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# A New $H^2$ Regularity Condition of the Solution to Dirichlet Problem of the Poisson Equation and Its Applications

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**Abstract** We study the regularity of the solution of Dirichlet problem of Poisson equations over a bounded domain. A new sufficient condition, uniformly positive reach is introduced. Under the assumption that the closure of the underlying domain of interest has a uniformly positive reach, the $H^2$ regularity of the solution of the Poisson equation is established. In particular, this includes all star-shaped domains whose closures are of positive reach, regardless if they are Lipschitz domains or non-Lipschitz domains. Application to the strong solution to the second order elliptic PDE in non-divergence form and the regularity of Helmholtz equations will be presented to demonstrate the usefulness of the new regularity condition.

**Keywords** Regularity, Poisson equations, uniformly positive reach, non-divergence form

**MR(2010) Subject Classification** 35B60, 35J15, 35D35

## 1 Introduction

Developing efficient numerical methods for solving second order elliptic equations, e.g., Poisson equation, has drawn a lot of interest for many years. For example, numerical methods for the following second order elliptic equations in non-divergence form are recently studied in [25, 30] and [21]: Find $u = u(x)$ satisfying

$$\sum_{i,j=1}^{n} a_{ij}\partial_{ij}^2 u = f, \quad \text{ in } \Omega \subset \mathbb{R}^n, \tag{1.1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega$ is an open bounded domain with a Lipschitz continuous boundary $\partial\Omega$, $a_{ij} \in L^\infty(\Omega)$, and $\partial_{ij}^2$ are standard second order differentiation operators. Because the coefficients $a_{ij}$ are only in $L^\infty(\Omega)$, the PDE in (1.1) cannot be rewritten in a divergence form. The study is motivated by numerical solution of Hamilton–Jacob–Bellman equation which characterizes the value

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functions of stochastic control problems for applications in engineering, physics, economics, and finance [26].

The study of the second order elliptic PDE in nondivergence form requires the $H^2$ regularity condition in order to have a strong solution. Usually, researchers use the convexity of the domain $\Omega \subset \mathbb{R}^n$ as an assumption, e.g. in [23, 25, 30], or assume a $C^2$ smooth boundary $\partial\Omega$ to ensure the $H^2$ regularity of the solution of the Poisson equation, e.g. in [6] or [12].

For many practical problems, the domain of interest does not have a $C^2$ boundary nor is convex. One definitely needs to know the $H^2$ regularity for more general domains to ensure the existence of a strong solution of the PDE in (1.1).

Our goal of this paper is to establish $H^2$ regularity for more general domains. Among many existing generalizations of convexity is the concept called positive reach first introduced by Federer [11]. See [28] for a recent survey.

**Definition 1.1** *Let $K \subset \mathbb{R}^n$ be a non-empty set. Let $r_K$ be the supremum of the number $r$ such that every points in*

$$P = \{\mathbf{x} \in \mathbb{R}^n : \mathrm{dist}(\mathbf{x}, K) < r\}$$

*has a unique projection in $K$. The set $K$ is said to have a positive reach if $r_K > 0$.*

It is easy to see that a closed convex set is of positive reach with $r = \infty$. We shall explain later that a domain with $C^2$ boundary has a positive reach. Figure 1 illustrates some non-convex planar sets with positive reach. As Figure 1 illustrates, sets of positive reach are much more general than convex sets.

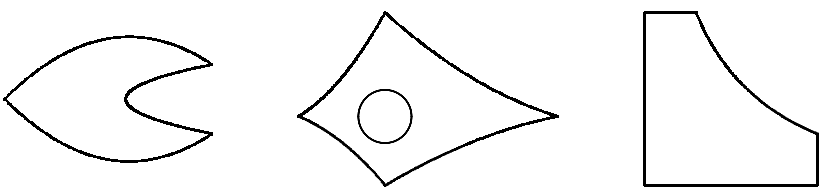

Figure 1 Domains (shaded) with positive reach

Next we introduce a new concept on domains of interest. Let $B(0, \epsilon)$ be the closed ball centering at 0 with radius $\epsilon > 0$, and let $K^c$ stand for the complement of the set $K$ in $\mathbb{R}^n$. For any $\epsilon > 0$, the set

$$E_\epsilon(K) := (K^c + B(0, \epsilon))^c \subset K \tag{1.3}$$

is called an $\epsilon$-erosion of $K$.

**Definition 1.2** *A set $K \subset \mathbb{R}^n$ is said to have a uniformly positive reach $r_0$ if there exists some $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0]$, $E_\epsilon(K)$ has a positive reach at least $r_0$.*

It is easy to check that any closed convex set has an uniformly positive reach. Indeed, any $\epsilon$-erosion of a convex set is also a closed convex set. Also, a domain with $C^2$ boundary has a uniformly positive reach. One can see that each of the three sets in Figure 1 has a uniformly positive reach.

Next let us give an example to show that a set $K$ has a positive reach, but not a uniformly positive reach.

**Example 1.3** Consider the closed set (on the left) in Figure 2. The set has a positive reach. However, at one of the boundary points, the center of the circle in red in Figure 2 violates the definition of the uniformly positive reach. As shown in Figure 2, the set $E_\epsilon(K)$ in red is an $\epsilon$-erosion of $K$. As $\epsilon \to 0$, the boundary of $E_\epsilon(K)$ is getting close to the boundary of $K$. The reach of $E_\epsilon(K)$ will go to 0. That is why $K$ does not have a uniformly positive reach.

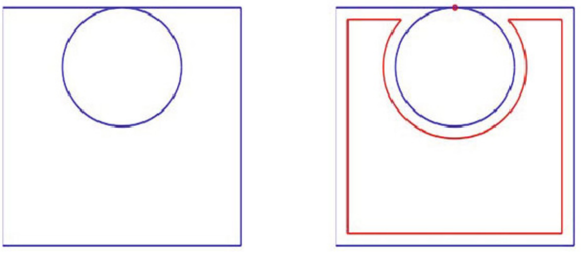

Figure 2 The set $K$ on the left (in-between the square and the circle) has a positive reach, but does not have uniformly positive reach. The graph (on right) shows the set $K$ (in blue), its $\epsilon$-erosion $E_\epsilon(K)$ (in red), and a small circle (in red). As $\epsilon$ becomes smaller, the reach of $E_\epsilon(K)$ gets smaller.

In addition, from the definition of the uniformly positive reach, we can see that if a bounded domain $\Omega \subset \mathbb{R}^n$ has a uniformly positive reach, then $\Omega$ is of positive reach as the $\epsilon$-erosions of $\Omega$ converge to $\Omega$. Thus, if a domain does not have a positive reach, it can not be of uniformly positive reach.

The main purpose of this paper is to establish the following

**Theorem 1.4** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose the closure of $\Omega$ is of uniformly positive reach $r_\Omega$. For any $f \in L^2(\Omega)$, let $u \in H_0^1(\Omega)$ be the unique weak solution of the Dirichlet problem:*

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

*Then $u \in H^2(\Omega)$ in the sense that*

$$\sum_{i,j=1}^{n} \int_\Omega \left(\frac{\partial^2}{\partial x_i \partial x_j} u\right)^2 \le C_0 \int_\Omega (f)^2 d\mathbf{x} \tag{1.5}$$

*for a positive constant $C_0$ dependent on $r_\Omega$, but independent of $f$ and $u$.*

We comment that the domain does not have to be a Lipschitz domain to have the $H^2$ regularity based on Theorem 1.4. To see this, we can look at the domain in the middle graph of Figure 1 and image that at one of the four tips, the two boundary curves have the same tangent line, and hence the domain is not Lipschitz. By Theorem 1.4, this domain has the $H^2$ regularity. Thus, a domain does not need to have a Lipschitz boundary in order to have the $H^2$ regularity.

Next we explain that the assumption of positive reach in Theorem 1.4 is necessary. An example is the solution $u = (1 - r^2)r^{2/3}\sin(2\theta/3)$ of the Poisson equation (1.4) over a domain $\Omega = \{(r, \theta) : 0 < r < 1, 0 < \theta < 2\pi/3\}$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. It is easy to see that $u$ is the solution of a Poisson equation with zero boundary and a uniform bounded function $f = 4(2/3 + 1)r^{2/3}\sin(2\theta/3) \in L^2(\Omega)$ satisfying (1.4). One can check that $u$ is not $H^2(\Omega)$, see, e.g. [4]. Also see [14] for another example. It is easy to see that the closure of the domain above does not have a positive reach nor uniform positive reach. For the higher dimensional setting, when a bounded domain has a sharp inward cusp, e.g. the well-known Lebesgue spine, the Dirichlet problem has no classic solution. For example, when $n = 3$, consider the following domain

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1,\ x^2 + y^2 > \mathrm{e}^{-1/(2z)} \text{ if } z > 0\}. \tag{1.6}$$

The inward cusp at $(0, 0, 0)$ is called a Lebesgue spine. At this cusp, the domain does not have a positive reach, and the corresponding solution does not have $H^2$ regularity. That is, in order to have $H^2$ regularity, the domain must have a positive reach.

Nevertheless, with some extra assumptions on the domain, one indeed may replace the assumption of uniform positive reach by simply positive reach. In particular, we will see in the next section that the conclusion of Theorem 1.4 remains true if $\Omega$ is star-shaped and the closure of $\Omega$ is of positive reach.

Before proving Theorem 1.4, let us review some classic results on the $H^2$ regularity property of the solution to Dirichlet problem of the Poisson equation first. In [15], the concept of domain with a cusp was introduced which is called turning points in [13]. An example from [13] was shown in Figure 3.

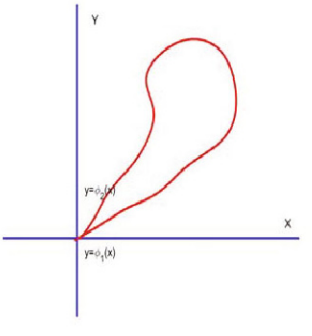


Figure 3 Domain with a turning point [13]

The following result was established in [13]. See also some similar result in [15].

