On the Number of Tight Wavelet Frame Generators associated with Multivariate Box Splines

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Abstract We study the number of Laurent polynomials in a sum of square magnitudes of a nonnegative Laurent polynomial associated with bivariate box splines on the three- and the four-direction meshes. In addition, we study the same problem associated with trivariate box splines. The number of Laurent polynomials in a sum of squares magnitude of a nonnegative Laurent polynomial determines the number of multivariate box spline tight wavelet frame generators under the simple construction method in [5]. As a result, the numbers of bivariate box spline tight wavelet frame generators on the three- and the four-direction meshes under certain condition and the one for trivariate box splines are much smaller than the number of multivariate box spline tight wavelet frame generators constructed via Kronecker poduct method in [2].

Keywords Sum of square magnitudes, Multivariate box splines, Tight wavelet frame generators, Nonnegative Laurent polynomials.

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1 Introduction

In [5], Lai and Stöckler introduced a simple constructive method for compactly supported multivariate tight wavelet frames. One crucial step of the Lai and Stöckler's method is whether one can find a finite number of Laurent polynomials whose sum of square magnitudes expresses a nonnegative Laurent polynomial $1-\sum_{\nu\in\{0,\pi\}^d}|P(\omega+\nu)|^2$, where Laurent polynomial P is a mask associated with a refinable function ϕ, e.g., a box spline in $\mathbb{R}^d$ with $d\geq 1$. That is, one needs to find Laurent polynomials $\widetilde{P}_1,\ldots,\widetilde{P}_N$, $N\geq 1$ such that

$$\sum_{\nu\in\{0,\pi\}^d}|P(\omega+\nu)|^2+\sum_{i=1}^N|\widetilde{P}_i(2\omega)|^2=1, \text{ for all } \omega\in\mathbb{R}^d. \tag{1}$$

This question is related to the 17th of Hilbert's 23 problems (see [7]). Recently Dritschel proved that a positive Laurent polynomial in multivariate settings can be written as a sum of square magnitudes of Laurent polynomials(see [3]). However, we do not know the answer for this question about a nonnegative Laurent polynomial. Relevant literatures both on mathematical theories and applications to this problem are summarized in [4] together with an elementary proof of Dritschel theorem.

Let us call the condition (1) the SOS(sum of square magnitudes) condition for further convenient mention. In this paper, we shall study the number of Laurent polynomials in the SOS condition for the Laruent polynomials associated with bivariate and trivariate box splines. We shall improve the results on the number appeared in Lemma 5.7 in [5] and extend the Lemma to the Laurent polynomial associated with trivariate box splines. Our main result in this paper is the number of tight wavelet frames associated with trivariate box splines. As a result, the numbers of tight wavelet frame generators for bivariate box splines on four-direction mesh and trivariate box splines constructed Lai and Stöckler's method in [5] are less than the ones constructed via Kronecker product method in [2]. The explicit Laurent polynomials satisfying the SOS condition in (1) for various orders of bivariate box splines over the three- and the four- direction meshes were found and many edge detected images using these box spline tight wavelet frames were demonstrated in [6].

In Section 2, we first introduce basic settings, terms and the result directly related to the number of Laurent polynomials in the SOS condition in [5]. In Section 3, we carefully analyze the structure of the Laurant polynomials and simplify the construction. We find out that the result of Lemma 5.7 in [5] can be improved. Our study explains several critical points to motivate a technical proof on the number of Laurent polynomials in the SOS condition and number of tight wavelet frame generators associated with trivariate box splines over a general direction mesh.

2 Preliminary

The box spline ϕ_D defined by a set of nonzero vectors $D \subset \mathbb{R}^d$ whose span is $\mathbf{R}^d$ is a piecewise polynomial of degree $|D| - d$ whose Fourier transformation is

$$\widehat{\phi}_D(\omega) = \prod_{\xi \in D} \frac{1-e^{-i\omega\cdot\xi}}{i\omega\cdot\xi},$$

where $|D|$ denotes the cardinality of D. Then $\widehat{\phi}_D(\omega) = P_D(\omega/2)\widehat{\phi}_D(\omega/2)$, where

$$P_D(\omega) = \prod_{\xi \in D} \frac{1+e^{-i\omega\cdot\xi}}{2}.$$

We call ϕ_D a refinable function and the Laurent polynomial P_D a mask or a filter in wavelet analysis or image processing terms, respectively.

Without loss of generality, we may assume that D contains all unit vectors in $\mathbf{R}^d$. In this case we have

$$|P_D(\omega)|^2 = \prod_{\xi \in D} \left| \cos\left(\xi \cdot \frac{\omega}{2}\right)\right|^2 \le \prod_{i=1}^{d} \cos^2(\omega_i) \le 1,$$

where $\omega = (\omega_1, \cdots, \omega_d)$. Then

$$\sum_{\nu \in \{0,\pi\}^d} |P_D(\omega+\nu)|^2 \le \prod_{i=1}^{d} (\cos^2(\omega_i) + \sin^2(\omega_i)) = 1.$$

We next state the following theorem in [5] that gives us the question of the number of Laurent polynomials for multivariate box splines satisfying the SOS condition in (1).

Theorem 3.4 of [5]. Suppose that a mask P of a refinable function satisfies

$$\sum_{\nu \in \{0,\pi\}^d} |P(\omega+\nu)|^2 \le 1$$

and there exist Laurent polynomials $\widetilde{P}_1, \ldots, \widetilde{P}_N$ such that

$$\sum_{\nu \in \{0,\pi\}^d} |P(\omega+\nu)|^2 + \sum_{i=1}^{N} |\widetilde{P}_i(2\omega)|^2 = 1.$$

Then there exist $2^d + N$ locally supported tight frame generators.

See [5] for a constructive proof of this theorem and see [6] for various examples and applications of bivariate box spline tight wavelet frames constructed by this method to image processing.

3 The SOS condition for Bivariate Box Splines

Recall that the Laurent polynomial

$$P_{\ell,m,n}(\omega_1,\omega_2) := \left(\tfrac{1+e^{i\omega_1}}{2}\right)^{\ell}\left(\tfrac{1+e^{i\omega_2}}{2}\right)^{m}\left(\tfrac{1+e^{i(\omega_1+\omega_2)}}{2}\right)^{n}$$

is the mask associated with the box spline function $\phi_{\ell,m,n}$ on the three-direction mesh, where $\ell, m, n \geq 1$. Similarly, the Laurent polynomial $P_{\ell,m,n,p}$

$$P_{\ell,m,n,p}(\omega_1,\omega_2) := \left(\tfrac{1+e^{i\omega_1}}{2}\right)^{\ell}\left(\tfrac{1+e^{i\omega_2}}{2}\right)^{m}\left(\tfrac{1+e^{i(\omega_1+\omega_2)}}{2}\right)^{n}\left(\tfrac{1+e^{i(\omega_1-\omega_2)}}{2}\right)^{p}$$

is the mask associated with the box spline function $\phi_{\ell,m,n,p}$ on the four-direction mesh, where $\ell, m, n, p \geq 1$. We first revisit the following Lemma in [5].
Lemma 5.7. of [5]. For the Laurent polynomial $P_{\ell,m,n}$ there exist Laurent polynomials $\widetilde{P}_1, \dots, \widetilde{P}_N$ with $N \leq 9$ such that

$$\sum_{\nu\in\{0,\pi\}^2} |P_{\ell,m,n}(\omega+\nu)|^2 + \sum_{i=1}^{N} |\widetilde{P}_i(2\omega)|^2 = 1\,,$$

and for the Laurent polynomial $P_{\ell,m,n,p}$ there exist Laurent polynomials $\widetilde{P}_1, \dots, \widetilde{P}_N$ with $N \leq 22$ such that

$$\sum_{\nu\in\{0,\pi\}^2} |P_{\ell,m,n,p}(\omega+\nu)|^2 + \sum_{i=1}^{N} |\widetilde{P}_i(2\omega)|^2 = 1\,.$$

We now prove Lemma 1 which is an improved version of above Lemma 5.7 on the number of Laurent polynomials in the SOS condition. Although our proof uses the same ideas of the proof of Lemma 5.7 in [5], we introduce a simple notation and refine the arguments to achieve the the improvements.

Lemma 1 *For the Laurent polynomial $P_{\ell,m,n}$ there exist Laurent polynomials $\widetilde{P}_1, \dots, \widetilde{P}_N$ with $N \leq 7$ such that*

$$\sum_{\nu\in\{0,\pi\}^2} |P_{\ell,m,n}(\omega+\nu)|^2 + \sum_{i=1}^{N} |\widetilde{P}_i(2\omega)|^2 = 1. \tag{2}$$

In particular, when $n = 1$, $N \leq 5$. Next for the Laurent polynomial $P_{\ell,m,n,p}$ associated with box splines on four direction mesh, there exist Laurent polynomials $\widetilde{P}_1, \dots, \widetilde{P}_N$ with $N \leq 15$ the above condition (2) holds. Especially, when $n = 1 = p$, $N \leq 8$.

