Error Bounds for Minimal Energy Interpolatory Spherical Splines

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Abstract. The convergence of the minimal energy interpolatory splines on the unit sphere is studied in this paper. An upper bound on the difference between a sufficiently smooth function and its interpolatory spherical spline in the infinity norm is given. The error bound is expressed in terms of a second order spherical Sobolev-type seminorm of the original function.

§1. Introduction

We study the minimal energy method for scattered data interpolation over the unit sphere. Mainly, we are interested in the convergence rate of the minimal energy method based on spherical spline functions. In the planar setting, such convergence rate was recently determined in [5]. It is natural to generalise the convergence result to the setting of spherical splines. This is the purpose of the paper.

To be more precise about what we will study in this paper, let us introduce some necessary notation and definition. Let $\mathbf{S}$ denote the unit sphere in $\mathbb{R}^3$. Given two points u, v on $\mathbf{S}$ that are not antipodal, the shortest curve connecting them is an arc $\widehat{uv}$ of the great circle through them. Given three points v_1, v_2 and v_3 on $\mathbf{S}$ such that v_1, v_2, v_3 form a basis for $\mathbb{R}^3$, a spherical triangle τ is a domain bounded by the arcs $\widehat{v_1 v_2}$, $\widehat{v_2 v_3}$ and $\widehat{v_3 v_1}$, which are called edges of the spherical triangle τ. The points v_1, v_2 and v_3 are called vertices of τ.

Given a set $\mathcal{V}$ of points on $\mathbf{S}$ we can form a triangulation Δ: a collection of spherical triangles. We will assume that the triangulation Δ is regular in the sense that any two triangles do not intersect each other, or share either a common vertex or a common edge and every edge of Δ is shared by exactly two triangles. Under the regularity assumption of Δ, the number of vertices $V = \#(\mathcal{V})$ and the number of triangles $T := \#(\{\tau \in \Delta\})$ are

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related as $T = 2(V-2)$. The number E of edges of Δ is related to the number of triangles as $E = 3T/2$.

Let

$$S_d^r(\Delta) = \{s \in C^r(\mathbf{S}), s|_\tau \in \mathcal{H}_d, \tau \in \Delta\}$$

be the homogeneous spherical spline space of degree d and smoothness r over a regular triangulation Δ whose vertices are the given scattered points on $\mathbf{S}$. Here $\mathcal{H}_d$ denotes the space of spherical homogeneous polynomials of degree d (cf. [2]).

Suppose we are given values $\{f(v), v \in \mathcal{V}\}$ of an unknown function f on the set $\mathcal{V}$. Let

$$U_f := \{s \in S_d^r(\Delta) : s(v) = f(v), v \in \mathcal{V}\}$$

be the set of all splines in $S_d^r(\Delta)$ that interpolate f at the points of $\mathcal{V}$. Then a commonly used way to create an approximation of f is to choose a spline $S_f \in S_d^r(\Delta)$ such that

$$\mathcal{E}(S_f) = \min_{s \in U_f} \mathcal{E}(s), \tag{1}$$

where $\mathcal{E}$ is an energy functional which will be defined later. We refer to S_f as the minimal energy interpolating spline.

It is interesting to see if S_f converges to f as the points in $\mathcal{V}$ become dense on $\mathbf{S}$. More precisely, we shall prove that for a spline space $S_d^r(\Delta)$ defined on a β-quasi-uniform triangulation Δ with size $|\Delta| \leq 1$ and $d \geq 3r+2$, there exist constants D_8, D_9, D_{10} depending only on d and β, such that the minimal energy interpolant S_f, defined in (1), satisfies

$$||f - S_f||_{\infty,\mathbf{S}} \leq D_8 (\tan \frac{|\Delta|}{2})^2 |f|_{2,\infty,\mathbf{S}}, \tag{2}$$

if $f \in C^2(\mathbf{S})$ for odd integer d and

$$||f - S_f||_{\infty,\mathbf{S}} \leq D_9 (\tan \frac{|\Delta|}{2})^2 |f|_{2,\infty,\mathbf{S}} + D_{10} (\tan \frac{|\Delta|}{2})^3 |f|_{3,\infty,\mathbf{S}}, \tag{3}$$

for all $f \in C^3(\mathbf{S})$ for even integer d.

The concept of quasi-uniform triangulations will be given in Section 2. The spaces $W^{k,p}(\mathbf{S})$ and associated norms $||\cdot||_{k,p,\mathbf{S}}$ and semi-norms $|\cdot|_{k,p,\mathbf{S}}$ will be defined in Section 3. In order to prove the convergence of S_f, we need several preliminary results concerning spherical triangulations in Sections 2, the approximation properties of interpolatory polynomials in Section 4, and the approximation properties of $S_d^r(\Delta)$ which are mainly based on the results in [8] in Section 5. Finally we prove the main result in Section 6. Some numerical examples are considered in Section 7.

§2. Natural Radial Projection

In order to obtain bounds on convergence of the minimal energy splines, we need to constrain spherical triangulations. Let us introduce a concept of a quasi-uniform triangulation on **S** similar to the planar case.

Define a diameter of a spherical cap C as $\sup_{u,v\in C} \arccos(u\cdot v)$. Given a spherical triangle τ, let $|\tau|$ denote the diameter of the smallest spherical cap containing τ, and let ρ_τ denote the diameter of the largest spherical cap contained in τ. Then

$$|\Delta| = \max\{|\tau|, \quad \tau\in\Delta\} \text{ and } \rho_\Delta = \min\{\rho_\tau, \quad \tau\in\Delta\}$$

are correspondingly the diameter of the largest triangle in Δ and the diameter of the smallest spherical cap inscribed in Δ.

Definition 1. *Let β be a positive real number. A triangulation Δ is said to be β-*quasi-uniform *provided that*

$$\frac{|\Delta|}{\rho_\Delta} \le \beta.$$

It is wellknown that in the planar case, the smallest angle of a quasi-uniform triangulation is bounded below by $1/\beta$ [7]. We make use of a concept of a natural radial projection developed in [8] to relate properties of planar quasi-uniform triangulations to the spherical ones.

Fix a spherical triangle τ with $|\tau| \le 1$. Define r_τ to be the center of a spherical cap of smallest possible radius containing τ, and let $\mathbf{T}_\tau$ be the tangent plane touching **S** at r_τ (cf. Figure 1). We define the radial projection from $\mathbf{T}_\tau$ into **S** by

$$w := R_\tau\bar{w} := \frac{\bar{w}}{|\bar{w}|} \in \mathbf{S}, \bar{w}\in\mathbf{T}_\tau.$$

Since R_τ is one-to-one, R_τ^{-1} is well-defined. Let $\bar\tau$ be the image of τ under R_τ^{-1}. It is not too difficult to check that

$$|\tau| \le |\bar\tau| \le K_1|\tau|, \text{ and } K_2^{-1}\rho_\tau \le \rho_{\bar\tau} \le K_2\rho_\tau, \tag{4}$$

for some positive constants K_1 and K_2. In this paper however we make use of the following

Lemma 1. *Let τ be a spherical triangle with $|\tau| \le 1$. Let $\bar\tau$ denote the image of τ under the map R_τ^{-1}. Then*

$$2\tan\frac{|\tau|}{2} = |\bar\tau| \tag{5}$$

and

$$2\tan\frac{\rho_\tau}{2} \le \rho_{\bar\tau}. \tag{6}$$

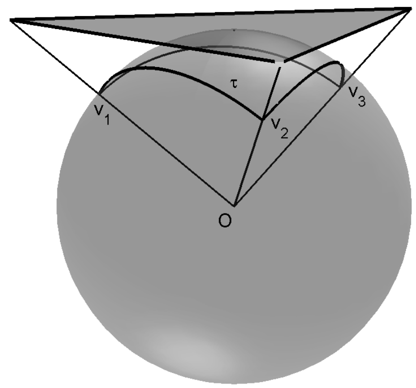

Fig. 1. Spherical triangle and its planar projection.

Proof: By the definition of R_τ, the center of the smallest spherical cap containing τ is the center of the circle outscribing $\bar{\tau}$. Let $\bar{v}$ be one of the vertices of $\bar{\tau}$. The center of the unit sphere O, $\bar{v}$ and r_τ form a right triangle with the leg Or_τ of length 1, the leg $\bar{v}r_\tau$ having length $\frac{|\bar{\tau}|}{2}$ and the angle $\angle\bar{v}Or_\tau$ having radian measurement $\frac{|\tau|}{2}$. Then (5) follows immediately.