**Theorem 1.5** *Let $f \in L^2(\Omega)$. Then there exists a unique $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying* (1.4) *provided*

$$\limsup_{x \to 0} \frac{(|\phi_1''(x)| + |\phi_2''(x)|)(\phi_2(x) - \phi_1(x))}{(\phi_2'(x) - \phi_1'(x))^2} < 2. \tag{1.7}$$

*with the assumption that* 0 *is a turning point, where $\phi_1, \phi_2$ are indicated in Figure* 3.

As explained in [13], when $\phi_1(x) \equiv 0$ and $\phi_2(x) = x^\alpha$ with $\alpha > 1$, the sufficient condition in (1.7) is satisfied. However, when $\phi_1(x) \equiv 0$ and $\phi_2(x) = x^2/4$, the sufficient condition (1.7) will not be fulfilled. Nevertheless, we can see that a domain with such a turning point as in Figure 3 has a uniformly positive reach and hence, our Theorem 1.4 can be used to establish the $H^2$ regularity even $\phi_2(x) = x^2/4$. That is, the condition in Theorem 1.4 is more general.

We next recall the following

**Theorem 1.6** (Adolfsson, 1992 [1]) *Suppose that $\Omega$ is a uniformly Lipschitz domain in $\mathbb{R}^n$ of finite width. If $\Omega$ satisfies an outer ball condition of uniform radius, then the unique solution $u \in H_0^1(\Omega)$ of the Dirichlet problem* (1.4) *has all its second order derivatives in $L^2(\Omega)$, i.e., $u \in H^2(\Omega)$.*

The outer ball condition mentioned above is that at each point $p$ on $\partial\Omega$ there exists an exterior ball $B$, i.e. $B \subset \Omega^c$ of radius $r$ touch at $p$, i.e. tangent to $\partial\Omega$, where $\Omega^c$ stands for the complement of $\Omega$. When $\Omega$ is bounded, $\Omega$ is of finite width. When $\Omega$ is a bounded Lipschitz domain, it is a uniformly Lipschitz domain. These explain all the notational questions in Theorem 1.6. It is easy to see that any convex domain satisfies the uniform outer ball condition. Also, a convex domain is a Lipschitz domain [2]. Thus, Theorem 1.6 is applicable to convex domains. Recently, the Lipschitz domains satisfying a uniform exterior ball condition is called semi-convex domains (e.g. [24] and [9]).

We will show that when $\overline{\Omega}$ is of positive reach, $\Omega$ satisfies an outer ball condition for a uniform radius as explained in Lemma 2.1 (see the following section). On the other hand, if $\Omega$ satisfies a uniform outer ball condition, $\overline{\Omega}$ may not have a uniformly positive reach. See the domain in the left panel of Figure 2. (Note that the domain is not Lipschitz. Both Theorems 1.4 and 1.6 fail to establish the regularity for such a domain.) These show that our condition and the one in Theorem 1.6 are two different conditions to ensure the $H^2$ regularity of the solution to Dirichlet problem.

Let us also note that if a set $K \subset \mathbb{R}^m \subset \mathbb{R}^n$ is of positive reach in $\mathbb{R}^n$, then it is also of positive reach in $\mathbb{R}^m$ for $m < n$. This is not the case for the outer ball condition: indeed, any planar set satisfies a uniform outer ball condition of any radius in $\mathbb{R}^3$. Let us also emphasize that the main difference between Theorems 1.4 and 1.6 is that in Theorem 1.4, the domain $\Omega$ does not need to have a Lipschitz boundary.

Because a set $\Omega$ satisfies an outer ball condition for a uniform radius whenever $\overline{\Omega}$ is of positive reach (Lemma 2.1) we may simply replace the outer ball condition in Theorem 1.6 by the positive reach condition, and restate the result as a part of Theorem 1.7. The other part of Theorem 1.7 is a precise relation of the constant $C_0$ in (1.5) with the positive reach $r_\Omega$ of $\overline{\Omega}$.

**Theorem 1.7** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary. Suppose that the closure of $\Omega$ has a positive reach $r_\Omega > 0$. For any $f \in L^2(\Omega)$, let $u \in H_0^1(\Omega)$ be the unique weak solution of the Dirichlet problem* (1.4). *Then $u \in H^2(\Omega)$ in the sense of* (1.5). *Moreover, the constant in* (1.5) *depends only on the positive reach $r_\Omega$.*

In Section 2, we first use the concept of star-shaped domain to establish a proof of $H^2$ regularity over a star-shaped domain which has a positive reach. See Theorem 2.7. Then, we devote our effort to proving Theorem 1.4. In Section 3.1, we address two applications of the new $H^2$ regularity condition.

## 2 Proofs

We shall prove Theorem 1.7 before proving Theorem 1.4. Let us begin with some properties of sets with positive reach.

**Lemma 2.1** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. If $\overline{\Omega}$ is of positive reach $r_\Omega$, then the following are true*:

(i) *For every $0 < r < r_\Omega$, and every $\mathbf{y} \in \mathbb{R}^n$ with $\operatorname{dist}(\mathbf{y}, \overline{\Omega}) < r_\Omega$, if $\mathbf{x}$ is the projection of $\mathbf{y}$ onto $\overline{\Omega}$, then, the closed ball $B(\mathbf{z}, r)$ centering at $\mathbf{z}$ with radius $r$ intersects $\overline{\Omega}$ precisely at $\mathbf{x}$, where*

$$\mathbf{z} = \mathbf{x} + \frac{r}{\|\mathbf{y} - \mathbf{x}\|}(\mathbf{y} - \mathbf{x}). \tag{2.1}$$

(ii) *For every $0 < r < r_\Omega$, every point on the boundary of $\Omega$ is touchable by a closed ball of radius $r$ from outside, that is, for every $0 < r < r_\Omega$ and every $\mathbf{x}_0 \in \partial\Omega$, there exists a $\mathbf{w} \in \mathbb{R}^n$, such that the closed ball $B(\mathbf{w}, r)$ intersects $\overline{\Omega}$ precisely at $\mathbf{x}_0$.*

*Proof* We prove (i) by contradiction. Suppose (i) is false, then there exist a $\mathbf{y} \in \mathbb{R}^n$ with $0 < \operatorname{dist}(\mathbf{y}, \Omega) < r_\Omega$, and an $\mathbf{x} \in \overline{\Omega}$ with $\|\mathbf{y} - \mathbf{x}\| = \operatorname{dist}(\mathbf{y}, \Omega)$, and $0 < r < r_\Omega$ such that the closed ball $B(\mathbf{z}, r)$ does not intersect $\overline{\Omega}$ uniquely at $\mathbf{x}$ with $\mathbf{z}$ in (2.1). Since $\overline{\Omega}$ is of positive reach, we must have

$$\|\mathbf{z} - \mathbf{x}\| > \operatorname{dist}(\mathbf{z}, \Omega). \tag{2.2}$$

This implies that

$$0 < 1 \le \tau := \sup\{t : \operatorname{dist}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}), \overline{\Omega}) = t\|(\mathbf{y} - \mathbf{x})\|\} \le \frac{r}{\|\mathbf{y} - \mathbf{x}\|} < \infty.$$

We have

$$r = \operatorname{dist}(\mathbf{z}, \mathbf{x}) = \left\| \frac{r}{\|\mathbf{y} - \mathbf{x}\|} \cdot (\mathbf{y} - \mathbf{x}) \right\| \ge \tau\|\mathbf{y} - \mathbf{x}\| \ge r_\Omega$$

by applying (6) of Theorem 4.8 in Federer [11]. This contradicts to the assumption $r < r_\Omega$. Hence, (i) is proved.

Now, we use (i) to prove (ii). Because $\mathbf{x}_0$ is on the boundary of $\Omega$, there exists $N_0 > 0$ such that for every integer $m \ge N_0$, we can choose a point $\mathbf{y}_m$ outside $\overline{\Omega}$ so that $\|\mathbf{y}_m - \mathbf{x}_0\| < r/m$. Let $\mathbf{x}_m$ be the projection of $\mathbf{y}_m$ onto $\overline{\Omega}$. Let

$$\mathbf{w}_m = \mathbf{x}_m + \frac{r}{\|\mathbf{y}_m - \mathbf{x}_m\|}(\mathbf{y}_m - \mathbf{x}_m).$$

By i), the closed ball $B(\mathbf{w}_m, r)$ intersects $\overline{\Omega}$ precisely at $\mathbf{x}_m$. Since

$$\|\mathbf{w}_m - \mathbf{x}_0\| \le \|\mathbf{w}_m - \mathbf{x}_m\| + \|\mathbf{x}_m - \mathbf{x}_0\| \le (1 + 1/m)r$$

for all $m \ge N_0$, the sequence $\{\mathbf{w}_m\}$ is bounded in $\mathbb{R}^n$. Hence it contains a subsequence that converges to some $\mathbf{w}_0 \in \mathbb{R}^n$. Clearly, we have

$$\|\mathbf{w}_0 - \mathbf{x}_0\| = r = \operatorname{dist}(\mathbf{w}_0, \overline{\Omega}).$$

Since $r < r_\Omega$, the closed ball $B(\mathbf{w}_0, r)$ intersects $\overline{\Omega}$ precisely at $\mathbf{x}_0$. This finishes the proof of (ii). □

We are now ready to establish the first part of Theorem 1.7. By using part (ii) of Lemma 2.1 above, we can see the positive reach $r_\Omega$ implies the outer ball condition of uniform radius $r_\Omega$

in Theorem 1.6. Thus, the first part of Theorem 1.7 is established by Theorem 1.6. We shall continue the investigation of the second part of Theorem 1.7 in the next. The proof is the same as the one for Theorem 1.4. We leave it to when we prove Theorem 1.4.

