - C\Delta t - Ch^\alpha T. \end{aligned}$$

Using Lemma 2.4 for the first term on the right-hand side of the inequality above, we let $h, \Delta t$ tend to zero in the fashion (3.19) to obtain the desired result. \square

Finally we are ready to prove the main result of this section.

THEOREM 3.8. *Suppose that $u_0 \in W^{1,1}(\Omega), f \in \text{Lip}(\alpha, L^2(\Omega))$. There exists a function U^* in $L^2(\Omega_T)$ so that $U_{N,M}$ converge to U^* weakly as $N, M \rightarrow \infty$ in the fashion (3.19) and U^* is the weak solution of (1.3).*

Proof. By Lemma 3.2, there exists a weakly convergent subsequence of $\{U_{N,M}, N \geq 1, M \geq 1\}$ in $L^2(\Omega_T)$. For convenience, we assume the whole sequence converges to $U^* \in L^2(\Omega_T)$ weakly. We now show U^* is the weak solution of the gradient flow as in Definition 2.1. As the weak solution is unique, the whole sequence $\{U_{N,M}, N \geq 1, M \geq 1\}$ converges weakly to U^* .

By using Theorem 2.2, we need to show that U^* satisfies the following inequality:

$$\begin{aligned} & \int_0^s \int_{\Omega} \frac{d}{dt} v(v - U^*) dxdt + \int_0^s (J(v) - J(U^*)) dt \\ & \geq \frac{1}{2} \left[\int_{\Omega} (v(x, s) - U^*(x, s))^2 dx - \int_{\Omega} (v(x, 0) - u_0(x, 0))^2 dx \right] \end{aligned} \quad (3.24)$$

for all $v \in L^1([0, T], W^{1,1}(\Omega))$ with $\frac{\partial}{\partial \mathbf{n}} v(x, t) = 0$ for all $(t, x) \in [0, T] \times \partial\Omega$, where

$$J(u) = \int_{\Omega} \sqrt{\epsilon + |\nabla u(x, t)|^2} dx + \frac{1}{2\lambda} \int_{\Omega} |f(x, t) - u(x, t)|^2 dx.$$

By the lower semi-continuity of J , Fatou's lemma and standard weak convergence, we have

$$\begin{aligned} & \int_0^s \int_{\Omega} \frac{d}{dt} v(v - U^*) dxdt + \int_0^s (J(v) - J(U^*)) dt \\ & \geq \liminf_{N, M \rightarrow \infty} \left[\int_0^s \int_{\Omega} \frac{d}{dt} v(v - U_{N,M}) dxdt + \int_0^s (J(v) - J(U_{N,M})) dt \right]. \end{aligned} \quad (3.25)$$

By the weak lower semi-continuity of the L^2 norm

$$\begin{aligned} & \liminf_{N, M \rightarrow \infty} \frac{1}{2} \left[\int_{\Omega} (v(x, s) - U_{N,M}(x, s))^2 dx - \int_{\Omega} (v(x, 0) - u_0(x, 0))^2 dx \right] \\ & \geq \frac{1}{2} \left[\int_{\Omega} (v(x, s) - U^*(x, s))^2 dx - \int_{\Omega} (v(x, 0) - u_0(x, 0))^2 dx \right]. \end{aligned} \quad (3.26)$$

We now prove the following inequality to finish the proof.

$$\begin{aligned} & \int_0^s \int_{\Omega} \frac{d}{dt} v(v - U_{N,M}) dxdt + \int_0^s (J(v) - J(U_{N,M})) dt \\ & \geq \frac{1}{2} \left[\int_{\Omega} (v(x, s) - U_{N,M}(x, s))^2 dx - \int_{\Omega} (v(x, 0) - u_0(x, 0))^2 dx \right] - \text{Error}_{N,M} \end{aligned}$$

where $\text{Error}_{N,M} > 0$ is an error term that goes to zero as $N, M \rightarrow \infty$. It's straightforward to verify(cf. [9]) that the above inequality is equivalent to

$$\int_0^s \int_{\Omega} \frac{d}{dt} U_{N,M}(v - U_{N,M}) dx dt + \int_0^s (J(v) - J(U_{N,M})) dt \geq -\text{Error}_{N,M}. \quad (3.27)$$

By Proposition 3.7, there exists a sequence $\{v_N\}$, so that

$$\lim_{N \rightarrow \infty} v_N = v \quad \text{in } L^1([0, T]; W^{1,1}(\Omega)),$$

and

$$\int_0^s \left[\int_{\Omega} \frac{d}{dt} U_{N,M}(v_N - U_{N,M}) dx + (J(v_N) - J(U_{N,M})) \right] dt \geq -\text{Err}_{N,M}$$

where $\text{Err}_{N,M}$ only depends on f and v , and tends to zero as N, M tend to infinity. We replace the original $W^{1,1}$ test function $v(\cdot, t)$ in (3.27) by v_N that is in $L^1([0, T], S_1^0(\Delta_N))$, therefore introduces an error $e_{N,M}$.