The largest spherical cap σ contained in τ is mapped onto an ellipse ϵ in the plane $\mathbf{T}_\tau$ which is contained in $\bar{\tau}$. The largest circle $\bar{\sigma}$ contained in $\bar{\tau}$ has a radius $\frac{\rho_{\bar{\tau}}}{2}$ greater than or equal to r_ϵ, the radius of the largest circle contained in the ellipse. Let o be the center of σ and v be any point on the boundary $\delta\sigma$ of the cap. Let $\bar{o}$ and $\bar{v}$ be the images of o and v under R_τ^{-1} correspondingly. Then r_ϵ can defined by $r_\epsilon := \min_{v\in\delta\sigma}\{|\bar{o}-\bar{v}|\}$. Note now that

$$|\bar{o}-\bar{v}| \geq \tan|o-v|, \forall v \in \delta\sigma.$$

Therefore

$$\frac{\rho_{\bar{\tau}}}{2} \geq r_\epsilon \geq \tan\frac{\rho_\tau}{2},$$

and we have (6). □

Note also that since great circles are mapped into straight lines under the inverse of the radial projection R_τ, any cluster of spherical triangles ω with $|\omega| \leq 1$ is mapped into a planar triangulation $\bar{\omega}$.

Lemma 2. *Let Δ be a β-quasi-uniform triangulation of the unit sphere with $|\Delta| \leq 1$. Let Θ_Δ denote the smallest angle of Δ. There exists a constant A_1 such that*

$$\Theta_\Delta \geq \frac{1}{A_1\beta}. \tag{7}$$

Proof: Fix a spherical triangle $\tau \in \Delta$, and construct the radial projection R_τ. By Lemma 1 we have

$$\frac{|\bar{\tau}|}{\rho_{\bar{\tau}}} \leq \frac{\tan\frac{|\tau|}{2}}{\tan\frac{\rho_\tau}{2}} \leq 2\tan\frac{1}{2}\ \beta.$$

Since $\bar{\tau}$ is a planar triangle, its every angle is bounded below by $\frac{1}{A_1\beta}$ with $A_1 := 2\tan\frac{1}{2}$. Since the corresponding spherical angles are even greater, (7) follows. We have thus established Lemma 2. $\square$

We will need another lemma comparing areas A_τ of spherical triangles to the size parameters $|\Delta|$ and ρ_Δ characterising spherical triangulations.

Lemma 3. *For every spherical triangle $\tau \in \Delta$ with $|\Delta| \leq 1$*

$$\frac{\pi\rho_\Delta^2}{5} \leq A_\tau \leq \frac{\pi|\Delta|^2}{4}. \tag{8}$$

Proof: The area A_τ of a spherical triangle is bounded above by the area of the smallest spherical cap containing τ. The diameter of this cap is $|\tau|$. Without loss of generality we assume that the center of this cap is located at the north pole. Then

$$A_\tau \leq \int_0^{2\pi}\int_0^{|\tau|/2} \sin\eta d\eta d\theta = 2\pi(1-\cos(|\tau|/2)) \leq \pi\frac{|\Delta|^2}{4}.$$

Here the last inequality holds since $|\tau| \leq |\Delta| \leq 1$. Similarly, A_τ is bounded below by the area of the largest spherical cap contained in τ, which by the definition has a diameter ρ_τ. Therefore

$$A_\tau \geq 2\pi(1-\cos(\rho_\tau/2)) \geq \frac{\pi\rho_\Delta^2}{5}. \ \square$$

Another result that we need concerning β-quasi-uniform triangulations is a bound on the number of triangles n_k in the k-th disk around τ. We denote $\mathrm{star}^1(v)$ the union of all triangles in Δ that share the vertex v, $\mathrm{star}^\ell(v) := \cup\{\mathrm{star}^1(w) : w \text{ is a vertex of } \mathrm{star}^{\ell-1}(v)\}$, $\ell > 1$, and $\mathrm{star}^\ell(\tau) := \cup\{\mathrm{star}^\ell(w) : w \text{ is a vertex of } \tau\}$, $\ell > 1$.

Lemma 4. *Suppose Δ is a β-quasi-uniform triangulation such that $|\Delta| \leq 1$. Then for any triangle $\tau \in \Delta$ and any $k \geq 0$, the number n_k of triangles in $\mathrm{star}^k(\tau)$ is*

$$n_k \leq \frac{5\beta^2}{4}(2k+1)^2. \tag{9}$$

If, in addition, Δ is regular, then

$$n_k \geq \frac{2}{\pi\beta^2}(2k+1)^2. \tag{10}$$

Proof: Note that $\mathrm{star}^k(\tau)$ is contained in a spherical cap of radius $R = (2k+1)\frac{|\Delta|}{2}$ with area denoted A_R. Also by Lemma 3 we have

$$\frac{\pi\rho_\Delta^2}{5} \le A_\tau.$$

Then

$$n_k\frac{\pi\rho_\Delta^2}{5} \le A_R = 2\pi(1-\cos(R)) \le \pi R^2.$$

Therefore

$$n_k \le \frac{5\beta^2(2k+1)^2}{4}.$$

If Δ is total, then $\mathrm{star}^k(\tau)$ contains a spherical cap of radius $r = (2k+1)\frac{\rho_\Delta}{2}$ with area denoted as A_r. Then by Lemma 3

$$2r^2 \le 2\pi(1-\cos(r)) = A_r \le n_k\frac{\pi|\Delta|^2}{4},$$

and therefore

$$n_k \ge \frac{2(2k+1)^2}{\pi\beta^2}.$$

This completes the proof of Lemma 4. □

§3. Spherical Sobolev Space Seminorms

In this section we start by following the construction in [8] to define Sobolev-type norms and seminorms for functions on the unit sphere. This construction uses a concept of a homogeneous extension.

Definition 2. *Given any spherical function f and any integer n, the homogeneous extension of f of degree n to $\mathbb{R}^3\backslash\{0\}$ is a function f_n defined by*

$$f_n(u) = |u|^n f(\frac{u}{|u|}). \tag{11}$$

We next recall that a trivariate function $f(v)$ is homogeneous of degree n if

$$f(\alpha v) = \alpha^n f(v), \forall\alpha\in\mathbb{R}. \tag{12}$$

Fix $0 \le p \le \infty$, k nonnegative integer and let B denote an open set in $\mathbb{R}^2$. Recall that the corresponding classical Sobolev space $W^{k,p}(B)$ is the space of functions on B whose derivatives up to order k belong to $L_p(B)$ [1]. A norm on $W^{k,p}(B)$ can be defined as

$$||g||_{k,p,B} := \sum_{\gamma_1+\gamma_2\le k} ||D_\xi^{\gamma_1} D_\eta^{\gamma_2} g||_{p,B}, \tag{13}$$

where $D_\xi^{\gamma_1} D_\eta^{\gamma_2} = \frac{\partial^{\gamma_1+\gamma_2}}{\partial\xi^{\gamma_1}\partial\eta^{\gamma_2}}$.

Let Ω be a subset of the unit sphere with $|\Omega| \leq 1$. Suppose that $\{(\Gamma_j, \phi_j)\}$ is an atlas for Ω. Let $\{\alpha_j\}$ be a partition of unity subordinate to the atlas. We define spherical Sobolev spaces $W^{k,p}(\Omega)$ as follows:

$$W^{k,p}(\Omega) := \{f : (\alpha_j f) \circ \phi_j^{-1} \in W^{k,p}(\phi_j(\Gamma_j)), \text{for all } j\}. \tag{14}$$

Let $f \in W^{k,p}(\Omega)$ and let f_{k-1} denote the unique homogeneous extension of f of degree $k-1$ as in Definition 2. Then

$$|f|_{k,p,\Omega} := \sum_{|\alpha|=k} ||D^\alpha f_{k-1}||_{p,\Omega} \tag{15}$$

is a Sobolev-type seminorm of f on $W^{k,p}(\Omega)$. Here $||D^\alpha f_{k-1}||_{p,\Omega}$ is understood as the L_p-norm of the restriction of the trivariate function $D^\alpha f_{k-1}$ to Ω. For $k = 0$ the above seminorm reduces to the usual L_p-norm.

§4. Basic Inequalities

Let $\mathcal{H}_d$ denote the space of trivariate homogeneous polynomials of degree d. It was shown in [2] that the set

$$B^d_{ijk}(v) = \frac{d!}{i!j!k!} b_1(v)^i b_2(v)^j b_3(v)^k, \qquad i+j+k=d \tag{16}$$

of Bernstein-Bézier basis polynomials of degree d forms a basis for $\mathcal{H}_d$. Here $b_1(v),\ b_2(v),\ b_3(v)$ are trihedral barycentric coordinates of a point $v \in \mathbb{R}^3$ satisfying and uniquely defined by

$$v = b_1(v)v_1 + b_2(v)v_2 + b_3(v)v_3$$

in terms of a triple of linearly independent unit vectors v_1, v_2 and v_3. The restrictions of the trihedral barycentric coordinates to a spherical triangle with the vertices $v_1,\ v_2$ and v_3 are called spherical barycentric coordinates. The restriction of a homogeneous Bernstein-Bézier polynomial of degree d to the points on the unit sphere is called a spherical Bernstein-Bézier (SBB-) polynomial of degree d. Any homogeneous polynomial P of degree d and its restriction to a spherical triangle τ have a Bernstein-Bézier (BB-) representation with respect to τ

$$P(v) = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(v). \tag{17}$$

Given a homogeneous trivariate polynomial P in BB form (17), let c be a vector of its coefficients. Let $||c||_{\infty,\tau}$ and $||c||_{p,\tau}$ denote its ℓ_∞ and ℓ_p norms on a spherical triangle τ respectively.