We now investigate how large the constant $C_0$ in (1.5) is. We shall connect it to the positive reach $r_\Omega$. To do so, we shall use the following standard formula [13, Theorem 3.1.1.1]: Suppose that $u \in H^2(\Omega) \cap H_0^1(\Omega)$ over a bounded domain $\Omega \subset \mathbb{R}^n$ with $C^{1,1}$ smooth boundary. Then we have

$$\begin{aligned}\int_\Omega |\text{div } \mathbf{v}|^2 d\mathbf{x} - \sum_{i,j=1}^n \int_\Omega \partial_i v_j \partial_j v_i d\mathbf{x} \\ = -2\int_{\partial\Omega} \mathbf{v}_T \nabla_T (\mathbf{v}\cdot\mathbf{n}) d\sigma - \int_{\partial\Omega} [\mathcal{B}(\mathbf{v}_T, \mathbf{v}_T) + \text{tr}(\mathcal{B})(\mathbf{v}\cdot\mathbf{n})^2],\end{aligned} \tag{2.3}$$

where $\mathcal{B}$ is a symmetric matrix of size $(n-1)\times(n-1)$, tr is the trace operator of the bilinear form: $\text{tr}(\mathcal{B}) = -\sum_{i=1}^{n-1} \frac{\partial \mathbf{n}}{\partial s_i}\cdot\tau_i$, and $T = [\tau_1, \ldots, \tau_{n-1}]$ and $\mathbf{n}$ are the tangent vectors and the outward normal direction vector of $\partial\Omega$, respectively. Letting $\mathbf{v} = \nabla u$ in (2.3) leads to

$$\begin{aligned}\sum_{i,j=1}^n \int_\Omega \left(\frac{\partial^2}{\partial x_i \partial x_j} u\right)^2 = \int_\Omega (\Delta u)^2 d\mathbf{x} + 2\int_{\partial\Omega} \nabla_T u \nabla_T (\nabla u\cdot\mathbf{n}) d\sigma \\ + \int_{\partial\Omega} [\mathcal{B}(\nabla u|_T, \nabla u|_T) + \text{tr}(\mathcal{B})(\nabla u\cdot\mathbf{n})^2] d\sigma.\end{aligned} \tag{2.4}$$

See a detailed proof of (2.4) in [13] or [5]. When the underlying domain $\Omega$ is convex, we have $\text{tr}(\mathcal{B}) \le 0$ and $\mathcal{B}(\mathbf{v}_T, \mathbf{v}_T) \le 0$. Thus we have $C_0 = 1$ in (1.5). We can also use the so-called Miranda–Talenti estimate. In [22] and [27] the following equality was calculated: for any $u \in H^2(\Omega)$,

$$\int_\Omega \sum_{i,j=1}^n \left(\frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_j^2} - \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2\right) d\mathbf{x} = -(n-1)\int_{\partial\Omega} H(\mathbf{x}) \|\nabla u\|^2 d\sigma, \tag{2.5}$$

where $H(\mathbf{x})$ is the mean curvature of $\partial\Omega$ which is $C^2$ boundary. In the setting of the convex domain $\Omega$, $H(\mathbf{x})$ is non-positive, we use (2.5) and the following identity:

$$\sum_{i,j=1}^n \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \sum_{i,j=1}^n \left(\frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_j^2} - \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2\right) = (\Delta u)^2 \tag{2.6}$$

to conclude the inequality in (1.5) with $C_0 \equiv 1$. When the domain $\Omega$ has a positive reach $r_\Omega$, the mean curvature $H(\mathbf{x})$ at each $\mathbf{x}$ can be bounded above by $(n-1)/r_\Omega$. This is quantitatively established in the following lemma.

**Lemma 2.2** *Suppose that $\Omega$ is a bounded open set with $C^{1,1}$ boundary $\partial\Omega$. Suppose that $\text{tr}(\mathcal{B}) \le c$ for a fixed real number $c > 0$, e.g. $c = (n-1)/r$ over $\partial\Omega$. Then the inequality in* (1.5) *holds for a constant $C_0 = (1+(cK)^2C^2)/(1-cK\epsilon)$, where $C$ is a constant in the Poincaré inequality* (2.7) *and $K$ is the constant in the trace inequality in* (2.10)*.*

*Proof* Using Poincaré inequality, we have

$$\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)} \tag{2.7}$$

for a positive constant $C$ dependent only on the size of $\Omega$. As $u$ is a weak solution, we have

$$\int_\Omega |\nabla u|^2 d\mathbf{x} = -\int_\Omega (\Delta u) u d\mathbf{x}.$$

We use Cauchy–Schwarz inequality to get

$$\|u\|^2_{L^2(\Omega)} \le C^2\|\nabla u\|^2_{L^2(\Omega)} \le C^2\|\Delta u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}.$$

That is, we have

$$\|u\|_{L^2(\Omega)} \le C^2\|\Delta u\|_{L^2(\Omega)} \tag{2.8}$$

and

$$\|\nabla u\|_{L^2(\Omega)} \le C\|\Delta u\|_{L^2(\Omega)}. \tag{2.9}$$

In addition, we need the standard Sobolev trace inequality:

$$\int_{\partial\Omega} |u|^2 d\sigma \le K\bigg(\epsilon\int_\Omega |\nabla u|^2 d\mathbf{x} + \frac{1}{\epsilon}\int_\Omega |u|^2 d\mathbf{x}\bigg) \tag{2.10}$$

for any $\epsilon \in (0,1)$, where $K > 0$ is a constant independent of $u$. The second term in (2.4) can be estimated by

$$\int_{\partial\Omega} \mathrm{tr}(\mathcal{B})(\nabla u\cdot\mathbf{n})^2 d\sigma \le c\int_{\partial\Omega}(\nabla u\cdot\mathbf{n})^2 d\sigma \le c\|\nabla u\|^2_{L^2(\partial\Omega)}.$$

We then use the trace inequality to the right-hand side of the above inequality to get

$$c\|\nabla u\|^2_{L^2(\partial\Omega)} \le cK\epsilon\sum_{i,j=1}^n\int_\Omega\bigg(\frac{\partial^2}{\partial x_i\partial x_j}u\bigg)^2 + \frac{cK}{\epsilon}\int_\Omega|\nabla u|^2 d\mathbf{x}. \tag{2.11}$$

Indeed, we first consider an open set $V_\epsilon \subset \Omega$ with $C^{1,1}$ boundary. Using the interior regularity of $u$, we have the above inequality with $\Omega$ replaced by $V_\epsilon$. Then we let $V_\epsilon \to \Omega$ to have (2.11).

Returning to (2.4), we have

$$\sum_{i,j=1}^n\int_\Omega\bigg(\frac{\partial^2}{\partial x_i\partial x_j}u\bigg)^2 \le \int_\Omega(\Delta u)^2 d\mathbf{x} + cK\epsilon\sum_{i,j=1}^n\int_\Omega\bigg(\frac{\partial^2}{\partial x_i\partial x_j}u\bigg)^2 + \frac{cK}{\epsilon}\int_\Omega|\nabla u|^2 d\mathbf{x}$$

By choosing $\epsilon = 1/(cK) < 1$ and using (2.9),

$$(1-\epsilon cK)\sum_{i,j=1}^n\int_\Omega\bigg(\frac{\partial^2}{\partial x_i\partial x_j}u\bigg)^2 \le \int_\Omega(\Delta u)^2 d\mathbf{x} + (cK)^2C^2\int_\Omega(\Delta u)^2 d\mathbf{x}.$$

The desired inequality (1.5) follows with $C_0 = (1+(cK)^2C^2)/(1-cK\epsilon)$. □

Let us make a remark that the result in the lemma above is not new. A similar result can be found in [18]. That is, when the domain with boundary $\partial\Omega$ consists of piecewise $C^2$ smooth surfaces with curvature bounded below by a constant $c > 0$, one has the property (1.5). See [18, Lemma 8.1]. When the domain $\Omega$ has a positive reach, we can estimate $c$ in terms of the reach $r = \mathrm{reach}(\Omega)$ as in the following lemma.

**Lemma 2.3** *Let $\Omega$ be a bounded set in $\mathbb{R}^n$ with a $C^{1,1}$ boundary. Suppose that the closure of $\Omega$ has a positive reach. If* $\mathrm{reach}(\Omega) \ge r$, *then* $\mathrm{tr}(\mathcal{B}) \le \frac{n-1}{r}$.

