Convergence Analysis of a Finite Difference Scheme for the Gradient Flow associated with the ROF Model

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Abstract

We present a convergent analysis of a finite difference scheme for the time dependent partial different equation called gradient flow associated with the Rudin-Osher-Fetami model. We devise an iterative algorithm to compute the solution of the finite difference scheme and prove the convergence of the iterative algorithm. Finally computational experiments are shown to demonstrate the convergence of the finite difference scheme. An application for image denoising is given.

1 Introduction

The well-known ROF model may be approximated in the following way

$$\min_{u\in \mathrm{BV}(\Omega)} \int_\Omega \sqrt{\epsilon + |\nabla u|^2}dx + \frac{1}{2\lambda}\int_\Omega |u-f|^2 dx. \tag{1}$$

As $\epsilon > 0$, the above minimizing functional is differentiable. Thus, the Euler-Lagrange equation associated with the above minimization is

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}}\right) - \frac{1}{\lambda}(u-f) = 0. \tag{2}$$

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Solution of this partial differential equation can be further approximated. Let us consider the time evolution version of the PDE:

$$\begin{cases} \frac{d}{dt}u = \text{ div } \left(\frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}} \right) - \frac{1}{\lambda}(u - f) & \in \Omega_T \\ \frac{\partial}{\partial \mathbf{n}} u = 0 & \text{on } \partial\Omega_T \\ u(\cdot, 0) = u_0(\cdot), & \Omega, \end{cases} \tag{3}$$

where f is given a noised image, $\Omega_T = [0, T) \times \Omega$, $\frac{\partial}{\partial \mathbf{n}}$ is the outward normal derivative operator. It is called the gradient flow of (1). When $\epsilon = 0$, it is called TV flow. Similar partial differential equations also appear in geometry analysis. See references, e.g., [12], [11], [2], [3], [4], and the references therein. The existence, uniqueness, stability of the weak solutions to these time dependent PDE were studied in the literature mentioned above. Numerical solution of the PDE (3) using finite elements has been discussed in [9] and [8]. In particular, the researchers showed that the finite element solution exists, is unique, is convergent to the weak solution of the PDE (3), the rate of convergence under some sufficient conditions, and the computation is stable. A fixed point iterative algorithm for the associated system of nonlinear equations was discussed in [16] and its convergence was studied in [6]. Although the finite difference solution of the time dependent PDE (3) has been the method of choice for image denoising (e.g. See [15]), no convergence of the finite difference solution to the weak solution of the PDE has been established in the literature so far to the best of the authors' knowledge. See also [7].

The purpose of this paper is to establish the convergence of the discrete solution obtained from a finite difference scheme for (3) to the weak solution. See our Theorem 3.1 in Section 3. Then we discuss how to numerically solve the time dependent PDE (3) by using our finite difference scheme. As the finite difference scheme is a system of nonlinear equations, we shall derive an iterative algorithm and show that the iterative solutions are convergent.

For convenience, let $\Omega = [0, 1] \times [0, 1]$. We let $N > 0$ be a positive integer and divide Ω by equally-spaced points $x_i = ih$ and $y_j = jh$ for $0 \leq i, j \leq N - 1$ where $h = 1/N$. For any $f(x, y)$ defined on Ω, let $f^h_{i,j} = f(x_i, y_j)$ if f is a continuous function on Ω. Otherwise, f^h will be defined as in (9). We shall use two different divided differences ∇^+ and ∇^- to approximate the gradient operator. That is,

$$\nabla^+ f^h_{i,j} = \left(\frac{f^h_{i+1,j} - f^h_{i,j}}{h}, \frac{f^h_{i,j+1} - f^h_{i,j}}{h} \right)$$

and

$$\nabla^- f^h_{i,j} = \left(\frac{f^h_{i,j} - f^h_{i-1,j}}{h}, \frac{f^h_{i,j} - f^h_{i,j-1}}{h} \right)$$

for all $0 \leq i, j \leq N-1$ with $f^h_{-1,j} = f^h_{0,j}, f^h_{N,j} = f^h_{N-1,j}$ for all j and $f^h_{i,-1} = f^h_{i,0}, f^h_{i,N} = f^h_{i,N-1}$ for all i. Furthermore, we define discrete divergence operators div^+ and div^- to approximate the continuous divergence operator, i.e.,

$$\operatorname{div}^+(f^h_{i,j}, g^h_{i,j}) = \begin{cases} f^h_{0,j}/h & i = 0, 0 \leq j \leq N-1 \\ (f^h_{i,j} - f^h_{i-1,j})/h & 0 < i < N-1, 0 \leq j \leq N-1 \\ -f^h_{i-2,j}/h & i = N-1, 0 \leq j \leq N-1 \end{cases}$$
$$+ \begin{cases} g^h_{i,0}/h & j = 0, 0 \leq i \leq N-1 \\ (g^h_{i,j} - g^h_{i,j-1})/h & 0 < j < N-1, 0 \leq i \leq N-1 \\ -g^h_{i,j-2}/h & j = N-1, 0 \leq i \leq N-1 \end{cases}$$

for all $0 \leq i, j \leq N-1$ and similarly for div^-. By their definitions, we have for every $p \in \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times N}$ and $u \in \mathbb{R}^{N\times N}$

$$\langle -\operatorname{div}^+ p, u\rangle = \langle p, \nabla^+ u\rangle, \qquad \langle -\operatorname{div}^- p, u\rangle = \langle p, \nabla^- u\rangle.$$

With these notations, we are able to define a finite difference scheme for numerical solution of the time dependent PDE (3).

$$\begin{cases} \frac{d}{dt} u_{i,j} = \frac{1}{2} \operatorname{div}^+ \left(\frac{\nabla^+ u_{i,j}}{\sqrt{\epsilon + |\nabla^+ u_{i,j}|^2}} \right) & \\ \qquad + \frac{1}{2} \operatorname{div}^- \left(\frac{\nabla^- u_{i,j}}{\sqrt{\epsilon + |\nabla^- u_{i,j}|^2}} \right) - \frac{1}{\lambda}(u_{i,j} - f^h_{i,j}) & 0 \leq i, j \leq N-1, t \in [0, T] \\ \frac{\partial}{\partial \mathbf{n}} u_{i,j} = 0 & i = 0, N, 0 \leq j \leq N-1; \\ & j = 0, N, 0 \leq i \leq N-1, \\ u(x_i, y_j, 0) = u^h_0(x_i, y_j), & 0 \leq i, j \leq N-1, \end{cases} \tag{4}$$

where u^h_0 is a discretization of the initial value u_0 according to (9). Next we discretize the time domain $[0, T]$ by equally-spaced points $t_k = k\Delta t$, $\Delta t = T/M$. We approximate the $\frac{d}{dt} u_{i,j}$ by $(u^k_{i,j} - u^{k-1}_{i,j})/\Delta t$ to have the fully discrete version of finite difference scheme:

$$\begin{cases} \frac{1}{\Delta t}(u^k_{i,j} - u^{k-1}_{i,j}) = \frac{1}{2} \operatorname{div}^+ \left(\frac{\nabla^+ u^k_{i,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i,j}|^2}} \right) & \\ \qquad + \frac{1}{2} \operatorname{div}^- \left(\frac{\nabla^- u^k_{i,j}}{\sqrt{\epsilon + |\nabla^- u^k_{i,j}|^2}} \right) - \frac{1}{\lambda}(u^k_{i,j} - f^h_{i,j}) & 0 \leq i, j \leq N-1, 1 \leq k \leq M \\ \frac{\partial}{\partial \mathbf{n}} u^k_{i,j} = 0 & i = 0, N, 0 \leq j \leq N-1; \\ & j = 0, N, 0 \leq i \leq N-1, 0 \leq k \leq M \\ u(x_i, y_j, 0) = u^h_0(x_i, y_j), & 0 \leq i, j \leq N-1. \end{cases} \tag{5}$$

Remark 1.1 *In our numerical scheme, the discrete variation for any array* $u^k := \{u^k_{ij}, 0 \le i, j \le N\}$ *is in fact defined by*

$$|u^k|_{\text{DBV}} = \frac{1}{2}\sum_{i,j}\sqrt{\epsilon + |\nabla^+ u^k_{i,j}|^2} + \frac{1}{2}\sum_{i,j}\sqrt{\epsilon + |\nabla^- u^k_{i,j}|^2}.$$

This way of defining discrete variation makes it possible to connect discrete and continuous variations by the observation that $|U|_{\text{BV}} = |u^k|_{\text{DBV}}$ *where* U *is a piecewise linear function obtained by interpolating* u^k *over grids on* Ω *which will be detailed in Section 3. The numerical scheme is constructed from the Euler-Lagrange equation of the following variation problem*

$$\arg\min E_k(v)$$

for each step k *where*

$$E_k(v) = \frac{1}{2}\sum_{i,j}\sqrt{\epsilon + |\nabla^+ v_{i,j}|^2}\, h^2 + \frac{1}{2}\sum_{i,j}\sqrt{\epsilon + |\nabla^- v_{i,j}|^2}\, h^2$$
$$+ \frac{1}{2\lambda}\sum_{i,j}(v_{i,j} - f^h_{i,j})^2\, h^2 + \frac{1}{2\Delta t}\sum_{i,j}(v_{i,j} - u^{k-1}_{i,j})^2\, h^2$$

for all arrays $\{v_{i,j}\}$, $0 \le i, j \le N-1$.

We shall first show that the above scheme has a uniqueness solution. Then we show the solution in (5) converges to the weak solution of time dependent PDE (3). These will be done in the next 2 sections. Next we shall explain how to numerically solve this system of nonlinear equations in §4. We report our computational results in §5. Finally we give a few remarks the last section.

2 Preliminary Results

We introduce a weak formulation of PDE (3) that is suggested by [9].

Definition 2.1 *We say that* $u \in L^1([0,T], \text{BV}(\Omega))$ *is a weak solution of (3) if* u *satisfies the initial value and boundary conditions in (3) and for any* $w \in L^1([0,T], W^{1,1}(\Omega))$ *with* $\frac{\partial}{\partial \mathbf{n}} w(x,t) = 0$ *for all* $(t,x) \in [0,T) \times \partial\Omega$,

$$\int_0^s \int_\Omega \frac{d}{dt} u w dx dt + \int_0^s \int_\Omega \frac{\nabla u \cdot \nabla w}{\sqrt{\epsilon + |\nabla u|^2}} + \frac{1}{\lambda}\int_0^s \int_\Omega (u - f) w dx dt = 0, \tag{6}$$

for any $s \in (0, T]$.

It is known (cf. [9]) there exists a unique weak solution U^* satisfying the above weak formulation. U^* is in fact in $L^\infty((0,T], \text{BV}(\Omega))$ if $u^0 \in \text{BV}(\Omega)$ and $f \in L^2(\Omega)$. Following the ideas in [12], the researchers in [9] further showed the weak solution can be characterized by the following inequality.