Lemma 5. *Any homogeneous polynomial P of degree d in Bernstein-Bézier form (17) with respect to a spherical triangle τ with $|\tau| \le 1$ satisfies the property*

$$A_2 \, ||c||_{\infty,\tau} \le ||P||_{\infty,\tau} \le A_3 ||c||_{\infty,\tau} \tag{18}$$

and

$$A_4 A_\tau^{1/p} \, ||c||_{p,\tau} \le ||P||_{p,\tau} \le A_5 A_\tau^{1/p} ||c||_{p,\tau} \tag{19}$$

for any $1 \le p < \infty$. Here A_2, A_3 and A_5 are positive constants independent of τ, P and p. A_4 depends d, p and the smallest angle of τ.

Proof: Proof of (18) can be found in [8]. For (19) fix $1 \le p < \infty$. By Lemma 4.4 in [8], there exists a positive constant K_3 depending on d, p and the smallest angle Θ_τ of τ such that

$$A_\tau^{-1/p} ||P||_{p,\tau} \le ||P||_{\infty,\tau} \le K_3 A_\tau^{-1/p} ||P||_{p,\tau}. \tag{20}$$

Then using (18) we get

$$\frac{A_\tau^{1/p}}{K_3} A_2 \binom{d+2}{2}^{-1/p} ||c||_{p,\tau} \le \frac{A_\tau^{1/p}}{K_3} A_2 ||c||_{\infty,\tau} \le \frac{A_\tau^{1/p}}{K_3} ||P||_{\infty,\tau} \le ||P||_{p,\tau}.$$

Similarly, by (20)

$$||P||_{p,\tau} \le A_\tau^{1/p} ||P||_{\infty,\tau} \le A_3 A_\tau^{1/p} ||c||_{\infty,\tau} \le A_3 A_\tau^{1/p} ||c||_{p,\tau}.$$

Therefore we obtain (19) with $A_4 := \frac{A_2}{K_3} \binom{d+2}{2}^{-1/p}$ and $A_5 := A_3$. $\square$

Next we need Markov-type inequalities.

Lemma 6. *Let P be a trivariate homogeneous polynomial of degree d defined on a spherical triangle τ with $|\tau| \le 1$. There exist constants A_6 depending on d and Θ_τ only, and A_7 depending on d, such that*

$$|P|_{k,\infty,\tau} \le \frac{A_6}{(\tan \frac{\rho_\tau}{2})^k} ||P||_{\infty,\tau}, \tag{21}$$

and

$$|P|_{k,p,\tau} \le \frac{A_7}{(\tan \frac{\rho_\tau}{2})^k} ||P||_{p,\tau} \tag{22}$$

for $1 \le p < \infty$. Here ρ_τ is a the diameter of the largest spherical cap contained in τ.

Proof: For (22) we follow the proof of Proposition 4.3 in [8]. Modify the proof by replacing (4) with (6). To prove (21), we apply (20) to both sides of (22) to get

$$|P|_{k,\infty,\tau} \le \frac{A_7 K_4}{(\tan \frac{\rho_\tau}{2})^k} ||P||_{\infty,\tau}$$

for some K_4 depending on $d-k$ and Θ_Δ. We obtain (21) with $A_6 = A_7 K_4$. $\square$

Finally we express a bound on the values of a smooth spherical function in terms of its 2nd Sobolev seminorm over a spherical triangle.

Lemma 7. *Let τ be a spherical triangle such that $|\tau| \leq 1$, and suppose $f \in W^{2,p}(\tau)$ vanishes at the vertices of τ, that is $f(v_i) = 0, i = 1,2,3$. Then for all $v \in \tau$,*

$$|f(v)| \leq A_8 (\tan \frac{|\tau|}{2})^2 |f|_{2,\infty,\tau} \tag{23}$$

for some positive constant A_8 independent of f and τ. Moreover, if f is a homogeneous polynomial of degree d, then

$$|f(v)| \leq A_9 A_\tau^{-1/p} (\tan \frac{|\tau|}{2})^2 |f|_{2,p,\tau} \tag{24}$$

for some positive constant A_9 dependent only on d, p and the smallest angle in τ.

Proof: Let R_τ be the radial projection defined before. Let $\bar{v}_i, i = 1,2,3$ denote the vertices of a planar triangle $\bar{\tau}$, which is the image of τ under the inverse of R_τ and $\bar{v} = R_\tau^{-1} v$ for $v \in \tau$. Recall that $|\bar{\tau}| = 2\tan\frac{|\tau|}{2}$ by Lemma 1.

Let $f_1(v) = |v| f(\frac{v}{|v|})$ be the linear homogeneous extension of f to $\mathbb{R}^3 \backslash \{0\}$ and let $\bar{f}_1$ denote its restriction to the planar triangle $\bar{\tau}$. By Lemma 3.2 in [8], $\bar{f}_1$ belongs to $W^{2,p}(\bar{\tau})$. Note also that $\bar{f}_1(\bar{v}_i) = |\bar{v}_i| f(v_i) = 0$, $i = 1,2,3$. Therefore by Lemma 6.1 in [5], we have for every $\bar{v} \in \bar{\tau}$

$$|\bar{f}_1(\bar{v})| \leq 12 |\bar{\tau}|^2 |\bar{f}_1|_{2,\infty,\bar{\tau}}. \tag{25}$$

Since $f(v) = \frac{\bar{f}_1(\bar{v})}{|\bar{v}|}$ and $|\bar{v}| \geq 1$ for all $\bar{v} \in \bar{\tau}$,

$$|f(v)| \leq |\bar{f}_1(\bar{v})| \leq 48 (\tan \frac{|\tau|}{2})^2 |\bar{f}_1|_{2,\infty,\bar{\tau}},$$

by (25). By Proposition 3.4 in [8], there exists a positive constant K_5 such that we get

$$|f(v)| \leq 48 K_5 (\tan \frac{|\tau|}{2})^2 |f|_{2,\infty,\tau},$$

and therefore $A_8 = 48 K_5$.

If f is a homogeneous polynomial of degree d, then its second derivatives are homogeneous polynomials of degree $d-2$, then by (20) we have

$$|f|_{2,\infty,\tau} \leq K_6 A_\tau^{-1/p} |f|_{2,p,\tau},$$

for some K_6 depending on d, p and the smallest angle in τ. Hence

$$|f(v)| \le 48K_5(\tan\frac{|\tau|}{2})^2|f|_{2,\infty,\tau} \le A_9 A_\tau^{-1/p}(\tan\frac{|\tau|}{2})^2|f|_{2,p,\tau}.$$

This completes the proof with $A_9 = 48K_5K_6$. □

§5. Stable Local Basis and Existence of a Quasi-Interpolant

We now describe the stable local bases that the spline spaces possess. We shall use the spline spaces that have a local basis to solve the interpolation problem on the sphere. Let

$$\mathcal{D} := \cup_{\tau\in\Delta}\{\xi^\tau_{ijk}, i+j+k=d\}, \tag{26}$$

with $\xi^\tau_{ijk} := \frac{iu+jv+kw}{d}$ for $\tau =< u,v,w >$ be the set of domain points associated with Δ and d. It is well known that each spline in $S^0_d(\Delta)$ is uniquely determined by associating one Bézier coefficient with each domain point. A subset $\mathcal{M} \subset \mathcal{D}$ is called a minimal determining set for $S^r_d(\Delta)$ if the values of the coefficients of $s \in S^r_d(\Delta)$ associated with domain points in $\mathcal{M}$ uniquely determine all of the coefficients of s.

Definition 3. *A basis $\{B_\xi\}_{\xi\in\mathcal{M}}$ for a space $\mathcal{S}$ of splines on a triangulation Δ is a stable local basis, if there exists an integer ℓ and constants $0 < C_1 < C_2 < \infty$ depending only on d and the smallest angle θ_Δ in the triangulation Δ such that*

1) *for each $\xi \in \mathcal{M}$, $supp(B_\xi) \subseteq star^\ell(v_\xi)$ for some v_ξ of Δ,*
2) *for all $\{c_\xi\}_{\xi\in\mathcal{M}}$,*

$$C_1 \max_{\xi\in\mathcal{M}}|c_\xi| \le ||\sum_{\xi\in\mathcal{M}} c_\xi B_\xi||_{\infty,\mathbf{S}} \le C_2 \max_{\xi\in\mathcal{M}}|c_\xi|. \tag{27}$$

A construction of a stable local basis using the Bernstein-Bézier representation of splines in $S^r_d(\Delta)$ when $d \ge 3r+2$ is outlined in [8] with a reference to [4]. Given a minimal determining set, we can construct a basis $\{B_\xi\}_{\xi\in\mathcal{M}}$ for $S^r_d(\Delta)$ by requiring

$$\mu_\eta B_\xi = \delta_{\xi,\eta}, \quad \eta \in \mathcal{M}, \tag{28}$$

where μ_η is the linear functional which picks the coefficient associated with the domain point η. In particular, B_ξ has the property that the coefficient associated with ξ is 1 while the coefficients associated with all other points in $\mathcal{M}$ are zero. The remaining coefficients of B_ξ are computed using smoothness conditions.