*Proof* Let $\Gamma$ be the boundary of $\Omega$. For any point $P$ on $\Gamma$, we choose a coordinate system $\{y_1, y_2, \dots, y_n\}$ with origin at $P$ such that the hyperplane $y_n = 0$ is tangent to $\Gamma$ at $P$, and choose a rectangular box

$$V = \{(y_1, y_2, \dots, y_n) : -a_j \le y_j \le a_j, 1 \le j \le n\},$$

and a function $\phi$ of class $C^{1,1}$ in the closure of

$$V' = \{(y_1, y_2, \dots, y_{n-1}) : -a_j \le y_j \le a_j, 1 \le j \le n-1\},$$

such that $|\phi(y_1, y_2, \dots, y_{n-1})| \le \frac{a_n}{2}$ for every $(y_1, y_2, \dots, y_{n-1}) \in V'$, and

$$\Omega \cap V = \{(y_1, y_2, \dots, y_n) \in V : y_n \le \phi(y_1, \dots, y_{n-1})\},$$
$$\Omega \cap \Gamma = \{(y_1, y_2, \dots, y_n) \in V : y_n = \phi(y_1, \dots, y_{n-1})\}.$$

Then,

$$\mathrm{tr}(\mathcal{B}) = \sum_{j=1}^{n-1} \frac{\partial^2 \phi}{\partial y_i^2}(0, 0, \dots, 0).$$

Note that by Lemma 2.1, for any $\delta < r$, there exists a ball of radius $\delta$ which intersects $\Omega$ only at the point $P$. Because the hyperplane $y_n = 0$ is the tangent plane of $\Omega$ at the origin $P$, this ball must lie entirely above the tangent hyperplane $y_n = 0$. Consequently, for $|h| < \min\{a_1, \delta\}$, we have $\phi(h, 0, \dots, 0) \le \delta - \sqrt{\delta^2 - h^2}$. Since $\phi(0, \dots, 0) = 0$ and $\frac{\partial \phi}{\partial y_1}(0, \dots, 0) = 0$, we obtain

$$\begin{aligned}\frac{\partial^2 \phi}{\partial y_1^2}(0, \dots, 0) &= \lim_{h \to 0} \frac{\phi(h, 0, \dots, 0) + \phi(-h, 0, \dots, 0)}{h^2} \\ &\le \lim_{h \to 0} \frac{2\delta - 2\sqrt{\delta^2 - h^2}}{h^2} \\ &= \frac{1}{\delta}.\end{aligned}$$

Similarly, we have $\frac{\partial^2 \phi}{\partial y_j^2}(0, \dots, 0) \le \frac{1}{\delta}$ for all $2 \le j \le n-1$. Because $\delta < r$ is arbitrary, the statement of the lemma follows. □

Combining the results above, we obtain the following:

**Corollary 2.4** *Let $\Omega$ be a bounded set in $\mathbb{R}^n$ with a $C^{1,1}$ boundary. Suppose that $\bar{\Omega}$ has a positive* reach$(\Omega) \ge r$. *Then*

$$\sum_{i,j=1}^{n} \int_\Omega \left(\frac{\partial^2}{\partial x_i \partial x_j} u\right)^2 \le C_1 \int_\Omega (\Delta u)^2 d\mathbf{x} \tag{2.12}$$

*for a constant $C_1 = (1 + ((n-1)K/r)^2 C^2)/(1 - \epsilon(n-1)K/r)$, where $C$ is the Poincaré constant in* (2.7) *and $K$ is the constant in the trace inequality in* (2.10).

Next we extend Theorem 1.7 to a more general setting. In particular, we need to remove the assumption of $C^{1,1}$ boundary in Corollary 2.4. Let us start with a well-known concept of star-shaped domains [4]. See Figure 4 for such an example.

**Definition 2.5** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say $\Omega$ is a star-shaped domain if there exists a point, say $\mathbf{x}_0 \in \Omega$ such that the line segment from $\mathbf{x}_0$ to any point $\mathbf{x} \in \Omega$ is completely contained in $\Omega$. $\mathbf{x}_0$ is called the center of $\Omega$.*

Clearly, we can extend the definition of star-shaped domains to the following way.

**Definition 2.6** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say $\Omega$ is a multiple-star-shaped domain if there exist finitely many points, say $\mathbf{x}_1, \dots, \mathbf{x}_k \in \Omega$ such that for any point $\mathbf{x} \in \partial\Omega$, there exists an index $i \in \{1, \dots, k\}$ such that the line segment $[\mathbf{x}, \mathbf{x}_i]$ is completely contained in $\Omega$. $\mathbf{x}_1, \dots, \mathbf{x}_k$ are called the centers of $\Omega$.*

Typically, a bounded domain with Lipschitz boundary is a multiple-star-shaped domain. A multiple-star-shaped domain may have a uniformly positive reach. See Figure 4 for an example.

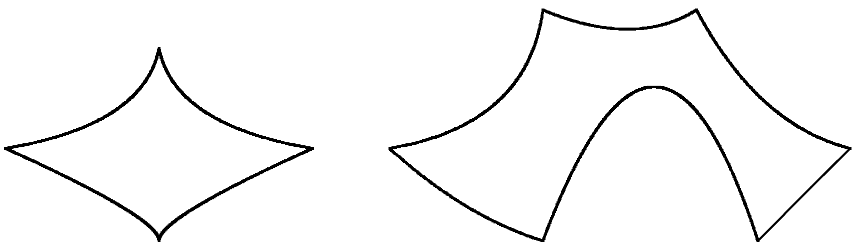

Figure 4 Left: A star-shaped domain which is not Lipschitz at one corner (the bottom one). Right: A multi-star shaped domain with uniform positive reach

We now state the $H^2$ regularity of the solution to the Dirichlet problem of Poisson equations over a general domain.

**Theorem 2.7** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded a star-shaped domain. Suppose that the closure of $\Omega$ is of positive reach $r$. For any $f \in L^2(\Omega)$, let $u \in H_0^1(\Omega)$ be the unique weak solution of the Dirichlet problem* (1.4)*. Then $u \in H^2(\Omega)$ satisfies* (1.5) *with constant $C_0$ dependent on $r$.*

To prove the above theorem, we need a few preparatory results. The lemma below shows that for any $\epsilon > 0$ there is an open subset $U_\epsilon \supset \Omega$ such that $U_\epsilon$ has a $C^{1,1}$ smooth boundary and $\text{dist}(U_\epsilon, \Omega) < \epsilon$.

**Lemma 2.8** *If $\Omega \subset \mathbb{R}^n$ is of positive reach $r_0 = \text{reach}(\Omega)$, then for any $0 < \epsilon < r_0$, the boundary of $\Omega_\epsilon := \Omega + B(0, \epsilon)$ containing $\Omega$ is of $C^{1,1}$. Furthermore, $\Omega_\epsilon$ has a positive reach $\geq r_0 - \epsilon$.*

*Proof* Because the boundary $B(0, \epsilon)$ is of $C^\infty$, it is known (see [16]) that the boundary of $\Omega_\epsilon$ is of $C^{1,1}$ if $\Omega$ is convex. We now extend the result to the setting that $\Omega$ is a domain whose closure has a positive reach.

For any $\mathbf{x} \in \partial\Omega_\epsilon$, there exists $\mathbf{y} \in \partial\Omega$ such that $\|\mathbf{x} - \mathbf{y}\| = \text{dist}(\mathbf{x}, \Omega) = \epsilon$ since $\Omega$ is of positive reach and $\epsilon < r_0$. Also, for any $\epsilon < r < r_0$, there exists a closed ball of radius $r$ that intersects $\Omega$ only at $\mathbf{y}$. Denote this ball by $B(\mathbf{c}, r)$. Then the ball $B(\mathbf{c}, r - \epsilon)$ intersects $\Omega_\epsilon$ only at $\mathbf{x}$.

On the other hand, the closed ball $B(\mathbf{y}, \epsilon)$ is contained in $\overline{\Omega}_\epsilon$ and intersect $\partial\Omega_\epsilon$ only at $\mathbf{x}$. Hence, $\partial\Omega_\epsilon$ intersects two tangent balls $B(\mathbf{y}, \epsilon)$ and $B(\mathbf{c}, r - \epsilon)$ at $\mathbf{x}$. Without loss of generality, we may assume $\mathbf{x} = 0$ and the tangent hyperplane at 0 is $x_n = 0$, where $\mathbf{x} = (x_1, \dots, x_n)$. The boundary of $\Omega_\epsilon$ can be expressed as $x_n = \phi(x_1, \dots, x_{n-1})$ by the implicit function theorem. Note that $\phi(0, 0, \dots, 0) = 0$. Letting $\bar{\mathbf{x}} = (x_1, \dots, x_{n-1})$ be the first part of $\mathbf{x} \in \partial\Omega$, for $\|\bar{\mathbf{x}}\| < \min\{\epsilon, r - \epsilon\}$, we have

$$\sqrt{\epsilon^2 - \|\bar{\mathbf{x}}\|^2} - \epsilon \leq \phi(\bar{\mathbf{x}}) = \phi(x_1, x_2, \dots, x_{n-1}) \leq r - \epsilon - \sqrt{(r - \epsilon)^2 - \|\bar{\mathbf{x}}\|^2}. \tag{2.13}$$

We claim that $\phi$ is differentiable at $\mathbf{0}$. Indeed, we have

$$\frac{\phi(\bar{\mathbf{x}}) - \phi(\mathbf{0})}{\|\bar{\mathbf{x}}\|} \le \frac{r - \epsilon - \sqrt{(r-\epsilon)^2 - \|\bar{\mathbf{x}}\|^2}}{\|\bar{\mathbf{x}}\|} = \frac{\|\bar{\mathbf{x}}\|}{r - \epsilon + \sqrt{(r-\epsilon)^2 - \|\bar{\mathbf{x}}\|^2}} \tag{2.14}$$

and

$$\frac{\phi(\bar{\mathbf{x}}) - \phi(\mathbf{0})}{\|\bar{\mathbf{x}}\|} \ge -\frac{\|\bar{\mathbf{x}}\|}{\epsilon + \sqrt{\epsilon^2 - \|\bar{\mathbf{x}}\|^2}} \tag{2.15}$$

by using the inequalities in (2.13). It follows that $\phi$ is differentiable at $\mathbf{0}$ and $\nabla\phi(\mathbf{0}) = \mathbf{0}$. Similar analysis for all point in $\Omega$ implies that $\Omega$ has a $C^1$ boundary.