Theorem 2.1 *Let u be a weak solution as in Definition 2.1. Then u satisfies the following inequality: for any $s \in (0,T]$,*

$$\begin{aligned} & \int_0^s \int_\Omega \frac{d}{dt} v(v-u) dx dt + \int_0^s (J(v) - J(u)) dt \\ \geq \ & \frac{1}{2} \left[\int_\Omega (v(x,s) - u(x,s))^2 dx - \int_\Omega (v(x,0) - u_0(x,0))^2 dx \right] \end{aligned} \tag{7}$$

for all $v \in L^1([0,T], W^{1,1}(\Omega))$ with $\frac{\partial}{\partial \mathbf{n}} v(x,t) = 0$ for all $(t,x) \in [0,T) \times \partial\Omega$, where

$$J(u) = \int_\Omega \sqrt{\epsilon + |\nabla u(x,t)|^2} dx + \frac{1}{2\lambda} \int_\Omega |f(x,t) - u(x,t)|^2 dx. \tag{8}$$

On the other hand, if a function $u \in L^1((0,T], \mathrm{BV}(\Omega))$ satisfies the above inequality (7), then u is a weak solution.

Regarding to the solution of finite difference scheme (5), we prove some basic properties in this section. To this end, we assume that the initial data for our numerical scheme $\{f^h_{i,j}, 0 \leq i,j \leq N-1\}$ is a discretization of the initial data for PDE (3). Specifically assuming the region $\Omega = [0,1] \times [0,1]$ is partitioned evenly into N by N grids with a grid size of $h = 1/N$, we discretize any function in $L^2(\Omega)$ by

$$f^h_{i,j} = \frac{1}{h^2} \int_{ih}^{(i+1)h} \int_{jh}^{(j+1)h} f(x)\, dx, \quad 0 \leq i,j \leq N-1 \tag{9}$$

and suppose that the pixel value on each grid at index (i,j) is $f^h_{i,j}$. In the next section we sometimes denote by $P_N f$ a related piecewise constant function defined on Ω for which

$$(P_N f)(x) = f^h_{i,j}, \qquad x \in [ih, (i+1)h] \times [jh, (j+1)h]. \tag{10}$$

Let us start with the following existence and uniqueness theorem.

Theorem 2.2 *Fix $N > 0$ and $M > 0$. There exists a unique array $u^k_{i,j}, 0 \leq i,j \leq N-1, 0 \leq k \leq M$ satisfying the above system (5) of nonlinear equations.*

Proof. We define a variation functional that is a discretized version of (8)

$$J^h(v) = \frac{1}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^+ v_{i,j}|^2}\, h^2 + \frac{1}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^- v_{i,j}|^2}\, h^2 + \frac{1}{2\lambda} \sum_{i,j} (v_{i,j} - f^h_{i,j})^2\, h^2, \tag{11}$$

and the discrete energy functional

$$E^h(v) = J^h(v) + \frac{1}{2\Delta t} \sum_{i,j} (v_{i,j} - u^{k-1}_{i,j})^2\, h^2 \tag{12}$$

for all arrays $v_{i,j}$, $0 \le i,j \le N-1$.

The Euler-Lagrange equation for the following minimization problem

$$\min_v E^h(v) \tag{13}$$

is

$$\frac{u^k_{i,j} - u^{k-1}_{i,j}}{\Delta t} - \frac{1}{2}\,\mathrm{div}^+\left(\frac{\nabla^+ u^k_{i,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i,j}|^2}}\right) - \frac{1}{2}\,\mathrm{div}^-\left(\frac{\nabla^- u^k_{i,j}}{\sqrt{\epsilon + |\nabla^- u^k_{i,j}|^2}}\right)$$
$$+ \frac{1}{\lambda}(u^k_{i,j} - f^h_{i,j}) = 0, \quad 0 \le i,j \le N-1, 1 \le k \le M \tag{14}$$

The existence and uniqueness of $u^k_{i,j}$ follows from the strict convexity of the functional J^h and E^h. ■

The following property is a characterization of the discrete solution of (5).

Lemma 2.1 *Suppose that array $\{u^k_{i,j}, 0 \le i,j \le N-1, 0 \le k \le M\}$ is a solution of the finite difference scheme (5). Then $u^k_{i,j}$ satisfies the following inequality*

$$\sum_{i,j} \frac{u^k_{i,j} - u^{k-1}_{i,j}}{\Delta t}(v_{i,j} - u^k_{i,j}) + \frac{1}{2}\left(\sum_{i,j}\sqrt{\epsilon + |\nabla^+ v_{i,j}|^2} - \sum_{i,j}\sqrt{\epsilon + |\nabla^+ u^k_{i,j}|^2}\right) +$$
$$\frac{1}{2}\left(\sum_{i,j}\sqrt{\epsilon + |\nabla^- v_{i,j}|^2} - \sum_{i,j}\sqrt{\epsilon + |\nabla^- u^k_{i,j}|^2}\right) + \frac{1}{2\lambda}\sum_{i,j}(v_{i,j} - f^h_{i,j})^2 - \frac{1}{2\lambda}\sum_{i,j}(u^k_{i,j} - f^h_{i,j})^2$$
$$\ge 0 \tag{15}$$

for all arrays $v_{i,j}$ that satisfies the Neumann boundary condition. On the other hand, if an array $\{u^k_{i,j}, 0 \le i,j \le N-1, 0 \le k \le M\}$ satisfies the above inequality for all $v_{i,j}$ satisfying the discrete Neumann boundary condition in (5), then array $\{u^k_{i,j}, 0 \le i,j \le N-1\}$ is a solution of (5).

Proof. From the Euler-Lagrange equation (14),

$$\begin{aligned}\frac{u^{k-1}_{i,j} - u^k_{i,j}}{\Delta t} &= -\frac{1}{2}\,\mathrm{div}^+\left(\frac{\nabla^+ u^k_{i,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i,j}|^2}}\right) - \frac{1}{2}\,\mathrm{div}^-\left(\frac{\nabla^- u^k_{i,j}}{\sqrt{\epsilon + |\nabla^- u^k_{i,j}|^2}}\right) \\ &\quad + \frac{1}{\lambda}(u^k_{i,j} - f^h_{i,j}) \\ &= \partial J^h(u^k_{i,j})\end{aligned}$$

The result follows from the definition of subgradient $\partial J^h(u^k)$. ■

The following result shows that the computation of finite difference scheme (5) is stable. For the simplicity of the notations, we define the discrete L^2 norms in analogue of standard L^2 norms. Assuming $\{u_{i,j}\}$ is an array, we define

$$\|u\| := \left\{ \sum_{i,j} (u_{i,j})^2 \, h^2 \right\}^{1/2}.$$

Theorem 2.3 *Let $\{u_f^k, 0 \le k \le M\}$ be the solution of the system of nonlinear equations (5) associated with f^h with initial value u_f^0. Similarly, let $\{u_g^k, 0 \le k \le M\}$ be the corresponding solution of (5) associated with g^h with initial value u_g^0. Then*

$$\|u_f^k - u_g^k\| \le \max\{\|u_f^0 - u_g^0\|, \|f^h - g^h\|\}, \qquad 1 \le k \le M. \tag{16}$$

Proof. We prove by induction. It is obvious true for $k = 0$. Assume the inequality holds for $k-1$. Rearrange the L^2 terms in (13). We have u_f^k is the minimizer of the following problem.

$$\min_v \frac{h^2}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^+ v_{i,j}|^2} + \frac{h^2}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^- v_{i,j}|^2} + (\mu_1 + \mu_2) \left\| v - \left(k_1 f^h + k_2 u_f^{k-1}\right) \right\|^2 \tag{17}$$

where $\mu_1 = 1/(2\lambda), \mu_2 = 1/2\Delta t$, and $k_1 = \mu_1/(\mu_1 + \mu_2), k_2 = \mu_2/(\mu_1 + \mu_2)$. By standard stability property of the minimization problem (17)(cf. [13])

$$\begin{aligned} \left\| u_f^k - u_g^k \right\| &\le \left\| \left(k_1 f^h + k_2 u_f^{k-1}\right) - \left(k_1 g^h + k_2 u_g^{k-1}\right) \right\| \\ &\le k_1 \|f^h - g^h\| + k_2 \left\| u_f^{k-1} - u_g^{k-1} \right\| \\ &\le \max\left\{ \|f^h - g^h\|, \left\| u_f^{k-1} - u_g^{k-1} \right\| \right\} \\ &\le \max\left\{ \|f^h - g^h\|, \left\| u_f^0 - u_g^0 \right\| \right\}. \end{aligned}$$

■

Remark 2.1 *As a direct deduction, if $g^h = u_g^0 = 0$, the solution u_g^k is also zero for all k, then*

$$\|u_f^k\| \le \max\{\|u_f^0\|, \|f^h\|\}, \qquad 1 \le k \le M. \tag{18}$$

3 Main Result and Its Proof

In this section, we shall show that the solution of the finite difference scheme (5) converges to the solution of the gradient flow (3). We suppose that the array $\{u_{i,j}^k, 0 \le i, j \le N-1, 0 \le k \le M\}$ is the solution of (5).

We first define a mapping of the array $\{u^k_{i,j}, 0 \le i, j \le N-1, 0 \le k \le M\}$ in the form of a piecewise linear interpolant of $u^k_{i,j}$.

Let Δ_N be the following type of triangulation of $\Omega = [0,1] \times [0,1]$ with vertices $((i+1/2)h, (j+1/2)h), 0 \le i, j \le N-1$. Suppose the base functions of the continuous linear finite element space $S^0_1(\Delta_N)$ are $\{\phi_{i,j}(x), (i,j) \in \mathbb{Z}^2\}$, where $\phi_{i,j}$ is a scaled and translated standard continuous linear box spline function $\phi(x)$ based on three directions $e_1 = (1,0), e_2(0,1)$ and $e_3 = (-1,1)$, i.e. $\phi_{i,j}(x) := \phi(x/h - (i+1/2, j+1/2))$ for any $(i,j) \in \mathbb{Z}^2$.