Proof: We prove for the Laurent polynomial $P_{\ell,m,n,p}$ associated with box splines on four direction mesh. For the Laurent polynomial $P_{\ell,m,n}$, one could apply the same steps as $P_{\ell,m,n,p}$ by setting $p = 0$. For convenience, we use 2ω instead of ω.

$$|P_{\ell,m,n,p}(2\omega)|^2 = \cos^{2\ell}(\omega_1)\cos^{2m}(\omega_2)\cos^{2n}(\omega_1+\omega_2)\cos^{2p}(\omega_1-\omega_2),$$

and

$$\begin{aligned}&\sum_{\nu\in\{0,\pi\}^2} |P_{\ell,m,n,p}(2\omega+\nu)|^2 \\ &= \cos^{2n}(\omega_1+\omega_2)\cos^{2p}(\omega_1-\omega_2)\Big(\cos^{2\ell}(\omega_1)\cos^{2m}(\omega_2)+\sin^{2\ell}(\omega_1)\sin^{2m}(\omega_2)\Big) \\ &+ \sin^{2n}(\omega_1+\omega_2)\sin^{2p}(\omega_1-\omega_2)\Big(\sin^{2\ell}(\omega_1)\cos^{2m}(\omega_2)+\cos^{2\ell}(\omega_1)\sin^{2m}(\omega_2)\Big).\end{aligned}$$

We convert each trigonometric function into a binomial form and write the sum of even orders and the sum of odd orders separately, i.e., for example

$$\begin{aligned}&\cos^{2\ell}(\omega_1) = \left(\frac{1+\cos(2\omega_1)}{2}\right)^\ell \\ =\ & \left(\frac{1}{2}\right)^\ell \left\{\sum_{0\le 2i\le \ell}\binom{\ell}{2i}\cos^{2i}(2\omega_1) + \sum_{0\le 2i+1\le \ell}\binom{\ell}{2i+1}\cos^{2i+1}(2\omega_1)\right\} \\ =:\ & E_\ell(\omega_1)+O_\ell(\omega_1).\end{aligned}$$

So $\sin^{2\ell}(\omega_1) = E_\ell(\omega_1) - O_\ell(\omega_1)$. Similar for each trigonometric function of order m, n, and p. That is $E_m(\omega_2), O_m(\omega_2), E_n(\omega_1+\omega_2), O_n(\omega_1+\omega_2), E_p(\omega_1-\omega_2)$, and $O_p(\omega_1-\omega_2)$. Note that all even order terms, e.g., $E_\ell(\omega_1)$ is nonnegative and can be written in terms of $\cos 4(\omega_1)$. By using the Riesz-Féjer lemma, $E_\ell(\omega_1) = |\widetilde{P}_\ell(4\omega_1)|^2$ for a polynomial in $e^{i4\omega_1}$. Similar for other even terms. For odd order terms, let us rewrite $O_\ell(\omega_1) = \widehat{O_\ell}(\omega_1)\cos(2\omega_1)$ so that $\widehat{O_\ell}(\omega_1)$ is nonnegative. Since $\widehat{O_\ell}(\omega_1)$ can be written in terms of $\cos(4\omega_1)$, $\widehat{O_\ell}(\omega_1) = |\tilde{q}_\ell(4\omega_1)|^2$ by Riesz-Féjer lemma for a polynomial $\tilde{q}_\ell$ in $e^{i4\omega_1}$. Similar for other three terms $O_m(\omega_2), O_n(\omega_1+\omega_2)$, and $O_p(\omega_1-\omega_2)$. Then we have

$$\begin{aligned}&\textstyle\sum_{\nu\in\{0,\pi\}^2} |P_{\ell,m,n,p}(2\omega+\nu)|^2 \\ &= (E_n(\omega_1+\omega_2)+O_n(\omega_1+\omega_2))(E_p(\omega_1-\omega_2)+O_p(\omega_1-\omega_2))\times \\ &\quad (2E_\ell(\omega_1)E_m(\omega_2)+2O_\ell(\omega_1)O_m(\omega_2)) \\ &\quad + (E_n(\omega_1+\omega_2)-O_n(\omega_1+\omega_2))(E_p(\omega_1-\omega_2)-O_p(\omega_1-\omega_2))\times \\ &\quad (2E_\ell(\omega_1)E_m(\omega_2)+2O_\ell(\omega_1)O_m(\omega_2)) \\ &= 4E_\ell(\omega_1)E_m(\omega_2)E_n(\omega_1+\omega_2)E_p(\omega_1-\omega_2)+4E_\ell(\omega_1)E_m(\omega_2)O_n(\omega_1+\omega_2)O_p(\omega_1-\omega_2) \\ &\quad + 4O_\ell(\omega_1)O_m(\omega_2)E_n(\omega_1+\omega_2)O_p(\omega_1-\omega_2)+4O_\ell(\omega_1)O_m(\omega_2)O_n(\omega_1+\omega_2)E_p(\omega_1-\omega_2).\end{aligned}$$

Note that since $1 = \sum_{\nu\in\{0,\pi\}^2} |P_{\ell,m,n,p}(\nu)|^2$, we have

$$1 = 4E_\ell(0)E_m(0)E_n(0)E_p(0) + 4E_\ell(0)E_m(0)O_n(0)O_p(0) + 4O_\ell(0)O_m(0)E_n(0)O_p(0) + 4O_\ell(0)O_m(0)O_n(0)E_p(0).$$

It follows that

$$\begin{aligned}
&1 - \textstyle\sum_{\nu\in\{0,\pi\}^2} |P_{\ell,m,n,p}(2\omega+\nu)|^2 \\
&= 4(E_\ell(0)E_m(0)E_n(0)E_p(0) - E_\ell(\omega_1)E_m(\omega_2)E_n(\omega_1+\omega_2)E_p(\omega_1-\omega_2)) \\
&\quad + 4(E_\ell(0)E_m(0)O_n(0)O_p(0) - E_\ell(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_1+\omega_2)\widehat{O_p}(\omega_1-\omega_2)) \\
&\quad + 4(O_\ell(0)O_m(0)E_n(0)O_p(0) - \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_1+\omega_2)\widehat{O_p}(\omega_1-\omega_2)) \\
&\quad + 4(O_\ell(0)O_m(0)O_n(0)E_p(0) - \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_1+\omega_2)E_p(\omega_1-\omega_2)) \\
&\quad + 4E_\ell(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_1+\omega_2)\widehat{O_p}(\omega_1-\omega_2)(1-\cos 2(\omega_1+\omega_2)\cos 2(\omega_1-\omega_2)) \\
&\quad + 4\widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_1+\omega_2)\widehat{O_p}(\omega_1-\omega_2)(1-\cos(2\omega_1)\cos(2\omega_2)\cos 2(\omega_1-\omega_2)) \\
&\quad + 4\widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_1+\omega_2)E_p(\omega_1-\omega_2)(1-\cos(2\omega_1)\cos(2\omega_2)\cos 2(\omega_1+\omega_2)) \\
&=: 4T_1 + 4T_2 + 4T_3 + 4T_4 + 4T_5 + 4T_6 + 4T_7,
\end{aligned}$$

where $T_1, T_2, \cdots$, and T_7 denote the seven terms above accordingly. Let us consider the last three terms first.

$$\begin{aligned}
T_5 &= E_\ell(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_1+\omega_2)\widehat{O_p}(\omega_1-\omega_2))(1-\cos 2(\omega_1+\omega_2)\cos 2(\omega_1-\omega_2)) \\
&= |\widetilde{P}_\ell(4\omega_1)|^2|\widetilde{P}_m(4\omega_2)|^2|\tilde{q}_n(4\omega_1+4\omega_2)|^2|\tilde{q}_p(4\omega_1-4\omega_2)|^2\times \\
&\qquad (1-\cos 2(\omega_1+\omega_2)\cos 2(\omega_1-\omega_2)).
\end{aligned}$$

A simple computation shows

$$1-\cos 2(\omega_1+\omega_2)\cos 2(\omega_1-\omega_2) = \tfrac{1}{8}\left|e^{i4\omega_1} - e^{-i4\omega_1}\right|^2 + \tfrac{1}{8}\left|e^{i4\omega_2} - e^{-i4\omega_2}\right|^2.$$

We know that $T_5 = |\widetilde{P}_1(4\omega_1, 4\omega_2)|^2 + |\widetilde{P}_2(4\omega_1, 4\omega_2)|^2$. Similar for T_6 and T_7 by using the following two identities:

$$1-\cos(2\omega_1)\cos(2\omega_2)\cos 2(\omega_1-\omega_2) = \tfrac{1}{4}\left|1 - \tfrac{1}{2}e^{i4\omega_1} - \tfrac{1}{2}e^{i4(\omega_1-\omega_2)}\right|^2 + \tfrac{3}{16}\left|1-e^{i4\omega_2}\right|^2,$$
$$1-\cos(2\omega_1)\cos(2\omega_2)\cos 2(\omega_1+\omega_2) = \tfrac{1}{4}\left|1 - \tfrac{1}{2}e^{i4\omega_1} - \tfrac{1}{2}e^{i4(\omega_1+\omega_2)}\right|^2 + \tfrac{3}{16}\left|1-e^{i4\omega_2}\right|^2,$$

