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On Bivariate Vertex Splines

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A vertex spline is one whose support contains exactly one interior vertex. Existence, construction, and approximation properties of bivariate vertex splines on triangulated regions are studied. Applications to interpolation and quasi-interpolation for unidiagonal triangulations are also discussed.

§1. Introduction

Suppose that we are given a triangulation Δ of a region D in the plane R^2 and are required to construct functions f whose restrictions to each triangular cell are bivariate polynomials of certain total degree so that they approximate the solution of a boundary value problem, say, or approximate and/or interpolate certain discrete data, we usually determine f as linear combinations of a certain basis $\{s_i\}$. In practice, it is most desirable, especially when finite element methods are used (cf. [6], p. 1129), that the supports of s_i are as small as possible, and in many situations, such as surface fitting problems, certain degree of smoothness is imposed on f. For instance, when continuous piecewise linear polynomials are used in the approximation problem, Courant basis elements [3] are usually chosen as the basis functions s_i.

It is clear that when higher degree polynomials are allowed, there is a better chance of achieving higher degree of smoothness and

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being able to construct s_i with smaller supports. However, although the smallest support is a single triangle, functions s_i with supports consisting of one or even two triangular cells are less useful since they, and any linear combination of them, have to vanish on all edges and vertices of the triangulation Δ. The smallest support without this "defect" is one which is a union of triangular cells sharing a common vertex as the only interior vertex of the support. We will call the basis functions with such supports vertex splines (or V-splines). Courant basis elements are certainly examples of V-splines. This paper is devoted to the study of existence, construction, and approximation properties of V-splines. Bézier representations of the polynomial pieces will be used, and a tool we develop to achieve our purposes efficiently is to express Taylor polynomials in terms of Bézier representations. This, together with the smoothing conditions, will be formulated in the next section. Existence theorems and characterizations in terms of areas of the appropriate triangles will be obtained in Section 3. The final section will be devoted to some important examples of V-splines, their applications to interpolation and quasi-interpolation, and the study of the order of approximation.

2. Preliminary results

Let $V_1 = (x_1,y_1)$, $V_2 = (x_2,y_2)$, and $V_3 = (x_3,y_3)$ be noncollinear points in the real plane R^2 and set

$$\delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}.$$

Then $\delta \neq 0$, and every point $X = (x,y)$ in R^2 can be expressed as

$$X = u_1V_1 + u_2V_2 + u_3V_3$$

where

$$u_1\delta = \begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \quad u_2\delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{vmatrix}, \quad u_3\delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix} .$$

It is clear that $u_1 + u_2 + u_3 = 1$ independent of (x,y), and (u_1,u_2,u_3) is usually called the barycentric coordinate representation of X with respect to the closed triangular region $T(V_1,V_2,V_3)$ with vertices at V_1, V_2, V_3.

Let P_n be a polynomial with total degree at most n. Then P_n may be expressed as a Taylor polynomial

(1) $$P_n(x,y) = \sum_{i+j\leq n} c_{ij}(x - x_1)^i(y - y_1)^j,$$

where $0 \leq i,\ j \leq n$, or as a Bézier polynomial

(2) $$P_n(u_1,u_2,u_3) = \sum_{i_1+i_2+i_3=n} a_{i_1i_2i_3}\, \phi^n_{i_1i_2i_3}(u_1,u_2,u_3) ,$$

where $0 \leq i_1,\ i_2,\ i_3 \leq n$ and

(3) $$\phi^n_{i_1i_2i_3}(u_1,u_2,u_3) = \frac{n!}{i_1!i_2!i_3!}\, u_1^{i_1} u_2^{i_2} u_3^{i_3} .$$

Note that $0 \leq u_1,\ u_2,\ u_3 \leq 1$ so that $\phi^n_{i_1i_2i_3} \geq 0$ if and only if $X = (x,y)$ lies on the closed triangular region $T(V_1,V_2,V_3)$. It is convenient to arrange the Bézier coefficients $a_{i_1i_2i_3}$ of $P_n(u_1,u_2,u_3)$ in a triangular array as shown in Fig. 1. For the same polynomial P_n, we will give the relationship between its Taylor coefficients c_{ij} and Bézier coefficients $a_{i_1i_2i_3}$. First, we need the following notation. Let $A = (a_1,a_2)$ and $B = (b_1,b_2)$. The derivative of a function f along the directed line segment $B-A = (b_1-a_1,\ b_2-a_2)$ at A is

(4) $$(D_{B-A}f)(A) = \frac{d}{dt} f(A + t(B-A))\Big|_{t=0} \text{ .}$$

It is clear that

(5) $$(D_{B-A}f)(A) = (b_1-a_1) \frac{\partial}{\partial x} f(A) + (b_2-a_2) \frac{\partial}{\partial y} f(A) \text{ .}$$

If f is the Bézier polynomial $P_n(u_1,u_2,u_3)$ given by (2), then it is also clear that

(6) $$(D_{V_2-V_1} P_n)(u_1,u_2,u_3) = \left(\frac{\partial}{\partial u_2} P_n - \frac{\partial}{\partial u_1} P_n\right)(u_1,u_2,u_3) \text{ .}$$

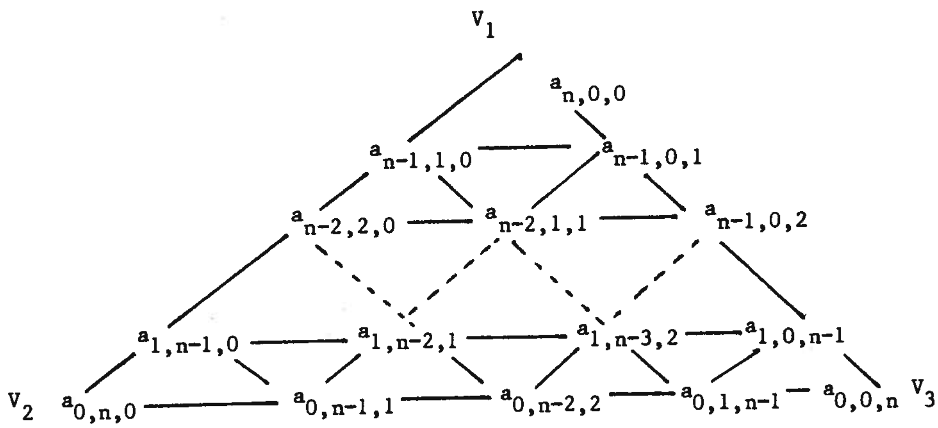

Fig. 1

Let Δ_{21}, Δ_{31}, and Δ_{32} be partial divided differences defined by

$$\Delta_{21}a_{i_1i_2i_3} = a_{i_1,i_2+1,i_3} - a_{i_1+1,i_2,i_3} \text{ ,}$$

$$\Delta_{31}a_{i_1i_2i_3} = a_{i_1,i_2,i_3+1} - a_{i_1+1,i_2,i_3} \text{ ,}$$

$$\Delta_{32}a_{i_1i_2i_3} = a_{i_1,i_2,i_3+1} - a_{i_1,i_2+1,i_3} \text{ .}$$

Then (6) becomes

(7) $$(D_{V_2-V_1}P_n)(u_1,u_2,u_3) = n \sum_{i_1+i_2+i_3=n-1} \Delta_{21}a_{i_1i_2i_3}\phi^{n-1}_{i_1i_2i_3}(u_1,u_2,u_3).$$

As usual, we also use the notation $(D^i_{B-A}D^j_{C-A}f)(A) = D_{B-A}(D^{i-1}_{B-A}D^j_{C-A}f)(A) = (D^i_{B-A}D_{C-A}D^{j-1}_{C-A}f)(A)$ with $(D^0_{B-A}f)(A) = f(A)$. We have the following result.

THEOREM 1. <u>Let</u> f <u>have continuous partial derivatives up to order</u> n <u>around</u> V_1. <u>Then the Taylor polynomial</u> P_n <u>of</u> f <u>at</u> V_1 <u>can be written as</u>

(8) $$P_n(x,y) = P_n(u_1,u_2,u_3) =$$

$$\sum_{i_1+i_2+i_3=n} \sum_{t=0}^{i_2} \binom{i_2}{t} \sum_{s=0}^{i_3} \binom{i_3}{s} \frac{(n-s-t)!}{n!} (D^t_{V_2-V_1} D^s_{V_3-V_1} f)(V_1)\phi^n_{i_1i_2i_3}(u_1,u_2,u_3).$$

REMARK 1. If $P_n(x,y)$ has the Taylor representation at $V_1 = (x_1,y_1)$ given by (1), then it is clear that

(9) $$(D^t_{V_2-V_1} D^s_{V_3-V_1} P_n)(V_1)$$

$$= \sum_{k=0}^{t+s} \sum_{i+j=k} \binom{t}{i}\binom{s}{j}(x_2-x_1)^i(y_2-y_1)^{t-i}(x_3-x_1)^j(y_3-y_1)^{s-j}k!(t+s-k)!c_{k,t+s-k}.$$

Hence, the Bézier coefficients can be expressed in terms of the Taylor coefficients at a vertex of the triangular region.