For any given spline space $S^r_d(\Delta)$, there are many possible choices for a minimal determining set $\mathcal{M}$. A choice of $\mathcal{M}$ presented in [4] leads to a basis with the following properties, where for each ξ, $\Omega_\xi := \mathrm{supp}(B_\xi)$ and τ_ξ is the triangle in which ξ lies.

Lemma 8. *Let $\{B_\xi\}_{\xi\in\mathcal{M}}$ be the basis for $S^r_d(\Delta)$ corresponding to the minimal determining set $\mathcal{M}$ described in [4]. Then there exist constants $C_3, ..., C_9$ depending only on d, p and the minimal angle in Δ such that for each $\xi \in \mathcal{M}$,*

1) *there exists a vertex $v_\xi \in \Delta$ such that $\Omega_\xi \subseteq star^3(v_\xi)$,*
2) *$||B_\xi||_{\infty,\mathbf{S}} \le C_3$,*
3) *$|\mu_\xi s| \le C_4 ||s||_{\infty,\tau_\xi}$, for all $s \in S^r_d(\Delta)$,*
4) *$|\mu_\xi s| \le C_5 A_{\tau_\xi}^{-1/p} ||s||_{p,\tau_\xi}$, for all $s \in S^r_d(\Delta)$, and for every $\tau \in \Delta$,*
5) *$||B_\xi||_{p,\tau} \le C_6 A_\tau^{1/p}$,*
6) *$\#I_\tau \le C_7$, where $I_\tau := \{\xi : \tau \subset \Omega_\xi\}$,*
7) *$|B_\xi|_{k,\infty,\tau} \le C_8 \rho_\tau^{-k}$, for all $0 \le k \le d$*
8) *$|B_\xi|_{k,p,\tau} \le C_9 \rho_\tau^{-k} A_\tau^{1/p}$, for all $0 \le k \le d$.*

The proof of the above lemma can be found in [8]. Furthermore the analysis of the proof of 8) of the above lemma allows the following change in 8). Using (22) instead of (4.3) in [8] one gets

$$|B_\xi|_{k,p,\tau} \le C_9 (\tan \frac{\rho_\tau}{2})^{-k} A_\tau^{1/p} \tag{29}$$

with $C_9 = A_7 C_6$.

It was shown in [8] that with the basis defined above one can construct a quasi-interpolation operator $Q : L_p(\mathbf{S}) \to S^r_d(\Delta)$ which achieves the optimal approximation property. Indeed, extend the linear functionals μ_ξ to all of $L_p(\mathbf{S})$ using Hahn-Banach theorem. Then for every $f \in L_p(\tau_\xi)$,

$$|\mu_\xi f| \le C_5 A_{\tau_\xi}^{-1/p} ||f||_{p,\tau_\xi}, \qquad \xi \in \mathcal{M}. \tag{30}$$

This inequality implies that for each ξ, the carrier of the extended functional μ_ξ is contained in τ_ξ, i.e., if $f \equiv 0$ on τ_ξ, then $\mu_\xi f = 0$. With (29) in mind we modify the proof of Proposition 5.2 in [8] accordingly to get the following

Lemma 9. *For each $f \in L_p(\mathbf{S})$, let*

$$Qf := \sum_{\xi\in\mathcal{M}} (\mu_\xi f) B_\xi. \tag{31}$$

Then $Qg = g$ for all $g \in \mathcal{H}_d(\mathbf{S})$. Moreover, there exists a constant C_{10} depending only on d, p and the smallest angle in Δ such that for each triangle $\tau \in \Delta$,

$$|Qf|_{k,p,\tau} \le C_{10}(\tan \frac{\rho_\tau}{2})^{-k} ||f||_{p,\Omega_\tau}, \tag{32}$$

where $\Omega_\tau := \cup_{\xi \in I_\tau} \Omega_\xi$ and $I_\tau := \{\xi : \tau \subset \Omega_\xi\}$.

Theorem 1. *Suppose $\tau \in \Delta$ is a spherical triangle with $|\tau| \le 1$. Let $f \in W^{m+1,p}(\tau)$ for $0 \le m \le d$ such that $(d-m)\text{mod } 2 = 0$. There exists a spherical homogeneous polynomial s of degree d such that for every $0 \le k \le m$*

$$|f - s|_{k,p,\tau} \le C_{11}(\tan \frac{|\tau|}{2})^{m+1-k} |f|_{m+1,p,\tau}. \tag{33}$$

Here C_{11} is a constant that depends on p, m and θ_Δ. Moreover

$$|f - s|_{k,p,\Omega_\tau} \le C_{11}(\tan \frac{|\Delta|}{2})^{m+1-k} |f|_{m+1,p,\Omega_\tau}. \tag{34}$$

Proof: Fix m. By Theorem 4.2 in [8], there exists a spherical homogeneous polynomial s' of degree m such that for every $0 \le k \le m$

$$|f - s'|_{k,p,\tau} \le C_{11}|\tau|^{m+1-k} |f|_{m+1,p,\tau}. \tag{35}$$

If we slightly modify the proof of Theorem 4.2 [8], i.e. replace (4) by (5), we can get

$$|f - s'|_{k,p,\tau} \le C_{11}(\tan \frac{|\tau|}{2})^{m+1-k} |f|_{m+1,p,\tau}. \tag{36}$$

Since $(d-m)\text{mod } 2 = 0$, $s = |v|^{d-m}s'$ is a homogeneous spherical polynomial of degree d. Since on the unit sphere $s' \equiv s$, their $k-1$-st extensions are the same, and we have (33). To get (34), sum (33) over triangles in Ω_τ. This completes the proof. □

Theorem 2. *Let Δ be a β-quasi-uniform spherical triangulation with $|\Delta| \le 1$. Let $1 \le p \le \infty$, $d \ge 3r+2$, and $0 \le k \le d$. Then there exists a constant C_{12} depending only on d, p and the smallest angle in Δ, such that*

$$|f - Qf|_{k,p,\tau} \le C_{12}(\tan \frac{|\Delta|}{2})^{m+1-k} |f|_{m+1,p,\Omega_\tau}, \tag{37}$$

for all $f \in W^{m+1,p}(\mathbf{S})$ and all $\tau \in \Delta$. Moreover, there exists a constant C_{13} such that

$$|f - Qf|_{k,p,\mathbf{S}} \le C_{13}(\tan \frac{|\Delta|}{2})^{m+1-k} |f|_{m+1,p,\mathbf{S}}, \tag{38}$$

for all $f \in W^{m+1,p}(\mathbf{S})$ and all $0 \le k \le d$ such that $Qf \in W^{k,p}(\mathbf{S})$. Here m is taken between 0 and d with $(d-m)$ mod $2 = 0$.

Proof: Let $\tau \in \Delta$ with $|\tau| \le 1$. By Theorem 1 there exists a spherical homogeneous polynomial s of degree d such that (33) holds. By the linearity of Q and the fact that Q reproduces polynomials of degree d, we can write

$$|f - Qf|_{k,p,\tau} \le |f - s|_{k,p,\tau} + |Q(f - s)|_{k,p,\tau}.$$

We now consider the last term in the above inequality. Since Δ is assumed to be β-quasi-uniform, $|\rho_\tau| \ge \frac{|\tau|}{\beta}$, and therefore

$$\tan \frac{\rho_\tau}{2} \ge \tan \frac{|\tau|}{2\beta} \ge \frac{1}{\beta^2} \tan \frac{|\tau|}{2}.$$

By (32) and Theorem 1

$$\begin{aligned} |Q(f-s)|_{k,p,\tau} &\le C_{10}(\tan \frac{\rho_\tau}{2})^{-k} ||f - s||_{p,\Omega_\tau} \\ &\le C_{10}C_{11}(\tan \frac{|\tau|}{2\beta})^{-k}(\tan \frac{|\Delta|}{2})^{m+1}|f|_{m+1,p,\Omega_\tau} \\ &\le C_{10}C_{11}(\beta)^{2k}(\tan \frac{|\Delta|}{2})^{m+1-k}|f|_{m+1,p,\Omega_\tau}. \end{aligned}$$

Therefore we get (37) with $C_{12} = C_{11}(1 + C_{10}\beta^{2k})$.