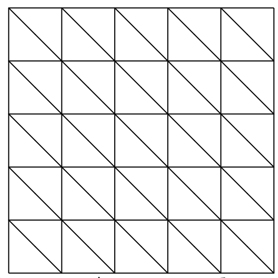

Fig. 1. A triangulation

For any k, we define piecewise linear function $U_{N,M}(x, t_k)$ on Ω by

$$U_{N,M}(x, t_k) = \sum_{i,j=0}^{N-1} u^k_{i,j} \phi_{i,j}(x).$$

Having defined $U_{N,M}(\cdot, t_k)$ for $k = 0, \cdots, M$ on Ω, we further define $U_{N,M}(\cdot, t)$ for $t_{k-1} \le t \le t_k$ by linear interpolating $U_{N,M}(\cdot, t_{k-1})$ and $U_{N,M}(\cdot, t_k)$ on interval $[t_{k-1}, t_k]$. That is,

$$U_{N,M}(\cdot, t) = \frac{t - t_{k-1}}{\Delta t} U_{N,M}(\cdot, t_k) + \frac{t_k - t}{\Delta t} U_{N,M}(\cdot, t_{k-1}).$$

We next need a sequence of useful lemmas in order to show that the solution of finite difference scheme (5) converges to the weak solution (6). We first show that the interpolant $U_{N,M}(\cdot, t)$ is TV monotone (We abuse a bit the notation of TV here since our $J(\cdot)$ includes not only a variation term but also an extra L^2 term). In the following discussion, we need to replace the L_2 integral in (8) by a discrete summation with some error. In addition to $P_N f$, the piecewise constant projection of f as defined in (10), we let $\tilde{P}_N u(\cdot, t_k)$ be a piecewise constant function defined by

$$\tilde{P}_N u(x, t_k) = u^k_{i,j} \qquad x \in [ih, (i+1)h] \times [jh, (j+1)h].$$

Replacing f and $U_{N,M}(\cdot, t_k)$ by $P_N f$ and $\tilde{P}_N U_{N,M}$ respectively, we have

$$\frac{1}{2\lambda} \int_\Omega (U_{N,M}(\cdot, t_k) - f)^2 = \frac{1}{2\lambda} \int_\Omega (\tilde{P} U_{N,M} - P_N f)^2 + \text{Err} = \sum_{i,j} (u^k_{i,j} - f^h_{i,j})^2 \, h^2 + \text{Err}$$

with error term Err. It is a standard analysis that the error term converges to zero as $P_N f \to f$ in $L_2(\Omega)$ and by using Lemma 3.5 to be given later in this section. We omit the detail here.

Lemma 3.1

$$J(U_{N,M}(\cdot,t_k))) \le J(U_{N,M}(\cdot,t)) + O(Err), \qquad t_{k-1} \le t \le t_k. \tag{19}$$

Proof. The proof is straightforward. First, it is easy to verify that the continuous variation $J(U_{N,M}(\cdot,t_k))$ equals the discrete variation $J^h(u^k)$ up to the error term Err. Define

$$u(t) := U_{N,M}(\cdot,t) = \frac{t-t_{k-1}}{\Delta t}u^k + \frac{t_k - t}{\Delta t}u^{k-1}, \qquad t_{k-1} \le t \le t_k.$$

To prove (19) is equivalent to prove

$$J^h(u^k) \le J^h(u(t)) + O(\mathrm{Err}), \qquad t_{k-1} \le t \le t_k. \tag{20}$$

Since u^k is the minimizer of the following functional

$$E^h(v) = J^h(v) + \frac{1}{2\Delta t}\|u^{k-1} - v\|^2$$

we have

$$J^h(u^k) + \frac{1}{2\Delta t}\|u^{k-1} - u^k\|^2 \le J^h(u(t)) + \frac{1}{2\Delta t}\|u^{k-1} - u(t)\|^2. \tag{21}$$

For each term in the summation of the L^2 square term

$$\begin{aligned}\left|u^{k-1} - u(t)\right| &= \left|u^{k-1} - \frac{t-t_{k-1}}{\Delta t}u^k + \frac{t_k - t}{\Delta t}u^{k-1}\right| \\ &= \frac{t-t_{k-1}}{\Delta t}\left|u^k - u^{k-1}\right| \le \left|u^k - u^{k-1}\right|.\end{aligned}$$

Then

$$\frac{1}{2\Delta t}\|u^{k-1} - u(t)\|^2 \le \frac{1}{2\Delta t}\|u^{k-1} - u^{k-1}\|^2.$$

We conclude (20) from (21) and thus,

$$J(U_{N,M}(\cdot,t_k)) = J^h(u^k) + \mathrm{Err} \le J^h(u(t)) + \mathrm{Err} = J(U_{N,M}(\cdot,t)) + O(\mathrm{Err}).$$

This completes the proof. ■

Lemma 3.2 *Suppose $u^0 \in W^{1,1}(\Omega), f \in L^2(\Omega)$. Then $\|\frac{d}{dt}U_{N,M}\|_{L^2(\Omega_T)} < C$ for a positive constant C only depending on u^0 and f.*

Proof. From the Euler-Lagrange equation (14)

$$\frac{u^{k-1} - u^k}{\Delta t} = \partial J^h(u^k)$$

The equation holds element-wise at each index (i, j). For the equation at each index (i, j), we multiply both sides by $u_{i,j}^{k-1} - u_{i,j}^k$ and add the equations for all (i, j). We use inner product notation to write the result in a concise way.

$$\left\langle \frac{u^{k-1} - u^k}{\Delta t}, u^{k-1} - u^k \right\rangle = \left\langle \partial J^h(u^k), u^{k-1} - u^k \right\rangle$$

By the definition of sub-differential $\partial J^h(u^k)$

$$\begin{aligned}\left\langle \frac{u^{k-1} - u^k}{\Delta t}, u^{k-1} - u^k \right\rangle &= \left\langle \partial J^h(u^k), u^{k-1} - u^k \right\rangle \\ &\leq J^h(u^{k-1}) - J^h(u^k).\end{aligned}$$

Note that

$$\frac{dU_{N,M}}{dt} = \frac{u^{k-1} - u^k}{\Delta t}, \qquad t^{k-1} < t < t_k.$$

We have

$$\frac{1}{\Delta t}\|u^{k-1} - u^k\|^2 \leq J^h(u^{k-1}) - J^h(u^k), \qquad 1 \leq k \leq M.$$

Add the above inequalities for $k = 1, \cdots, M$,

$$\sum_{k=1}^{M} \frac{1}{\Delta t}\|u^{k-1} - u^k\|^2 \leq J^h(u^0) - J^h(u^M). \tag{22}$$

It equals

$$\left\| \frac{dU_{N,M}}{dt} \right\|_{L^2(\Omega_T)}^2 \leq J^h(u^0) - J^h(u^M) \leq J^h(u^0).$$

Here $u^0 = u_0^h$ by the initial values. Note that $J^h(u_0^h)$ is bounded by a positive constant independent of h when $u_0 \in W^{1,1}(\Omega)$. This completes the proof. ■

Lemma 3.3 *Suppose $u^0, f \in L^2(\Omega)$. Then $\|U_{N,M}\|_{L^2(\Omega_T)} \leq C$ for a constant C only dependent on f and u^0. Furthermore, $\|U_{N,M}(\cdot, t)\|_{L^2(\Omega)} \leq C$ for a positive constant C independent of $t \in [0, T]$.*

Proof. We use (18) to bound $\|U_{N,M}\|_{L^2(\Omega_T)}$ and $\|U_{N,M}(\cdot, t)\|_{L^2(\Omega)}$. Recall $u_f^0 = u^0$. Letting

$C = \max\{\|u_f^0\|, \|f^h\|\}$, we have

$$\begin{aligned}
\|U_{N,M}\|^2_{L^2(\Omega_T)} &= \int_0^T \|U_{N,M}(\cdot,t)\|^2_{L^2(\Omega)}\, dt \\
&= \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \left\| \frac{(t-t_{k-1})U_{N,M}(\cdot,t_k) + (t_k - t)U_{N,M}(\cdot,t_{k-1})}{\Delta t} \right\|^2_{L^2(\Omega)} dt \\
&\leq \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \|U_{N,M}(\cdot,t_k)\|^2_{L^2(\Omega)} + \|U_{N,M}(\cdot,t_{k-1})\|^2_{L^2(\Omega)}\, dt \\
&\leq \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \|u^k\|^2 + \|u^{k-1}\|^2\, dt \leq 2TC^2.
\end{aligned}$$

As discussed above, for each $t \in [0,T]$, the integrand is $\|U_{N,M}(\cdot,t)\|^2_{L^2(\Omega)}$ which is less than or equal to $2C^2$ by (18). These complete the proof. ■

In image analysis, the input image usually does not have much regularity. For example, most natural images do not even have weak derivatives. Therefore, to model images, we introduce the notation of Lipschitz space, and treat images as elements in this space.

Definition 3.1 *Let $\alpha \in (0,1]$ be a real number. A function $f \in Lip(\alpha, L^2(\Omega))$ if $f \in L^2(\Omega)$ and the following quantity*

$$|f|_{\mathrm{Lip}(\alpha,L^2(\Omega))} := \sup_{|h|\leq 1} \frac{\|f(\cdot) - f(\cdot + h)\|_{L^2(\Omega_h)}}{|h|^\alpha} \tag{23}$$

is finite, where $\Omega_h := \{x \in \Omega, x + th \in \Omega, \forall t \in [0,1]\}$. We let $\|f\|_{\mathrm{Lip}(\alpha,L^2(\Omega))} = \|f\|_{L^2(\Omega)} + |f|_{\mathrm{Lip}(\alpha,L^2(\Omega))}$.

The parameter α is related to the "smoothness" of functions in the Lipschitz space. Smoother functions belong to Lipschitz spaces with larger α values. For example, a function of bounded variation is a function in $Lip(1, L^2(\Omega))$.

Lemma 3.4 *Define translation operators $T_{1,0}$ and $T_{0,1}$ by*

$$\begin{aligned}
(T_{1,0}u^k)_{i,j} &= u^k_{i+1,j} \qquad 0 \leq i,j \leq N-1 \\
(T_{0,1}u^k)_{i,j} &= u^k_{i,j+1} \qquad 0 \leq i,j \leq N-1
\end{aligned}$$

Then

$$\left\|T_{1,0}u^k - u^k\right\| \leq (\|u^0\|_{\mathrm{Lip}(\alpha,L^2)} + \|f\|_{\mathrm{Lip}(\alpha,L^2)})h^\alpha$$

and similarly

$$\left\|T_{0,1}u^k - u^k\right\| \leq (\|u^0\|_{\mathrm{Lip}(\alpha,L^2)} + \|f\|_{\mathrm{Lip}(\alpha,L^2)})h^\alpha.$$

Proof. We only prove the first inequality. Recall the Euler-Lagrange equation that

$$\frac{u^{k-1} - u^k}{\Delta t} = \partial J^h(u^k).$$

We write the equation element-wisely as

$$\frac{u^k_{i,j} - u^{k-1}_{i,j}}{\Delta t} = \frac{1}{2}\operatorname{div}^+\left(\frac{\nabla^+ u^k_{i,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i,j}|^2}}\right) + \frac{1}{2}\operatorname{div}^-\left(\frac{\nabla^- u^k_{i,j}}{\sqrt{\epsilon + |\nabla^- u^k_{i,j}|^2}}\right) - \frac{1}{\lambda}(u^k_{i,j} - f^h_{i,j}).$$

Then subtract the equation at index $(i+1, j)$ from the same equation at index (i, j) for $0 \le i \le N-2$.