<u>Proof of Theorem 1</u>. We will use induction on the degree n. For $n = 0$, (8) is trivially true. Let $Q_{n-1}(x,y)$ be the Taylor polynomial of total degree $n-1$ of $(D_{V_2-V_1}f)(x,y)$ at V_1, and assume that (8) holds for $Q_{n-1}(x,y)$ with f replaced by $D_{V_2-V_1}f$. Let $R_n(x,y)$ be the

polynomial on the right-hand side of (8) then $R_n(V_1) = R_n(1,0,0) = f(V_1)$, and by (7),

$$(D_{V_2-V_1}R_n)(x,y)$$

$$= n \sum_{i_1+i_2+i_3=n-1} \Delta_{21}\Big(\sum_{t=0}^{i_2} \binom{i_2}{t} \sum_{s=0}^{i_3} \binom{i_3}{s} \frac{(n-s-t)!}{n!}(D^t_{V_2-V_1}D^s_{V_3-V_1}f)(V_1)\Big)$$

$$\cdot\ \phi^{n-1}_{i_1 i_2 i_3}(u_1,u_2,u_3)\ .$$

We first note that

$$n\Delta_{21}\Big(\sum_{t=0}^{i_2} \binom{i_2}{t} \sum_{s=0}^{i_3} \binom{i_3}{s} \frac{(n-s-t)!}{n!}(D^t_{V_2-V_1}D^s_{V_3-V_1}f)(V_1)\Big)$$

$$= \sum_{t=0}^{i_2+1} \binom{i_2+1}{t} \sum_{s=0}^{i_3} \binom{i_3}{s} \frac{(n-s-t)!}{(n-1)!}(D^t_{V_2-V_1}D^s_{V_3-V_1}f)(V_1)$$

$$- \sum_{t=0}^{i_2} \binom{i_2}{t} \sum_{s=0}^{i_3} \binom{i_3}{s} \frac{(n-s-t)!}{(n-1)!}(D^t_{V_2-V_1}D^s_{V_3-V_1}f)(V_1)$$

$$= \sum_{s=0}^{i_3} \binom{i_3}{s} \frac{(n-s-i_2-1)!}{(n-1)!}(D^{i_2+1}_{V_2-V_1}D^s_{V_3-V_1}f)(V_1)$$

$$+ \sum_{t=1}^{i_2} \frac{i_2!}{(t-1)!(i_2-t+1)!} \sum_{s=0}^{i_3} \binom{i_3}{s} \frac{(n-s-t)!}{(n-1)!}(D^t_{V_2-V_1}D^s_{V_3-V_1}f)(V_1)$$

$$= \sum_{t=0}^{i_2} \binom{i_2}{t} \sum_{s=0}^{i_3} \binom{i_3}{s} \frac{((n-1)-s-t)!}{(n-1)!}(D^t_{V_2-V_1}D^s_{V_3-V_1}(D_{V_2-V_1}f))(V_1)\ .$$

That is, $(D_{V_2-V_1}R_n)(x,y) = Q_{n-1}(x,y)$ by the induction hypothesis. Since $Q_{n-1}(x,y)$ is the (n-1)st degree Taylor polynomial of $(D_{V_2-V_1}f)(x,y)$ at V_1, we have

$$(D^i_{V_2-V_1} D^j_{V_3-V_1} Q_{n-1})(V_1)$$

$$= (D^i_{V_2-V_1} D^j_{V_3-V_1} D_{V_2-V_1} f)(V_1) = (D^{i+1}_{V_2-V_1} D^j_{V_3-V_1} f)(V_1) .$$

Hence, it follows that

$$(D^{i+1}_{V_2-V_1} D^j_{V_3-V_1} R_n)(V_1) = (D^{i+1}_{V_2-V_1} D^j_{V_3-V_1} f)(V_1)$$

for all i and j with $0 \leq i,j \leq n-1$ and $0 \leq i+j \leq n-1$. Similarly, we also have

$$(D^i_{V_2-V_1} D^{j+1}_{V_3-V_1} R_n)(V_1) = (D^i_{V_2-V_1} D^{j+1}_{V_3-V_1} f)(V_1)$$

for the same values of i and j. That is, the right-hand side expression $R_n(x,y)$ in (8) is the nth degree Taylor polynomial of $f(x,y)$ at V_1. This completes the proof of the theorem.

Now, let $T_1 = T_1(V_1,V_2,V_3)$ and $T_2 = T_2(V_1,V_4,V_2)$ be two closed triangular regions with common edge V_1V_2 (cf. Fig. 2). Consider a function F in $C^r(T_1 \cup T_2)$, $r \geq 0$, such that

$$F\big|_{T_1} = P_n \quad \text{and} \quad F\big|_{T_2} = \hat{P}_n$$

where P_n and $\hat{P}_n$ are polynomials of total degree at most n, and denote their Bézier polynomials by (2) and

(10) $$\hat{P}_n(v_1,v_2,v_3) = \sum_{i_1+i_2+i_3=n} b_{i_1i_2i_3} \phi^n_{i_1i_2i_3}(v_1,v_2,v_3)$$

respectively, with Bézier coefficients listed in triangular arrays as shown in Fig. 2.

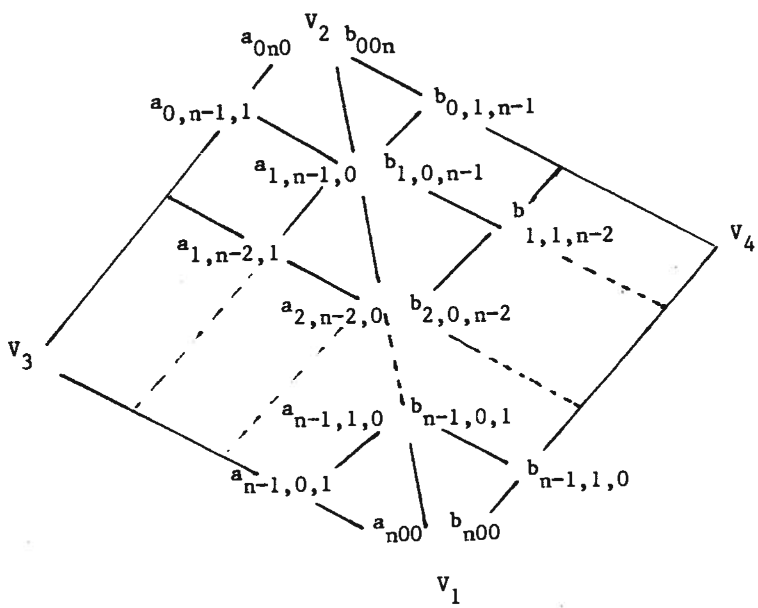

Fig. 2

Let (u_1^0, u_2^0, u_3^0) be the barycentric coordinate representation of V_4 with respect to T_1; that is, $V_4 = u_1^0 V_1 + u_2^0 V_2 + u_3^0 V_3$. Then it follows from (6) that

$$(D_{V_4 - V_1} \hat{P}_n)(v_1, v_2, v_3) = (\frac{\partial}{\partial v_2} \hat{P}_n - \frac{\partial}{\partial v_1} \hat{P}_n)(v_1, v_2, v_3) \; . \tag{11}$$