We then have

$$T_6 = \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_1+\omega_2)\widehat{O_p}(\omega_1-\omega_2))(1-\cos(2\omega_1)\cos(2\omega_2)\cos 2(\omega_1-\omega_2))$$

$$
\begin{aligned}
&= |\widetilde{q}_\ell(4\omega_1)|^2|\widetilde{q}_m(4\omega_2)|^2|\widetilde{P}_n(4\omega_1+4\omega_2)|^2|\widetilde{q}_p(4\omega_1-4\omega_2)|^2\times \\
&\quad \left(\tfrac{1}{4}\left|1-\tfrac{1}{2}e^{i4\omega_1}-\tfrac{1}{2}e^{i4(\omega_1-\omega_2)}\right|^2+\tfrac{3}{16}\left|1-e^{i4\omega_2}\right|^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
T_7 &= \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_1+\omega_2)E_p(\omega_1-\omega_2))(1-\cos(2\omega_1)\cos(2\omega_2)\cos 2(\omega_1+\omega_2)) \\
&= |\widetilde{q}_\ell(4\omega_1)|^2|\widetilde{q}_m(4\omega_2)|^2|\widetilde{q}_n(4\omega_1+4\omega_2)|^2|\widetilde{P}_p(4\omega_1-4\omega_2)|^2\times \\
&\quad \left(\tfrac{1}{4}\left|1-\tfrac{1}{2}e^{i4\omega_1}-\tfrac{1}{2}e^{i4(\omega_1+\omega_2)}\right|^2+\tfrac{3}{16}\left|1-e^{i4\omega_2}\right|^2\right).
\end{aligned}
$$

That is, $T_5+T_6+T_7=\sum_{i=1}^{6}|\widetilde{P}_i(4\omega_1,4\omega_2)|^2$. Observe when $n=p=1$, we have $E_n(\omega_1+\omega_2)=1=\widehat{O_n}(\omega_1+\omega_2)$ and $E_p(\omega_1-\omega_2)=1=\widehat{O_p}(\omega_1-\omega_2)$. Then

$$
\begin{aligned}
T_6+T_7 &= |\widetilde{q}_\ell(4\omega_1)|^2|\widetilde{q}_m(4\omega_2)|^2\times \\
&\quad \left(\tfrac{1}{4}\left|1-\tfrac{1}{2}e^{i4\omega_1}-\tfrac{1}{2}e^{i4(\omega_1-\omega_2)}\right|^2+\tfrac{6}{16}\left|1-e^{i4\omega_2}\right|^2+\tfrac{1}{4}\left|1-\tfrac{1}{2}e^{i4\omega_1}-\tfrac{1}{2}e^{i4(\omega_1+\omega_2)}\right|^2\right)
\end{aligned}
$$

Thus we have $T_5+T_6+T_7=\sum_{i=1}^{5}|\widetilde{P}_i(4\omega_1,4\omega_2)|^2$ instead. Next we consider $T_1,\cdots,T_4$.

$$
\begin{aligned}
T_1 &= E_\ell(0)E_m(0)E_n(0)E_p(0)-E_\ell(\omega_1)E_m(\omega_2)E_n(\omega_1+\omega_2)E_p(\omega_1-\omega_2)) \\
&= (E_\ell(0)-E_\ell(\omega_1))E_m(0)E_n(0)E_p(0) \\
&\quad +E_\ell(\omega_1)(E_m(0)-E_m(\omega_2))E_n(0)E_p(0) \\
&\quad +E_\ell(\omega_1)E_m(\omega_2)(E_n(0)-E_n(\omega_1+\omega_2))E_p(0) \\
&\quad +E_\ell(\omega_1)E_m(\omega_2)E_n(\omega_1+\omega_2)(E_p(0)-E_p(\omega_1-\omega_2)) \\
&=: T_{1,1}+T_{1,2}+T_{1,3}+T_{1,4}.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
T_2 &= E_\ell(0)E_m(0)O_n(0)O_p(0)-E_\ell(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_1+\omega_2)\widehat{O_p}(\omega_1-\omega_2) \\
&= (E_\ell(0)-E_\ell(\omega_1))E_m(0)O_n(0)O_p(0) \\
&\quad +E_\ell(\omega_1)(E_m(0)-E_m(\omega_2))O_n(0)O_p(0) \\
&\quad +E_\ell(\omega_1)E_m(\omega_2))(O_n(0)-\widehat{O_n}(\omega_1+\omega_2))O_p(0) \\
&\quad +E_\ell(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_1+\omega_2)(O_p(0)-\widehat{O_p}(\omega_1-\omega_2)) \\
&=: T_{2,1}+T_{2,2}+T_{2,3}+T_{2,4},
\end{aligned}
$$

$$
\begin{aligned}
T_3 &= O_\ell(0)O_m(0)E_n(0)O_p(0)-\widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_1+\omega_2)\widehat{O_p}(\omega_1-\omega_2) \\
&= (O_\ell(0)-\widehat{O_\ell}(\omega_1))O_m(0)E_n(0)O_p(0) \\
&\quad +\widehat{O_\ell}(\omega_1)(O_m(0)-\widehat{O_m}(\omega_2))E_n(0)O_p(0) \\
&\quad +\widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)(E_n(0)-E_n(\omega_1+\omega_2))O_p(0)
\end{aligned}
$$

$$
\begin{aligned}
&+\widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_1+\omega_2))(O_p(0)-\widehat{O_p}(\omega_1-\omega_2)) \\
=: &\ T_{3,1}+T_{3,2}+T_{3,3}+T_{3,4},
\end{aligned}
$$

and

$$
\begin{aligned}
T_4 &= O_\ell(0)O_m(0)O_n(0)E_p(0)-\widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_1+\omega_2)E_p(\omega_1-\omega_2) \\
&= (O_\ell(0)-\widehat{O_\ell}(\omega_1))O_m(0)O_n(0)E_p(0) \\
&\quad +\widehat{O_\ell}(\omega_1)(O_m(0)-\widehat{O_m}(\omega_2))O_n(0)E_p(0) \\
&\quad +\widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)(O_n(0)-\widehat{O_n}(\omega_1+\omega_2))E_p(0) \\
&\quad +\widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_1+\omega_2))(E_p(0)-E_p(\omega_1-\omega_2)) \\
&=: T_{4,1}+T_{4,2}+T_{4,3}+T_{4,4}.
\end{aligned}
$$

Note that $T_{i,1}\geq 0$ is a univariate Laurent polynomial of $e^{i4\omega_1}$ for each i and hence $\sum_{i=1}^4 T_{i,1}\geq 0$. By Riesz-Féjer lemma, there exists a polynomial $\widetilde{P}_7$ in $e^{i4\omega_1}$ such that

$$
\sum_{i=1}^4 T_{i,1} = |\widetilde{P}_7(4\omega_1)|^2.
$$

Note that $E_\ell(\omega_1)\geq 0$ which can be rewritten in terms of $\cos(4\omega_1)$. Also $E_m(0)-E_m(\omega_2)\geq 0$ is a function of $\cos(4\omega_2)$. Hence, by Riesz-Féjer lemma,

$$
\begin{aligned}
T_{1,2}+T_{2,2} &= E_\ell(\omega_1)(E_m(0)-E_m(\omega_2))(E_n(0)E_p(0)+O_n(0)O_p(0)) \\
&= |\widetilde{P}_\ell(4\omega_1)|^2|\widetilde{P}_{8,1}(4\omega_2)|^2 =: |\widetilde{P}_8(4\omega_1,4\omega_2)|^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
T_{3,2}+T_{4,2} &= \widehat{O_\ell}(\omega_1)(O_m(0)-\widehat{O_m}(\omega_2))(E_n(0)O_p(0)+O_n(0)E_p(0)) \\
&= |\tilde{q}_\ell(4\omega_1)|^2|\widetilde{P}_{9,1}(4\omega_2)|^2 =: |\widetilde{P}_9(4\omega_1,4\omega_2)|^2,
\end{aligned}
$$

$$
\begin{aligned}
&T_{13}+T_{2,3} \\
= &\ E_\ell(\omega_1)E_m(\omega_2)\Big[(E_n(0)-E_n(\omega_1+\omega_2))E_p(0)+(O_n(0)-\widehat{O_n}(\omega_1+\omega_2))O_p(0)\Big] \\
= &\ |\widetilde{P}_\ell(4\omega_1)|^2|\widetilde{P}_m(4\omega_2)|^2|\widetilde{P}_{10,1}(4\omega_1+4\omega_2)|^2 \\
=: &\ |\widetilde{P}_{10}(4\omega_1,4\omega_2)|^2.
\end{aligned}
$$

and

$$
\begin{aligned}
&T_{3,3}+T_{4,3} \\
= &\ \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\Big[(E_n(0)-E_n(\omega_1+\omega_2))O_p(0)+(O_n(0)-\widehat{O_n}(\omega_1+\omega_2))E_p(0)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= |\tilde{q}_\ell(4\omega_1)|^2|\tilde{q}_m(4\omega_2)|^2|\widetilde{P}_{11,1}(4\omega_1+4\omega_2)|^2 \\
&=: |\widetilde{P}_{11}(4\omega_1,4\omega_2)|^2.
\end{aligned}
$$