$$\begin{aligned}\frac{u^k_{i+1,j} - u^k_{i,j}}{\Delta t} - \frac{u^{k-1}_{i+1,j} - u^{k-1}_{i,j}}{\Delta t} &= F(\nabla^+ u^k_{i+1,j}, \nabla^+ u^k_{i,j}) + F(\nabla^- u^k_{i+1,j}, \nabla^- u^k_{i,j}) \\ &\quad - \frac{1}{\lambda}(u^k_{i+1,j} - u^k_{i,j}) + \frac{1}{\lambda}(f^h_{i+1,j} - f^h_{i,j}) \end{aligned} \tag{24}$$

where $F(\nabla^+ u^k_{i+1,j}, \nabla^+ u^k_{i,j})$ is defined by

$$F(\nabla^+ u^k_{i+1,j}, \nabla^+ u^k_{i,j}) = \frac{1}{2}\operatorname{div}^+\left(\frac{\nabla^+ u^k_{i+1,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i+1,j}|^2}}\right) - \frac{1}{2}\operatorname{div}^+\left(\frac{\nabla^+ u^k_{i,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i,j}|^2}}\right).$$

Equation (24) only holds for $0 \le i \le N-2$, $0 \le j \le N-1$. Although equation (24) is not defined for $i = N-1$, we can set $u^k_{N+1,j} = u^k_{N,j}$ and $f_{N+1,j} = f_{N,j}$, and equation (24) still holds. We multiply (24) by $u^k_{i+1,j} - u^k_{i,j}$ and add all resulting equations for $0 \le i, j \le N-1$ to have

$$\begin{aligned}&\frac{1}{\Delta t}\sum_{i,j=0}^{N-1}(u^k_{i+1,j} - u^k_{i,j})^2 \\ &= \frac{1}{\Delta t}\sum_{i,j=0}^{N-1}(u^{k-1}_{i+1,j} - u^{k-1}_{i,j})(u^k_{i+1,j} - u^k_{i,j}) \\ &\quad + \sum_{i,j=0}^{N-1} F(\nabla^+ u^k_{i+1,j}, \nabla^+ u^k_{i,j})(u^k_{i+1,j} - u^k_{i.j}) + \sum_{i,j=0}^{N-1} F(\nabla^- u^k_{i+1,j}, \nabla^- u^k_{i,j})(u^k_{i+1,j} - u^k_{i.j}) \\ &\quad - \sum_{i,j=0}^{N-1}\frac{1}{\lambda}(u^k_{i+1,j} - u^k_{i,j})^2 + \sum_{i,j=0}^{N-1}\frac{1}{\lambda}(f^h_{i+1,j} - f^h_{i,j})(u^k_{i+1,j} - u^k_{i,j}).\end{aligned}$$

We show next that the second term is no greater than zero. The third term can be proved to be nonpositive similarly. By definition of F,

$$\sum_{i,j=0}^{N-1} F(\nabla^+ u^k_{i+1,j}, \nabla^+ u^k_{i,j})(u^k_{i+1,j} - u^k_{i,j})$$
$$= \sum_{i,j=0}^{N-1} \frac{1}{2} \operatorname{div}^+ \left(\frac{\nabla^+ u^k_{i+1,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i+1,j}|^2}} \right) (u^k_{i+1,j} - u^k_{i,j}) - \sum_{i,j=0}^{N-1} \frac{1}{2} \operatorname{div}^+ \left(\frac{\nabla^+ u^k_{i,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i,j}|^2}} \right) (u^k_{i+1,j} - u^k_{i,j}).$$

We use the discrete divergence operators and gradient operators to get

$$\sum_{i.j=0}^{N-1} F(\nabla^+ u^k_{i+1,j}, \nabla^+ u^k_{i,j})(u^k_{i+1,j} - u^k_{i,j})$$
$$= \frac{1}{2} \sum_{i,j=0}^{N-1} \left(\operatorname{div}^+ \left(\frac{\nabla^+ u^k_{i+1,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i+1,j}|^2}} \right) - \operatorname{div}^+ \left(\frac{\nabla^+ u^k_{i,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i,j}|^2}} \right) \right) (u^k_{i+1,j} - u^k_{i,j})$$
$$= -\frac{1}{2} \sum_{i,j=0}^{N-1} \left(\left(\frac{\nabla^+ u^k_{i+1,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i+1,j}|^2}} \right) - \left(\frac{\nabla^+ u^k_{i,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i,j}|^2}} \right) \right) \cdot (\nabla^+ u^k_{i+1,j} - \nabla^+ u^k_{i,j})$$
$$- \sum_{j=0}^{N-1} \frac{|\nabla^+ u^k_{0,j}|^2}{\sqrt{\epsilon + |\nabla^+ u^k_{0,j}|^2}}$$

Each term in the first sum is non-negative due to the fact that for any $x, y \in \mathbf{R}^2$,

$$\left(\frac{x}{\sqrt{\epsilon + |x|^2}} - \frac{y}{\sqrt{\epsilon + |x|^2}} \right) \cdot (x - y) \geq 0$$

By similar arguments, one has

$$\sum_{i,j=0}^{N-1} F(\nabla^- u^k_{i+1,j}, \nabla^- u^k_{i,j})(u^k_{i+1,j} - u^k_{i,j}) \leq 0$$

It follows

$$\frac{1}{\Delta t} \sum_{i,j=0}^{N-1} (u^k_{i+1,j} - u^k_{i,j})^2$$
$$\leq \frac{1}{\Delta t} \sum_{i,j=0}^{N-1} (u^{k-1}_{i+1,j} - u^{k-1}_{i,j})(u^k_{i+1,j} - u^k_{i,j})$$
$$- \sum_{i,j=0}^{N-1} \frac{1}{\lambda} (u^k_{i+1,j} - u^k_{i,j})^2 + \sum_{i,j=0}^{N-1} \frac{1}{\lambda} (f^h_{i+1,j} - f^h_{i,j})(u^k_{i+1,j} - u^k_{i,j}).$$

We rewrite the sums in form of discrete integrals and discrete inner products, and apply Cauchy-Schwarz inequality

$$\begin{aligned}
&\frac{1}{\Delta t}\|T_{1,0}u^k - u^k\|^2 \\
&\leq \frac{1}{\Delta t}\left\langle T_{1,0}u^{k-1} - u^{k-1}, T_{1,0}u^k - u^k\right\rangle \\
&\quad - \frac{1}{\lambda}\|T_{1,0}u^k - u^k\|^2 + \frac{1}{\lambda}\left\langle T_{1,0}f - f, T_{1,0}u^k - u^k\right\rangle \\
&\leq \frac{1}{2\Delta t}\|T_{1,0}u^{k-1} - u^{k-1}\|^2 + \frac{1}{2\Delta t}\|T_{1,0}u^k - u^k\|^2 \\
&\quad - \frac{1}{2\lambda}\|T_{1,0}u^k - u^k\|^2 + \frac{1}{2\lambda}\|T_{1,0}f - f\|^2.
\end{aligned}$$

Rearrange and combine similar terms to have

$$(\frac{1}{\Delta t} + \frac{1}{\lambda})\|T_{1,0}u^k - u^k\|^2 \leq \frac{1}{\Delta t}\|T_{1,0}u^{k-1} - u^{k-1}\|^2 + \frac{1}{\lambda}\|T_{1,0}f - f\|^2. \tag{25}$$

We now prove the following inequality by induction

$$\|T_{1,0}u^k - u^k\|^2 \leq \max\{\|T_{1,0}u^0 - u^0\|^2, \|T_{1,0}f - f\|^2\}. \tag{26}$$

It is obvious true for $k = 0$. Assuming the inequality holds for $k-1$, one can easily see that it also holds for k by (25). Therefore, one has

$$\begin{aligned}
\|T_{1,0}u^k - u^k\| &\leq \|T_{1,0}u^0 - u^0\| + \|T_{1,0}f - f\| \\
&\leq (\|u^0\|_{\text{Lip}(\alpha,L^2)} + \|f\|_{\text{Lip}(\alpha,L^2)})h^\alpha.
\end{aligned}$$

This completes the proof. ■

We also define a piecewise constant function $\overline{U}_{N,M}(\cdot,t)$ in a similar way to the definition of $U_{N,M}(\cdot,t)$. First we define for $k = 0,\cdots,M$

$$\overline{U}_{N,M}(x,t_k) = u^k_{i,j}, \qquad \forall x \in [ih,(i+1)h] \times [jh,(j+1)h].$$

Then we define $\overline{U}_{N,M}(\cdot,t)$ for $t_{k-1} \leq t \leq t_k$ by interpolating $\overline{U}(\cdot,t_{k-1})$ and $\overline{U}(\cdot,t_k)$:

$$\overline{U}(\cdot,t) = \frac{t-t_{k-1}}{\Delta t}\overline{U}(\cdot,t_k) + \frac{t_k - t}{\Delta t}\overline{U}(\cdot,t_{k-1}).$$

We are now ready to show

Lemma 3.5 *Suppose $f, u_0 \in \text{Lip}(\alpha, L^2(\Omega))$. Then*

$$\|U_{N,M} - \overline{U}_{N,M}\|_{L^2(\Omega_T)} \leq C\sqrt{T}(\|u^0\|_{\text{Lip}(\alpha,L^2)} + \|f\|_{\text{Lip}(\alpha,L^2)})h^\alpha$$

for a positive constant C dependent only on f and u_0.

Proof. Let $g(x,t) = U_{N,M}(x,t) - \overline{U}_{N,M}(x,t)$. For any x, $g(x,t)$ is a linear function of t. A direct calculation shows

$$\int_{t_{k-1}}^{t_k} \|g(x,t)\|^2_{L^2(\Omega)}\, dt \le \frac{1}{2}\left(\|g(x,t_k)\|^2_{L^2(\Omega)} + \|g(x,t_{k-1})\|^2_{L^2(\Omega)}\right)(t_k - t_{k-1}).$$

Adding the inequality for $k = 1, \cdots, M$, we have

$$\int_0^T \|g(x,t)\|^2_{L^2(\Omega)}\, dt \le \Delta t \sum_{k=0}^{M} \|g(x,t_k)\|^2_{L^2(\Omega)}. \tag{27}$$

Then we only need to bound $\|g(x,t_k)\|$. We note that $g(x,t)$ is a piecewise linear function of x on each sub-grid $\Omega_{i,j} := [ih,(i+1)h] \times [jh,(j+1)h]$, $0 \le i,j \le N-1$ for any t. Tedious calculation gives

$$\begin{aligned}
\|g(x,t_k)\|^2_{L^2(\Omega)} &= \sum_{i,j} \int_{\Omega_{i,j}} |U_{N,M}(x,t_k) - \overline{U}_{N,M}(x,t_k)|^2 \\
&\le \sum_{i,j} Ch^2 \left(\left|u^k_{i+1,j} - u^k_{i,j}\right|^2 + \left|u^k_{i,j+1} - u^k_{i,j}\right|^2 + \left|u^k_{i-1,j} - u^k_{i,j}\right|^2 + \left|u^k_{i,j-1} - u^k_{i,j}\right|^2 \right) \\
&\le C\left(\left\|T_{1,0}u^k - u^k\right\|^2 + \left\|T_{0,1}u^k - u^k\right\|^2 \right) \\
&\le 2C(\|f\|_{\mathrm{Lip}(\alpha,L^2)} + \|u_0\|_{\mathrm{Lip}(\alpha,L^2)})^2 h^{2\alpha}.
\end{aligned}$$

The last line follows from Lemma 3.4. We substitute the bound for the $\|g(x,t_k)\|_{L^2(\Omega)}$ in inequality (27) to complete the proof. ■

Finally we are ready to prove the main result of this section.