Since $V_4 - V_1 = u_2^0(V_2 - V_1) + u_3^0(V_3 - V_1)$ and F is in $C^r(T_1 \cup T_2)$, we have

$$(D^\ell_{V_4 - V_1} \hat{P}_n)\big|_{V_1V_2} = (u_2^0 D_{V_2 - V_1} + u_3^0 D_{V_3 - V_1})^\ell \, P_n\big|_{V_1V_2} \; ,$$

so that by (7) and (11),

$$\frac{n!}{(n-\ell)!} \sum_{i_1+i_2+i_3=n-\ell} \Delta_{21}^{\ell}\, b_{i_1 i_2 i_3} \phi_{i_1 i_2 i_3}^{n-\ell}(v_1,v_2,v_3)\,\Big|_{V_1V_2}$$

$$= \sum_{j=0}^{\ell} \binom{\ell}{j} (u_2^0)^j (u_3^0)^{\ell-j} D_{V_2-V_1}^{j} D_{V_3-V_1}^{\ell-j} P_n \Big|_{V_1V_2}$$

$$= \sum_{j=0}^{\ell} \binom{\ell}{j} (u_2^0)^j (u_3^0)^{\ell-j} \frac{n!}{(n-j)!} \frac{(n-j)!}{(n-j-\ell+j)!}$$

$$\cdot \sum_{i_1+i_2+i_3=n-j-\ell+j} \Delta_{31}^{\ell-j} \Delta_{21}^{j}\, a_{i_1 i_2 i_3}\, \phi_{i_1 i_2 i_3}^{n-j-\ell+j}(u_1,u_2,u_3)\Big|_{V_1V_2}$$

$$= \frac{n!}{(n-\ell)!} \sum_{j=0}^{\ell} \binom{\ell}{j} (u_2^0)^j (u_3^0)^{\ell-j}$$

$$\cdot \sum_{i_1+i_2+i_3=n-\ell} \Delta_{31}^{\ell-j} \Delta_{21}^{j} a_{i_1 i_2 i_3} \phi_{i_1 i_2 i_3}^{n-\ell}(u\ ,u\ ,u\ )\Big|_{V_1V_2} ;$$

or equivalently, using $u_1 = v_1$, $u_2 = v_3$, and $u_3 = v_2 = 0$ on V_1V_2, we obtain

$$\sum_{i_1+i_2+i_3=n-\ell} \Delta_{21}^{\ell}\, b_{i_1 i_2 i_3} \phi_{i_1 i_2 i_3}^{n-\ell}(u_1,0,u_2)$$

$$= \sum_{i_1+i_2+i_3=n-\ell} \sum_{j=0}^{\ell} \binom{\ell}{j} (u_2^0)^j (u_3^0)^{\ell-j} \Delta_{31}^{\ell-j} \Delta_{21}^{j} a_{i_1 i_2 i_3} \phi_{i_1 i_2 i_3}^{n-\ell}(u_1,u_2,0).$$

From the linear independence of $\phi_{i_1 i_2 i_3}^{n-\ell}$, it follows that, for $i = 0,\dots,n-\ell$ and $\ell = 0,\dots,r$,

(12) $$\Delta_{21}^{\ell} b_{i,0,n-\ell-i}$$

$$= \sum_{j=0}^{\ell} \binom{\ell}{j} (u_2^0)^j (u_3^0)^{\ell-j} \Delta_{31}^{\ell-j} \Delta_{21}^{j}\ a_{i,n-\ell-i,0}$$

$$= (u_2^0 \Delta_{21} + u_3^0 \Delta_{31})^{\ell}\ a_{i,n-\ell-i,0}\ .$$

For instance,

(13a) $$b_{i,0,n-i} = a_{i,n-i,0}$$

(13b) $$b_{i,1,n-i-1} - b_{i+1,0,n-i-1} = u_2^0(a_{i,n-i,0} - a_{i+1,n-i-1,0}) + u_3^0(a_{i,n-i-1,1} - a_{i+1,n-i-1,0})$$

by setting $\ell = 0$ and 1 respectively. We summarize the above result in the following

LEMMA. Let F be defined on $T_1 \cup T_2$ by $F|_{T_1} = P_n$, $F|_{T_2} = \hat{P}_n$ where P_n and $\hat{P}_n$ are as given in (2) and (10) respectively. Then $F \in C^r(T_1 \cup T_2)$ if and only if the conditions in (12) hold for $i = 0,\ldots,n-\ell$ and $\ell = 0,\ldots,r$.

REMARK 2. The above procedure to arrive at the necessary conditions (12) also shows that the conditions are sufficient. The formulation of these smoothing conditions is somewhat different from that of Farin [4].

REMARK 3. Let P_n and $\hat{P}_n$ be two nth degree Bézier polynomials on T_1 and T_2 given by (2) and (10) respectively, such that

$$(D^i_{V_2-V_1} D^j_{V_3-V_1} P_n)(V_1) = (D^i_{V_2-V_1} D^j_{V_3-V_1} f)(V_1)$$

and

$$(D^i_{V_2-V_1} D^j_{V_4-V_1} \hat{P}_n)(V_1) = (D^i_{V_2-V_1} D^j_{V_4-V_1} f)(V_1)$$

for $i+j \leq r \leq n$ and some function f which has continuous partial derivatives up to order r around V_1. Then the smoothing conditions (12) for $n-r \leq i \leq n-\ell$ and $\ell = 0,\ldots,r$ automatically hold. Hence, in constructing Hermite interpolants of class $C^r(T_1 \cup T_2)$ at V_1, the only remaining smoothing conditions in (12) for the Bézier coefficients of P_n and $\hat{P}_n$ to satisfy are those with $0 \le i \le n-r-1$.

REMARK 4. The results in both Theorem 1 and the above lemma can be easily generalized to the m-dimensional setting for any $m \ge 2$. More precisely, if $P_n(u_1,\ldots,u_{m+1})$ and $\hat{P}_n(v_1,\ldots,v_{m+1})$ are Bézier polynomials on the simplices $T(V_1,\ldots,V_{m+1})$ and $T(V_1,V_{m+2},V_2,\ldots,V_m)$ with common plane $V_1\ldots V_m$, then a generalization of (8) to R^m is

$$P_n(u_1,\ldots,u_{m+1})$$

$$= \sum_{i_1+\ldots+i_{m+1}=n} \sum_{t_2=0}^{i_2} \binom{i_2}{t_2} \cdots \sum_{t_{m+1}=0}^{i_{m+1}} \binom{i_{m+1}}{t_{m+1}} \frac{(n-i_2-\ldots-i_{m+1})!}{n!}$$

$$\cdot \left(D_{V_2-V_1}^{t_2} \cdots D_{V_{m+1}-V_1}^{t_{m+1}} f\right)(V_1)\phi^n_{i_1\ldots i_{m+1}}(u_1,\ldots,u_{m+1})$$

and a generalization of (12) is

$$\Delta_{21}^{\ell} b_{i_1,0,i_3,\ldots,i_{m+1}} = \left(\sum_{j=2}^{m+1} u_j^0 \Delta_{j1}\right)^{\ell} a_{i_1,i_3,\ldots,i_{m+1},0}$$

where $V_{m+2} = u_1^0 V_1 + \ldots + u_{m+1}^0 V_{m+1}$.

3. V-splines

Let Δ be a triangulation of a region D in R^2, r and n nonnegative integers, and $S_n^r = S_n^r(\Delta) = S_n^r(\Delta,D)$ be the space of all functions f in $C^r(D)$ whose restrictions to each cell of the triangulation Δ of D are polynomials of total degree at most n. We may assume, without loss of generality, that the interior vertices of Δ lie in D and that the triangulation Δ itself has been extended so that the closure of the union of all triangular regons contains D. We introduce the following

DEFINITION 1. An S^r_n vertex spline (or simply, a V-spline) is a function in S^r_n whose support contains exactly one vertex of Δ in its interior.

To facilitate our argument and description, we need the following notation. Let V be an interior vertex of Δ surrounded by the vertices $V_1,\ldots,V_m$, in the counterclockwise direction, in the sense that the closed triangular regions $T_1 = T_1(V,V_1,V_2)$, $T_2 = T_2(V,V_2,V_3),\ldots,$ $T_m = T_m(V,V_m,V_1)$ are cells of the triangulation Δ and share the common vertex V. On each triangular region T_j, $j = 1,\ldots,m$, the Bézier polynomial of degree n defined there will always be denoted by

$$P^j_n = P^j_n(u^j_1,u^j_2,u^j_3) = \sum_{i_1+i_2+i_3=n} a^j_{i_1i_2i_3}\,\phi^n_{i_1i_2i_3}(u^j_1,u^j_2,u^j_3),$$

where $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ are the barycentric coordinates of the vertices V, V_j, and V_{j+1}, respectively (with $V_{m+1} := V_1$), of the triangle T_j. We have the following result.

THEOREM 2. There exist S^1_n V-splines if and only if $n \geq 4$.

Proof. There are certainly no S^1_0 and S^1_1 V-splines. For $n = 2$, if $s \in S^1_2$ is supported by $T_1 \cup \ldots \cup T_m$ and $s|_{T_j} = P^j_2$, $j = 1,\ldots,m$, then since s vanishes outside $T_1 \cup \ldots \cup T_m$, we have $a^j_{i_1i_2i_3} = 0$ for all $j = 1,\ldots,m$ $i_1 = 0,1$ and all i_2, i_3 with $i_1 + i_2 + i_3 = 2$ by using (13a) and (13b). Choose j so that V_{j-1}, V_j, and V_{j+1} are noncollinear. Then by comparing P^j_2 with P^{j+1}_2 $(P^{n+1}_n := P^1_n)$ and using the same smoothing conditions, we also have $a^j_{200} = a^{j+1}_{200} = 0$, so that $a^1_{200} = \ldots = a^m_{200}$. That is, s is identically zero. For $n = 3$, the same argument yields $a^j_{i_1i_2i_3} = 0$ for all $j = 1,\ldots,m$ and $i_1 = 0,1$. By comparing P^j_2 with P^{j+1}_2 using (13a) and (13b), where

V_{j-1}, V_j, V_{j+1} are noncollinear, we also have $a^j_{201} = a^{j+1}_{210}$, $j = 1,\ldots,m$, $(a^{m+1}_{i_1i_2i_3} := a^1_{i_1i_2i_3})$. The same argument again gives $a^j_{300} = 0$, $j = 1,\ldots,m$. That is, s is again identically zero.