To prove (38), we sum (37) over all triangles in Δ.

$$\begin{aligned} |f - Qf|_{k,p,\mathbf{S}} &= \sum_{\tau\in\Delta} |f - Qf|_{k,p,\tau} \le C_{12}(\tan \frac{|\Delta|}{2})^{m+1-k} \sum_{\tau\in\Delta} |f|_{m+1,p,\Omega_\tau} \\ &\le C_{12}(\tan \frac{|\Delta|}{2})^{m+1-k} \sum_{\tau\in\Delta} \sum_{\tau'\subset\Omega_\tau} |f|_{k,p,\tau'} \\ &= C_{12}(\tan \frac{|\Delta|}{2})^{m+1-k} \sum_{\tau'\in\Delta} \#\{\tau : \tau' \subset \Omega_\tau\}|f|_{m+1,p,\tau'} \\ &\le C_{12}K_7(\tan \frac{|\Delta|}{2})^{m+1-k} \sum_{\tau'\in\Delta} |f|_{m+1,p,\tau'}. \end{aligned}$$

Here $K_7 := \max\{\#\{\tau : \tau' \subset \Omega_\tau\}, \tau' \in \Delta\}$ which is bounded by Lemma 4. Therefore (38) holds with $C_{13} = C_{12}K_7$. This completes the proof. □

§6. Minimal Energy Interpolating Splines

Suppose we are given values $\{f(v), v \in \mathcal{V}\}$ of an unknown function f at a set $\mathcal{V}$ of scattered points on the unit sphere. To approximate f, we choose a linear space $\mathcal{S} \subseteq S_d^r(\Delta)$ of polynomial splines of degree d defined on a triangulation Δ with vertices at the points of $\mathcal{V}$. Recall that

$$U_f := \{s \in \mathcal{S} : s(v) = f(v), v \in \mathcal{V}\}$$

is the set of all splines in $\mathcal{S}$ that interpolate f at the points of $\mathcal{V}$. We assume that $\mathcal{S}$ is big enough so that U_f is not empty. Recall a commonly used way to create an approximation of f is to choose a spline S_f such that

$$\mathcal{E}(S_f) = \min_{s \in U_f} \mathcal{E}(s), \tag{39}$$

where for a spherical triangle $\tau \in \Delta$

$$\mathcal{E}_\tau(s) := \sum_{|\alpha|=2} ||D^\alpha s_1||_{2,\tau}^2 \text{ and } \mathcal{E}(s) := \sum_{\tau \in \Delta} \mathcal{E}_\tau(s). \tag{40}$$

Here s_1 is the linear homogeneous extension of s to $\mathbb{R}^3 \backslash \{0\}$, α is a triple index with entries running through x, y, z, e.g., $D^{(1,1,1)} = D_x D_y D_z$. That is, S_f is the minimal energy interpolating spline.

Let

$$\mathcal{X} := \{f \in B(\mathbf{S}) : f|_\tau \in C^2(\tau), \forall \tau \in \Delta\},$$

where $B(\mathbf{S})$ is the set of all bounded real-valued functions on the sphere. For each triangle $\tau \in \Delta$, let

$$\langle f, g \rangle_\tau := \int_\tau \sum_{|\alpha|=2} D^\alpha f_1 \ D^\alpha g_1.$$

Then

$$\langle f, g \rangle := \langle f, g \rangle_{\mathbf{S}} = \sum_{\tau \in \Delta} \langle f, g \rangle_\tau$$

is a semidefinite inner product on $\mathcal{X}$. Let $||f||_\tau$ and $||f||$ be the associated seminorms. We refer to them as energy or $\mathcal{X}$-norms.

It is easy to see that $< \cdot, \cdot >$ is an inner product on the linear space

$$\mathcal{W} := \{s \in \mathcal{S} : s(v) = 0, v \in \mathcal{V}\}. \tag{41}$$

Indeed, if $< w, w >= 0$ for some $w \in \mathcal{W}$, then w is a linear homogeneous polynomial on Δ and since w vanishes at all vertices, $w \equiv 0$. Since $\mathcal{W}$ is finite-dimensional, it follows that $\mathcal{W}$ equipped with the inner product $< \cdot, \cdot >$ is a Hilbert space.

Given f, suppose s_f is any spline in the set U_f defined above. Then it is easy to see that the solution S_f to the minimal energy problem is equal to $s_f - \mathcal{P}s_f$, where $\mathcal{P}$ is the linear projector $\mathcal{P} : \mathcal{X} \to \mathcal{W}$ defined by

$$\mathcal{E}(f - \mathcal{P}f) = \min_{w \in \mathcal{W}} \mathcal{E}(f - w), \tag{42}$$

for all $f \in \mathcal{X}$. Since $\mathcal{W}$ is a Hilbert space with respect to $\langle \cdot, \cdot \rangle$, $\mathcal{P}f$ is uniquely defined and is characterised by

$$\langle f - \mathcal{P}f, w \rangle = 0, \qquad \forall w \in \mathcal{W}. \tag{43}$$

Moreover

$$\mathcal{E}(\mathcal{P}f) \le \mathcal{E}(f) \tag{44}$$

for all $f \in \mathcal{X}$.

We now establish a lemma showing the equivalence of certain semi-norms on the space $\mathcal{X}$ defined above.

Lemma 10. *Let τ be a spherical triangle with $|\tau| \le 1$ and $f \in \mathcal{X}$. Let $\mathcal{E}_\tau$ be the functional defined in (40). There exists a constant D_1 such that*

$$D_1 |f|^2_{2,2,\tau} \le \mathcal{E}_\tau(f) \le |f|^2_{2,2,\tau}. \tag{45}$$

Proof: By the definition

$$|f|^2_{2,2,\tau} = (\sum_{|\alpha|=2} ||D^\alpha f_1||_{2,\tau})^2 \ge \sum_{|\alpha|=2} ||D^\alpha f_1||^2_{2,\tau} = \mathcal{E}_\tau(f).$$

Since the number of elements in the sum (40) is 8,

$$|f|^2_{2,2,\tau} = (\sum_{|\alpha|=2} ||D^\alpha f_1||_{2,\tau})^2 \le 8 \sum_{|\alpha|=2} ||D^\alpha f_1||^2_{2,\tau} \le 8\ \mathcal{E}_\tau(f).\ \square$$

Next we establish reproductive property of the energy functional $\mathcal{E}_\tau$.

Lemma 11. *Let τ be a spherical triangle with $|\tau| \le 1$. Suppose $f \in \mathcal{X}$. Then $\mathcal{E}_\tau(f) = 0$ if and only if f is a trivariate homogeneous linear polynomial on τ.*

Proof: Apply the definition of $\mathcal{E}_\tau$. $\square$

In addition to Lemma 11 we need to establish the equivalence of energy and L_2 norms on the Hilbert space $\mathcal{W}$.

Theorem 3. *Suppose $\mathbf{S} \subseteq S^0_d(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ with $|\Delta| \le 1$, and let $\mathcal{W}$ be the associated Hilbert space (41). Then there exist constants $0 < D_2 \le D_3 < \infty$ depending only d and β such that*

$$D_2 \mathcal{E}(f) \le (\tan \frac{|\Delta|}{2})^{-4} ||f||^2_{2,\mathbf{S}} \le D_3 \mathcal{E}(f), \tag{46}$$

for all $f \in \mathcal{W}$, where $\|f\|^2_{2,\mathbf{S}} := \int_{\mathbf{S}} |f|^2$.

Proof: By Lemmas 7 and 10 for every $f \in \mathcal{W}$,

$$\int_\tau |f|^2 \le A_9^2 (\tan \frac{|\tau|}{2})^4 |f|^2_{2,2,\tau} \le {D_1}^{-1} A_9^2 (\tan \frac{|\tau|}{2})^4 \mathcal{E}_\tau(f).$$

Summing over all $\tau \in \Delta$, we get

$$||f||_{2,\mathbf{S}}^2 = \int_{\mathbf{S}} |f|^2 \le {D_1}^{-1} A_9^2 (\tan \frac{|\Delta|}{2})^4 \mathcal{E}(f).$$

By Lemma 10 and Lemma 6,

$$\mathcal{E}_\tau(f) \le |f|_{2,2,\tau}^2 \le \frac{A_7^2}{(\tan \frac{\rho_\tau}{2})^4} ||f||_{2,\tau}^2.$$

Sum over $\tau \in \Delta$ to get

$$\mathcal{E}(f) \le \frac{A_7^2}{(\tan \frac{\rho_\Delta}{2})^4} ||f||_{2,\mathbf{S}}^2.$$

Since Δ is β-quasi-uniform, $|\rho_\Delta| \ge \frac{|\Delta|}{\beta}$, and therefore

$$\tan \frac{\rho_\Delta}{2} \ge \tan \frac{|\Delta|}{2\beta} \ge \frac{1}{\beta^2} \tan \frac{|\Delta|}{2}.$$

Then

$$\mathcal{E}(f) \le \frac{A_7^2 \beta^8}{(\tan \frac{|\Delta|}{2})^4} ||f||_{2,\mathbf{S}}^2.$$

Let $D_3 := D_1^{-1} A_9^2$ and $D_2 := {A_7}^{-2} \beta^{-8}$ to get the result. $\square$

Next we want to show that under certain conditions on $\mathcal{S}$, the $\mathcal{X}$-norm on the Hilbert space $\mathcal{W}$ is also equivalent to a certain coefficient norm.