Theorem 3.1 *Suppose that $u_0 \in W^{1,1}(\Omega), f \in L^2(\Omega)$. Furthermore, suppose that $f \in Lip(\alpha, L^2(\Omega))$. If we choose $\Delta t = o(h^\alpha)$, then there exists a function U^* in $L^2(\Omega_T)$ so that $U_{N,M}$ converge to U^* weakly as $N, M \to \infty$ and U^* is the weak solution of (3).*

Proof. By Lemma 3.3, there exists a weakly convergent subsequence of $\{U_{N,M}, N \ge 1, M \ge 1\}$ in $L^2(\Omega_T)$. For convenience, we assume the whole sequence converges to $U^* \in L^2(\Omega_T)$ weakly. We now show U^* is the weak solution of the gradient flow as in Definition 2.1. As the weak solution is unique, the whole sequence $\{U_{N,M}, N \ge 1, M \ge 1\}$ converges weakly to U^*.

Let us outline the main ideas of the proof. By using Theorem 2.1, we need to show that U^* satisfies the following inequality:

$$\begin{aligned}
&\int_0^s \int_\Omega \frac{d}{dt} v(v - U^*) dx dt + \int_0^s (J(v) - J(U^*)) dt \\
\ge\ & \frac{1}{2}\left[\int_\Omega (v(x,s) - U^*(x,s))^2 dx - \int_\Omega (v(x,0) - u_0(x,0))^2 dx \right]
\end{aligned} \tag{28}$$

for all $v \in L^1([0,T], W^{1,1}(\Omega))$ with $\frac{\partial}{\partial \mathbf{n}} v(x,t) = 0$ for all $(t,x) \in [0,T) \times \partial\Omega$, where

$$J(u) = \int_\Omega \sqrt{\epsilon + |\nabla u(x,t)|^2} dx + \frac{1}{2\lambda} \int_\Omega |f(x,t) - u(x,t)|^2 dx.$$

By the lower semi-continuity of J, Fatou's lemma and standard weak convergence, we have

$$\begin{aligned} & \int_0^s \int_\Omega \frac{d}{dt} v(v - U^*) dx dt + \int_0^s (J(v) - J(U^*)) dt \\ \geq \ & \liminf_{N,M\to\infty} \left[\int_0^s \int_\Omega \frac{d}{dt} v(v - U_{N,M}) dx dt + \int_0^s (J(v) - J(U_{N,M})) dt \right]. \end{aligned} \tag{29}$$

By the Banach-Steinhaus theorem,

$$\begin{aligned} & \liminf_{N,M\to\infty} \frac{1}{2} \left[\int_\Omega (v(x,s) - U_{N,M}(x,s))^2 dx - \int_\Omega (v(x,0) - u_0(x,0))^2 dx \right] \\ \geq \ & \frac{1}{2} \left[\int_\Omega (v(x,s) - U^*(x,s))^2 dx - \int_\Omega (v(x,0) - u_0(x,0))^2 dx \right]. \end{aligned} \tag{30}$$

Indeed, in (30), $v(x,s) - U_{N,M}(x,s)$ is convergent weakly to $v(x,s) - U^*(x,s)$ in $L^2(\Omega_T)$ and for all $s \in [0,T]$, $v(x,s) - U_{N,M}(x,s)$ is convergent weakly to $v(x,s) - U^*(x,s)$ in $L^2(\Omega)$. They define linear functionals on $L^2(\Omega)$ for all $s \in [0,T]$. By the Banach-Steinhaus theorem, the norm of the linear function satisfies the following inequality

$$\int_\Omega (v(x,s) - U^*(x,s))^2 dx \leq \liminf_{N,M\to\infty} \int_\Omega (v(x,s) - U_{N,M}(x,s))^2 dx$$

for almost all $s \in [0,T]$.

We now prove the following inequality to finish the proof.

$$\begin{aligned} & \int_0^s \int_\Omega \frac{d}{dt} v(v - U_{N,M}) dx dt + \int_0^s (J(v) - J(U_{N,M})) dt \\ & \geq \frac{1}{2} \left[\int_\Omega (v(x,s) - U_{N,M}(x,s))^2 dx - \int_\Omega (v(x,0) - u_0(x,0))^2 dx \right] - \text{Error}_{N,M} \end{aligned}$$

where $\text{Error}_{N,M} > 0$ is an error term that goes to zero as $N, M \to \infty$. It's straightforward to verify(cf. [9]) that the above inequality is equivalent to

$$\int_0^s \int_\Omega \frac{d}{dt} U_{N,M}(v - U_{N,M}) dx dt + \int_0^s (J(v) - J(U_{N,M})) dt \geq -\text{Error}_{N,M}. \tag{31}$$

Recall that $S_1^0(\Delta_N)$ is the finite element space associated with triangulation Δ_N. We replace the original $W^{1,1}$ test function $v(\cdot,t)$ in (31) with a test function $v'(\cdot,t)$ that is in $L^1([0,T], S_1^0(\Delta_N))$ which introduces another error $e_{N,M}$.

$$e_{N,M} = \int_0^s \int_\Omega \frac{d}{dt} U_{N,M}(v - v') + J(v) - J(v').$$

It is easy to show $e_{N,M}$ tends to zero as N, M go to infinity by standard density argumentation based on linear finite element approximation property (cf. Theorem 4.4.20 in [5]). Thus we only need to prove

$$\int_0^s \left[\int_\Omega \frac{d}{dt} U_{N,M}(v - U_{N,M}) dx + (J(v) - J(U_{N,M})) \right] dt \geq -\text{Error}_{N,M} \tag{32}$$

for all test functions v in $L^1([0,T], S_1^0(\Delta_N))$ where $\text{Error}_{N,M}$ tends to zero as $N, M \to \infty$.

Let us verify the key inequality (32). Consider the integrand of the left side of (32) for $t = t_k$. For a continuous piecewise linear function $v(\cdot, t_k) \in S_1^0(\Delta_N)$, assuming $v(\cdot, t_k) = \sum_{i,j} v_{i,j}^k \phi_{i,j}$, we have

$$J(v(\cdot, t_k)) = \frac{h^2}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^+ v_{i,j}^k|^2} + \frac{h^2}{2} \sum_{i,j} \sqrt{\epsilon + |\nabla^- v_{i,j}^k|^2} + \frac{1}{2\lambda} \int_\Omega (v(\cdot, t_k) - f)^2 \tag{33}$$

We need to replace the continuous integral in (33) by a discrete summation with some error. Let $\tilde{P}_N v(\cdot, t_k)$ be a piecewise constant function defined by

$$\tilde{P}_N v(x, t_k) = v_{i,j}^k \qquad x \in [ih, (i+1)h] \times [jh, (j+1)h]$$

as before and $P_N f$ be the piecewise constant projection of f as defined in (10). Replacing f and v by $P_N f$ and $\tilde{P}_N v$ respectively, we have

$$\frac{1}{2\lambda} \int_\Omega (v(\cdot, t_k) - f)^2 = \frac{1}{2\lambda} \sum_{i,j} (v_{i,j}^k - f_{i,j}^h)^2 \, h^2 + \frac{1}{2\lambda} \int_\Omega ((v(\cdot, t_k) - f)^2 - (\tilde{P}_N v(\cdot, t_k) - P_N f)^2).$$

It is a standard analysis that the second term on the right-hand side converges to zero as $P_N f \to f$ and $\tilde{P}_N v \to v$ in $L^2(\Omega)$. We omit the detail here. Thus, we write

$$\frac{1}{2\lambda} \int_\Omega (v(\cdot, t_k) - f)^2 = \frac{1}{2\lambda} \sum_{i,j} (v_{i,j}^k - f_{i,j}^h)^2 \, h^2 + \text{Err}_1,$$

where Err_1 denotes the error dependent on N that is convergent to zero when $N \to \infty$. Similarly, we have

$$\frac{1}{2\lambda} \int_\Omega (U_{N,M}(\cdot, t_k) - f)^2 = \frac{1}{2\lambda} \sum_{i,j} (u_{i,j}^k - f_{i,j}^h)^2 \, h^2 + \text{Err}_2.$$

with error term Err_2 that converges to zero as $N \to \infty$ by Lemma 3.5.

We remind the reader that that for $t \in (t_{k-1}, t_k)$,

$$\frac{d}{dt} U_{N,M}(\cdot, t) = \frac{U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})}{\Delta t}. \tag{34}$$

Then

$$\int_\Omega \frac{d}{dt}U_{N,M}(v-U_{N,M})dx = \int_\Omega \frac{U_{N,M}(\cdot,t_k)-U_{N,M}(\cdot,t_{k-1})}{\Delta t}(v(\cdot,t_k)-U_{N,M}(\cdot,t_k))dx.$$

Replacing $v, U_{N,M}$ by $\tilde{P}_N v, \overline{U}_{N,M}$ respectively, we have

$$\begin{aligned}
& \int_\Omega \frac{U_{N,M}(\cdot,t_k)-U_{N,M}(\cdot,t_{k-1})}{\Delta t}(v(\cdot,t_k)-U_{N,M}(\cdot,t_k)) \\
= & \int_\Omega \frac{\overline{U}_{N,M}(\cdot,t_k)-\overline{U}_{N,M}(\cdot,t_{k-1})}{\Delta t}(\tilde{P}_N v(\cdot,t_k)-\overline{U}_{N,M}(\cdot,t_k))dx + \frac{\text{Err}_3}{\Delta t} \\
= & \sum_{i,j} \frac{u^k_{i,j}-u^{k-1}_{i,j}}{\Delta t}(v_{i,j}-u^k_{i,j})\, h^2 + \frac{\text{Err}_3}{\Delta t},
\end{aligned}$$

where Err_3 stands for another error term that can be bounded by Lemma 3.5. Note that we have to use Cauchy-Schwarz inequality to show that Err_3 goes to zero at the rate of h^α. By one of the assumptions, we know $\text{Err}_3/\Delta t \to 0$ when $N \to \infty$.