The existence of S^1_n V-splines for $n \geq 4$ is quite wasy to see since the number of restrictions governed by the smoothing conditions (13a) and (13b) is less than the number of parameters; namely, the Bézier coefficients. In fact, for $n = 4$ the difference is exactly one, and it is even larger for $n > 4$. This completes the proof of the theorem.

We remark, however, that it is quite possible that the only S^1_4 V-splines are those which vanish at their own (interior) vertices. This is certainly the case for nonuniform (or irregular) unidiagonal (or type-1) triangulation, as can be seen as a consequence of Theorem 3 that we will establish below. When this occurs, V-splines are then not very useful in applications, especially in interpolation at the vertices, since any linear combination of such V-splines has to vanish at all the vertices. For this reason, we introduce the following

DEFINITION 2. An S^r_n vertex spline s is called a V_ℓ-spline if s does not vanish at its (interior) vertex, and a V_ℓ-spline is called a $V_{\ell h}$-spline if, in addition, both its first x and y partial derivatives vanish at its vertex.

Here, we have used the subscript ℓ and the second subscript h to remind ourselves that these vertex splines are useful in Lagrange and Hermite interpolations, respectively.

To describe the existence result of S^1_4 V_ℓ-splines, we need the following notation. Let $V = (a,b)$, $V_j = (a_j,b_j)$ (with $V_{m+\ell} := V_\ell$, $a_{m+\ell} := a_\ell$, and $b_{m+\ell} := b_\ell$) denote vertices of Δ as described previously, and consider the determinants

$$\eta_{jk} = \begin{vmatrix} 1 & a & b \\ 1 & a_j & b_j \\ 1 & a_k & b_k \end{vmatrix}$$

and

$$\mu_j = \begin{vmatrix} 1 & a_j & b_j \\ 1 & a_{j+1} & b_{j+1} \\ 1 & a_{j+2} & b_{j+2} \end{vmatrix} .$$

Observe that halves of the above quantities are the "directed" areas of the triangular regions $T(V, V_j, V_k)$ and $T(V_j, V_{j+1}, V_{j+2})$ respectively. Now, consider an $s \varepsilon S_4^1$ supported by $T_1 \cup \ldots \cup T_m$, and let $s|_{T_j} = P_4^j$ as described earlier. Of course, in order to satisfy (13a) and (13b) across the exterior edges V_jV_{j+1}, we must have

$$a_{004}^j = \ldots = a_{040}^j = 0,$$

$$a_{103}^j = \ldots = a_{130}^j = 0,$$

and

$$a_{220}^j = a_{202}^j = 0 ,$$

for $j = 1,\ldots,m$. The remaining Bézier coefficients which still have to satisfy (13b) across the interior edges VV_j, $j = 1,\ldots,m$, are

$$\alpha := a_{400}^1 = \ldots = a_{400}^m ,$$

$$\beta_{j+1} := a_{310}^{j+1} = a_{301}^j$$

and

$$\gamma_j := a_{211}^j ,$$

$j = 1,\ldots,m$, with $\beta_1 := \beta_{m+1}$. There are two smoothing conditions across each VV_j that they have to satisfy, namely:

$$\beta_j - \alpha = \frac{\eta_{j,j+2}}{\eta_{j+1}} (\beta_{j+1} - \alpha) - \frac{\eta_j}{\eta_{j+1}} (\beta_{j+2} - \alpha) \tag{14}$$

and

$$\gamma_j - \beta_{j+1} = - \frac{\eta_{j,j+2}}{\eta_{j+1}} \beta_{j+1} - \frac{\eta_j}{\eta_{j+1}} (\gamma_{j+1} - \beta_{j+1}), \tag{15}$$

where $\eta_{j,\ell} := \eta_{j,m+\ell}$ and $\eta_j := \eta_{j,j+1}$, $\eta_1 := \eta_{m+1}$.

REMARK 5. Since $s(V) = \alpha$, s is a V_ℓ-spline if and only if $\alpha \neq 0$. In addition, from Theorem 1 it follows that a V_ℓ-spline s is a $V_{\ell h}$-spline if and only if $\beta_j = \alpha$ for all $j = 1,\ldots,m$. Hence an S_4^1 $V_{\ell h}$-spline s satisfying $s(V) = 1$ exists if and only if the linear system

(16) $$\gamma_j' + \frac{\eta_j}{\eta_{j+1}} \gamma_{j+1}' = -\eta_{j,j+2}/\eta_{j+1}$$

has a solution in $\gamma_j' := \gamma_j - 1$, $j = 1,\ldots,m$.

We have the following result.

THEOREM 3. *For odd* m, *there exists exactly one* S_4^1 $V_{\ell h}$*-spline* s *satisfying* $s(V) = 1$. *For even* m, S_4^1 $V_{\ell h}$*-splines exist if and only if the condition*

(17) $$\sum_{j=1}^{m} (-1)^{j+1} \frac{\eta_{j,j+2}}{\eta_j \eta_{j+1}} = 0$$

is satisfied. Again for even m, S_4^1 V_ℓ*-splines exist if and only if the condition*

(18) $$\sum_{j=1}^{m} (-1)^{j+1} \frac{\mu_j}{\eta_j \eta_{j+1}} \beta_{j+1} = 0$$

is satisfied for some solution $\beta_1,\ldots,\beta_m$ *of the linear system* (14) *with* $\alpha \neq 0$.

REMARK 6. Although the above result is an existence theorem, the V_ℓ and $V_{\ell h}$-splines can actually be computed by solving (14) and (15) with $\alpha = 1$, say.

We consider the following three examples.

Example (a). For $m = 3$, there are exactly 3 linearly independent S_4^1 V-splines s_1, s_2, and s_3 satisfying

$$
(19)\qquad
\begin{aligned}
&s_1(V) = 1\ , \quad \frac{\partial}{\partial x} s_1(V) = 0, \quad \frac{\partial}{\partial y} s_1(V) = 0\\
&s_2(V) = 0\ , \quad \frac{\partial}{\partial x} s_2(V) = 1, \quad \frac{\partial}{\partial y} s_2(V) = 0\\
&s_3(V) = 0\ , \quad \frac{\partial}{\partial x} s_3(V) = 0, \quad \frac{\partial}{\partial y} s_3(V) = 1\ .
\end{aligned}
$$

Hence, there are exactly 3 V_ℓ-splines s_1, s_1+s_2, and s_1+s_3, where s_1 is a $V_{\ell h}$-spline.

Example (b). For $m = 4$, S_4^1 $V_{\ell h}$-splines exist if and only if the areas of the appropriate triangles satisfy:

$$(20)\qquad \eta_{13}(\eta_1\eta_2 - \eta_3\eta_4) - \eta_{24}(\eta_2\eta_3 - \eta_1\eta_4) = 0\ .$$

If (20) is satisfied but

$$(21)\qquad \eta_2\eta_4 - \eta_1\eta_3 + \eta_{13}\eta_{24} = 0$$

is not, then every V_ℓ-spline is a $V_{\ell h}$-spline. If all of (20), (21), and

$$(22)\qquad \mu_1\beta_2\eta_3\eta_4 - \mu_2\beta_3\eta_4\eta_1 + \mu_3\beta_4\eta_1\eta_2 - \mu_4\beta_1\eta_2\eta_3 = 0$$

are satisfied, then there are exactly 3 linearly independent S_4^1 V-splines s_1, s_2, s_3 satisfying (19). Hence, under these 3 assumptions, there are 3 linearly independent V_ℓ-splines s_1, s_1+s_2, and s_1+s_3. In the particular case when V, V_1, V_3 are collinear but V, V_2, V_4 are not, then a $V_{\ell h}$-spline does not exist whenever $\eta_2\eta_3 \neq \eta_1\eta_4$.