We next claim that the gradient of $\phi$ is bounded from above and below. Using the notations above, we have

$$\frac{\phi(\bar{\mathbf{x}}) + \phi(-\bar{\mathbf{x}}) - \phi(\mathbf{0})}{2\|\bar{\mathbf{x}}\|^2} \le \frac{r - \epsilon - \sqrt{(r-\epsilon)^2 - \|\bar{\mathbf{x}}\|^2}}{\|\bar{\mathbf{x}}\|^2} = \frac{1}{r - \epsilon + \sqrt{(r-\epsilon)^2 - \|\bar{\mathbf{x}}\|^2}} \tag{2.16}$$

and similarly,

$$\frac{\phi(\bar{\mathbf{x}}) + \phi(-\bar{\mathbf{x}}) - \phi(\mathbf{0})}{2\|\bar{\mathbf{x}}\|^2} \ge -\frac{1}{\epsilon + \sqrt{\epsilon^2 - \|\bar{\mathbf{x}}\|^2}}. \tag{2.17}$$

Therefore, for all $\|\bar{\mathbf{x}}\| \le \min\{r-\epsilon, \epsilon\}$, we have

$$-\frac{1}{\epsilon} < \frac{\phi(\bar{\mathbf{x}}) + \phi(-\bar{\mathbf{x}}) - 2\phi(\mathbf{0})}{2\|\bar{\mathbf{x}}\|^2} < \frac{1}{r-\epsilon}. \tag{2.18}$$

We now apply a known result Lemma 2.9 to finish the proof of $C^{1,1}$.

Finally, we show $\Omega$ is of reach $r - \epsilon$. For any point $q \notin \Omega_\epsilon$, if $\mathrm{dist}(q, \Omega_\epsilon) < r - \epsilon$, we know $\mathrm{dist}(q, \Omega) = \delta < r$. Since the reach of $\Omega$ is at least $r$, there exists a closed ball centering at $q$ with radius $\delta > 0$ which intersects $\Omega$ only at one point $p$, say. Now the closed ball centered at $q$ with radius $\delta - \epsilon$ intersects $\Omega_\epsilon$ only at one point, namely the point $p + \epsilon(q-p)/\delta$. Therefore, $\Omega_\epsilon$ is of reach $\delta - \epsilon$. This holds for all $\delta < r$. Thus, $\Omega_\epsilon$ has a reach of $\delta - \epsilon$. In general, $r - \epsilon$ is the best one can hope for. Indeed, if $\Omega$ is the complement of an open ball with radius $r$ which has a positive reach $r$, then $\Omega_\epsilon$ is the complement of an open ball with radius $r - \epsilon$ which has a reach $r - \epsilon$. □

**Lemma 2.9** *Assume that the function $f$ is bounded on a neighborhood of $\mathbf{x}_0 \in \partial\Omega$. $f$ is of class $C^{1,1}$ at $\mathbf{x}_0$ if and only if there exists a neighborhood $U$ of $\mathbf{x}_0$ such that the central difference*

$$\Delta_2 f(\mathbf{x}; h) = f(\mathbf{x} + h\mathbf{d}) - 2f(\mathbf{x}) + f(\mathbf{x} - h\mathbf{d}) \tag{2.19}$$

*is bounded on $U$ for all $h \in (-\delta, \delta)$ for a fixed $\delta > 0$ and $\mathbf{d} \in \mathcal{S}^{n-1}$ which is the unit sphere in $\mathbb{R}^n$.*

*Proof* See [29, Corollary 2.1] for a proof. □

Next we show that if $\Omega$ is a star-shaped domain whose closure is of positive reach, then for any $\epsilon > 0$, there exists an open set $U_\epsilon \subset \Omega$ with $C^{1,1}$ boundary with $\mathrm{dist}(\Omega, U_\epsilon) < \epsilon$.

**Lemma 2.10** *Let $\Omega$ be a bounded domain with a positive reach in $\mathbb{R}^n$. Suppose that $\Omega$ is a star-shaped domain. Then, for each $\epsilon > 0$, there exists an open set $U_\epsilon$ with $C^{1,1}$ smooth boundary such that $U_\epsilon \subset \Omega$ and $\mathrm{dist}(\Omega, U_\epsilon) \le \epsilon$.*

*Proof* Let $\mathbf{x}_0 \in \Omega$ be the center of the star-shaped domain $\Omega$. By Lemma 2.8, we have $\Omega_\epsilon$ containing $\Omega$ with $C^{1,1}$ boundary for any $\epsilon < r$, where $r$ stands for the reach of $\Omega$. Letting $\partial\Omega_\epsilon$ be the boundary of $\Omega_\epsilon$, we define

$$U_\epsilon = \left\{ \mathbf{y} = \mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0) : 0 \le t \le \left(1 - \frac{2\epsilon}{\|\mathbf{x} - \mathbf{x}_0\|}\right), \mathbf{x} \in \partial\Omega_\epsilon \right\}. \tag{2.20}$$

We claim that $U_\epsilon \subset \Omega$ is a domain with $C^{1,1}$ boundary. Indeed, at the center $\mathbf{x}_0 \in \Omega$, we fix a spherical coordinate system. For any point $\mathbf{x} \in \partial\Omega_\epsilon$, we define a function $\phi$ of the angle of the ray from $\mathbf{x}_0$ to $\mathbf{x}$ with value to be the length $\|\mathbf{x} - \mathbf{x}_0\|$. Then $\phi$ is a function describing the boundary of $\Omega_\epsilon$. Since $\Omega_\epsilon$ has a $C^{1,1}$ boundary by Lemma 2.8, $\phi$ is a $C^{1,1}$ function. Next we define a new function $\psi$ over the same spherical coordinate system as $\phi$ by

$$\psi\left(\mathbf{x}_0 + \left(1 - \frac{2\epsilon}{\|\mathbf{x} - \mathbf{x}_0\|}\right)(\mathbf{x} - \mathbf{x}_0)\right) = \phi(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega_\epsilon$$

which is the function which can describe the boundary of $U_\epsilon$. Clearly, $\psi(\mathbf{x})$ is a $C^{1,1}$ function for $\mathbf{x} \in U_\epsilon$ and hence, $U_\epsilon$ has a $C^{1,1}$ boundary. □

We are now ready to prove Theorem 2.7 under an assumption that $\Omega$ is a star-shaped domain.

*Proof of Theorem* 2.7 We use Lemma 2.10 to select a sequence of sets $U_\epsilon \subset \Omega$ with $\mathrm{dist}(\Omega, U_\epsilon) \le \epsilon$ with $\epsilon \to 0$.

For a function $f \in L^2(\Omega)$. Then, we define $u_\epsilon \in H_0^1(U_\epsilon)$ to be the weak solution of

$$\begin{cases} -\Delta u = f, & \text{in } U_\epsilon, \\ u = 0, & \text{on } \partial U_\epsilon. \end{cases} \tag{2.21}$$

We claim that $u_\epsilon \in H^2(U_\epsilon)$. Indeed, using Lemma 2.10 again, we take an open set $V_\eta \subset U_\epsilon$. The interior regularity of $u_\epsilon$ [10] and the $C^{1,1}$ boundary of $V_\eta$ allows us to have

$$\begin{aligned} \sum_{i,j=1}^n \int_{V_\eta} \left(\frac{\partial^2}{\partial x_i \partial x_j} u_\epsilon\right)^2 = \int_{V_\eta} f^2 d\mathbf{x} + \int_{\partial V_\eta} \nabla_T u_\epsilon \nabla_T (\nabla u_\epsilon \cdot \mathbf{n}) d\sigma \\ + \int_{\partial V_\eta} [\mathcal{B}(\nabla u_\epsilon|_T, \nabla u_\epsilon|_T) + \mathrm{tr}(\mathcal{B})(\nabla u_\epsilon \cdot \mathbf{n})^2] d\sigma \end{aligned} \tag{2.22}$$

by using (2.4), where $T$ and $\mathbf{n}$ stand for the tangential and normal direction of $\partial V_\eta$. As the above equation in (2.22) holds for all $V_\eta$, hence it holds for $U_\epsilon$. Using the boundary condition $u_\epsilon|_{\partial\Omega} = 0$, we have $\nabla_T u_\epsilon = 0$ and

$$\sum_{i,j=1}^n \int_{U_\epsilon} \left(\frac{\partial^2}{\partial x_i \partial x_j} u_\epsilon\right)^2 = \int_{U_\epsilon} f^2 d\mathbf{x} + \int_{\partial U_\epsilon} \mathrm{tr}(\mathcal{B})(\nabla u_\epsilon \cdot \mathbf{n})^2 d\sigma, \tag{2.23}$$

Since the domain $U_\epsilon$ has $C^{1,1}$ boundary and a positive reach, we use Corollary 2.4, i.e. (2.12) to have

$$\sum_{i,j=1}^n \int_{U_\epsilon} \left(\frac{\partial^2}{\partial x_i \partial x_j} u_\epsilon\right)^2 \le C_1 \|f\|_{L^2(\Omega)}^2$$

with $C_1 \ge 1$ being a positive constant dependent on the reach.

Let us extend $u_\epsilon$ by zero outside $U_\epsilon$ to $\Omega$, also denote it by $u_\epsilon$. Then $u_\epsilon$ belongs to $H^2(\Omega)\cap H_0^1(\Omega)$ for all $\epsilon>0$ sufficiently small and

$$\|u_\epsilon\|_{H^2(U_\epsilon)} \le C_2\|f\|_{L^2(\Omega)} \tag{2.24}$$

for another positive constant $C_2$. By Rellich's theorem, there is $\tilde{u}\in H^2(\Omega)$ and a subsequence, without loss of generality, we may assume that $u_\epsilon \to \tilde{u}$ strongly in $L^2(\Omega)$, strongly in $H^1(\Omega)$, and weakly in $H^2(\Omega)$.