We put all the estimates above together to have

$$\begin{aligned}
& \int_\Omega \frac{d}{dt}U_{N,M}(\cdot,t_k)(v(\cdot,t_k)-U_{N,M}(\cdot,t_k))dx + J(v(\cdot,t_k)) - J(U_{N,M}(\cdot,t_k)) \\
= & \sum_{i,j} \frac{u^k_{i,j}-u^{k-1}_{i,j}}{\Delta t}(v^k_{i,j}-u^k_{i,j})\, h^2 + \\
& \frac{1}{2}\sum_{i,j}\sqrt{\epsilon+|\nabla^+ v^k_{i,j}|^2}\, h^2 + \frac{1}{2}\sum_{i,j}\sqrt{\epsilon+|\nabla^- v^k_{i,j}|^2}\, h^2 + \frac{1}{2\lambda}\sum_{i,j}(v^k_{i,j}-f^h_{i,j})^2\, h^2 \\
& -\frac{1}{2}\sum_{i,j}\sqrt{\epsilon+|\nabla^+ u^k_{i,j}|^2}\, h^2 - \frac{1}{2}\sqrt{\epsilon+|\nabla^- u^k_{i,j}|^2}\, h^2 - \frac{1}{2\lambda}\sum_{i,j}(u^k_{i,j}-f^h_{i,j})^2\, h^2 \\
& +\frac{\text{Err}_3}{\Delta t} + \text{Err}_1 - \text{Err}_2
\end{aligned}$$

Note that the summation of the first 7 terms on the right-hand side above are nonnegative by inequality (15) in Lemma 2.1. Then

$$\begin{aligned}
& \int_\Omega \frac{d}{dt}U_{N,M}(\cdot,t_k)(v(\cdot,t_k)-U_{N,M}(\cdot,t_k))dx + J(v(\cdot,t_k)) - J(U_{N,M}(\cdot,t_k)) \\
\geq & \frac{\text{Err}_3}{\Delta t} + \text{Err}_1 - \text{Err}_2.
\end{aligned}$$

Thus the integrand on the left-hand side of the inequality (32) for $t = t_k$ is bigger than a small term which will go to zero. .

Now we consider the integrand on the left-hand side of the inequality when $t \in (t_{k-1}, t_k)$. Note that

$$U_{N,M}(\cdot,t) = U_{N,M}(\cdot,t_{k-1})(t_k-t)/\Delta t + U_{N,M}(\cdot,t_k)(t-t_{k-1})/\Delta t$$

Without loss of generality, we consider the integration over $[0, T]$ instead of $[0, s]$. By using (34), we have

$$\begin{aligned} & \int_0^T \int_\Omega \frac{d}{dt} U_{N,M}(v(\cdot,t) - U_{N,M}(\cdot.t))\, dxdt \\ = & \sum_{k=1}^M \int_{t_k-1}^{t_k} \int_\Omega \frac{U_{N,M}(\cdot,t_k) - U_{N,M}(\cdot,t_{k-1})}{\Delta t} (v(\cdot,t) - U_{N,M}(\cdot,t))\, dxdt \\ = & \sum_{k=1}^M \int_{t_k-1}^{t_k} \int_\Omega \frac{U_{N,M}(\cdot,t_k) - U_{N,M}(\cdot,t_{k-1})}{\Delta t} (v(\cdot,t) - U_{N,M}(\cdot,t_k)\, dxdt + \mathrm{Err}_4. \end{aligned}$$

We bound Err_4 by

$$\begin{aligned} & |\mathrm{Err}_4| \\ & \leq 2\sum_{k=1}^M \int_{t_k-1}^{t_k} \int_\Omega \left| \frac{U_{N,M}(\cdot,t_k) - U_{N,M}(\cdot,t_{k-1})}{\Delta t} (U_{N,M}(\cdot,t_k) - U_{N,M}(\cdot,t_{k-1})) \frac{t_k - t}{\Delta t} \right| dxdt. \\ & \leq \sum_{k=1}^M \int_\Omega \left| \frac{U_{N,M}(\cdot,t_k) - U_{N,M}(\cdot,t_{k-1})}{\Delta t} (U_{N,M}(\cdot,t_k) - U_{N,M}(\cdot,t_{k-1}))\, dx \right| \Delta t \\ & = \Delta t \left\| \frac{dU_{N,M}}{dt} \right\|_{L^2(\Omega_T)}^2 \leq C\Delta t, \end{aligned}$$

where the last inequality comes from Lemma 3.2.

For the variation term, we write

$$\begin{aligned} & \int_0^T J(v(\cdot,t)) - J(U_{N,M}(\cdot,t)) \\ & = \sum_{k=1}^M \int_{t_{k-1}}^{t_k} J(v(\cdot,t)) - J(U_{N,M}(\cdot,t))\, dt \\ & = \sum_{k=1}^M \int_{t_{k-1}}^{t_k} J(v(\cdot,t)) - J(U_{N,M}(\cdot,t_k))\, dt + \mathrm{Err}_5 \end{aligned}$$

with

$$\mathrm{Err}_5 = \sum_{k=1}^M \int_{t_{k-1}}^{t_k} J(U_{N,M}(\cdot,t)) - J(U_{N,M}(\cdot,t_k))\, dt.$$

To bound Err_5, we use the convexity of J and the monotonicity of the variation term described in Lemma 3.1,

$$
\begin{aligned}
&|\mathrm{Err}_5| \\
&\leq \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \left| \frac{t-t_{k-1}}{\Delta t} J(U_{N,M}(\cdot,t_k)) + \frac{t_k - t}{\Delta t}(J_{N,M}(\cdot,t_{k-1})) - J(U_{N,M}(\cdot,t_k)) \right| dt \\
&= \sum_{k=1}^{M} |J(U_{N,M}(\cdot,t_{k-1}) - J(U_{N,M}(\cdot,t_k))| \int_{t_{k-1}}^{t_k} \frac{t_k - t}{\Delta t}\, dt \\
&\leq \sum_{k=1}^{M} 2|\mathrm{Err}|\Delta t = 2T|\mathrm{Err}|.
\end{aligned}
$$

We conclude that Err_5 tends to zero as Err approaches zero. Collecting these results together, we proved inequality (32). Indeed, the detail with $s = T$ can be explained as follows. Letting

$$
\mathrm{Err}_6 := \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \left[\int_\Omega \frac{d}{dt} U_{N,M}(\cdot,t_k)(v(\cdot,t) - v(\cdot,t_k)) dx + J(v(\cdot,t)) - J(v(\cdot,t_k)) \right] dt,
$$

the left-hand side of (32) with $s = T$ is written

$$
\begin{aligned}
&\int_0^T \int_\Omega \frac{d}{dt} U_{N,M}(v - U_{N,M}) dx dt + \int_0^T (J(v) - J(U_{N,M})) dt \\
=& \sum_{k=1}^{M} \int_{t_k-1}^{t_k} \int_\Omega \frac{U_{N,M}(\cdot,t_k) - U_{N,M}(\cdot,t_{k-1})}{\Delta t} (v(\cdot,t) - U_{N,M}(\cdot,t_k)\, dx dt + \mathrm{Err}_4 \\
&+ \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} J(v(\cdot,t)) - J(U_{N,M}(\cdot,t_k))\, dt + \mathrm{Err}_5 \\
=& \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \left[\int_\Omega \frac{d}{dt} U_{N,M}(\cdot,t_k)(v(\cdot,t_k) - U_{N,M}(\cdot,t_k)) dx + J(v(\cdot,t_k)) - J(U_{N,M}(\cdot,t_k)) \right] dt + \mathrm{Err}_4 \\
&+ \mathrm{Err}_5 + \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \left[\int_\Omega \frac{d}{dt} U_{N,M}(\cdot,t_k)(v(\cdot,t) - v(\cdot,t_k)) dx + J(v(\cdot,t)) - J(v(\cdot,t_k)) \right] dt \\
\geq& T \frac{\mathrm{Err}_3}{\Delta t} + T\mathrm{Err}_1 - T\mathrm{Err}_2 + \mathrm{Err}_4 + \mathrm{Err}_5 + \mathrm{Err}_6.
\end{aligned}
$$

As we have already shown that the first 5 terms go to zero as $N, M \to \infty$. The remaining part is to show that $\text{Err}_6 \to 0$. To this end, by using Cauchy-Schwarz's inequality,

$$\text{Err}_6 \leq \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \|\frac{d}{dt} U_{N,M}(\cdot, t_k)\|_2 \|v(\cdot, t) - v(\cdot, t_k)\|_2 dt$$
$$+ \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \left[\int_{\Omega} |\nabla v(\cdot, t) - \nabla v(\cdot, t_k)| dx + \frac{C}{\lambda} \|v(\cdot, t) - v(\cdot, t_k)\|_{L^2(\Omega)} \right] dt, \quad (35)$$

where C is a constant dependent on f and $\|U_{N,M}(\cdot, t)\|_{L^2(\Omega)}$ which is bounded independent of t by Lemma 3.3. Now the first summation in the above (35) is

$$\sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \|\frac{d}{dt} U_{N,M}(\cdot, t_k)\|_2 \|v(\cdot, t) - v(\cdot, t_k)\|_2 dt$$
$$\leq \sum_{k=1}^{M} \|u^k - u^{k-1}\|_2 \frac{1}{\triangle t} \int_{t_{k-1}}^{t_k} \|v(\cdot, t) - v(\cdot, t_k)\|_2 dt$$
$$\leq \sqrt{M \triangle t} \|\frac{d}{dt} U_{N,M}\|_2 \max_{1 \leq k \leq M} \frac{1}{\triangle t} \int_{t_{k-1}}^{t_k} \|v(\cdot, t) - v(\cdot, t_k)\|_2 dt$$
$$\leq \sqrt{T} C \max_{1 \leq k \leq M} \frac{1}{\triangle t} \int_{t_{k-1}}^{t_k} \|v(\cdot, t) - v(\cdot, t_k)\|_2 dt \to 0$$

for all $v \in C([0,T], S_1^0(\triangle_N))$, where we have used Lemma 3.2. The two terms in the second summation in the (35) go to zero for all $v \in C([0,T], S_1^0(\triangle_N))$ because they are approximated by their piecewise constant functions. That is, Err_6 goes to zero for all $v \in L^1([0,T], S_1^0(\triangle_N))$ and hence, (32) holds for all such functions. These complete the proof. ∎

4 Numerical Solution of Our Finite Difference Scheme

The system (5) of nonlinear equations has been solved by many methods as explained in [16]. In [6], the researchers provided an analysis of a fixed point method proposed in [16] based on auxiliary variable and functionals and proved that the iterative method converges. In this section, we mainly present another method to show the convergence of the fixed point method. From notation simplicity, we assume the grid size $h = 1$ in this section that has no influence in the convergence analysis of our algorithm.