Example (c). Consider the type-1 triangulations as shown in Fig. 3 below. From (17) and (18) it can be shown that an S_4^1

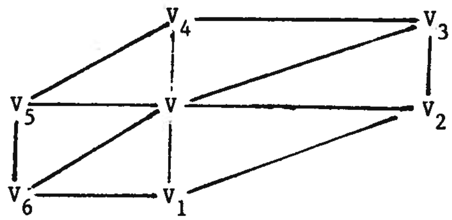

Fig. 3

V_ℓ-spline exists if and only if the two lines V_1V_2 and V_5V_4 are parallel. If V_1V_2 and V_5V_4 are parallel, then there are exactly 3 linearly independent S_4^1 V-splines s_1, s_2, and s_3 satisfying (19). If, however, V_1V_2 and V_5V_4 are not parallel, then there are 2 linearly independent S_4^1 V-splines both of which vanish at the vertex V.

Proof of Theorem 3. By Remark 5, we note that the existence of $V_{\ell h}$-splines is determined by the nonhomogeneous linear equations in (16). The coefficient matrix in (16) can be reduced by Gauss elimination to an upper-triangular matrix with all diagonal elements equal to 1 except one which is

$$1 - \prod_{i=1}^{m} (-\eta_i/\eta_{i+1}) = 1 - (-1)^m .$$

Hence, (16) has a unique solution when m is odd. For even m, the same row-reduction process of the augmented matrix yields (17) in order to preserve consistency.

To study the existence of V_ℓ-splines, we rewrite (15) into the form

$$\gamma_j + \frac{\eta_j}{\eta_{j+1}}\gamma_{j+1} = (\eta_{j+1} - \eta_{j,j+2} + \eta_j)\beta_{j+1}/\eta_{j+1}$$

$$= \frac{\mu_j}{\eta_{j+1}}\beta_{j+1} .$$

Hence, the same row-reduction procedure above yields (18) to preserve consistency. This completes the proof of the theorem.

REMARK 7. It is important to observe that the subspace generated by all S_4^1 V-splines does not contain the constant function. Indeed, if $P_4(u_1,u_2,u_3)$ with Bézier coefficients a_{i_1,i_2,i_3} is the restriction of any V-spline on any triangular region, it is necessary that $a_{220} = a_{202} = 0$. For this reason, it is sometimes essential to use higher degree V-splines.

Let r be any nonnegative integer. We now study V-splines in the space S_{4r+1}^r. We will use the convention:

$$D^\beta = \left(\frac{\partial}{\partial x}\right)^{\beta_1}\left(\frac{\partial}{\partial y}\right)^{\beta_2}, \quad \beta = (\beta_1,\beta_2),$$

and $|\beta| = \beta_1 + \beta_2$. Also, let $s^\alpha = s_V^\alpha$, $\alpha = (\alpha_1,\alpha_2)$, denote an S_{4r+1}^r V-spline satisfying

(23) $$D^\beta s^\alpha(V) = \begin{cases} 0 & \text{if} \quad \beta \neq \alpha \\ 1 & \text{if} \quad \beta = \alpha \end{cases} .$$

We have the following existence result.

THEOREM 4. <u>For each ordered pair</u> α <u>with</u> $0 \leq |\alpha| \leq 2r$, <u>there exists an</u> S_{4r+1}^r <u>V-spline</u> $s^\alpha = s_V^\alpha$ <u>satisfying</u> (23).

<u>Proof</u>. Let $s \in S_{4r+1}^r$ be supported by the polygonal region $T_1 \cup \ldots \cup T_m$ with interior vertex V as described previously, and let $s|_{T_j} = P_{4r+1}^j := P^j$, with Bézier coefficients $a_{i_1i_2i_3}^j$, $i_1+i_2+i_3 = 4r+1$,

$j = 1,\ldots,m$. By Theorem 1, the coefficients $a^j_{i_1 i_2 i_3}$ with

(i) $0 \le i_2 + i_3 \le 2r$,

(ii) $0 \le i_1 + i_3 \le 2r$,

and

(iii) $0 \le i_1 + i_2 \le 2r$,

are uniquely determined by the Hermite data (23), $D^\beta s^\alpha(V_j) = 0$, and $D^\beta s^\alpha(V_{j+1}) = 0$, for $0 \le |\beta| \le 2r$, respectively. Applying the previous lemma to the edge $V_j V_{j+1}$, we also have

(iv) $a^j_{k,4r-k-i+1,i} = 0$

for $0 \le i \le 4r - k + 1$ and $k = 0, \ldots, r$. Furthermore, the coefficients $a^j_{k,\ell,i}$ where

(v) $r+1 \le k \le 2r \le \ell \le 3r$, $k+\ell+i = 4r+1$, $i = 1,\ldots,r$

and

(vi) $r+1 \le k \le 2r \le i \le 3r$, $k+\ell+i = 4r+1$, $\ell = 1,\ldots,r$

are governed by the smoothing conditions across the interior edges VV_j and VV_{j+1}, respectively, using the lemma. The remaining coefficients $a^j_{i_1 i_2 i_3}$ with

(vii) $r+1 \le i_1, i_2, i_3 \le 2r+1$, $i_1+i_2+i_3 = 4r+1$

are free parameters. Since the seven groups of coefficients in (i) - (vii) are pairwise disjoint, the existence of s^α_V is guaranteed.

REMARK 8. Let $\{f_{\alpha,V}$: V an interior vertex of Δ and $\alpha \in \Lambda\}$, where Λ is a subset of the set of ordered pairs β with $0 \le |\beta| \le 2r$ be given. Then the bivariate spline

$$s = \sum_V \sum_{\alpha\in\Lambda} f_{\alpha,V} s^\alpha_V \tag{24}$$

interpolates the given date in the sense that

$$D^{\alpha}s(V) = f_{\alpha,V}$$

for all $\alpha \in \Lambda$ and all interior vertices V.

We will study the order of approximation of these interpolants for some specific settings in the next section.

4. Applications

Let D be a rectangular region $[a,b] \times [c,d]$ in R^2, and let $a = x_0 < \dots < x_{p+1} = b$, $c = y_0 < y_1 < \dots < y_{q+1} = d$, and Δ_{pq} be a so-called unidiagonal (or type-1) triangulation of D, which is the triangulation of the $(p+1)(q+1)$ rectangular cells with edges defined by $x = x_i$ and $y = y_j$, $i = 1,\dots,p$ and $j = 1,\dots,q$, by drawing in all $(p+1)(q+1)$ diagonals with positive slopes. If we have $x_{i+1} - x_i = x_i - x_{i-1} = h$ and $y_{j+1} - y_j = y_j - y_{j-1} = k$, $i = 1,\dots,p$ and $j = 1,\dots,q$, the triangulation is said to be uniform or regular. Note that Δ_{pq} is also assumed to have been extended, by adding, say, $x = x_{-1} < a$, $x = x_{p+2} > b$, $y = y_{-1} < c$, $y = y_{q+2} > d$, and the appropriate diagonals, so that Δ_{pq} has $(p+2)(q+2)$ interior vertices. Note also that the support of each vertex spline in $S_n^r(\Delta_{pq})$ consists of $m = 6$ triangles.

In Example (c) of the previous section, we see that if the triangulation Δ_{pq} is uniform, then there are three V-splines in S_4^1 associated with each vertex. Since they satisfy (19), they can be applied to interpolating Hermite data that are sampled at the vertices. We give their Bézier coefficients (on each triangle) in Figures 4 (a), (b), (c) below.

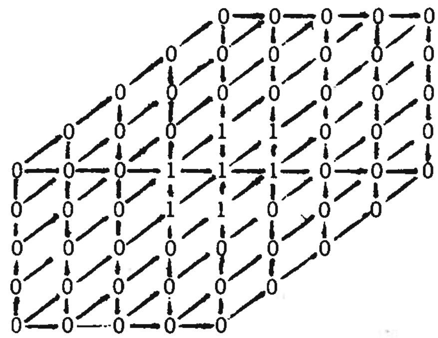

Fig 4 (a)

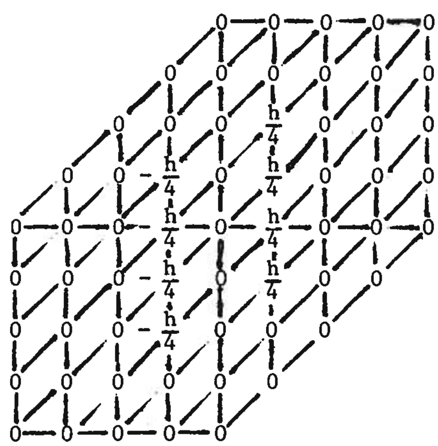

Fig 4(b)

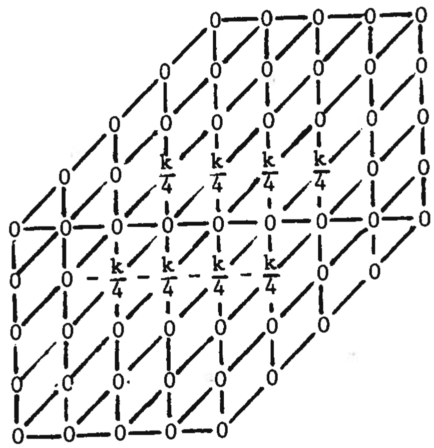

Fig 4(c)