We now claim that $u=\tilde{u}$. Since $u\in H_0^1(\Omega)$, for any $\phi\in C_0^\infty(\Omega)$, we have

$$\int_\Omega \nabla u\nabla\phi d\mathbf{x} = -\int_\Omega f\phi d\mathbf{x}. \tag{2.25}$$

Since $\phi$ is compactly supported in $\Omega$, Lemma 2.10 shows that there is $\epsilon>0$ such that supp($\phi$) $\subset U_\epsilon$ for all $\epsilon>0$ sufficiently small. Since $u_\epsilon\in H^2(U_\epsilon)\cap H_0^1(U_\epsilon)$ solves the Poisson equation over $U_\epsilon$ and $u_\epsilon\to\tilde{u}$ in $H_0^1(\Omega)$ weakly, we get

$$\begin{aligned}\int_\Omega \nabla\tilde{u}\nabla\phi d\mathbf{x} &= \lim_{\epsilon\to 0_+}\int_\Omega \nabla u_\epsilon\nabla\phi d\mathbf{x}\\ &= \lim_{\epsilon\to 0_+} -\int_{U_\epsilon} f\phi d\mathbf{x}\\ &= -\int_\Omega f\phi d\mathbf{x}\\ &= \int_\Omega \nabla u\nabla\phi d\mathbf{x}\end{aligned} \tag{2.26}$$

by using (2.25). It follows that $\nabla u\equiv\nabla\tilde{u}$ since (2.26) holds for all $\phi\in C_0^\infty(\Omega)$. Then $u\equiv\tilde{u}$ because of the zero boundary condition for both $u$ and $\tilde{u}$. Therefore, $u\in H^2(\Omega)$.

**Remark 2.11** Based on the proof above, we can remove the $C^{1,1}$ assumption in Corollary 2.4 to have (2.12) when the underlying domain $\Omega$ is a star-shaped domain.

Finally let us proceed to establish Theorem 1.4.

*Proof of Theorem* 1.4 The proof of Theorem 1.4 is similar to that of Theorem 2.7 in the following senses. Indeed, instead of defining $U_\epsilon\subset\Omega$ by using the formula in (2.20), we let $E_\epsilon(\Omega)=(\Omega^c+B(0,\epsilon))^c\subset\Omega$ be an $\epsilon$-erosion of $\Omega$ and

$$U_\epsilon = E_{2\epsilon}(\Omega)+B(0,\epsilon)\subset\Omega. \tag{2.27}$$

By the assumption of Theorem 1.4, i.e. the uniformly positive reach, we know the domain $E_{2\epsilon}(\Omega)$ has a positive reach $r_0$ for sufficiently small $\epsilon$. Then by Lemma 2.8, $U_\epsilon$ has $C^{1,1}$ boundary. Then the rest of the proof is the same as that of Theorem 2.7. These finish the proof. □

## 3 Some Applications

In this section, we apply the new regularity conditions to two examples of PDE and establish the $H^2$ regularity.

### 3.1 The Strong Solution to Second Order PDE in Non-divergence Form

In this section we study the solution to the second order PDE in non-divergence form in (1.1). For simplicity, we let

$$\mathcal{L}(u) = \sum_{i,j=1}^{n} a_{i,j}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} u \tag{3.1}$$

be the differential operator associated with the model problem (1.1). Next we assume that

**Definition 3.1** *The PDE coefficients $a_{ij}$ $(i, j = 1, \ldots, n)$ satisfy the Cordés condition* [23]:

$$\frac{\sum_{i,j=1}^{n} a_{ij}^2}{(\sum_{i=1}^{n} a_{ii})^2} \leq \frac{1}{n-1+\epsilon}, \quad \text{in } \Omega, \tag{3.2}$$

*for a positive number $\epsilon \in (0, 1]$.*

Following the studies in [25, 26], and [23], we assume that

$$\gamma = \frac{\sum_{i=1}^{n} a_{ii}}{\sum_{i,j=1}^{n} a_{ij}^2} > 0. \tag{3.3}$$

For example, the ellipticity of the PDE in (1.1) will imply (3.3) when $n = 2$. The following result is known.

**Lemma 3.2** ([23, 25]) *Suppose that the PDE coefficients $a_{ij}, (i, j = 1, \ldots, n)$ satisfy the Cordés condition. Then*

$$|\gamma \mathcal{L}(u) - \Delta u| \leq \sqrt{1-\epsilon} \left( \sum_{i,j=1}^{n} \left( \frac{\partial^2}{\partial x_i \partial x_j} u \right)^2 \right)^{1/2}. \tag{3.4}$$

With this preparation, we are able to prove one of our main results in this section.

**Theorem 3.3** *Suppose that $\Omega$ has a uniformly positive reach. If the PDE coefficients $a_{ij}$ satisfy the Cordés condition with an $\epsilon$ close to 1 such that $\sqrt{(1-\epsilon)C_0} < 1$, then there exists a unique strong solution $u \in H^2(\Omega)$ to the PDE in* (1.1)*, where $C_0$ appears in Theorem* 1.4*, i.e. in* (1.5)*.*

*Proof* We mainly follow the approach in [25]. First of all, it is easy to see that $\|\Delta u\|_{L^2(\Omega)}$ is a norm on $H$. By Theorem 1.4, $\|\Delta u\|_{L^2(\Omega)} = 0$ implies

$$\sum_{i,j=1}^{n} \left( \frac{\partial^2}{\partial x_i \partial x_j} u \right)^2 = 0$$

by using (1.5). Thus, $u$ is a linear polynomial over $\Omega$. The zero boundary condition of $u$ implies $u \equiv 0$. The other norming properties can be established in a standard fashion.

Next let us consider an equivalent PDE: find $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying the following:

$$\gamma \mathcal{L}(u) = \gamma f, \quad \in \Omega. \tag{3.5}$$

Write $H = H^2(\Omega) \cap H_0^1(\Omega)$ and define a bilinear form [25].

$$A(u, v) = \int_{\Omega} \gamma \mathcal{L}(u) \Delta v d\mathbf{x} \tag{3.6}$$

We now claim that the bilinear form $A(u, v)$ is continuous in the sense that there is a positive constant $\beta > 0$ such that

$$|A(u, v)| \leq \beta \|\Delta u\|_{L^2(\Omega)} \|\Delta v\|_{L^2(\Omega)} \tag{3.7}$$

for all $u, v \in H$ and coercive, i.e.

$$A(u,u) \geq \alpha \|\Delta u\|^2_{L^2(\Omega)}, \quad \forall u \in H \tag{3.8}$$

for $\epsilon \in (0,1)$ large enough, where $\alpha > 0$ is a constant independent of $u$. Indeed, we use Lemma 3.2 to have

$$\begin{aligned}\left| A(u,u) - \int_\Omega (\Delta u)^2 d\mathbf{x} \right| &\leq \int_\Omega |\gamma \mathcal{L}(u) - \Delta u||\Delta u| d\mathbf{x} \\ &\leq \int_\Omega \sqrt{1-\epsilon} \left( \sum_{i,j=1}^n \left( \frac{\partial^2}{\partial x_i \partial x_j} u \right)^2 \right)^{1/2} |\Delta u| d\mathbf{x} \\ &\leq \sqrt{1-\epsilon} C_0^{1/2} \int_\Omega (\Delta u)^2 d\mathbf{x}\end{aligned}$$

by Cauchy–Schwarz inequality and the inequality in (2.12). If $\epsilon \in (0,1)$ is large enough such that $\sqrt{C_0}\sqrt{1-\epsilon} < 1$, we have (3.7) and (3.8) with $\beta = 1 + \sqrt{1-\epsilon}\sqrt{C_0}$ and $\alpha = 1 - \sqrt{1-\epsilon}\sqrt{C_0}$.

Now Lax–Milgram theorem implies that there exists a unique weak solution in $H$ satisfying

$$A(u,v) = \int_\Omega \gamma f \Delta v d\mathbf{x}.$$

That is, $u$ is a weak solution in $H$ satisfying (3.5). Since $u \in H^2(\Omega)$, $u$ is a strong solution. This finishes the proof of Theorem 3.3. □

## 3.2 The Regularity of the Solution to the Helmholtz Problem

We are interested in the regularity of the solution of the following Helmholtz problem:

$$\begin{cases} -\Delta u - k^2 u = f, & \text{in } \Omega \subset \mathbb{R}^2 \\ \dfrac{\partial}{\partial \mathbf{n}} u + \mathbf{i} k u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where $\Omega$ is a bounded domain with Lipschitz boundary, $\mathbf{i} = \sqrt{-1}$ denotes the imaginary unit, $\mathbf{n}$ is the normal to $\partial\Omega$, and $k \geq 1$ is the wave number. This Helmholtz problem arises from many application areas: acoustic scattering, electromagnetic fields, etc. In the literature, the solution to Helmholtz problem (3.9) will be $H^2$ if $\Omega$ is convex or $\Omega$ has a $C^2$ smooth boundary (see [8]). We now show that the unique solution $u$ is of $H^2$ regularity when $\Omega$ has a uniform positive reach.

**Theorem 3.4** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose the closure of $\Omega$ is of uniformly positive reach $r_\Omega > 0$. For any $f \in L^2(\Omega)$, let $u \in H^1_0(\Omega)$ be the unique weak solution of the Helmholtz problem* (3.9)*. Then $u \in H^2(\Omega)$ satisfying* (1.5) *for a positive constant $C_0$ dependent on $r_\Omega$, but independent of $f$ and $u$.*

*Proof* The ideas for this proof are similar to that of Theorem 1.4. We first prove that the solution $u$ satisfies (1.5) if $u \in H^2(\Omega)$. Then we find a sequence of subdomains $U_\epsilon \subset \Omega$ which has $C^{1,1}$ boundary with uniform positive reach $r_\Omega/2$ and solve $u_\epsilon \in H^2(\Omega_\epsilon)$ satisfying (1.5) with a constant $C_0$ dependent on $r_\Omega/2$. Finally, we use Rellich theorem to find a limit $u \in H^2(\Omega)$ of $u_\epsilon$ strongly in $L^2(\Omega)$ and strong in $H^1(\Omega)$ and weakly in $H^2(\Omega)$.