$$e_{N,M} = \int_0^s \int_{\Omega} \frac{d}{dt} U_{N,M}(v - v_N) + J(v) - J(v_N).$$

It is easy to show $e_{N,M}$ tends to zero as N, M go to infinity by Lemmas 3.1 and 3.4. Thus we complete the proof. \square

4. Numerical Solution of Our Finite Difference Scheme. The system (1.5) of nonlinear equations has been solved by many methods as explained in [17]. In [6], the researchers provided an analysis of a fixed point method proposed in [17] based on auxiliary variable and functionals and proved that the iterative method converges. In this section, we mainly present another method to show the convergence of the fixed point method. From notation simplicity, we assume the grid size $h = 1$ in this section that has no influence in the convergence analysis of our algorithm.

First of all, let us explain the fixed point method. Recall that we need to solve $\{u_{i,j}^k, 0 \leq i, j \leq N-1\}$ from the following equations

$$\begin{aligned} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} - \frac{1}{2} \text{div}^+ \left(\frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) - \frac{1}{2} \text{div}^- \left(\frac{\nabla^- u_{i,j}^k}{\sqrt{\epsilon + |\nabla^- u_{i,j}^k|^2}} \right) \\ + \frac{1}{\lambda} (u_{i,j}^k - f_{i,j}^h) = 0, \quad 0 \leq i, j \leq N-1, \end{aligned}$$

assuming that we have the solution $\{u_{i,j}^{k-1}, 0 \leq i, j \leq N-1\}$. Let us define an iterative algorithm to compute $u_{i,j}^k$.

ALGORITHM 4.1. *Starting with $v_{i,j}^0 = u_{i,j}^{k-1}, 0 \leq i, j \leq N-1$, for $\ell = 1, 2, \dots$, we compute array $\{v_{i,j}^{\ell}, 0 \leq i, j \leq N-1\}$ by*

$$\begin{aligned} \frac{v_{i,j}^{\ell} - u_{i,j}^{k-1}}{\Delta t} = \frac{1}{2} \text{div}^+ \left(\frac{\nabla^+ v_{i,j}^{\ell}}{\sqrt{\epsilon + |\nabla^+ v_{i,j}^{\ell-1}|^2}} \right) + \frac{1}{2} \text{div}^- \left(\frac{\nabla^- v_{i,j}^{\ell}}{\sqrt{\epsilon + |\nabla^- v_{i,j}^{\ell-1}|^2}} \right) \\ - \frac{1}{\lambda} (v_{i,j}^{\ell} - f_{i,j}^h), \quad 0 \leq i, j \leq N-1, \end{aligned} \quad (4.1)$$

together with boundary conditions in (1.5).

We now show that the iterative solutions $\{v_{i,j}^\ell, 0 \leq i, j \leq N-1\}, \ell \geq 0$ converge. Indeed, we first have LEMMA 4.1. *There exists a positive constant C dependent only on f and initial values $u_{i,j}^{k-1}$ such that*

$$\|v^\ell\|^2 := \sum_{i,j} |v_{i,j}^\ell|^2 \leq C \quad (4.2)$$

for all $\ell \geq 1$.

Proof. Multiplying $v_{i,j}^\ell$ to the equation (4.1) and summing over $i, j = 0, \dots, N-1$, we have

$$\begin{aligned} \frac{\|v^\ell\|^2}{\Delta t} &= \frac{1}{\Delta t} \sum_{i,j} u_{i,j}^{k-1} v_{i,j}^\ell - \frac{1}{2} \sum_{i,j} \frac{\nabla^+ v_{i,j}^\ell \nabla^+ v_{i,j}^\ell}{\sqrt{\epsilon + |\nabla^+ v_{i,j}^\ell|^2}} \\ &\quad - \frac{1}{2} \sum_{i,j} \frac{\nabla^- v_{i,j}^\ell \nabla^- v_{i,j}^\ell}{\sqrt{\epsilon + |\nabla^- v_{i,j}^\ell|^2}} - \frac{1}{\lambda} \|v^\ell\|^2 + \frac{1}{\lambda} \sum_{i,j} f_{i,j}^h v_{i,j}^\ell. \end{aligned}$$

By using the Cauchy-Schwarz equality, it follows that

$$\left(\frac{1}{\Delta t} + \frac{1}{\lambda}\right) \|v^\ell\|^2 \leq \frac{1}{\Delta t} \|u_{i,j}^{k-1}\| \|v^\ell\| + \frac{1}{\lambda} \|f^h\| \|v^\ell\|.$$