For nonuniform triangulations Δ_{pq}, we need S_5^1 vertex splines (the lowest degree being 5). In this case, it is possible to handle Hermite data up to order two. Note that in the proof of Theorem 4, with $r = 1$, although the coefficient set in (vii) is empty, there are two Bézier coefficients in each of the six triangles, one from (v) and the other from (vi), that are governed by six smoothing conditions across the interior edges. Hence, there are still six free parameters, one for each triangle, corresponding to each V-spline $s_{ij}^{\alpha} := s^{\alpha}(V_{ij})$, $|\alpha| \leq 2$, satisfying (23) at its vertex where $V_{ij} := (x_i, y_j)$. Since each triangular cell is the intersection of the supports of 18 V-splines, 3 for each $\alpha = (\alpha_1, \alpha_2)$, it is possible to choose these 18 parameters on each triangle so that the approximation order of the Hermite interpolants

$$(25) \qquad s_f(x,y) = \sum_{i,j} \sum_{|\alpha| \leq 2} (D^{\alpha} f(x_i, y_j)) s_{ij}^{\alpha}(x,y)$$

to a sufficiently smooth function f on D is maximized. We determine 15 of these parameters by requiring $s_g \equiv g$ for all polynomials g with total degree ≤ 4 and the other 3 parameters using some natural conditions on symmetry. The six vertex splines s_{ij}^{α}, where $\alpha = (0,0)$, (1,0), (0,1), (2,0), (1,1), (0,2), at each vertex $V_{ij} = (x_i, y_j)$ are listed in Figures 5 (a) - (f), and their corresponding 3-dimensional pictures with $x_{i-1} = -1$, $x_i = 0$, $x_{i+1} = 1.2$, $y_{j-1} = -1$, $y_j = 0$, $y_{j+1} = .9$ are shown in Figures 6(a) - (f) below. Here, we have again given the Bézier coefficients of each polynomial piece and used the notation