First of all, let us explain the fixed point method. Recall that we need to solve $\{u_{i,j}^k, 0 \leq$

$i, j \leq N-1\}$ from the following equations

$$\frac{u^k_{i,j} - u^{k-1}_{i,j}}{\Delta t} - \frac{1}{2}\,\mathrm{div}^+\left(\frac{\nabla^+ u^k_{i,j}}{\sqrt{\epsilon + |\nabla^+ u^k_{i,j}|^2}}\right) - \frac{1}{2}\,\mathrm{div}^-\left(\frac{\nabla^- u^k_{i,j}}{\sqrt{\epsilon + |\nabla^- u^k_{i,j}|^2}}\right)$$
$$+ \frac{1}{\lambda}(u^k_{i,j} - f^h_{i,j}) = 0, \quad 0 \leq i, j \leq N-1,$$

assuming that we have the solution $\{u^{k-1}_{i,j}, 0 \leq i, j \leq N-1\}$. Let us define an iterative algorithm to compute $u^k_{i,j}$.

Algorithm 4.1 *Starting with* $v^0_{i,j} = u^{k-1}_{i,j}, 0 \leq i, j \leq N-1$, *for* $\ell = 1, 2, \cdots,$, *we compute*

$$\frac{v^\ell_{i,j} - u^{k-1}_{i,j}}{\Delta t} = \frac{1}{2}\,\mathrm{div}^+\left(\frac{\nabla^+ v^\ell_{i,j}}{\sqrt{\epsilon + |\nabla^+ v^{\ell-1}_{i,j}|^2}}\right) + \frac{1}{2}\,\mathrm{div}^-\left(\frac{\nabla^- v^\ell_{i,j}}{\sqrt{\epsilon + |\nabla^- v^{\ell-1}_{i,j}|^2}}\right)$$
$$- \frac{1}{\lambda}(v^\ell_{i,j} - f^h_{i,j}), \quad 0 \leq i, j \leq N-1, \tag{36}$$

together with boundary conditions in (5).

We now show that the iterative solutions $\{v^\ell_{i,j}, 0 \leq i, j \leq N-1\}, \ell \geq 0$ converge. Indeed, we first have

Lemma 4.1 *There exists a positive constant* C *dependent only on* f *and initial values* $u^{k-1}_{i,j}$ *such that*

$$\|v^\ell\|^2 := \sum_{i,j} |v^\ell_{i,j}|^2 \leq C \tag{37}$$

for all $\ell \geq 1$.

Proof. Multiplying $v^\ell_{i,j}$ to the equation (36) and summing over $i, j = 0, \cdots, N-1$, we have

$$\begin{aligned}\frac{\|v^\ell\|^2}{\Delta t} &= \frac{1}{\Delta t}\sum_{i,j} u^{k-1}_{i,j} v^\ell_{i,j} - \frac{1}{2}\sum_{i,j}\frac{\nabla^+ v^\ell_{i,j}\nabla^+ v^\ell_{i,j}}{\sqrt{\epsilon + |\nabla^+ v^{\ell-1}_{i,j}|^2}} \\ &\quad - \frac{1}{2}\sum_{i,j}\frac{\nabla^- v^\ell_{i,j}\nabla^- v^\ell_{i,j}}{\sqrt{\epsilon + |\nabla^- v^{\ell-1}_{i,j}|^2}} - \frac{1}{\lambda}\|v^\ell\|^2 + \frac{1}{\lambda}\sum_{i,j} f^h_{i,j} v^\ell_{i,j}.\end{aligned}$$

By using the Cauchy-Schwarz equality, it follows that

$$(\frac{1}{\Delta t} + \frac{1}{\lambda})\|v^\ell\|^2 \leq \frac{1}{\Delta t}\|u^{k-1}_{i,j}\|\|v^\ell\| + \frac{1}{\lambda}\|f^h\|\|v^\ell\|.$$

Hence, $\|v^\ell\|$ is bounded by a constant C independent of ℓ. ■

It follows that the sequence of vectors $\{v^\ell_{i,j}, 0 \le i,j \le N-1\}, \ell \ge 1$ contains a convergent subsequence. Let us say the vectors $v^{\ell_k}_{i,j}, 0 \le i,j \le N-1$ converge to $v^*_{i,j}, 0 \le i,j \le N-1$. Next we claim that the whole sequence converges. To prove this claim, we recall the energy functional

$$E^h(v) = J^h(v) + \frac{1}{2\Delta t}\sum_{i,j}(v_{i,j} - u^{k-1}_{i,j})^2. \tag{38}$$

where

$$J^h(v) = \frac{1}{2}\sum_{i,j}\sqrt{\epsilon + |\nabla^+ v_{i,j}|^2} + \frac{1}{2}\sum_{i,j}\sqrt{\epsilon + |\nabla^- v_{i,j}|^2} + \frac{1}{2\lambda}\sum_{i,j}(v_{i,j} - f^h_{i,j})^2. \tag{39}$$

Let us prove the following lemma

Lemma 4.2 *For all $\ell \ge 1$, we have*

$$\frac{1}{2\lambda}\|v^\ell - v^{\ell-1}\|^2 \le E(v^{\ell-1}) - E(v^\ell).$$

Proof. Fix $\ell \ge 1$. For the terms in $E(v^{\ell-1}) - E(v^\ell)$, we first consider

$$\begin{aligned}
&\frac{1}{2\Delta t}\sum_{i,j}(v^{\ell-1}_{i,j} - u^{k-1}_{i,j})^2 - \frac{1}{2\Delta t}\sum_{i,j}(v^{\ell}_{i,j} - u^{k-1}_{i,j})^2 \\
= \; &\frac{1}{2\Delta t}\sum_{i,j}(v^{\ell-1}_{i,j} - v^{\ell}_{i,j})^2 + \frac{1}{\Delta t}\sum_{i,j}(v^{\ell}_{i,j} - u^{k-1}_{i,j})(v^{\ell-1}_{i,j} - v^{\ell}_{i,j}).
\end{aligned} \tag{40}$$

To estimate the second term on the right-hand side of the equation above, we multiply $v^{\ell-1}_{i,j} - v^\ell_{i,j}$ to the equation (36) and sum over $i,j = 0, \cdots, N-1$ to have

$$\begin{aligned}
&\frac{1}{\Delta t}\sum_{i,j}(v^{\ell}_{i,j} - u^{k-1}_{i,j})(v^{\ell-1}_{i,j} - v^{\ell}_{i,j}) \\
= \; &-\frac{1}{2}\sum_{i,j}\frac{\nabla^+ v^\ell_{i,j}\nabla^+(v^{\ell-1}_{i,j} - v^\ell_{i,j})}{\sqrt{\epsilon + |\nabla^+ v^{\ell-1}_{i,j}|^2}} - \frac{1}{2}\sum_{i,j}\frac{\nabla^- v^\ell_{i,j}\nabla^-(v^{\ell-1}_{i,j} - v^\ell_{i,j})}{\sqrt{\epsilon + |\nabla^- v^{\ell-1}_{i,j}|^2}} - \frac{1}{\lambda}\sum_{i,j}(v^\ell_{i,j} - f^h_{i,j})(v^{\ell-1}_{i,j} - v^\ell_{i,j}).
\end{aligned}$$

Note that it is easy to see

$$\begin{aligned}
&-\frac{1}{2}\sum_{i,j}\frac{\nabla^+ v^\ell_{i,j}\nabla^+(v^{\ell-1}_{i,j} - v^\ell_{i,j})}{\sqrt{\epsilon + |\nabla^+ v^{\ell-1}_{i,j}|^2}} \\
\ge \; &-\frac{1}{4}\sum_{i,j}\frac{\nabla^+ v^{\ell-1}_{i,j}\nabla^+ v^{\ell-1}_{i,j}}{\sqrt{\epsilon + |\nabla^+ v^{\ell-1}_{i,j}|^2}} + \frac{1}{4}\sum_{i,j}\frac{\nabla^+ v^{\ell}_{i,j}\nabla^+ v^{\ell}_{i,j}}{\sqrt{\epsilon + |\nabla^+ v^{\ell-1}_{i,j}|^2}}.
\end{aligned}$$

Similar for other term involving ∇^-.

Next we consider

$$\begin{aligned} & \frac{1}{2\lambda}\sum_{i,j}(v_{i,j}^{\ell-1}-f_{i,j}^h)^2-\frac{1}{2\lambda}\sum_{i,j}(v_{i,j}^{\ell}-f_{i,j}^h)^2 \\ = & \frac{1}{2\lambda}\sum_{i,j}(v_{i,j}^{\ell-1}-v_{i,j}^{\ell})(v_{i,j}^{\ell-1}+v_{i,j}^{\ell}-2f_{i,j}^h) \\ = & \frac{1}{2\lambda}\sum_{i,j}(v_{i,j}^{\ell-1}-v_{i,j}^{\ell})^2+\frac{1}{\lambda}\sum_{i,j}(v_{i,j}^{\ell}-f_{i,j}^h)(v_{i,j}^{\ell-1}-v_{i,j}^{\ell}). \end{aligned} \tag{41}$$

Finally we can easily verify the following inequality

$$\begin{aligned} & \frac{1}{2}\sum_{i,j}\sqrt{\epsilon+|\nabla^+v_{i,j}^{\ell-1}|^2}-\frac{1}{2}\sum_{i,j}\sqrt{\epsilon+|\nabla^+v_{i,j}^{\ell}|^2} \\ \geq & \frac{1}{4}\sum_{i,j}\frac{\nabla^+v_{i,j}^{\ell-1}\nabla^+v_{i,j}^{\ell-1}}{\sqrt{\epsilon+|\nabla^+v_{i,j}^{\ell-1}|^2}}-\frac{1}{4}\sum_{i,j}\frac{\nabla^+v_{i,j}^{\ell}\nabla^+v_{i,j}^{\ell}}{\sqrt{\epsilon+|\nabla^+v_{i,j}^{\ell-1}|^2}}. \end{aligned} \tag{42}$$

Similar for the terms involving ∇^-. We now add all equalities and inequalities (40), (41), (42) together to have

$$E(v^{\ell-1})-E(v^{\ell})\geq\frac{1}{2\lambda}\sum_{i,j}(v_{i,j}^{\ell-1}-v_{i,j}^{\ell})^2. \tag{43}$$

This completes the proof. ■

We are now ready to prove the main result in this subsection.