Corollary 1. *Suppose $\mathcal{S} \subseteq S_d^r(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ, and that $\{B_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}$ defined in Lemma 8. Then $\{B_\xi\}_{\xi \in \mathcal{N}}$ is a Riesz basis (with respect to the $\mathcal{X}$-norm) for the linear space $\mathcal{W}$ defined in (41). Here $\mathcal{N}$ is the subset of the minimal determining set $\mathcal{M}$ excluding the set of vertices $\mathcal{V}$ of Δ. In particular, there exist constants D_4, D_5 depending on d, β such that*

$$D_4 \sum_{\xi \in \mathcal{N}} |c_\xi|^2 \le (\tan \frac{|\Delta|}{2})^{-2} || \sum_{\xi \in \mathcal{N}} c_\xi B_\xi ||_{2,\mathbf{S}}^2 \le D_5 \sum_{\xi \in \mathcal{N}} |c_\xi|^2, \tag{47}$$

for all $\{c_\xi\}_{\xi \in \mathcal{N}}$.

Proof: Let us note first that for any spline $s \in \mathcal{W}$, $s = \sum_{\xi \in \mathcal{M}} c_\xi B_\xi = \sum_{\xi \in \mathcal{N}} c_\xi B_\xi$ due to the zero interpolating conditions and (28). By Lemma 8, 4) there exists a positive constant C_5 depending only on d and θ_Δ, such that on each triangle $\tau \in \Delta$ and for all domain point $\xi \in \mathcal{N}$ which are on τ,

$$\sum_{\xi \in \mathcal{N} \cap \tau} |c_\xi|^2 \le \binom{d+2}{2} C_5^2 A_\tau^{-1} \int_\tau | \sum_{\xi \in \mathcal{N}_\tau} c_\xi B_\xi |^2,$$

where $\mathcal{N}_\tau = \{\xi \in \mathcal{N}, B_\xi|_\tau \neq 0\}$. Then summing over $\tau \in \Delta$

$$\frac{1}{\binom{d+2}{2} C_5^2} \pi \rho_\Delta^2 \sum_{\xi \in \mathcal{N}} |c_\xi|^2 \leq \int_{\mathbf{S}} |\sum_{\xi \in \mathcal{N}} c_\xi B_\xi|^2. \tag{48}$$

Similarly, by Lemma 8, 5) there exists a positive constant C_6, depending only on d and θ_Δ, such that

$$\int_\tau |\sum_{\xi \in \mathcal{N}_\tau} c_\xi B_\xi|^2 \leq \int_\tau \sum_{\xi \in \mathcal{N}_\tau} |c_\xi|^2 \sum_{\xi \in \mathcal{N}_\tau} |B_\xi|^2 \leq \binom{d+2}{2} n_3 C_6^2 A_\tau \sum_{\xi \in \mathcal{N}_\tau} |c_\xi|^2$$

by using Lemma 8,1). Then

$$\int_{\mathbf{S}} |\sum_{\xi \in \mathcal{N}} c_\xi B_\xi|^2 \leq \binom{d+2}{2} C_6^2 n_3 \sum_{\tau \in \Delta} A_\tau \sum_{\xi \in \mathcal{N}_\tau} |c_\xi|^2. \tag{49}$$

By Lemmas 3 and 8, 6), we get

$$\begin{aligned} \frac{\pi}{\binom{d+2}{2} C_5^2} \rho_\Delta^2 \qquad & \sum_{\xi \in \mathcal{N}} |c_\xi|^2 \leq || \sum_{\xi \in \mathcal{N}} c_\xi B_\xi ||_{2,\mathbf{S}}^2 \\ & \leq D_3 \binom{d+2}{2}^2 C_6^2 n_3^2 (\tan(\frac{|\Delta|}{2})^2 C_7 \sum_{\xi \in \mathcal{N}} |c_\xi|^2. \end{aligned}$$

Therefore, we obtain (47) with

$$D_4 = \frac{1}{\binom{d+2}{2} C_5^2} \frac{1}{\beta^2}$$

and

$$D_5 = D_3 n_3^2 \binom{d+2}{2}^2 C_6^2 C_7. \;\square$$

Next, we estimate $\mathcal{X}$-norm of the projection operator $\mathcal{P}$ in (43) outside of τ - the support of an interpolant $f \in \mathcal{X}$. Here we follow a similar result for bivariate splines that can be found in [6], making several adjustments for the spherical splines. Before we proceed with the result we need the following lemma, which can be found in [3].

Lemma 12. *If the sequence* $\{a_i\}_{i=1}^\infty$ *satisfies* $|a_m| \geq \gamma \sum_{j \geq m} |a_j|$ *for all* $m \geq 0$ *and some* $\gamma \in (0,1)$, *then* $|a_m| \leq a_0 \frac{(1-\gamma)^m}{\gamma}$.

Proof: See [3]. $\square$

It is established in Section 5 of [8] that $\{B_\xi\}_{\xi \in \mathcal{M}}$ is a local basis with a local support size ℓ equal to 3. The following theorem, however, holds in general for any fixed ℓ.

Theorem 4. *There exist constants $0 \le \sigma \le 1$ and D_6, depending only on ℓ, d, β, such that for any triangle $T \in \Delta$ and any function $f \in \mathcal{X}$ with supp$(f) \subseteq T$*

$$\mathcal{E}_\tau(\mathcal{P}f) \le D_6 \sigma^k \mathcal{E}(f), \tag{50}$$

whenever $\tau \in star^{2(k+2)\ell+1}(T) \backslash star^{2(k+1)\ell+1}(T)$ with $k \ge 1$.

Proof: Let

$$\begin{array}{rcl} \mathcal{M}_0^T: & = & \{\xi \in \mathcal{M} : \mathrm{supp}(B_\xi) \cap T \ne \emptyset\}, \\ \mathcal{M}_k^T: & = & \{\xi \in \mathcal{M} : \mathrm{supp}(B_\xi) \cap \mathrm{star}^{2k\ell}(T) \ne \emptyset\}, \\ \mathcal{N}_0^T: & = & \mathcal{M}_0^T, \\ \mathcal{N}_k^T: & = & \mathcal{M}_k^T \backslash \mathcal{M}_{k-1}^T. \end{array}$$

Suppose $\mathcal{P}f = \sum_{\xi \in \mathcal{M}} c_\xi B_\xi$, and let

$$u_k := \sum_{\xi \in \mathcal{M}_k^T} c_\xi B_\xi, \quad w_k := \mathcal{P}f - u_k, \quad a_k := \sum_{\xi \in \mathcal{N}_k^T} c_\xi^2,$$

for $k \ge 0$. Since $\mathcal{P}f \in \mathcal{W}$, by Corollary 1

$$\sum_{j \ge k+1} a_j = \sum_{\xi \notin \mathcal{M}_k^T} c_\xi^2 \le (\tan \frac{|\Delta|}{2})^{-2} {D_4}^{-1} ||w_k||_{2,\mathbf{s}}^2 \le (\tan \frac{|\Delta|}{2})^2 \frac{D_3}{D_4} \mathcal{E}(w_k).$$

Note that $w_k \in \mathcal{W}$ as well, then using (43) we have $< f - \mathcal{P}f, w_k >= 0$. Moreover, $< f, w_k >= 0$, since supp$(f) \subseteq T$ and supp(w_k) lies outside T. In fact, supp$(w_k) \cap \cup_{\xi \in \mathcal{M}_{k-1}^T} \mathrm{supp}(B_\xi) = \emptyset$ for $k \ge 1$, it follows that

$$\begin{array}{rcl} \mathcal{E}(w_k) & = & \langle \mathcal{P}f - u_k, w_k \rangle = \langle f - u_k, w_k \rangle = -\langle u_k, w_k \rangle \\ & = & -\langle \sum_{\xi \in \mathcal{N}_k^T} c_\xi B_\xi, w_k \rangle \le \mathcal{E}(\sum_{\xi \in \mathcal{N}_k^T} c_\xi B_\xi)^{1/2} \ \mathcal{E}(w_k)^{1/2}, \end{array}$$

and therefore by (46) and (47)

$$\mathcal{E}(w_k) \le \mathcal{E}(\sum_{\xi \in \mathcal{N}_k^T} c_\xi B_\xi) \le \frac{1}{D_2 (\tan \frac{|\Delta|}{2})^4} || \sum_{\xi \in \mathcal{N}_k^T} c_\xi B_\xi ||_{2,\mathbf{s}}^2 \le \frac{D_5}{D_2} (\tan \frac{|\Delta|}{2})^{-2} a_k.$$