(26) $$h_i = x_{i+1} - x_i, \quad k_j = y_{j+1} - y_j$$

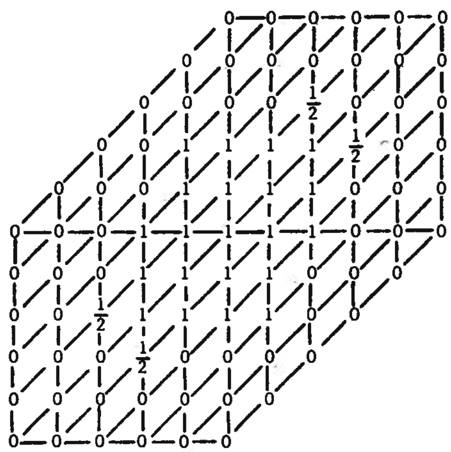

Fig 5(a): $s_{ij}^{(0,0)}$

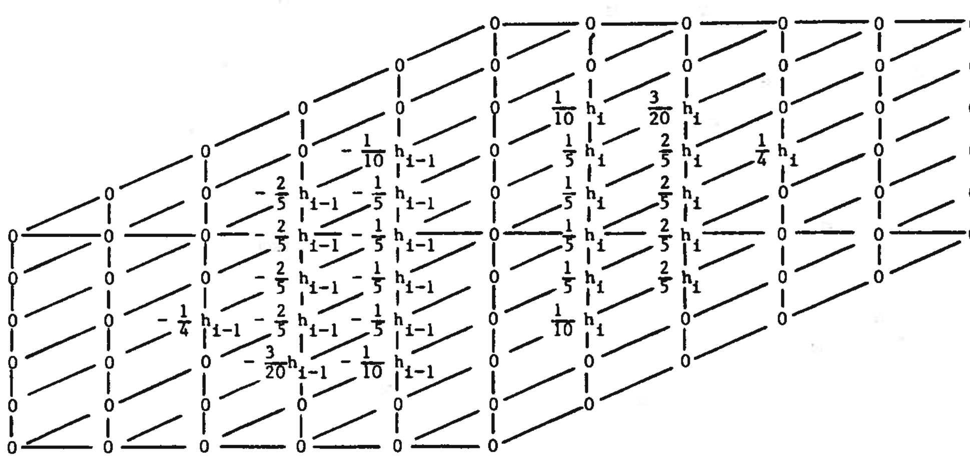

Fig 5(b): $s_{ij}^{(1,0)}$

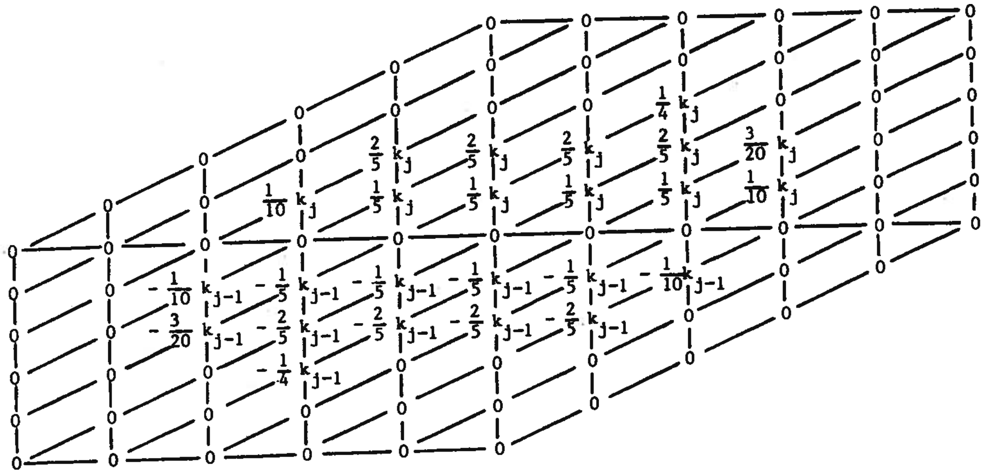

Fig 5(c): $s_{ij}^{(0,1)}$

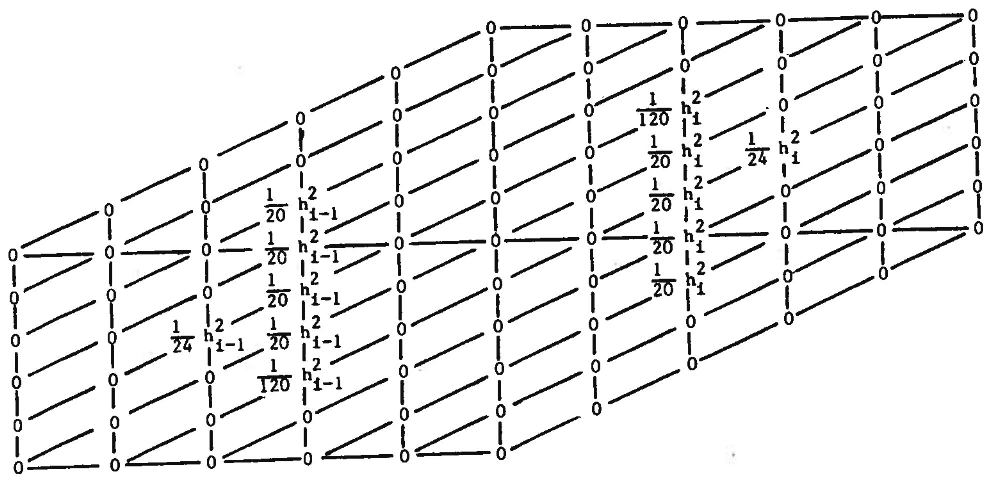

Fig. 5(d): $s_{ij}^{(2,0)}$

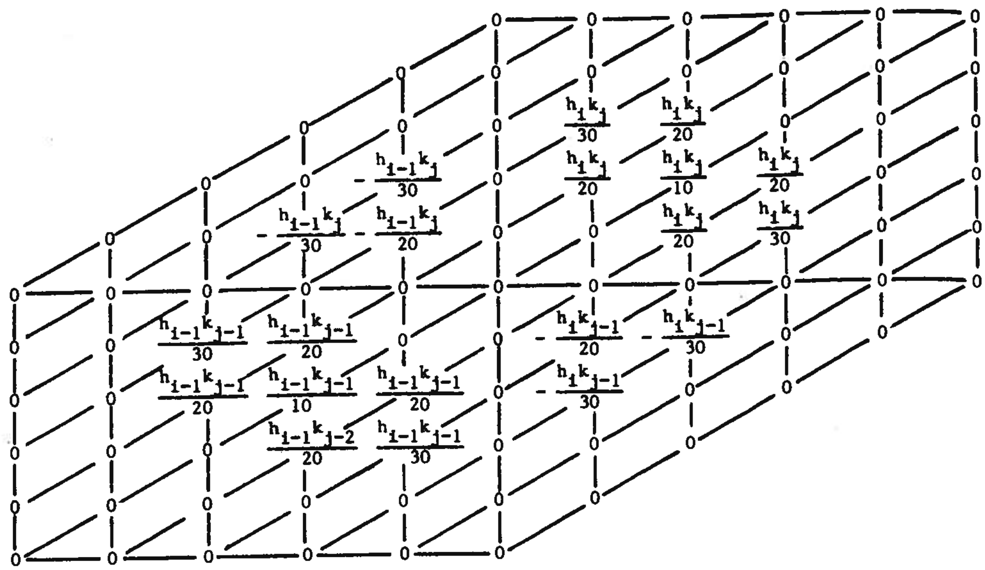

Fig 5(e): $s_{ij}^{(1,1)}$

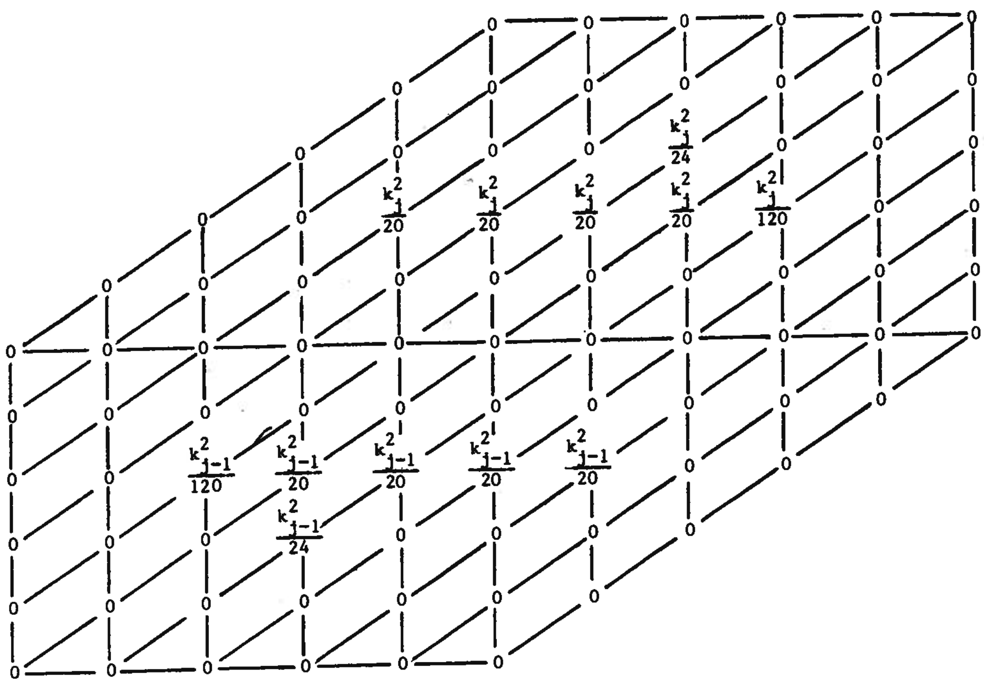

Fig 5(f): $s_{ij}^{(0,2)}$

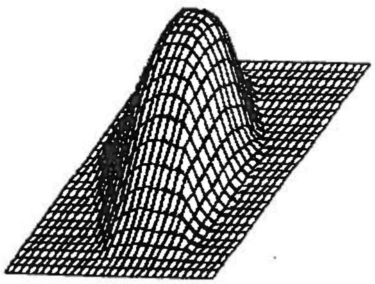

Fig 6(a): $s_{00}^{(0,0)}$

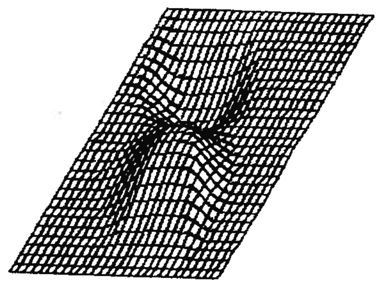

Fig 6(b): $s_{00}^{(1,0)}$

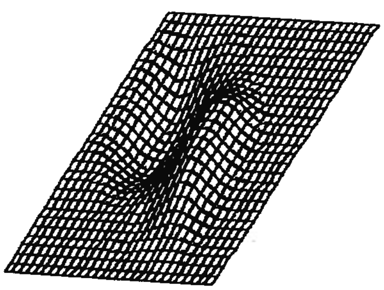

Fig 6(c): $s_{00}^{(0,1)}$

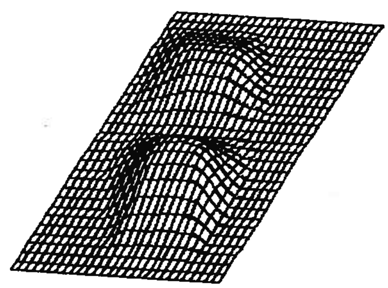

Fig 6(d): $s_{00}^{(2,0)}$

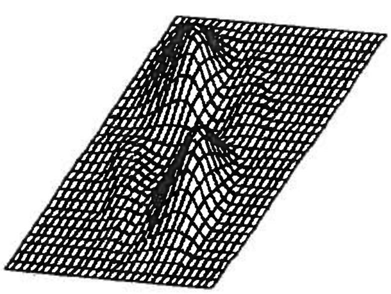

Fig 6(e): $s_{00}^{(1,1)}$

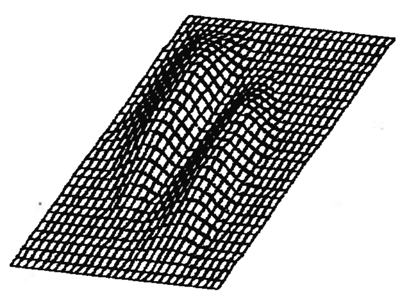

Fig 6(f): $s_{00}^{(0,2)}$

To study the order of approximation of the Hermite interpolation scheme given by (25), we use the notation

$$\|D^k f\|_D = \max\{|D^\alpha f(x,y)| : |\alpha| = k, \quad (x,y) \in D\}$$

and obtain the following result.

THEOREM 5. *Let* $f \in C^5(D)$ *and* s_f *be the bivariate Hermite spline interpolant of* f *defined by* (25). *Then*

$$|s_f(x,y) - f(x,y)| \leq 21\|D^5 f\|_D\, \eta^5 \tag{27}$$

for all $(x,y) \in D$ *where*

$$\eta = \max_{i,j}\{h_i, k_j\} . \tag{28}$$

Proof. Since $s_g = g$ where g is the constant function we have

$$s_f(x,y) - f(x,y) = \sum_{i,j} \sum_{|\alpha|\leq 2} [D^\alpha f(x_i,y_j) - D^\alpha f(x,y)] s_{ij}^\alpha(x,y)$$

$$+ \sum_{i,j} \sum_{1\leq|\alpha|\leq 2} D^\alpha f(x,y) s_{ij}^\alpha(x,y) .$$

By using Taylor's formula at $X = (x,y) \in D,$ this expression can be written as

$$s_f(X) - f(X) = \sum_{i,j} \sum_{|\alpha|\leq 2} \sum_{1\leq|\beta|\leq 4-|\alpha|} D^{\alpha+\beta} f(X)(X_{ij} - X)^\beta s_{ij}^\alpha(X)/\beta! \tag{29}$$

$$+ \sum_{i,j} \sum_{1\leq|\alpha|\leq 2} D^\alpha f(X) s_{ij}^\alpha(X) + R(X) ,$$

with

$$R(X) = \sum_{i,j} \sum_{|\alpha|\leq 2} \sum_{\substack{|\beta|=5 \\ \beta\geq\alpha}} D^\beta f(Z_{ij\beta})(X_{ij} - X)^{\beta-\alpha}\, s_{ij}^\alpha(X)/(\beta-\alpha)! , \tag{30}$$

for some $Z_{ij\beta}$ that lies on the line segment joining X and X_{ij}. Here and throughout, we use the usual multivariate notation: $X^\beta = x^{\beta_1}y^{\beta_2}$, $\beta! = \beta_1!\beta_2!$, $\alpha \geq \beta$ if and only if $\alpha_1 \geq \beta_1$ and $\alpha_2 \geq \beta_2$. Also, we have set $X = (x,y)$, $X_{ij} = (x_i,y_j)$, $\alpha = (\alpha_1,\alpha_2)$, and $\beta = (\beta_1,\beta_2)$. The expression (29) can be simplified by a change of index and interchanging orders of summation to be