The major proof is to show the boundedness of $u$ in $H^2(\Omega)$ semi-norm. Let us start with the the following identity [13]: when $\Omega$ has $C^{1,1}$ boundary and $u \in \mathbb{H}^2(\Omega)$, $u$ satisfies

$$\sum_{i,j=1}^{n} \int_\Omega \left(\frac{\partial^2}{\partial x_i \partial x_j} u\right)^2 = \int_\Omega (\Delta u)^2 d\mathbf{x} + \int_\Gamma \nabla_T u \nabla_T (\nabla u \cdot \mathbf{n}) d\sigma + \int_\Gamma [\mathcal{B}(\nabla u|_T, \nabla u|_T) + \mathrm{tr}(\mathcal{B})(\nabla u \cdot \mathbf{n})^2] d\sigma \tag{3.10}$$

for $n \geq 2$, where tr is the standard trace operator, $\mathcal{B}$ is a bilinear form associated with the curvature of boundary surface $\Gamma$ of $\Omega$ as defined in [13], and $\mathbf{n}$ is the unit outward normal direction of $\Gamma = \partial\Omega$ and $T$ is the unit tangential direction of $\Gamma$.

Since $u$ is the solution of (3.9), we use the boundary condition to get

$$\int_\Gamma \nabla_T u \nabla_T (\nabla u \cdot \mathbf{n}) d\sigma = -k \int_\Gamma \nabla_T u \nabla_T u d\sigma = -k \int_\Gamma |\nabla_T u|^2 d\sigma \leq 0. \tag{3.11}$$

Next we split $\Gamma$ into three portions according the symmetric Hessian matrix $\mathcal{B}$, i.e. $\Gamma = \Gamma_p \cup \Gamma_n \cup \Gamma_i$ such that $\mathcal{B} \geq 0$ for $\mathbf{x} \in \Gamma_p$, $\mathcal{B} < 0$ for $\mathbf{x} \in \Gamma_n$ and $\mathcal{B}$ is indefinite when $\mathbf{x} \in \Gamma_i$. The last term in (3.10) can be rewritten as

$$\int_\Gamma [\mathcal{B}(\nabla u|_T, \nabla u|_T) + \mathrm{tr}(\mathcal{B})(\nabla u \cdot \mathbf{n})^2] d\sigma \leq \int_{\Gamma_p \cup \Gamma_i} [\mathcal{B}(\nabla u|_T, \nabla u|_T) + \mathrm{tr}(\mathcal{B})(\nabla u \cdot \mathbf{n})^2] d\sigma.$$

When the underlying domain $\Omega$ has a positive reach $r_\Omega$, we have $\mathrm{tr}(\mathcal{B}) \leq (n-1)/r_\Omega$ over the boundary $\Gamma$ by Lemma 2.3. Also, over $\Gamma_p$, we $\mathcal{B}(\nabla u|_T, \nabla u|_T) \leq \|\nabla u\|_\Gamma^2 \|\mathcal{B}\| \leq |\nabla u\|_\Gamma^2 (n-1)/r_\Omega$. Thus,

$$\int_{\Gamma_p} [\mathcal{B}(\nabla u|_T, \nabla u|_T) + \mathrm{tr}(\mathcal{B})(\nabla u \cdot \mathbf{n})^2] d\sigma \leq \frac{n-1}{r_\Omega} \int_{\Gamma_p} [|\nabla u_T|^2 + (\nabla u \cdot \mathbf{n})^2] d\sigma.$$

Over $\Gamma_i$, we rewrite $\mathcal{B} = \mathbb{B}_p - \mathbb{B}_n$ with symmetric nonnegative definite matrices $\mathbb{B}_p$ and $\mathbb{B}_n$. Thus,

$$\int_{\Gamma_i} \left[\mathcal{B}(\nabla u|_T, \nabla u|_T) + \mathrm{tr}(\mathcal{B})(\nabla u \cdot \mathbf{n})^2\right] d\sigma \leq \int_{\Gamma_i} \mathbb{B}_p(\nabla u|_T, \nabla u|_T) d\sigma + \int_{\Gamma_i} \mathrm{tr}(\mathbb{B}_p)(\nabla u \cdot \mathbf{n})^2 d\sigma.$$

By the same argument as the proof of Lemma 2.3, we have $\mathrm{tr}(\mathbb{B}_p) \leq (n-1)/r_\Omega$ and hence, $\|\mathcal{B}_p\| \leq \mathrm{tr}(\mathbb{B}_p)$. That is, we also have

$$\int_{\Gamma_i} \mathbb{B}_p(\nabla u|_T, \nabla u|_T) d\sigma + \int_{\Gamma_i} \mathrm{tr}(\mathbb{B}_p)(\nabla u \cdot \mathbf{n})^2 d\sigma \leq \frac{n-1}{r_\Omega} \int_{\Gamma_p} [|\nabla u_T|^2 + (\nabla u \cdot \mathbf{n})^2] d\sigma.$$

We summarize the discussion above to have

$$\begin{aligned}
\sum_{i,j=1}^{2} \int_\Omega \left(\frac{\partial^2}{\partial x_i \partial x_j} u\right)^2 &\leq \int_\Omega (\Delta u)^2 d\mathbf{x} + \frac{n-1}{r_\Omega} \left(\int_{\Gamma_p} + \int_{\Gamma_i}\right) [|\nabla u_T|^2 + (\nabla u \cdot \mathbf{n})^2] d\sigma \\
&= \int_\Omega (\Delta u)^2 d\mathbf{x} + \frac{n-1}{r_\Omega} \int_\Gamma |\nabla u|^2 d\sigma \\
&\leq \int_\Omega (f - k^2 u)^2 + cK\epsilon \sum_{i,j=1}^{n} \int_\Omega \left(\frac{\partial^2}{\partial x_i \partial x_j} u\right)^2 + \frac{cK}{4\epsilon} \int_\Omega |\nabla u|^2 d\mathbf{x}
\end{aligned}$$

for some $\epsilon > 0$, where $c = (n-1)/r_\Omega$ for convenience. Choosing $\epsilon K c < 1/2$, we obtain

$$\sum_{i,j=1}^{2} \int_\Omega \left(\frac{\partial^2}{\partial x_i \partial x_j} u\right)^2 \leq 4\|f\|^2 + 4k^4 \int_\Omega u^2 + \frac{cK}{2\epsilon} \int_\Omega |\nabla u|^2 d\mathbf{x}. \tag{3.12}$$

Next we use [19, Theorem 1] to have

$$\sum_{i,j=1}^{2}\int_{\Omega}\left(\frac{\partial^2}{\partial x_i\partial x_j}u\right)^2 \le C(1+k^2)\|f\|^2 \tag{3.13}$$

for a positive constant $C$ independent of $f$ and $k$. Therefore, in the semi-norm of $\mathbb{H}^2(\Omega)$, $|u|_{2,\Omega} \le C(1+k)\|f\|$. We have thus established Theorem 3.4. □

## 4 Remarks

We present a few remarks on domains with uniformly positive reach. Let us start with the following example

**Remark 4.1** There is a bounded domain $\Omega$ which satisfies the uniform outer ball condition, but does not have a positive reach. See Figure 5.

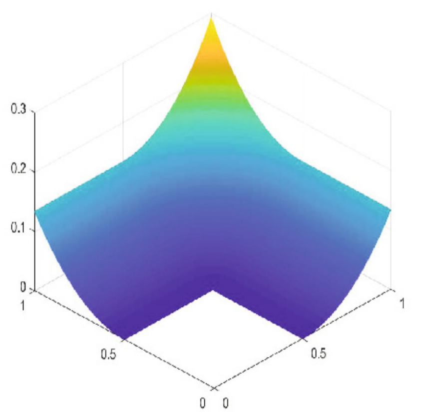


Figure 5 The solid domain has a uniform outer ball condition while does not have a positive reach.

However, it is not a Lipschitz boundary and hence, Theorem 1.6 can not be applied.

**Remark 4.2** It is known that a domain $\Omega \subset \mathbb{R}^n$ with $C^2$ boundary, $\Omega$ has a positive reach by [17, Lemma 1.2.5]. Also, in [17], there is an example, a $C^{1,\alpha}$ domain does not have a positive reach for some $\alpha > 0$.

**Remark 4.3** Let us give a sufficient condition to ensure that the uniform outer ball condition implies the condition of positive reach.

**Lemma 4.4** *If a set $\Omega$ satisfies a uniform outer ball condition with radius $r_0$ and $\Omega$ is a $C^1$ boundary, then it is of positive reach $r_0$.*

*Proof* For any $p \in \bar{\Omega}^c$ with $0 < \mathrm{dist}(p, \bar{\Omega}) < r_0$, let $q \in \bar{\Omega}$ be a projection of $p$ on $\partial\Omega$. Because $\bar{\Omega}$ has an out-ball of radius $r_0$ at $q$, let $B(w, r_0)$ be the ball touched at $q$. Then $p$ must lie on the line segment between $w$ and $q$. This is because the both vectors $\vec{qp}$ and $\vec{qw}$ orthogonal to the tangent hyperplane of $\Omega$ at $q$ and $\mathrm{dist}(p, \bar{\Omega}) < r_0 = \mathrm{dist}(w, q)$. Now $B(p, \|p-q\|) \subset B(w, r_0)$, $B(p, \|p-q\|)$ can only touch $(\bar{\Omega})$ at $q$. Hence, $p$ has a unique projection. That is, the reach of $\bar{\Omega}$ is greater or equal to $r_0$. □

**Remark 4.5** It is known that if $\Omega$ has a $C^2$ boundary, then $\Omega$ has a positive reach by [17, Lemma 1.2.5]. We now show that $\Omega$ has a uniformly positive reach.

**Lemma 4.6** *Suppose that a bounded domain $\Omega$ has a $C^2$ boundary. Then $\Omega$ has a uniformly positive reach.*

*Proof* By [17, Lemma 1.2.5], $\Omega$ has a positive reach. In fact, according to the proof, there exists an open set $V$ containing the boundary $\partial\Omega$ such that any point $v \in V$ has a unique projection on $\partial\Omega$. That is, both $\overline{\Omega}$ and $\overline{\Omega^c}$ have a positive reach.