For $T_{i,4}, i=1,\cdots,4$, we can see that they are nonnegative and can be written in terms of $\cos(4\omega_1), \cos(4\omega_2)$, $\cos(4\omega_1+4\omega_2)$ and $\cos(4\omega_1-4\omega_2)$. That is, there exist $\widetilde{P}_{12}, \widetilde{P}_{13}, \widetilde{P}_{14}$ and $\widetilde{P}_{15}$ such that $T_{i,4} = |\widetilde{P}_{11+i}(4\omega_1,4\omega_2)|^2, i=1,\cdots,4$. Hence,

$$
1 - \sum_{j\in\{0,\pi\}^2} |P_{\ell,m,n,p}(2\omega+\nu)|^2 = \sum_{i=1}^{15} |\widetilde{P}_i(4\omega_1,4\omega_2)|^2.
$$

When $n=p=1$, we can see that $E_n(\omega_1+\omega_2) = 1 = \widehat{O_n}(\omega_1+\omega_2)$ and $E_p(\omega_1-\omega_2) = 1 = \widehat{O_p}(\omega_1-\omega_2)$. Hence, $\widetilde{P}_6 = 0$ and $\widetilde{P}_{10} = \cdots = \widetilde{P}_{15} = 0$. Thus, in this case,

$$
1 - \sum_{j\in\{0,\pi\}^2} |P_{\ell,m,n,p}(2\omega+\nu)|^2 = \sum_{i=1}^{8} |\widetilde{P}_i(4\omega_1,4\omega_2)|^2.
$$

This completes the proof of Lemma 1. $\square$

Therefore, there are at most 11 tight frame generators for bivariate box splines on the three-direction mesh and at most 19 tight frame generators for bivariate box splines on the four-direction mesh by Theorem 3.4 in [5]. In particular, when $n=p=1$, the number of tight wavelet frame generators for bivariate box splines on the four direction mesh is 12. This improves the number of tight wavelet frames associated with box splines in [5]. Also, we have less number of bivariate box spline tight wavelet frame generators on the four-direction mesh than the one constructed using Kronecker product operator in [2] which is 15 when $n=p=1$. We shall continue on finding the number of Laurent polynomials in the SOS condition in (1) for the Laurent polynomial associated with trivariate box splines in the next section.

4 The SOS condition for Trivariate Box Splines

We let ϕ_S be a trivariate box spline function with a directional set

$$
S := \{(1,0,0),(0,1,0),(0,0,1),(1,1,1),(0,1,1),(1,0,1)\}.
$$

Then the Laurent polynomial

$$
P_S(\omega) = \left(\tfrac{1+e^{i\omega_1}}{2}\right)^\ell \left(\tfrac{1+e^{i\omega_2}}{2}\right)^m \left(\tfrac{1+e^{i\omega_3}}{2}\right)^n \left(\tfrac{1+e^{i(\omega_1+\omega_2+\omega_3)}}{2}\right)^p \left(\tfrac{1+e^{i(\omega_2+\omega_3)}}{2}\right)^q \left(\tfrac{1+e^{i(\omega_1+\omega_3)}}{2}\right)^r
$$

be the mask associated with the box spline function ϕ_S over the directional set S, where $\ell, m, n, p, q, r \geq 1$ and $\omega := (\omega_1,\omega_2,\omega_3)$. This particular direction set S with even numbers

ℓ, m, n, p, q, r and at most three zeros among the orders for trivariate box splines is generally used because of the shifts of the trivariate box spline on direction set S are linearly independent. The linear independence of a box spline shifts is directly related to cardinal interpolation. (See [1]).

We conclude this paper by giving the following lemma without proof for the number of Laurent polynomials satisfying the SOS condition in (1) for the Laurent polynomial P_S of the trivariate box spline ϕ_S. We include a proof of Lemma 2 in Appendix.

Lemma 2 *For a Laurent polynomial P_S there exist Laurent polynomials $\widetilde{P}_1, \ldots, \widetilde{P}_N$ with $N \leq 48$ such that*

$$\sum_{\nu \in \{0,\pi\}^3} |P_S(\omega + \nu)|^2 + \sum_{i=1}^{N} |\widetilde{P}_i(2\omega)|^2 = 1$$

holds. When $\ell = m = q = r = 1$, $N \leq 19$, where each index ℓ, m, q, r is the order of each variable ω_1, ω_2, $\omega_2 + \omega_3$, and $\omega_1 + \omega_3$ respectively.

Therefore, as a parallel version of Theorem 3.4 in [5], we write down the following theorem for trivariate box spline tight frame generators on the direction set S.

Theorem 1 *For the mask P_S associated with trivariate box splines on the direction set*

$$S = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1), (0,1,1), (1,0,1)\},$$

there exist Laurent polynomials $\widetilde{P}_1, \cdots, \widetilde{P}_N$, $N \leq 48$ satisfying

$$\sum_{\nu \in \{0,\pi\}^3} |P_S(\omega + \nu)|^2 + \sum_{i=1}^{N} |\widetilde{P}_i(2\omega)|^2 = 1.$$

Thus there exist at most $2^3 + 48 = 56$ compactly supported trivariate box spline tight frame generators on the direction set S. In particular, when $\ell = m = q = r = 1$, $N \leq 19$ and the number of tight wavelet frames is no more than 27.

We have found the number of trivariate box spline tight frame generators on the direction set S using Lai and Stöckler's method is 56 which is less than the one using the Kronecker product operation in [2] which is $2^6 - 1 = 63$.

As a further research question, it is worthy to study on the minimum number of Laurent polynomials in the SOS condition in (1).

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5 Appendix

Lemma 2 For a Laurent polynomial P_S there exist Laurent polynomials $\widetilde{P}_1, \ldots, \widetilde{P}_N$ with $N \leq 48$ such that

$$\sum_{\nu \in \{0,\pi\}^3} |P_S(\omega+\nu)|^2 + \sum_{i=1}^{N} |\widetilde{P}_i(2\omega)|^2 = 1$$

holds. When $\ell = m = q = r = 1$, $N \leq 19$, where each index ℓ, m, q, r is the order of each variable ω_1, ω_2, $\omega_2 + \omega_3$, and $\omega_1 + \omega_3$ respectively.

Proof: We follow the same process in the proof of Lemma 1. We also use 2ω instead of ω for convenience. Then the square magnitude of Laurent polynomial P_S associated with trivariate box spline is

$$|P_S(2\omega)|^2 = \cos^{2q}(\omega_2+\omega_3)\cos^{2r}(\omega_1+\omega_3)\cos^{2\ell}(\omega_1)\cos^{2m}(\omega_2)\cos^{2n}(\omega_3)\cos^{2p}(\omega_1+\omega_2+\omega_3),$$

and

$$\begin{aligned}
&\textstyle\sum_{\nu \in \{0,\pi\}^3} |P_S(2\omega+\nu)|^2 = \\
&\cos^{2q}(\omega_2+\omega_3)\cos^{2r}(\omega_1+\omega_3)\Big(\cos^{2\ell}(\omega_1)\cos^{2m}(\omega_2)\cos^{2n}(\omega_3)\cos^{2p}(\omega_1+\omega_2+\omega_3) \\
&\qquad\qquad + \sin^{2\ell}(\omega_1)\sin^{2m}(\omega_2)\sin^{2n}(\omega_3)\sin^{2p}(\omega_1+\omega_2+\omega_3)\Big) \\
&+ \cos^{2q}(\omega_2+\omega_3)\sin^{2r}(\omega_1+\omega_3)\Big(\sin^{2\ell}(\omega_1)\cos^{2m}(\omega_2)\cos^{2n}(\omega_3)\sin^{2p}(\omega_1+\omega_2+\omega_3) \\
&\qquad\qquad + \cos^{2\ell}(\omega_1)\sin^{2m}(\omega_2)\sin^{2n}(\omega_3)\cos^{2p}(\omega_1+\omega_2+\omega_3)\Big) \\
&+ \sin^{2q}(\omega_2+\omega_3)\cos^{2r}(\omega_1+\omega_3)\Big(\cos^{2\ell}(\omega_1)\sin^{2m}(\omega_2)\cos^{2n}(\omega_3)\sin^{2p}(\omega_1+\omega_2+\omega_3) \\
&\qquad\qquad + \sin^{2\ell}(\omega_1)\cos^{2m}(\omega_2)\sin^{2n}(\omega_3)\cos^{2p}(\omega_1+\omega_2+\omega_3)\Big) \\
&+ \sin^{2q}(\omega_2+\omega_3)\sin^{2r}(\omega_1+\omega_3)\Big(\cos^{2\ell}(\omega_1)\cos^{2m}(\omega_2)\sin^{2n}(\omega_3)\sin^{2p}(\omega_1+\omega_2+\omega_3) \\
&\qquad\qquad + \sin^{2\ell}(\omega_1)\sin^{2m}(\omega_2)\cos^{2n}(\omega_3)\cos^{2p}(\omega_1+\omega_2+\omega_3)\Big).
\end{aligned}$$