Hence, $\|v^\ell\|$ is bounded by a constant C independent of ℓ . \square

It follows that the sequence of vectors $\{v_{i,j}^\ell, 0 \leq i, j \leq N-1\}, \ell \geq 1$ contains a convergent subsequence. Let us say the vectors $v_{i,j}^{\ell_k}, 0 \leq i, j \leq N-1$ converge to $v_{i,j}^*$, $0 \leq i, j \leq N-1$. Next we claim that the whole sequence converges. To prove this claim, we recall the energy functional

$$E^h(v) = J^h(v) + \frac{1}{2\Delta t} \sum_{i,j} (v_{i,j} - u_{i,j}^{k-1})^2. \quad (4.3)$$

where

$$J^h(v) = \frac{1}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^+ v_{i,j}|^2} + \frac{1}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^- v_{i,j}|^2} + \frac{1}{2\lambda} \sum_{i,j} (v_{i,j} - f_{i,j}^h)^2. \quad (4.4)$$

Let us prove the following lemma

LEMMA 4.2. *Given v^ℓ defined in Algorithm 4.1, we have for all $\ell \geq 1$*

$$\frac{1}{2\lambda} \|v^\ell - v^{\ell-1}\|^2 \leq E(v^{\ell-1}) - E(v^\ell).$$

Proof. Fix $\ell \geq 1$. For the terms in $E(v^{\ell-1}) - E(v^\ell)$, we first consider

$$\begin{aligned} &\frac{1}{2\Delta t} \sum_{i,j} (v_{i,j}^{\ell-1} - u_{i,j}^{k-1})^2 - \frac{1}{2\Delta t} \sum_{i,j} (v_{i,j}^\ell - u_{i,j}^{k-1})^2 \\ &= \frac{1}{2\Delta t} \sum_{i,j} (v_{i,j}^{\ell-1} - v_{i,j}^\ell)^2 + \frac{1}{\Delta t} \sum_{i,j} (v_{i,j}^\ell - u_{i,j}^{k-1})(v_{i,j}^{\ell-1} - v_{i,j}^\ell). \end{aligned} \quad (4.5)$$

To estimate the second term on the right-hand side of the equation above, we multiply $v_{i,j}^{\ell-1} - v_{i,j}^\ell$ to the equation (4.1) and sum over $i, j = 0, \dots, N-1$ to have

$$\begin{aligned} &\frac{1}{\Delta t} \sum_{i,j} (v_{i,j}^\ell - u_{i,j}^{k-1})(v_{i,j}^{\ell-1} - v_{i,j}^\ell) \\ &= -\frac{1}{2} \sum_{i,j} \frac{\nabla^+ v_{i,j}^\ell \nabla^+ (v_{i,j}^{\ell-1} - v_{i,j}^\ell)}{\sqrt{\epsilon + |\nabla^+ v_{i,j}^{\ell-1}|^2}} - \frac{1}{2} \sum_{i,j} \frac{\nabla^- v_{i,j}^\ell \nabla^- (v_{i,j}^{\ell-1} - v_{i,j}^\ell)}{\sqrt{\epsilon + |\nabla^- v_{i,j}^{\ell-1}|^2}} - \frac{1}{\lambda} \sum_{i,j} (v_{i,j}^\ell - f_{i,j}^h)(v_{i,j}^{\ell-1} - v_{i,j}^\ell). \end{aligned}$$

Using an elementary inequality $a(b-a) \leq b^2/2 - a^2/2$, we can easily see

$$-\frac{1}{2} \sum_{i,j} \frac{\nabla^+ v_{i,j}^\ell \nabla^+ (v_{i,j}^{\ell-1} - v_{i,j}^\ell)}{\sqrt{\epsilon + |\nabla^+ v_{i,j}^{\ell-1}|^2}} \geq -\frac{1}{4} \sum_{i,j} \frac{\nabla^+ v_{i,j}^{\ell-1} \nabla^+ v_{i,j}^{\ell-1}}{\sqrt{\epsilon + |\nabla^+ v_{i,j}^{\ell-1}|^2}} + \frac{1}{4} \sum_{i,j} \frac{\nabla^+ v_{i,j}^\ell \nabla^+ v_{i,j}^\ell}{\sqrt{\epsilon + |\nabla^+ v_{i,j}^{\ell-1}|^2}}. \quad (4.6)$$

Similar for other term involving ∇^- .