Next we claim that if $\overline{\Omega}$ is of positive reach $r_0$ and $\overline{\Omega^c}$ is of positive reach $r_1$, then for any $\varepsilon < r_1$, the $\epsilon$-erosion $\overline{\Omega_\varepsilon}$ has a positive reach $\geq r_0 + \varepsilon$. Indeed, for any $x \in \partial\Omega_\varepsilon$, by the definition of $\Omega_\varepsilon$, there exists $p \in \partial\Omega$ such that $\|x - p\| = \text{dist}(x, \partial\Omega) = \varepsilon$. Because $\Omega$ has a tangent hyperplane at $p$, we have $x = p + \varepsilon\vec{n_p}$, where $\vec{n_p}$ is the unit inward normal vector of $\overline{\Omega}$ at $p$. On the other hand, for any $p \in \partial\Omega$, the point $p + \varepsilon\vec{n_p}$ is clearly at a distance $\varepsilon$ away from $\Omega^c$. Because $\overline{\Omega^c}$ is of positive reach $r_1 > \varepsilon$, the distance between $p + \varepsilon\vec{n_p}$ and any point $q \in \partial\Omega \setminus \{p\}$ is larger than $\varepsilon$, thus we must have $p + \varepsilon\vec{n_p} \in \partial\Omega_\varepsilon$. Therefore, we have the following characterization of $\partial\Omega_\varepsilon$:

$$\partial\Omega_\varepsilon = \{p + \varepsilon\vec{n_p} : p \in \partial\Omega\}.$$

For any $w$ satisfying $\text{dist}(w, \overline{\Omega_\varepsilon}) \leq r_0 + \varepsilon$. If $w \in \overline{\Omega}$, we let $p$ be a point on $\overline{\Omega}$ that is closest to $w$. Because $\text{dist}(w, \partial\Omega) \leq \varepsilon$, by the assumption that $\overline{\Omega^c}$ is of positive reach $r_1 > \varepsilon$, the point $p$ is uniquely determined by $w$. We show that $x = p + \varepsilon\vec{n_p}$ is the unique point on $\overline{\Omega}$ that is closest to $p$. Indeed, suppose $u \in \overline{\Omega_\varepsilon}$ satisfies $\|u - p\| \leq \varepsilon$. Since $\text{dist}(\overline{\Omega_\varepsilon}, \partial\Omega) = \varepsilon$, $p$ is a point on $\partial\Omega$ that is closest to $u$. This means that $u = p + \varepsilon\vec{n_p} = x$. Hence, $x = p + \varepsilon\vec{n_p}$ is the unique point on $\overline{\Omega}$ that is closest to $p$. In particular, this implies that the closed ball centering at $p$ with radius $\varepsilon$ intersects $\overline{\Omega_\varepsilon}$ only at the point $x$. Since $w$ lies between $p$ and $x$, the closed ball centering at $w$ with radius $\|x - w\|$ intersects $\overline{\Omega_\varepsilon}$ only at the point $x$. Thus, $x$ is the unique point in $\overline{\Omega_\varepsilon}$ that is closest to $w$.

If $w \notin \overline{\Omega}$. Let $x$ be a point in $\overline{\Omega_\varepsilon}$ that is closest to $w$. We show that $x$ is uniquely determined by $w$. Indeed, let $p$ be the intersection of the line segment $xw$ with $\partial\Omega$. Because for any $u \in \overline{\Omega_\varepsilon}$,

$$\|u - p\| + \|p - w\| \geq \|u - w\| \geq \|x - w\| = \|x - p\| + \|p - w\|,$$

$x$ is a point on $\overline{\Omega}$ that is closest to $p$. By the characterization proved above, we have $x = p + \varepsilon\vec{n_p}$. Consequently, $w = p - \|w - p\|\vec{n_p}$. Since

$$\|w - p\| = \|w - x\| - \|p - x\| = \|w - x\| - \varepsilon = \text{dist}(w, \overline{\Omega_\varepsilon}) - \varepsilon < r_0,$$

by the assumption that $\overline{\Omega}$ is of positive reach $r_0$, the point $p$ is uniquely determined by $w$. Hence, $x = p + \varepsilon\vec{n_p}$ is also uniquely determined by $w$. Therefore, $\overline{\Omega_\varepsilon}$ is of positive reach $\geq r_0 + \varepsilon$ and the claim is prove. □

Thus, the solution to the Dirichlet problem of the Poisson equation is in $H^2(\Omega)$ with constant $C_0$ in (1.5) dependent on the positive reach. When the boundary is $C^2$, we can find the positive reach $r_\Omega$ and hence we know how large the regularity constant $C_0$ in (1.5).

## References

[1] Adolfsson, V.: $L^2$ integrability of second order derivatives for Poisson equations in nonsmooth domain. *Math. Scan.*, **70**, 146–160 (1992)

[2] Agranovich, M. S.: Sobolev Spaces, Their Generalizations and Elliptic Problems in Smooth and Lipschitz Domains, Springer Monographs in Mathematics. Springer-Verlag, 2015

[3] Awanou, G., Lai, M. J., Wenston, P.: The multivariate spline method for numerical solution of partial differential equations, in: Wavelets and Splines, Nashboro Press, Brentwood, 24–74 (2006)

[4] Brenner, S. C., Scott, L. R.: The Mathematical Theory of Finite Element Methods, Springer-Verlag, New York, 1994

[5] Caccioppoli, R.: Limitazioni integrali per le soluzioni di unequazione linear ellittica a derivate parziali. *Giornale di Bataglini*, **80**, 186–212 (1950–51)

[6] Calderón, A. P., Zygmund, A.: On the existence of certain singular integrals. *Acta Math.*, **88**, 85–139 (1952)

[7] Cockreham, J., Gao, F.: Metric entropy of classes of sets with positive reach. *Constructive Approximation*, **47**, 357–371 (2018)

[8] Cummings, P., Feng, X. B.: Shape regularity coefficient estimates for complex-valued acoustic and elastic Helmholtz equations. *Math. Models Methods in Applied Sciences*, **16**, 139–160 (2006)

[9] Duong, X., Hofmann, S., Mitrea, D., et al.: Hardy spaces and regularity for the inhomogeneous Dirichlet and Neumann problems. *Rev. Mat. Iberoam.*, **29**, 183–236 (2013)

[10] Evans, L.: Partial Differential Equations, American Math. Society, Providence, 1998

[11] Federer, H.: Curvature measures. *Trans. Am. Math. Soc.*, **93**, 418–491 (1959)

[12] Gilbarg, D., Trudinger, N. S.: Elliptic Partial Differential Equations of Second Order. Springer-Verlag, Berlin, 1998

[13] Grisvard, P.: Elliptic Problems in Nonsmooth Domains, Pitman, Boston, 1985

[14] Hu, X., Han, D., Lai, M. J.: Bivariate Splines of Various Degrees for Numerical Solution of PDE. *SIAM Journal of Scientific Computing*, **29**, 1338–1354 (2007)

[15] Khelif, A.: Problèmes aux limites pour le Laplacien dans un domaine à points cuspides. *C.R.A.S. Paris*, **287**, 1113–1116 (1978)

[16] Krantz, S. G., Parks, H. R.: On the vector sum of two convex sets in space. *Canad. J. Math.*, **43**, 347–355 (1991)

[17] Krantz, S. G., Parks, H. R.: Geometry of Domains in Space, Springer-Verlag, 1999

[18] Ladyzhenskaya, O. A., Ural'tseva, N. N.: Linear and Quasi-linear Elliptic Equations, Academic Press, New York, 1968

[19] Lai, M. J., Mersmann, C.: Bivariate splines for numerical solution of Helmholtz equation with large wave number, submitted, 2019

[20] Lai, M. J., Schumaker, L. L.: Spline Functions over Triangulations, Cambridge University Press, 2007

[21] Lai, M. J., Wang, C. M.: A bivariate spline method for 2nd order elliptic equations in non-divergence form. *Journal of Scientific Computing*, 803–829 (2018)

[22] Miranda, C.: Alcune limitazioni integrali per le soluzioni delle equazioni lineari ellittiches del secondo ordine. *Ann. Mat. Pura Appl.*, **64**, 353–384 (1963)

[23] Maugeri, A., Palagachev, D. K., Softova, L. G.: Elliptic and Parabolic Equations with Discontinuous Coefficients, **109** Mathematical Research, Wiley-VCH Verlag, Berlin, 2000

[24] Mitrea, D., Mitrea, M., Yan, L.: Boundary value problems for the Laplacian in convex and semiconvex domains. *Journal of Functional Analysis*, **258**, 2507–2585 (2010)

[25] Smears, I., Süli, E.: Discontinuous Galerking finite element approximation of nondivergence form elliptic equations with Cordés coefficients. *SIAM J Numer. Anal.*, **51**, 2088–2106 (2013)

[26] Smears, I., Süli, E.: Discontinuous Galerkin finite element approximation of Hamilton–Jacobi–Bellman equations with Cordes coefficients. *SIAM J. Numer. Anal.*, **52**, 993–1016 (2014)

[27] Talenti, G.: Sopra una classse di equazioni ellittiche a coefficienti misurabili. *Ann. Mat. Pura Appl.*, **69**, 285–304 (1969)

[28] Thäle, C.: 50 years sets with positive reach. *Surveys in Mathematics and its Applications*, **3**, 123–165 (2008)

[29] Torre, D. L., Rocca, M.: $C^{1,1}$ functions and optimality conditions. *J. Concr. Appl. Math.*, **3**, 41–54 (2005)

[30] Wang, C., Wang, J.: A primal-dual weak Galerkin finite element method for second order elliptic equations in non-divergence form. *Math. Comp.*, **87**, 515–545 (2018)