As the proof in Lemma 1, we use binomial expansions to each power of the trigonometric function and separate it into the sum of even order terms and the sum of odd order terms in the above equation and we denote $\cos^{2\ell}(\omega_1) = E_\ell(\omega_1) + O_\ell(\omega_1)$ and $\sin^{2\ell}(\omega_1) = E_\ell(\omega_1) - O_\ell(\omega_1)$. Similar for $E_m(\omega_2), O_m(\omega_2), E_n(\omega_3), O_n(\omega_3), E_p(\omega_1+\omega_2+\omega_3), O_p(\omega_1+\omega_2+\omega_3), E_q(\omega_2+\omega_3), O_q(\omega_2+\omega_3), E_r(\omega_1+\omega_3)$ and $O_r(\omega_1+\omega_3)$. Note that all even order terms, e.g., $E_\ell(\omega_1)$ is nonnegative and can be written in terms of $\cos 4(\omega_1)$. By using the Riesz-Féjer lemma, $E_\ell(\omega_1) = |\widetilde{P}_\ell(4\omega_1)|^2$ for a polynoimal in $e^{i4\omega_1}$. Similar for other even terms. For odd order terms, let us rewrite $O_\ell(\omega_1) = \widehat{O_\ell}(\omega_1)\cos(2\omega_1)$ so that $\widehat{O_\ell}(\omega_1)$

is nonnegative. Since $\widehat{O_\ell}(\omega_1)$ can be written in terms of $\cos(4\omega_1)$, $\widehat{O_\ell}(\omega_1) = |\tilde{q}_\ell(4\omega_1)|^2$ by Riesz-Féjer lemma for a polynomial $\tilde{q}_\ell$ in $e^{i4\omega}$. Similar for other terms $O_m(\omega_2), O_n(\omega_3)$, $O_p(\omega_1+\omega_2+\omega_3)$, $O_q(\omega_2+\omega_3)$ and $O_r(\omega_1+\omega_3)$. Then we have

$$\begin{aligned}
&\textstyle\sum_{\nu\in\{0,\pi\}^3} |P_S(2\omega+\nu)|^2 \\
&= 8E_\ell(\omega_1)E_m(\omega_2)E_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)E_r(\omega_1+\omega_3) \\
&\quad + 8O_\ell(\omega_1)O_m(\omega_2)O_n(\omega_3)O_p(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)E_r(\omega_1+\omega_3) \\
&\quad + 8E_\ell(\omega_1)E_m(\omega_2)O_n(\omega_3)O_p(\omega_1+\omega_2+\omega_3)O_q(\omega_2+\omega_3)O_r(\omega_1+\omega_3) \\
&\quad + 8O_\ell(\omega_1)O_m(\omega_2)E_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)O_q(\omega_2+\omega_3)O_r(\omega_1+\omega_3) \\
&\quad + 8E_\ell(\omega_1)O_m(\omega_2)E_n(\omega_3)O_p(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)O_r(\omega_1+\omega_3) \\
&\quad + 8O_\ell(\omega_1)E_m(\omega_2)O_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)O_r(\omega_1+\omega_3) \\
&\quad + 8E_\ell(\omega_1)O_m(\omega_2)O_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)O_q(\omega_2+\omega_3)E_r(\omega_1+\omega_3) \\
&\quad + 8O_\ell(\omega_1)E_m(\omega_2)E_n(\omega_3)O_p(\omega_1+\omega_2+\omega_3)O_q(\omega_2+\omega_3)E_r(\omega_1+\omega_3)
\end{aligned}$$

Note that since $1 = \sum_{\nu\in\{0,\pi\}^3} |P_S(\nu)|^2$, we have

$$\begin{aligned}
1 &= 8E_\ell(0)E_m(0)E_n(0)E_p(0)E_q(0)E_r(0) + 8O_\ell(0)O_m(0)O_n(0)O_p(0)E_q(0)E_r(0) \\
&\quad + 8E_\ell(0)E_m(0)O_n(0)O_p(0)O_q(0)O_r(0) + 8O_\ell(0)O_m(0)E_n(0)E_p(0)O_q(0)O_r(0) \\
&\quad + 8E_\ell(0)O_m(0)E_n(0)O_p(0)E_q(0)O_r(0) + 8O_\ell(0)E_m(0)O_n(0)E_p(0)E_q(0)O_r(0) \\
&\quad + 8E_\ell(0)O_m(0)O_n(0)E_p(0)O_q(0)E_r(0) + 8O_\ell(0)E_m(0)E_n(0)O_p(0)O_q(0)E_r(0).
\end{aligned}$$

We rewrite and rearrange $1 - \sum_{\nu\in\{0,\pi\}^3} |P_S(2\omega+\nu)|^2$ by using above two equalities as follows. For convenience we separate it into two parts and denote by A and B.

$$1 - \sum_{\nu\in\{0,\pi\}^3} |P_S(2\omega+\nu)|^2 =: A + B,$$

where A consists of eight terms denoted by A_1, A_2, A_3, A_4, A_5, A_6, A_7, and A_8 and B consists of seven terms $B_1, B_2, B_3, B_4, B_5, B_6$, and B_7 accordingly. They are given below.

$$\begin{aligned}
A &= 8(E_\ell(0)E_m(0)E_n(0)E_p(0)E_q(0)E_r(0) \\
&\qquad - E_\ell(\omega_1)E_m(\omega_2)E_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)E_r(\omega_1+\omega_3)) \\
&\quad + 8(O_\ell(0)O_m(0)O_n(0)O_p(0)E_q(0)E_r(0) \\
&\qquad - \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)E_r(\omega_1+\omega_3)) \\
&\quad + 8(E_\ell(0)E_m(0)O_n(0)O_p(0)O_q(0)O_r(0) \\
&\qquad - E_\ell(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)\widehat{O_r}(\omega_1+\omega_3)) \\
&\quad + 8(O_\ell(0)O_m(0)E_n(0)E_p(0)O_q(0)O_r(0) \\
&\qquad - \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)\widehat{O_r}(\omega_1+\omega_3))
\end{aligned}$$

$$
\begin{aligned}
&+ 8(E_\ell(0)O_m(0)E_n(0)O_p(0)E_q(0)O_r(0) \\
&\quad - E_\ell(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)\widehat{O_r}(\omega_1+\omega_3)) \\
&+ 8(O_\ell(0)E_m(0)O_n(0)E_p(0)E_q(0)O_r(0) \\
&\quad - \widehat{O_\ell}(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_3)E_p(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)\widehat{O_r}(\omega_1+\omega_3)) \\
&+ 8(E_\ell(0)O_m(0)O_n(0)E_p(0)O_q(0)E_r(0) \\
&\quad - E_\ell(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3)E_p(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)E_r(\omega_1+\omega_3)) \\
&+ 8(O_\ell(0)E_m(0)E_n(0)O_p(0)O_q(0)E_r(0) \\
&\quad - \widehat{O_\ell}(\omega_1)E_m(\omega_2)E_n(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)E_r(\omega_1+\omega_3)) \\
&=: 8A_1 + 8A_2 + 8A_3 + 8A_4 + 8A_5 + 8A_6 + 8A_7 + 8A_8,
\end{aligned}
$$

$$
\begin{aligned}
B = {} & 8\widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)E_r(\omega_1+\omega_3)\times \\
&\quad (1-\cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_3)\cos 2(\omega_1+\omega_2+\omega_3)) \\
&+ 8E_\ell(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)\widehat{O_r}(\omega_1+\omega_3)\times \\
&\quad (1-\cos(2\omega_3)\cos 2(\omega_1+\omega_2+\omega_3)\cos 2(\omega_2+\omega_3)\cos 2(\omega_1+\omega_3)) \\
&+ 8\widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)\widehat{O_r}(\omega_1+\omega_3)\times \\
&\quad (1-\cos(2\omega_1)\cos(2\omega_2)\cos 2(\omega_2+\omega_3)\cos 2(\omega_1+\omega_3)) \\
&+ 8E_\ell(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)\widehat{O_r}(\omega_1+\omega_3))\times \\
&\quad (1-\cos(2\omega_2)\cos 2(\omega_1+\omega_2+\omega_3)\cos 2(\omega_1+\omega_3)) \\
&+ 8\widehat{O_\ell}(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_3)E_p(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)\widehat{O_r}(\omega_1+\omega_3))\times \\
&\quad (1-\cos(2\omega_1)\cos(2\omega_3)\cos 2(\omega_1+\omega_3)) \\
&+ 8E_\ell(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3)E_p(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)E_r(\omega_1+\omega_3))\times \\
&\quad (1-\cos(2\omega_2)\cos(2\omega_3)\cos 2(\omega_2+\omega_3)) \\
&+ 8\widehat{O_\ell}(\omega_1)E_m(\omega_2)E_n(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)E_r(\omega_1+\omega_3))\times \\
&\quad (1-\cos(2\omega_1)\cos 2(\omega_1+\omega_2+\omega_3)\cos 2(\omega_2+\omega_3)) \\
&=: 8B_1 + 8B_2 + 8B_3 + 8B_4 + 8B_5 + 8B_6 + 8B_7.
\end{aligned}
$$