Hence

$$\sum_{j \ge k+1} a_j \le \frac{D_5 D_3}{D_4 D_2} a_k.$$

Let $\gamma := \frac{D_4 D_2}{D_4 D_2 + D_5 D_3}$. Then by Lemma 12

$$a_k \le a_0 \frac{(1-\gamma)^k}{\gamma} = \frac{a_0}{\gamma} \sigma^{2k},$$

with $\sigma := \sqrt{1-\gamma}$. It is easy to see that by our assumption on Δ and definition of D_4 and D_5, both γ and σ are positive and bounded above by 1. Since (44) holds for f, by Corollary 1 we have

$$a_0 \leq \sum_{j\geq 0} a_j = \sum_{\xi\in\mathcal{M}} c_\xi^2 \leq \frac{(\tan\frac{|\Delta|}{2})^{-2}}{D_4}||\mathcal{P}f||_{2,\mathbf{S}}^2 \leq \frac{(\tan\frac{|\Delta|}{2})^{2}}{D_4}D_3\mathcal{E}(f).$$

Let $\tau \in \operatorname{star}^{2(k+2)\ell+1}(T)\backslash\operatorname{star}^{2(k+1)\ell+1}(T)$ for some $k \geq 1$. If $\xi \in \mathcal{M}_k^T$, then $\operatorname{supp}(B_\xi) \subseteq \operatorname{star}^{2(k+1)\ell}(T)$, and therefore $\tau \cap \operatorname{supp}(B_\xi) = \emptyset$. Using (47) again,

$$\mathcal{E}_\tau(\mathcal{P}f) \leq \frac{1}{D_2(\tan\frac{|\Delta|}{2})^4}||\mathcal{P}f\chi_\tau||_{2,\mathbf{S}}^2 = \frac{1}{D_2(\tan\frac{|\Delta|}{2})^4}||\sum_{\xi\notin\mathcal{M}_k^\tau} c_\xi B_\xi||_{2,\mathbf{S}}^2 \leq$$

$$\frac{D_5}{D_2}(\tan\frac{|\Delta|}{2})^{-2}\sum_{\xi\notin\mathcal{M}_k^\tau} c_\xi^2 = \frac{D_5}{D_2}(\tan\frac{|\Delta|}{2})^{-2}\sum_{j\geq k+1} a_j \leq \frac{D_5 D_3}{\gamma D_4 D_2}\sigma^{2k}\mathcal{E}(f).$$

We obtained (50) with $D_6 = \frac{D_5 D_3}{\gamma D_4 D_2}$. □

As a consequence of the last result, we can now compare Sobolev semi-norms of $\mathcal{P}f$ and f. Analogous result for bivariate polynomials can be found in [5], and a similar proof holds.

Theorem 5. *There exists a constant D_7 depending only on d, ℓ and β, such that for every $f \in \mathcal{X}$*

$$|\mathcal{P}f|_{2,\infty,\mathbf{S}} \leq D_7|f|_{2,\infty,\mathbf{S}}. \tag{51}$$

Proof: Let τ be a fixed triangle in Δ, and let

$$\Omega_0^\tau := \operatorname{star}^{4\ell+1}(\tau), \quad \Omega_k^\tau := \operatorname{star}^{2(k+2)\ell+1}(\tau)\backslash\operatorname{star}^{2(k+1)\ell+1}(\tau).$$

Let n_k denote the number of triangles in $\Omega_k^\tau,\ k \geq 0$. For any homogeneous polynomial P of degree with d we have, by Lemma 10 and (20)

$$\mathcal{E}_T(P) \geq D_1|P|_{2,2,T}^2 \geq \frac{D_1 A_T}{K_6^2}|P|_{2,\infty,T}^2,$$

where T is a triangle in Δ. Similarly, for any function $f \in \mathcal{X}$, by Lemma 10, we have

$$\mathcal{E}_T(f) \leq A_T|f|_{2,\infty,T}^2. \tag{52}$$

Write $f = \sum_{T\in\Delta} f_T$ with $\operatorname{supp}(f_T) \subseteq T$. Since $\mathcal{P}$ is a linear operator,

$$|\mathcal{P}f|_{2,\infty,\tau} \leq \sum_{T\in\Delta}|\mathcal{P}f_T|_{2,\infty,\tau} \leq \frac{K_6}{(D_1 A_\tau)^{1/2}}\sum_{T\in\Delta}\mathcal{E}_\tau(\mathcal{P}f_T)^{1/2}.$$

Then by (50) and (52)

$$\begin{aligned}
|\mathcal{P}f|_{2,\infty,\tau} &\le \frac{K_6}{(D_1 A_\tau)^{1/2}} \sum_{k\ge 0} \sum_{T\in\Omega_k^\tau} \mathcal{E}_\tau(\mathcal{P}f_T)^{1/2} \\
&\le \frac{K_6 D_6}{(D_1 A_\tau)^{1/2}} \Big(\sum_{T\in\Omega_0^\tau} \mathcal{E}(f_T)^{1/2} + \sum_{k\ge 1}\sum_{T\in\Omega_k^\tau} \sigma^k \mathcal{E}(f_T)^{1/2} \Big) \\
&\le \frac{K_6 D_6}{(D_1 A_\tau)^{1/2}} (\max_{\tau\in\Delta} A_T^{1/2})(n_0 + \sum_{k\ge 1} \sigma^k n_k) |f|_{2,\infty,\mathbf{S}}.
\end{aligned}$$

Since $\sigma < 1$, $\sum_{k\ge 1} \sigma^k n_k < \infty$, and

$$\frac{\max_{\tau\in\Delta} A_\tau^{1/2}}{\min_{\tau\in\Delta} A_\tau^{1/2}} \le \sqrt{\frac{5}{4}} \frac{|\Delta|}{\rho_\Delta} \le \sqrt{\frac{5}{4}}\beta,$$

by Lemma 3, (51) follows by taking the supremum over all $\tau \in \Delta$. $\square$

We are finally in a position to prove the main result of this paper.

Theorem 6. *Suppose $\mathcal{S} \subseteq S_d^r(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ with $|\Delta| \le 1$ and $d \ge 3r+2$. For d odd there exists a constant D_8 depending only on d and β, such that the minimal energy interpolant S_f, defined in (1), satisfies*

$$||f - S_f||_{\infty,\mathbf{S}} \le D_8 (\tan \frac{|\Delta|}{2})^2 |f|_{2,\infty,\mathbf{S}}, \tag{53}$$

for all $f \in C^2(\mathbf{S})$. For d even there exist constants D_9 and D_{10} depending only on d and β, such that the minimal energy interpolant S_f satisfies

$$||f - S_f||_{\infty,\mathbf{S}} \le D_9 (\tan \frac{|\Delta|}{2})^2 |f|_{2,\infty,\mathbf{S}} + D_{10} (\tan \frac{|\Delta|}{2})^3 |f|_{3,\infty,\mathbf{S}}, \tag{54}$$

for all $f \in C^3(\mathbf{S})$.

Proof: Given a function $f \in \mathcal{X}$, let $s_f \in U_f$ be the quasi-interpolant defined in Section 5. If d is odd, by Theorem 2 there exists a constant K_7 depending on d and the smallest angle of Δ such that

$$||f - s_f||_{\infty,\mathbf{S}} \le K_7 (\tan \frac{|\Delta|}{2})^2 |f|_{2,\infty,\mathbf{S}}, \tag{55}$$

and

$$|f - s_f|_{2,\infty,\mathbf{S}} \le K_7 |f|_{2,\infty,\mathbf{S}}.$$

Then

$$|s_f|_{2,\infty,\mathbf{S}} \le |f - s_f|_{2,\infty,\mathbf{S}} + |f|_{2,\infty,\mathbf{S}}$$

$$\leq \quad (K_7+1)|f|_{2,\infty,\mathbf{S}}. \tag{56}$$

Since $\mathcal{P}s_f = s_f - S_f$, by Theorem 5

$$|s_f - S_f|_{2,\infty,\mathbf{S}} = |\mathcal{P}s_f|_{2,\infty,\mathbf{S}} \leq D_7|s_f|_{2,\infty,\mathbf{S}}, \tag{57}$$

and by (56)

$$|s_f - S_f|_{2,\infty,\mathbf{S}} \leq D_7(K_7+1)|f|_{2,\infty,\mathbf{S}}.$$

Since both S_f and s_f interpolate f, their difference satisfies the hypothesis of Lemma 7 and thus

$$\begin{aligned} ||s_f - S_f||_{\infty,\mathbf{S}} &\leq A_8(\tan\frac{|\Delta|}{2})^2|s_f - S_f|_{2,\infty,\mathbf{S}} \\ &\leq A_8D_7(K_7+1)(\tan\frac{|\Delta|}{2})^2|f|_{2,\infty,\mathbf{S}}. \end{aligned}$$