$$s_f(X) - f(X) - R(X) = \sum_{i,j} \sum_{\alpha \leq 2} \sum_{\substack{1 \leq \beta \leq 4 \\ \beta \geq \alpha}} D^\beta f(X)(X_{ij} - X)^{\beta-\alpha} s_{ij}^\alpha(X)/(\beta-\alpha)!$$

$$= \sum_{1 \leq \beta \leq 4} D^\beta f(X)\Big[\sum_{i,j} \sum_{\substack{\alpha \leq \beta \\ \alpha \leq 2}} (X_{ij} - X)^{\beta-\alpha} s_{ij}^\alpha(X)/(\beta-\alpha)!\Big]$$

$$= \sum_{1 \leq \beta \leq 4} D^\beta f(X)\Big[\sum_{i,j} \sum_{\substack{\alpha \leq \beta \\ \alpha \leq 2}} \{D^\alpha (X_{ij} - X)^\beta\} s_{ij}^\alpha(X)/\beta!\Big]$$

$$= 0 ,$$

since $s_g = g$ for all polynomials of total degrees ≤ 4. That is, $s_f(X) - f(X) = R(X)$ for all X. Let X be fixed. Then there are only 3 terms in the summation $\sum_{i,j}$ in (30). Since

$$\sum_{i,j} \left|s_{ij}^\alpha(X)\right| \leq \eta^{\alpha} \tag{31}$$

for each α, we have

$$\left|R(X)\right| \leq 21\|D^5 f\|_D\, \eta^5.$$

This completes the proof of the theorem.

We note that the identity $s_f - f = R$ in the above proof does not depend on the grid partition and the basis $\{s_{ij}^\alpha\}$ with the exception that the operator s_f preserves polynomials of degree ≤ 4. Hence, the same proof yields the following more general result.

THEOREM 6. *Let* L *be a linear operator on* $C^k(\Omega)$, Ω *being any convex region in* R^m, $m \geq 1$, *defined by*

$$(Lf)(X) = \sum_{i=1}^{N} \sum_{\alpha \leq k} D^\alpha f(X_i) b_i^\alpha(X),$$

where α *is an* m*-tuple of nonnegative integers and* $X_1, \ldots, X_N$ *are distinct points in* Ω, *such that* $Lg = g$ *for all polynomials* g *of total degree* $\leq M$, $M \geq k$. *Then for any* $f \in C^{M+1}(\Omega)$ *and* $X \in \Omega$,

$$(Lf)(X) - f(X) = \sum_{i=1}^{N} \sum_{|\alpha| \leq k} \sum_{\substack{|\beta| = M+1 \\ \beta \geq \alpha}} D^\beta f(Z_{i\alpha\beta})(X_i - X)^{\beta-\alpha} b_i^\alpha(X),$$

where $Z_{i\alpha\beta}$ *lies in the line segment joining* X_i *and* X.

In most applications, Hermite data are not available. For this reason, we give the following so-called quasi-interpolation scheme that preserves the same polynomials, as s_f in (25) does, using only function values.

THEOREM 7. *There exist constants* $\{a_{ijuv}^\alpha\}$ *such that the linear operator* Q *defined by*

$$(32) \qquad (Qf)(x,y) = \sum_{i,j} \sum_{\alpha \leq 2} \sum_{u+v \leq 4} a_{ijuv}^\alpha f(x_{i+u}, y_{j+v}) s_{ij}^\alpha(x,y),$$

where $u, v \geq 0$, *preserves all polynomials of total degree* ≤ 4.

Proof. Let $\phi_{\mu\nu}(x,y) = x^\mu y^\nu$, and consider the equations

$$(Q\phi_{\mu\nu})(x,y) = s_{\phi_{\mu\nu}}(x,y)$$

where $\mu, \nu \geq 0$ and $\mu+\nu \leq 4$. Since $\{s_{ij}^\alpha\}$ is a linearly independent set, we have the linear system

$$(33) \qquad \sum_{u+v\leq 4} a^{\alpha}_{ijuv}\phi_{\mu\nu}(x_{i+u},y_{j+v}) = D^{\alpha}\phi_{\mu\nu}(x_i,y_j), \quad \mu+\nu \leq 4 ,$$

where $i = 0, \ldots, p+1$, $j = 0, \ldots, q+1$, and $|\alpha| \leq 2$. Since the coefficient matrix has determinant

$$\det[\phi_{\mu\nu}(x_{i+u},y_{j+v})] = \prod_{0\leq t<s\leq 4} (x_{i+s} - x_{i+t})^{5-s} \prod_{0\leq t<s\leq 4} (y_{j+s} - y_{j+t})^{5-s}$$

which is different from zero, the coefficients a^{α}_{ijuv} in (33) can be uniquely determined using $s_g = g$ for all polynomials of degree ≤ 4, we have completed the proof of the theorem.

REMARK 9. In the above theorem, a forward-sampled scheme was used to determine the quasi-interpolant Q. This can be generalized to other choices of (x_{i+u}, y_{j+v}) as long as the corresponding matrix in (33) is non-singular. For instance, the criterion given in [5] can be applied. More detail in this direction will be considered in a later paper.

When the triangulation Δ_{pq} is uniform we have the following order of approximation.

PROPOSITION. <u>Let</u> Q <u>be the quasi-interpolation operator defined by</u> (32) <u>where</u> Δ_{pq} <u>is a uniform unidiagonal triangulation of</u> D <u>with</u> $x_{i+1} - x_i = y_{j+1} - y_j = \eta$ <u>for all</u> i <u>and</u> j. <u>Then</u>

$$\max_{(x,y)\varepsilon D} \left|(Qf - f)(x,y)\right| = O(\eta^5)$$

<u>as</u> $\eta \to 0$ <u>for every</u> $f \varepsilon C^5(D)$.

<u>Proof</u>. Fix (i,j). We borrow a notation from [2, p. 737], setting

$$(34) \qquad \phi_{st}(x,y) = \frac{\pi_{st}(x,y)}{\pi_{st}(x_s,y_t)}$$

as the quartic polynomials satisfying

$$\psi_{st}(x_{i+u},\ y_{j+v}) = \delta_{su}\delta_{tv} ,$$

where δ_{jk} is the kronecker delta, and π_{st} is the appropriate product of four of the linear polynomials $y - y_j, \ldots, y - y_{j+3}$, $x - x_i, \ldots, x_{i+3}, (x+y) - (x_i+y_j) - \eta, \ldots, (x+y) - (x_i+y_j) - 4\eta$. Since each ψ_{st} is a linear combination of the $\phi_{\mu\nu}$'s, (33) becomes

$$\sum_{u+v\leq 4} a^{\alpha}_{ijuv}\psi_{st}(x_{i+u},y_{j+v}) = D^{\alpha}\psi_{st}(x_i,y_j) ,$$

or

$$a^{\alpha}_{ijst} = D^{\alpha}\psi_{st}(x_i,y_j) .$$

Using (34), we arrive at the estimate

$$a^{\alpha}_{ijst} = O(\eta^{-\alpha}) .$$

Let $(x,y) \in D$. Suppose that (x,y) lies in the triangle T_0 with (x_{i_0},y_{j_0}) as a vertex. By using Taylor's expansion at (x_{i_0},y_{j_0}), we may write $f = P_4 + R_f$ where P_4 is a polynomial of degree ≤ 4 and $R_f(x,y) = O(\eta^5)$. Hence, we have, using (31),

$$\begin{aligned}(Qf - f)(x,y) &= (QR_f - R_f)(x,y)\\ &= \sum{}' \sum_{|\alpha|\leq 2} a^{\alpha}_{ijst}R_f(x_{i+s},y_{j+t})s^{\alpha}_{ij}(x,y) + O(\eta^5)\\ &= O\Big(\sum_{|\alpha|\leq 2} \eta^{-|\alpha|}\eta^5\eta^{|\alpha|}\Big) + O(\eta^5) = O(\eta^5)\end{aligned}$$

where Σ' indicates the sum over (i,j) where each (x_i,y_j) is a vertex of the triangle T_0. This completes the proof of the proposition.

REMARK 10. The assumption on the uniformity of Δ_{pq} can be weakened by using a more careful estimate.

REMARK 11. Results in this section can also be formulated in the criss-cross (or type-2) triangulation. In fact, computational algorithms and some analogous formulas on Hermite interpolation and quasi-interpolation, as well as their order of approximation, can be obtained. These results will appear in a later paper. When $r = 1$ and all Hermite data are given, (that is, Λ is the set of all β with $0 \le |\beta| \le 2$), an application of (24) also yields an S_5^1 interpolation scheme similar to those considered in [1].

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