Let us consider the part B first. Note that $\widehat{O_\ell}(\omega_1) \geq 0$ can be written in terms of $\cos(4\omega_1)$ and similar for $\widehat{O_m}(\omega_2)$, $\widehat{O_n}(\omega_3)$, $\widehat{O_p}(\omega_1+\omega_2+\omega_3)$, $\widehat{O_q}(\omega_2+\omega_3)$, and $\widehat{O_r}(\omega_1+\omega_3)$. By Riesz-Féjer's lemma, $\widehat{O_\ell}(\omega_1) = |\tilde{q}_\ell(4\omega_1)|^2$, $\widehat{O_m}(\omega_2) = |\tilde{q}_m(4\omega_2)|^2$, $\widehat{O_n}(\omega_3) = |\tilde{q}_n(4\omega_3)|^2$, $\widehat{O_p}(\omega_1+\omega_2+\omega_3) = |\tilde{q}_p(4\omega_1+4\omega_2+4\omega_3)|^2$, $\widehat{O_q}(\omega_2+\omega_3) = |\tilde{q}_q(4\omega_2+4\omega_3)|^2$ and $\widehat{O_r}(\omega_1+\omega_3) =$

$|\tilde{q}_r(4\omega_1+4\omega_3)|^2$. We show the first term B_1 in detail.

$$\begin{aligned}B_1 &= \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)E_r(\omega_1+\omega_3)\times\\ &\quad (1-\cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_3)\cos 2(\omega_1+\omega_2+\omega_3))\\ &= |\tilde{q}_\ell(4\omega_1)|^2|\tilde{q}_m(4\omega_2)|^2|\tilde{q}_n(4\omega_3)|^2|\tilde{q}_p(4\omega_1+4\omega_2+4\omega_3)|^2|\widetilde{P}_q(4\omega_2+4\omega_3)|^2\times\\ &\quad |\widetilde{P}_r(4\omega_1+4\omega_3)|^2(1-\cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_3)\cos 2(\omega_1+\omega_2+\omega_3)).\end{aligned}$$

A simple computation shows the following term can be expressed as a sum of square magnitudes of three Laurent polynomials.

$$\begin{aligned}&1-\cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_3)\cos 2(\omega_1+\omega_2+\omega_3)\\ &= \left|a_1+a_2e^{i4\omega_1}+a_3e^{i4(\omega_1+\omega_2)}+a_4e^{i4(\omega_1+\omega_2+\omega_3)}\right|^2\\ &\quad+\left|a_1+a_2e^{i4\omega_2}+a_3e^{i4(\omega_2+\omega_3)}+a_4e^{i4(\omega_1+\omega_2+\omega_3)}\right|^2\\ &\quad+\left|a_1+a_2e^{i4\omega_3}+a_3e^{i4(\omega_1+\omega_3)}+a_4e^{i4(\omega_1+\omega_2+\omega_3)}\right|^2,\end{aligned}$$

where

$$\begin{aligned}a_1 &= \tfrac{1}{60}\sqrt{\tfrac{3}{2}-\sqrt{\tfrac{2}{3}}+\tfrac{1}{2}\sqrt{-27+12\sqrt{6}}}\,(-12-3\sqrt{6}+7\sqrt{-27+12\sqrt{6}}+9\sqrt{-18+8\sqrt{6}})\\ a_2 &= \tfrac{1}{40}\sqrt{\tfrac{3}{2}-\sqrt{\tfrac{2}{3}}+\tfrac{1}{2}\sqrt{-27+12\sqrt{6}}}\,(1+4\sqrt{6}-\sqrt{-27+12\sqrt{6}}-2\sqrt{-18+8\sqrt{6}})\\ a_3 &= \tfrac{1}{4}\sqrt{\tfrac{3}{2}-\sqrt{\tfrac{2}{3}}+\tfrac{1}{2}\sqrt{-27+12\sqrt{6}}}\\ a_4 &= -\tfrac{1}{120}\sqrt{\tfrac{3}{2}-\sqrt{\tfrac{2}{3}}+\tfrac{1}{2}\sqrt{-27+12\sqrt{6}}}\,(9+6\sqrt{6}+11\sqrt{-27+12\sqrt{6}}+12\sqrt{-18+8\sqrt{6}})\end{aligned}$$

Thus we have $B_1=|\widetilde{P}_1(4\omega_1,4\omega_2,4\omega_3)|^2+|\widetilde{P}_2(4\omega_1,4\omega_2,4\omega_3)|^2+|\widetilde{P}_3(4\omega_1,4\omega_2,4\omega_3)|^2$. Similar for B_2, B_3, B_4, B_5, B_6 and B_7 by using the following six equalities

$$\begin{aligned}&1-\cos(2\omega_3)\cos 2(\omega_1+\omega_2+\omega_3)\cos 2(\omega_2+\omega_3)\cos 2(\omega_1+\omega_3) = \tfrac{1}{6}\left|1-e^{i4\omega_3}\right|^2\\ &+\tfrac{1}{16}\left|1-e^{i4(\omega_1+\omega_2+\omega_3)}-e^{i4(\omega_1+\omega_2+2\omega_3)}+\tfrac{1}{4}e^{i4(\omega_1+\omega_3)}\right|^2+\left|\tfrac{1}{2\sqrt{3}}+\tfrac{1}{4\sqrt{3}}e^{i4\omega_3}-\tfrac{\sqrt{3}}{4}e^{i4(\omega_1+\omega_3)}\right|^2,\\ &1-\cos(2\omega_1)\cos(2\omega_2)\cos 2(\omega_2+\omega_3)\cos 2(\omega_1+\omega_3) = \tfrac{1}{8}\left|e^{i4(\omega_1+\omega_3)}-e^{i4(\omega_2+\omega_3)}\right|^2\\ &+\left|b_1+b_2e^{i4\omega_1}+b_3e^{i4\omega_2}+b_4e^{i4(\omega_1+\omega_2+\omega_3)}\right|^2+\left|b_1+b_2e^{i4(\omega_1+\omega_3)}+b_3e^{i4(\omega_2+\omega_3)}+b_4e^{i4\omega_3}\right|^2\end{aligned}$$

where $b_1 = -2\sqrt{\frac{1}{8} - \frac{\sqrt{3}}{16}} - \sqrt{3(\frac{1}{8} - \frac{\sqrt{3}}{16})}$, and

$$b_2 = b_3 = \frac{1}{2}\sqrt{\frac{1}{8} - \frac{\sqrt{3}}{16}} + \sqrt{3(\frac{1}{8} - \frac{\sqrt{3}}{16})}, \quad b_4 = \sqrt{\frac{1}{8} - \frac{\sqrt{3}}{16}},$$

$$1 - \cos(2\omega_2)\cos 2(\omega_1 + \omega_2 + \omega_3)\cos 2(\omega_1 + \omega_3) = \frac{1}{4}\left|1 - \frac{1}{2}e^{i4\omega_2} - \frac{1}{2}e^{i4(\omega_1+\omega_2+\omega_3)}\right|^2 + \frac{3}{16}\left|1 - e^{i4(\omega_1+\omega_3)}\right|^2,$$

$$1 - \cos(2\omega_1)\cos(2\omega_3)\cos 2(\omega_1 + \omega_3) = \frac{1}{4}\left|1 - \frac{1}{2}e^{i4\omega_1} - \frac{1}{2}e^{i4(\omega_1+\omega_3)}\right|^2 + \frac{3}{16}\left|1 - e^{i4\omega_3}\right|^2,$$

$$1 - \cos(2\omega_2)\cos(2\omega_3)\cos 2(\omega_2 + \omega_3) = \frac{1}{4}\left|1 - \frac{1}{2}e^{i4\omega_2} - \frac{1}{2}e^{i4(\omega_2+\omega_3)}\right|^2 + \frac{3}{16}\left|1 - e^{i4\omega_3}\right|^2,$$

$$1 - \cos(2\omega_1)\cos 2(\omega_1 + \omega_2 + \omega_3)\cos 2(\omega_2 + \omega_3) = \frac{1}{4}\left|1 - \frac{1}{2}e^{i4\omega_1} - \frac{1}{2}e^{i4(\omega_1+\omega_2+\omega_3)}\right|^2 + \frac{3}{16}\left|1 - e^{i4(\omega_2+\omega_3)}\right|^2.$$