Then by (55)

$$\begin{aligned} ||f - S_f||_{\infty,\mathbf{S}} &\leq ||f - s_f||_{\infty,\mathbf{S}} + ||s_f - S_f||_{\infty,\mathbf{S}} \\ &\leq K_7(\tan\frac{|\Delta|}{2})^2|f|_{2,\infty,\mathbf{S}} + A_8D_7(K_7+1)(\tan\frac{|\Delta|}{2})^2|f|_{2,\infty,\mathbf{S}} \\ &= D_8(\tan\frac{|\Delta|}{2})^2|f|_{2,\infty,\mathbf{S}} \end{aligned}$$

With $D_8 = K_7 + A_8D_7(K_7+1)$ we get (53). Similarly, if d is even, we have to consider even m in Theorem 2 and hence, there exists a constant K_8 depending on d and the smallest angle of Δ such that

$$||f - s_f||_{\infty,\mathbf{S}} \leq K_8(\tan\frac{|\Delta|}{2})^3|f|_{3,\infty,\mathbf{S}}, \tag{58}$$

and

$$|f - s_f|_{2,\infty,\mathbf{S}} \leq K_8(\tan\frac{|\Delta|}{2})|f|_{3,\infty,\mathbf{S}}.$$

Then

$$|s_f|_{2,\infty,\mathbf{S}} \leq K_8(\tan\frac{|\Delta|}{2})|f|_{3,\infty,\mathbf{S}} + |f|_{2,\infty,\mathbf{S}}, \tag{59}$$

and by (57) and (59)

$$|s_f - S_f|_{2,\infty,\mathbf{S}} \leq D_7(K_8(\tan\frac{|\Delta|}{2})|f|_{3,\infty,\mathbf{S}} + |f|_{2,\infty,\mathbf{S}}).$$

Since both S_f and s_f interpolate f, their difference satisfies the hypothesis of Lemma 7 and thus

$$||s_f - S_f||_{\infty,\mathbf{S}} \quad \leq \quad A_8(\tan\frac{|\Delta|}{2})^2D_7(K_8(\tan\frac{|\Delta|}{2})|f|_{3,\infty,\mathbf{S}} + |f|_{2,\infty,\mathbf{S}}).$$

Then by (58)

$$||f - S_f||_{\infty,\mathbf{S}} \quad \leq \quad D_9(\tan\frac{|\Delta|}{2})^2|f|_{2,\infty,\mathbf{S}} + D_{10}(\tan\frac{|\Delta|}{2})^3|f|_{3,\infty,\mathbf{S}}.$$

We have thus established the result of our main theorem in this paper. $\square$

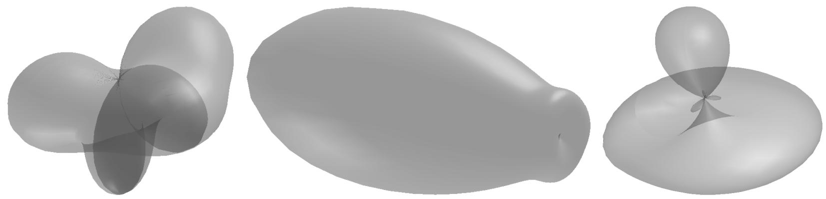

Fig. 2. Test functions f_1, f_2, f_3.

§7. Numerical Experiments

We have implemented spherical splines of arbitrary degree and arbitrary smoothness over any given spherical triangulation in MATLAB. In particular, we have coded the minimal energy method for scattered data interpolation using spherical splines. In this section we provide a description of three sets of experiments. 1) We numerically test the convergence rate of the minimal energy method over a sequence of uniformly refined triangulations. Our test confirms the second order convergence rate. 2) We do the same test as in 1) for spherical splines of various degrees for a single function to show that the convergence rate is the same no matter which degree is used. 3) We demonstrate a spherical spline interpolation to a given set of scattered data which represent the geopotential of the earth.

Let us describe our experiments in detail. The initial triangulation Δ_1 in the first two examples consists of 8 triangles with vertices being unit coordinate vectors and their antipodes. Its first refinement Δ_2 is obtained by connecting the midpoints of all edges in Δ_1 such that each triangle is split in four subtriangles. Similarly Δ_3 and Δ_4 are obtained from Δ_2 and Δ_3 correspondingly. Note that with such a refinement the size $|\Delta_{i+1}|$ of a refinement is not a half of $|\Delta_i|, i = 1, 2, 3$ as it happens in the planar case. Instead $\tan \frac{|\Delta_i|}{2}, i = 1, 2, 3$ is reduced in half as illustrated in Table 1.

i	1	2	3	4
$\lvert\Delta\rvert$	1.91063	1.23095	0.67967	0.34994
$(\tan \frac{\lvert\Delta\rvert}{2})^2$	2	0.5	0.125	0.03125

Tab. 1. Triangulation parameters.

In the following example we shall test several functions:

$$f_1(x, y, z) = x^2 - (y^3 + z^7)$$

$$f_2(x, y, z) = 0.1x^8 + e^{2y^3}$$

$$f_3(x, y, z) = \ln(2 + x^2) - \sin(4z - y).$$

These functions are displayed in Figure 2.

Example 1. We use spherical splines of degree 5 and smoothness 1 to find the minimal energy interpolants over the triangulation $\Delta_1, \Delta_2, \Delta_3$ and Δ_4. Then we evaluate the splines at 5120 evenly spaced points w and record the relative errors for these three test functions. The relative error on Δ_i is defined by $e(\Delta_i) := \frac{||s(w)-f(w)||_\infty}{||f(w)||_\infty}$, $s \in S_d^r(\Delta_i)$. The errors are listed in Table 2. In Table 3 we list ratios of the form $\frac{e(\Delta_i)}{e(\Delta_{i+1})}, i = 1, 2, 3$ for all three functions. The numerical convergence rates are close to the convergence rate we derived in the previous section.

$f \backslash i$	1	2	3	4
f_1	$1.0317e-00$	$1.8164e-01$	$2.8386e-02$	$2.8290e-03$
f_2	$0.3834e-00$	$0.6344e-01$	$1.7466e-02$	$2.5500e-03$
f_3	$1.0496e-00$	$4.3803e-01$	$0.5142e-01$	$0.5150e-02$

Tab. 2. Experimental errors for C^1 quintic splines.

$f \backslash i$	1	2	3
f_1	5.6799	6.3989	10.0339
f_2	6.0435	3.6322	6.8494
f_3	2.3962	8.5187	9.9845

Tab. 3. Convergence rates of the C^1 quintic splines.

Example 2. In this example we work with one function only and vary the degree d of the spline space. That is, we use $S_d^1(\Delta_i)$, $d = 3, 4, 5, 6, 7$, $i = 1, 2, 3, 4$. Even though we cannot apply Theorem 6 in $S_3^1(\Delta)$ and $S_4^1(\Delta)$, the experimental result are similar to the ones we obtain for higher degrees. Errors are computed as in Example 1 and are recorded in Table 4. The corresponding convergence rates are displayed in Table 5. The numerical rates show that increasing the degree of the spline space will not result in a better rate.

$d \backslash i$	1	2	3	4
5	$3.8334e-01$	$0.6344e-01$	$1.7466e-02$	$2.5500e-03$
6	$3.8057e-01$	$0.6315e-01$	$1.8148e-02$	$2.6871e-03$
7	$3.8286e-01$	$0.6254e-01$	$1.8429e-02$	$2.7489e-03$

Tab. 4. Splines of various degrees interpolating $f_2(x, y, z) = 0.1x^8 + e^{2y^3}$.

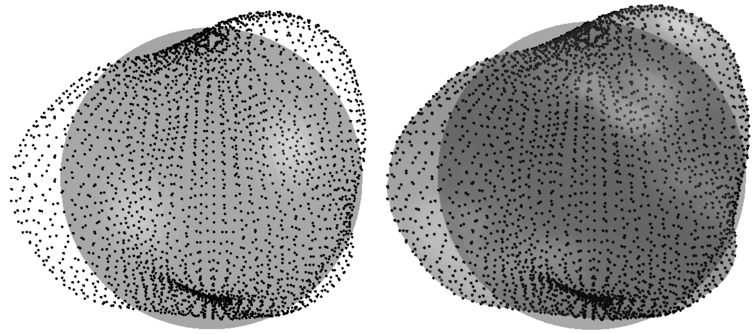

Fig. 3. Geopotential data and minimal energy C^1 cubic interpolant.

$d\backslash i$	1	2	3
5	6.0435	3.6322	6.8494
6	6.0265	3.4797	6.7538
7	6.1225	3.3934	6.7045

Tab. 5. Convergence rates for splines of various degrees interpolating f_2.

Example 3. We present an example of scattered data interpolation over the earth. We are given a set of locations with geopotential values collected by a satellite. The total amount of data values is 5760. The left graph in Figure 3 shows the set of scattered data and the right graph shows the minimal energy interpolatory spherical spline surface. Both the original data and the spline solution are scaled in this figure for convenience. We use C^1 cubic spherical splines since the data set is very large. The spline surface represents the given data quite well by visual inspection.

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