Theorem 4.1 *The iterative solutions defined in Algorithm 4.1 converge to the solution of (5) for any fixed $k\geq 1$.*

Proof. We have already shown that the iterative solution vectors $\{v_{i,j}^{\ell}, 0\leq i,j\leq N-1\}$ have a convergent subsequence $\{v_{i,j}^{\ell_k}, 0\leq i,j\leq N-1\}, k=1,2,\cdots$ to a vector v^*. It is easy to see that the energies $E(v^{\ell_k}), k\geq 1$ are also convergent to $E(v^*)$. By Lemma 4.2, we know that energies $E(v^{\ell})$ are decreasing for all ℓ and hence, $E(v^{\ell_k+1})$ decrease to $E(v^*)$. By using Lemma 4.2 again, we see $\|v^{\ell_k+1}-v^{\ell_k}\|^2\leq 2\lambda(E(v^{\ell_k}-E(v^{\ell_k+1})\to 0$. Thus, $v^{\ell_k+1}, k\geq 1$ are also convergent to v^*. The uniqueness of the solution of (5) implies that v^* is the solution vector $\{u_{i,j}^k, 0\leq i,j\leq N-1\}$. ■

5 Computational Results

We have implemented our iterative algorithm in the previous section in MATLAB. Let us report two numerical examples.

Example 5.1 *In this example, we tested the proposed algorithm on two exact functions:*

$$u_1(x,t) = \frac{100\cos(\pi x/50)\cos(\pi y/50)}{t+1}$$

and

$$u_2(x,t) = \frac{100\cos(\pi x/50)\cos(\pi y/50)}{e^{0.2t}}.$$

It is a straightforward to calculate from the time dependent PDE to find out the corresponding function f *with* $\lambda > 0$. *They are*

$$f_1(x,t) = \frac{100\cos(\pi x/50)\cos(\pi y/50)}{1+t} - \frac{\lambda 100\cos(\pi x/50)\cos(\pi y/50)}{(1+t)^2}$$
$$+ \frac{\lambda 100(\pi/50)^2\cos(\pi x/50)\cos(\pi y/50)/(1+t)\left(2\epsilon + \frac{(2\pi)^2}{(1+t)^2}(\sin^2(\pi x/50) + \sin^2(\pi y/50))\right)}{\left(\epsilon + (2\pi)^2/(1+t)^2(\sin^2(\pi x/50)\cos^2(\pi y/50) + \cos^2(\pi x/50)\sin^2(\pi y/50))\right)^{3/2}}$$

for $(x,y) \in [0,50] \times [0,50]$, *and* $t \in [0,19]$ *and*

$$f_2(x,t) = \frac{100\cos(\pi x/50)\cos(\pi y/50)}{e^{0.2t}} - \frac{\lambda 20\cos(\pi x/50)\cos(\pi y/50)}{e^{0.2t}}$$
$$+ \frac{\lambda 100(\pi/50)^2\cos(\pi x/50)\cos(\pi y/50)e^{-0.2t}\left(2\epsilon + (2\pi)^2 e^{-0.4t}(\sin^2(\pi x/50) + \sin^2(\pi y/50))\right)}{\left(\epsilon + (2\pi)^2 e^{-0.4t}(\sin^2(\pi x/50)\cos^2(\pi y/50) + \cos^2(\pi x/50)\sin^2(\pi y/50))\right)^{3/2}}$$

with $(x,y) \in [0,50] \times [0,50]$, *and* $t \in [0,15]$. *For discretization of the space domain, a uniform mesh with* $\Delta x = \Delta y = 1$ *was used, leading to a total number* 50×50 *grids points. On the time domain, we used a uniform step size* $\Delta t = 0.019$, *which leads to a total number of 1000 steps. We use* $u_1(x,y,0) = 100\cos(\pi x/50)\cos(\pi y/50)$ *as an initial value. We choose the final time* $T_1 = 19$ *with* $u_1(x,y,T_1) = 5\cos(\pi x/50)\cos(\pi y/50)$. *We use a uniform time step with step size* $\Delta_t = 0.019$ *and do 1000 steps. In each step, we do 10 iterations. Similarly, for* u_2, *we use the final time* $T_2 = 15$ *with the same initial value and do 1000 steps in time. In Figure 1, we show the graph of the function* $5\cos(\pi x/50)\cos(\pi y/50)$ *which is the final time for* u_1 *and* u_2. *In Figure 2, we show the relative and maximum errors of numerical solutions from Algorithm 4.1 and the exact solution* $u_1(x,t)$ *and* $u_2(x,t)$.

Example 5.2 *In this Example, we use the algorithm to remove the noised from images. For comparison, we also provide denoised images by using a standard Perona-Malik PDE method with diffusivity function* $c(s) = 1/\sqrt{1+s}$. *A Gaussian noise with* $\sigma^2 = 20$ *is added to the clean image of LENA and BARBARA. The PSNR of the noised images is 22.11. The two denoised images are shown in Figures 3 and 4. The left one is done by the PM method and the right one is based on our finite difference scheme. To test our method we in fact divided each of noised image into several small blocks, denoise each block and add them together. From these examples, we can see that our finite difference scheme works as the same or slightly better than the Perona-Malik method.*

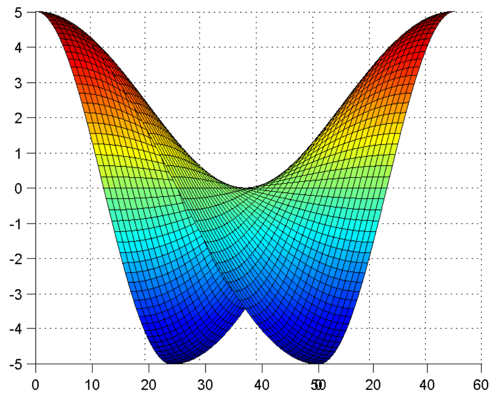

Figure 1: Plot of the function $u(x, y) = 5\cos(\pi x/50)\cos(\pi y/50)$ at the final time.

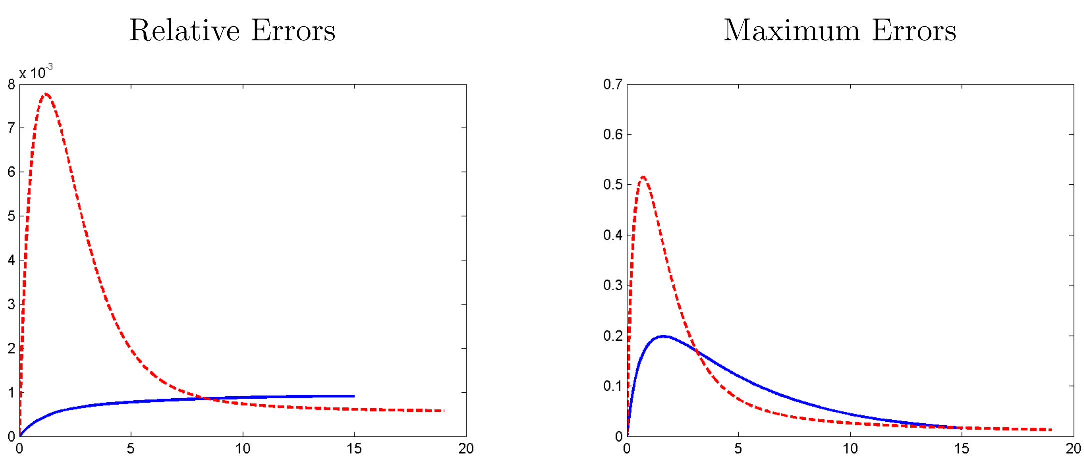

Figure 2: The error curves of red dash lines are associated with u_1 and the error curve of blue solid lines are associated with u_2.

Figure 3: The denoised images by the PM method and the denoised image (right) by our finite difference scheme

Figure 4: The denoised images by the PM method and the denoised image (right) by our finite difference scheme

6 Remarks

We end this paper with a few remarks.

Remark 6.1 *P. Perona and J. Malik proposed a non-stationary PDE model in [14] to remove noises by using anisotropic diffusion. For a given noised image f, we find an improved image u by solving the following non-stationary PDE model with initial value f over time $t \in [0, T)$:*

$$\begin{cases} \frac{\partial u}{\partial t} & = div(c(|\nabla u|^2)\nabla u), \ \ in \ \Omega \times (0, T) \\ \frac{\partial u}{\partial n} & = 0, \ \ on \ \partial\Omega \times (0, T), \\ u(0, x) & = f(x), \ \ in \ \Omega, \end{cases} \tag{44}$$

where $c(s) : [0, +\infty) \mapsto [0, +\infty)$ is a diffusive function which is a decreasing function satisfying $c(0) = 1$ and $\lim_{s\to+\infty} c(s) = 0$. The following is a list of commonly used diffusive functions:

- $c(s) = 1/\sqrt{1 + s/\lambda}$ *which is called Charbonnier diffusivity.*
- $c(s) = 1/(1 + s/\lambda)$ *which was used in [14]. We may call it Perona-Malik diffusivity.*
- $c(s) = \exp(-c/\lambda)$ *which is the standard Gaussian diffusivity function.*
- $c(s) = (1 + s/\lambda)^{\beta-1/2}$ *for $\beta \in (0, 1/2)$.*

For a fixed $c(s)$, we solve $u(T, x)$ for a large T such that the restored image $u(T, x)$ is a satisfactory one. If we let T sufficiently large, $u(T, x)$ starts a degradation such as some edges are lost or severally blurred.

It is clear when using the Charbonnier diffusivity, i.e., $c(s) = 1/\sqrt{\epsilon + s}$, the PDE in (44) is very similar to the one in (3) with two distinct differences: one is λ in (3) is ∞ and the other one is to use the noised image f as an initial value. Our convergence analysis discussed in the previous two sections can be applied to the PM model with the special diffusive function $c(s)$. In addition, the convexity of anti-derivative of $c(s)$ plays a significant role in our analysis. For other diffusive functions, e.g., Perona-Malik diffusive function, we notice that the function $C(s)$ such that $C'(s) = c(s) = 1/(1 + s/\lambda)$ is not convex when $s > \lambda$. When $s \leq \lambda$, i.e. $|\nabla u| \leq \lambda$ for $t \in [0, T]$ for some $T > 0$, $C(s)$ is convex and our analysis can be used to show that the corresponding finite difference method is convergent.

Remark 6.2 *Our convergence analysis is independent of ϵ. Thus, we can let $\epsilon = 0$. Also, we can replace the integral with coefficient $1/(2\lambda)$ by the boundary integral. Then the time dependent PDE is associated with evolutionary surfaces with prescribed mean curvature as in [12] and [11]. Our analysis can be used to show that the corresponding finite difference method for evolutionary surfaces of prescribed mean curvature is convergent to the pseudo-solution.*

Remark 6.3 *It is interesting to know the convergence rate of the finite difference solution to the weak solution of (3). The convergence rate of the fully discrete finite element solution was established in [8] under a high regularity assumption on the noised image* f*, i.e.,* $f \in L^\infty((0,T]; W^{1,\infty}(\Omega))$ *and a very high regularity condition on domain* Ω*, i.e.* $\partial\Omega \in C^3$*. In general, an image function may not have such a high regularity. We hope to reduce the assumption on the regularities and give an estimate of convergence rate for the finite element solutions. These have to be left to the interested reader.*

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