That is, $\sum_{i=1}^{7} B_i = \sum_{i=1}^{17} |\widetilde{P}_i(4\omega_1, 4\omega_2, 4\omega_3)|^2$. When $\ell = m = q = r = 1$ we can see that $E_\ell(\omega_1) = 1 = \widehat{O_\ell}(\omega_1)$ and $E_m(\omega_2) = 1 = \widehat{O_m}(\omega_2)$, $E_q(\omega_2 + \omega_3) = 1 = \widehat{O_q}(\omega_2 + \omega_3)$, $E_r(\omega_1 + \omega_3) = 1 = \widehat{O_r}(\omega_1 + \omega_3)$. In addition to that, the fifth equation and the sixth equation have the same term $\frac{3}{16}\left|1 - e^{i4\omega_3}\right|^2$. Thus B_6 and B_7 can be combined as follows.

$$\begin{aligned}
B_6 + B_7 &= \widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1 + \omega_2 + \omega_3)(1 - \cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_3)\cos 2(\omega_1 + \omega_3)) \\
&\quad + \widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1 + \omega_2 + \omega_3)(1 - \cos(2\omega_1)\cos(2\omega_2)\cos(2\omega_3)\cos 2(\omega_2 + \omega_3)) \\
&= \widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1 + \omega_2 + \omega_3)\left(\frac{1}{4}\left|1 - \frac{1}{2}e^{i4\omega_1} - \frac{1}{2}e^{i4(\omega_1+\omega_3)}\right|^2 + \frac{3}{16}\left|1 - e^{i4\omega_3}\right|^2\right) \\
&\quad + \widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1 + \omega_2 + \omega_3)\left(\frac{1}{4}\left|1 - \frac{1}{2}e^{i4\omega_1} - \frac{1}{2}e^{i4(\omega_2+\omega_3)}\right|^2 + \frac{3}{16}\left|1 - e^{i4\omega_3}\right|^2\right) \\
&= \widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1 + \omega_2 + \omega_3)\times \\
&\quad \left(\frac{1}{4}\left|1 - \frac{1}{2}e^{i4\omega_1} - \frac{1}{2}e^{i4(\omega_1+\omega_3)}\right|^2 + \frac{6}{16}\left|1 - e^{i4\omega_3}\right|^2 + \frac{1}{4}\left|1 - \frac{1}{2}e^{i4\omega_1} - \frac{1}{2}e^{i4(\omega_2+\omega_3)}\right|^2\right) \\
&= |\tilde{q}_n(4\omega_3)|^2|\tilde{q}_p(4\omega_1 + 4\omega_2 + 4\omega_3)|^2\times \\
&\quad \left(\frac{1}{4}\left|1 - \frac{1}{2}e^{i4\omega_1} - \frac{1}{2}e^{i4(\omega_1+\omega_3)}\right|^2 + \frac{6}{16}\left|1 - e^{i4\omega_3}\right|^2 + \frac{1}{4}\left|1 - \frac{1}{2}e^{i4\omega_1} - \frac{1}{2}e^{i4(\omega_2+\omega_3)}\right|^2\right)
\end{aligned}$$

Then we have $\sum_{i=1}^{7} B_i = \sum_{i=1}^{16} |\widetilde{P}_i(4\omega_1, 4\omega_2, 4\omega_3)|^2$. Next we consider the part A. We rewrite each A_i , $i = 1, \dots, 8$ into six terms as follows

$$\begin{aligned}
A_1 &= E_\ell(0)E_m(0)E_n(0)E_p(0)E_q(0)E_r(0) \\
&\qquad - E_\ell(\omega_1)E_m(\omega_2)E_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)E_r(\omega_1+\omega_3) \\
&= (E_\ell(0) - E_\ell(\omega_1))E_m(0)E_n(0)E_p(0)E_q(0)E_r(0) \\
&\quad + E_\ell(\omega_1)(E_m(0) - E_m(\omega_2))E_n(0)E_p(0)E_q(0)E_r(0) \\
&\quad + E_\ell(\omega_1)E_m(\omega_2)(E_n(0) - E_n(\omega_3))E_p(0)E_q(0)E_r(0) \\
&\quad + E_\ell(\omega_1)E_m(\omega_2)E_n(\omega_3)(E_p(0) - E_p(\omega_1+\omega_2+\omega_3))E_q(0)E_r(0) \\
&\quad + E_\ell(\omega_1)E_m(\omega_2)E_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)(E_q(0) - E_q(\omega_2+\omega_3))E_r(0) \\
&\quad + E_\ell(\omega_1)E_m(\omega_2)E_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)(E_r(0) - E_r(\omega_1+\omega_3)) \\
&=: A_{1,1} + A_{1,2} + A_{1,3} + A_{1,4} + A_{1,5} + A_{1,6},
\end{aligned}$$

$$\begin{aligned}
A_2 &= O_\ell(0)O_m(0)O_n(0)O_p(0)E_q(0)E_r(0) \\
&\qquad - \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)E_r(\omega_1+\omega_3) \\
&= (O_\ell(0) - \widehat{O_\ell}(\omega_1))O_m(0)O_n(0)O_p(0)E_q(0)E_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)(O_m(0) - \widehat{O_m}(\omega_2))O_n(0)O_p(0)E_q(0)E_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)(O_n(0) - \widehat{O_n}(\omega_3))O_p(0)E_q(0)E_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3))(O_p(0) - \widehat{O_p}(\omega_1+\omega_2+\omega_3))E_q(0)E_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3))\widehat{O_p}(\omega_1+\omega_2+\omega_3)(E_q(0) - E_q(\omega_2+\omega_3))E_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3))\widehat{O_p}(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)(E_r(0) - E_r(\omega_1+\omega_3)) \\
&=: A_{2,1} + A_{2,2} + A_{2,3} + A_{2,4} + A_{2,5} + A_{2,6},
\end{aligned}$$

$$\begin{aligned}
A_3 &= E_\ell(0)E_m(0)O_n(0)O_p(0)O_q(0)O_r(0) \\
&\qquad - E_\ell(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)\widehat{O_r}(\omega_1+\omega_3)) \\
&= (E_\ell(0) - E_\ell(\omega_1))E_m(0)O_n(0)O_p(0)O_q(0)O_r(0) \\
&\quad + E_\ell(\omega_1)(E_m(0) - E_m(\omega_2))O_n(0)O_p(0)O_q(0)O_r(0) \\
&\quad + E_\ell(\omega_1)E_m(\omega_2)(O_n(0) - \widehat{O_n}(\omega_3))O_p(0)O_q(0)O_r(0) \\
&\quad + E_\ell(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_3)(O_p(0) - \widehat{O_p}(\omega_1+\omega_2+\omega_3))O_q(0)O_r(0) \\
&\quad + E_\ell(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)(O_q(0) - \widehat{O_q}(\omega_2+\omega_3))O_r(0) \\
&\quad + E_\ell(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)(O_r(0) - \widehat{O_r}(\omega_1+\omega_3)) \\
&=: A_{3,1} + A_{3,2} + A_{3,3} + A_{3,4} + A_{3,5} + A_{3,6},
\end{aligned}$$

$$
\begin{aligned}
A_4 &= O_\ell(0)O_m(0)E_n(0)E_p(0)O_q(0)O_r(0) \\
&\qquad - \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)\widehat{O_r}(\omega_1+\omega_3)) \\
&= (O_\ell(0) - \widehat{O_\ell}(\omega_1))O_m(0)E_n(0)E_p(0)O_q(0)O_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)(O_m(0) - \widehat{O_m}(\omega_2))E_n(0)E_p(0)O_q(0)O_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)(E_n(0) - E_n(\omega_3))E_p(0)O_q(0)O_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_3)(E_p(0) - E_p(\omega_1+\omega_2+\omega_3))O_q(0)O_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)(O_q(0) - \widehat{O_q}(\omega_2+\omega_3))O_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_3)E_p(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)(O_r(0) - \widehat{O_r}(\omega_1+\omega_3)) \\
&=: A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4} + A_{4,5} + A_{4,6}, \\
A_5 &= E_\ell(0)O_m(0)E_n(0)O_p(0)E_q(0)O_r(0) \\
&\qquad - E_\ell(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)\widehat{O_r}(\omega_1+\omega_3) \\
&= (E_\ell(0) - E_\ell(\omega_1))O_m(0)E_n(0)O_p(0)E_q(0)O_r(0) \\
&\quad + E_\ell(\omega_1)(O_m(0) - \widehat{O_m}(\omega_2))E_n(0)O_p(0)E_q(0)O_r(0) \\
&\quad + E_\ell(\omega_1)\widehat{O_m}(\omega_2)(E_n(0) - E_n(\omega_3))O_p(0)E_q(0)O_r(0) \\
&\quad + E_\ell(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_3)(O_p(0) - \widehat{O_p}(\omega_1+\omega_2+\omega_3))E_q(0)O_r(0) \\
&\quad + E_\ell(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)(E_q(0) - E_q(\omega_2+\omega_3))O_r(0) \\
&\quad + E_\ell(\omega_1)\widehat{O_m}(\omega_2)E_n(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)(O_r(0) - \widehat{O_r}(\omega_1+\omega_3)) \\
&=: A_{5,1} + A_{5,2} + A_{5,3} + A_{5,4} + A_{5,5} + A_{5,6}, \\
A_6 &= O_\ell(0)E_m(0)O_n(0)E_p(0)E_q(0)O_r(0) \\
&\qquad - \widehat{O_\ell}(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_3)E_p(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)\widehat{O_r}(\omega_1+\omega_3) \\
&= (O_\ell(0) - \widehat{O_\ell}(\omega_1))E_m(0)O_n(0)E_p(0)E_q(0)O_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)(E_m(0) - E_m(\omega_2))O_n(0)E_p(0)E_q(0)O_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)E_m(\omega_2)(O_n(0) - \widehat{O_n}(\omega_3))E_p(0)E_q(0)O_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_3)(E_p(0) - E_p(\omega_1+\omega_2+\omega_3))E_q(0)O_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_3)E_p(\omega_1+\omega_2+\omega_3)(E_q(0) - E_q(\omega_2+\omega_3))O_r(0) \\
&\quad + \widehat{O_\ell}(\omega_1)E_m(\omega_2)\widehat{O_n}(\omega_3)E_p(\omega_1+\omega_2+\omega_3)E_q(\omega_2+\omega_3)(O_r(0) - \widehat{O_r}(\omega_1+\omega_3)) \\
&=: A_{6,1} + A_{6,2} + A_{6,3} + A_{6,4} + A_{6,5} + A_{6,6},
\end{aligned}
$$