Next we have

$$\begin{aligned} & \frac{1}{2\lambda} \sum_{i,j} (v_{i,j}^{\ell-1} - f_{i,j}^h)^2 - \frac{1}{2\lambda} \sum_{i,j} (v_{i,j}^\ell - f_{i,j}^h)^2 \\ &= \frac{1}{2\lambda} \sum_{i,j} (v_{i,j}^{\ell-1} - v_{i,j}^\ell)(v_{i,j}^{\ell-1} + v_{i,j}^\ell - 2f_{i,j}^h) \\ &= \frac{1}{2\lambda} \sum_{i,j} (v_{i,j}^{\ell-1} - v_{i,j}^\ell)^2 + \frac{1}{\lambda} \sum_{i,j} (v_{i,j}^\ell - f_{i,j}^h)(v_{i,j}^{\ell-1} - v_{i,j}^\ell). \end{aligned} \quad (4.7)$$

Finally we need another elementary inequality: for any real numbers a, b and $\epsilon > 0$,

$$2\sqrt{\epsilon + b^2} - 2\sqrt{\epsilon + a^2} \geq \frac{b^2}{\sqrt{\epsilon + b^2}} - \frac{a^2}{\sqrt{\epsilon + b^2}}.$$

This inequality can be proved as follows. By the arithmetic-geometric inequality, we have

$$2\sqrt{\epsilon + a^2} \sqrt{\epsilon + b^2} \leq 2\epsilon + a^2 + b^2.$$

Rearranging the terms, we get

$$b^2 - a^2 \leq 2(\epsilon + b^2) - 2\sqrt{\epsilon + a^2} \sqrt{\epsilon + b^2}.$$

Now dividing $\sqrt{\epsilon + b^2}$ both sides, we obtain the desired inequality.

Using the above inequality, we can easily verify the following inequality

$$\frac{1}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^+ v_{i,j}^{\ell-1}|^2} - \frac{1}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^+ v_{i,j}^\ell|^2} \geq \frac{1}{4} \sum_{i,j} \frac{\nabla^+ v_{i,j}^{\ell-1} \nabla^+ v_{i,j}^{\ell-1}}{\sqrt{\epsilon + |\nabla^+ v_{i,j}^{\ell-1}|^2}} - \frac{1}{4} \sum_{i,j} \frac{\nabla^+ v_{i,j}^\ell \nabla^+ v_{i,j}^\ell}{\sqrt{\epsilon + |\nabla^+ v_{i,j}^{\ell-1}|^2}}. \quad (4.8)$$

Similar for the terms involving ∇^- . We now add all equalities and inequalities (4.5), (4.7) and (4.8) together to have

$$E(v^{\ell-1}) - E(v^\ell) \geq \frac{1}{2\lambda} \sum_{i,j} (v_{i,j}^{\ell-1} - v_{i,j}^\ell)^2. \quad (4.9)$$

This completes the proof. \square

We are now ready to prove the main result in this subsection.

THEOREM 4.3. *The iterative solutions defined in Algorithm 4.1 converge to the solution of (1.5) for any fixed $k \geq 1$.*

Proof. We have already shown that the iterative solution vectors $\{v_{i,j}^\ell, 0 \leq i, j \leq N-1\}$ have a convergent subsequence $\{v_{i,j}^{\ell_k}, 0 \leq i, j \leq N-1\}, k = 1, 2, \dots$ to a vector v^* . It is easy to see that the energies $E(v^{\ell_k}), k \geq 1$ are also convergent to $E(v^*)$. By Lemma 4.2, we know that energies $E(v^\ell)$ are decreasing for all ℓ and hence, $E(v^{\ell_{k+1}})$ decrease to $E(v^*)$. By using Lemma 4.2 again, we see $\|v^{\ell_{k+1}} - v^{\ell_k}\|^2 \leq 2\lambda(E(v^{\ell_k}) - E(v^{\ell_{k+1}})) \rightarrow 0$. Thus, $v^{\ell_{k+1}}, k \geq 1$ are also convergent to v^* . The uniqueness of the solution of (1.5) implies that v^* is the solution vector $\{u_{i,j}^k, 0 \leq i, j \leq N-1\}$. \square

5. Computational Results. We have implemented our iterative algorithm in the previous section in MATLAB. Let us report one numerical example.

EXAMPLE 5.1. *In this Example, we use the algorithm to remove the noised from images. For comparison, we also provide denoised images by using a standard Perona-Malik PDE method with diffusivity function $c(s) = 1/\sqrt{1+s}$ (cf. [15]). A Gaussian noise with $\sigma^2 = 20$ is added to the clean image of LENA and BARBARA. The PSNR of the noised images is 22.11. PSNR of the recovered images are shown on the top of the images. The two denoised images are shown in Figures 5.1 and 5.2. The left one is done by the PM method and the right one is based on our finite difference scheme. From these examples, we can see that our finite difference scheme works as the same or slightly better than the Perona-Malik method.*



FIG. 5.1. *The denoised images by the PM method and the denoised image (right) by our finite difference scheme*



FIG. 5.2. *The denoised images by the PM method and the denoised image (right) by our finite difference scheme*

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