$$\begin{aligned}
A_7 &= E_\ell(0)O_m(0)O_n(0)E_p(0)O_q(0)E_r(0)\\
&\qquad - E_\ell(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3)E_p(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)E_r(\omega_1+\omega_3)\\
&= (E_\ell(0)-E_\ell(\omega_1))O_m(0)O_n(0)E_p(0)O_q(0)E_r(0)\\
&\quad + E_\ell(\omega_1)(O_m(0)-\widehat{O_m}(\omega_2))O_n(0)E_p(0)O_q(0)E_r(0)\\
&\quad + E_\ell(\omega_1)\widehat{O_m}(\omega_2)(O_n(0)-\widehat{O_n}(\omega_3))E_p(0)O_q(0)E_r(0)\\
&\quad + E_\ell(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3)(E_p(0)-E_p(\omega_1+\omega_2+\omega_3))O_q(0)E_r(0)\\
&\quad + E_\ell(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3)E_p(\omega_1+\omega_2+\omega_3)(O_q(0)-\widehat{O_q}(0\omega_2+\omega_3))E_r(0)\\
&\quad + E_\ell(\omega_1)\widehat{O_m}(\omega_2)\widehat{O_n}(\omega_3)E_p(\omega_1+\omega_2+\omega_3)\widehat{O_q}(0\omega_2+\omega_3)(E_r(0)-E_r(\omega_1+\omega_3))\\
&=: A_{7,1}+A_{7,2}+A_{7,3}+A_{7,4}+A_{7,5}+A_{7,6},
\end{aligned}$$

and

$$\begin{aligned}
A_8 &= O_\ell(0)E_m(0)E_n(0)O_p(0)O_q(0)E_r(0)\\
&\qquad - \widehat{O_\ell}(\omega_1)E_m(\omega_2)E_n(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)E_r(\omega_1+\omega_3)\\
&= (O_\ell(0)-\widehat{O_\ell}(\omega_1))E_m(0)E_n(0)O_p(0)O_q(0)E_r(0)\\
&\quad + \widehat{O_\ell}(\omega_1)(E_m(0)-E_m(\omega_2))E_n(0)O_p(0)O_q(0)E_r(0)\\
&\quad + \widehat{O_\ell}(\omega_1)E_m(\omega_2)(E_n(0)-E_n(\omega_3))O_p(0)O_q(0)E_r(0)\\
&\quad + \widehat{O_\ell}(\omega_1)E_m(\omega_2)E_n(\omega_3)(O_p(0)-\widehat{O_p}(\omega_1+\omega_2+\omega_3))O_q(0)E_r(0)\\
&\quad + \widehat{O_\ell}(\omega_1)E_m(\omega_2)E_n(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)(O_q(0)-\widehat{O_q}(\omega_2+\omega_3))E_r(0)\\
&\quad + \widehat{O_\ell}(\omega_1)E_m(\omega_2)E_n(\omega_3)\widehat{O_p}(\omega_1+\omega_2+\omega_3)\widehat{O_q}(\omega_2+\omega_3)(E_r(0)-E_r(\omega_1+\omega_3))\\
&=: A_{8,1}+A_{8,2}+A_{8,3}+A_{8,4}+A_{8,5}+A_{8,6}.
\end{aligned}$$

Note that $A_{i,1} \geq 0$ for each i and hence $\sum_{i=1}^{8} A_{i,1} \geq 0$. By Riesz-Féjer lemma, there exists a polynomial $\widetilde{P}_{18}$ in $e^{i4\omega_1}$ such that

$$\sum_{i=1}^{8} A_{i,1} = |\widetilde{P}_{18}(4\omega_1)|^2.$$

Note that $E_\ell(\omega_1) \geq 0$ which can be rewritten in terms of $\cos(4\omega_1)$. Also $E_m(0)-E_m(\omega_2) \geq 0$

and $O_m(0) - \widehat{O_m}(\omega_2) \geq 0$ are functions of $\cos(4\omega_2)$. Hence, by Riesz-Féjer lemma,

$$\begin{aligned}
A_{1,2} + A_{3,2} + A_{5,2} + A_{7,2} &= E_\ell(\omega_1)\Big((E_m(0) - E_m(\omega_2))E_n(0)E_p(0)E_q(0)E_r(0) \\
&\quad + (E_m(0) - E_m(\omega_2))O_n(0)O_p(0)O_q(0)O_r(0) \\
&\quad + (O_m(0) - \widehat{O_m}(\omega_2))E_n(0)O_p(0)E_q(0)O_r(0) \\
&\quad + (O_m(0) - \widehat{O_m}(\omega_2))O_n(0)E_p(0)O_q(0)E_r(0)\Big) \\
&= |\widetilde{P}_\ell(4\omega_1)|^2 |\widetilde{P}_{19,1}(4\omega_2)|^2 =: |\widetilde{P}_{19}(4\omega_1, 4\omega_2)|^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
A_{2,2} + A_{4,2} + A_{6,2} + A_{8,2} &= \widehat{O_\ell}(\omega_1)\Big((O_m(0) - \widehat{O_m}(\omega_2))O_n(0)O_p(0)E_q(0)E_r(0) \\
&\quad + (O_m(0) - \widehat{O_m}(\omega_2))E_n(0)E_p(0)O_q(0)O_r(0) \\
&\quad + (E_m(0) - E_m(\omega_2))O_n(0)E_p(0)E_q(0)O_r(0) \\
&\quad + (E_m(0) - E_m(\omega_2))E_n(0)O_p(0)O_q(0)E_r(0)\Big) \\
&= |\widetilde{q}_\ell(4\omega_1)|^2 |\widetilde{P}_{20,1}(4\omega_2)|^2 =: |\widetilde{P}_{20}(4\omega_1, 4\omega_2)|^2.
\end{aligned}$$

We also can apply Riesz-Féjer lemma to the following four equations since $E_m(\omega_2) \geq 0$, which is a function of $\cos(4\omega_2)$ and $E_n(0) - E_n(\omega_3) \geq 0$ and $O_n(0) - \widehat{O_n}(\omega_3) \geq 0$ which are a function of $\cos(4\omega_3)$.

$$\begin{aligned}
A_{1,3} + A_{3,3} &= E_\ell(\omega_1)E_m(\omega_2)\Big((E_n(0) - E_n(\omega_3))E_p(0)E_q(0)E_r(0) \\
&\quad + (O_n(0) - \widehat{O_n}(\omega_3))O_p(0)O_q(0)O_r(0)\Big) \\
&= |\widetilde{P}_\ell(4\omega_1)|^2 |\widetilde{P}_m(4\omega_2)|^2 |\widetilde{P}_{21,1}(4\omega_3)|^2 =: |\widetilde{P}_{21}(4\omega_1, 4\omega_2, 4\omega_3)|^2,
\end{aligned}$$

$$\begin{aligned}
A_{2,3} + A_{4,3} &= \widehat{O_\ell}(\omega_1)\widehat{O_m}(\omega_2)\Big((O_n(0) - \widehat{O_n}(\omega_3))O_p(0)E_q(0)E_r(0) \\
&\quad + (E_n(0) - E_n(\omega_3))E_p(0)O_q(0)O_r(0)\Big) \\
&= |\widetilde{q}_\ell(4\omega_1)|^2 |\widetilde{q}_m(4\omega_2)|^2 |\widetilde{P}_{22,1}(4\omega_3)|^2 =: |\widetilde{P}_{22}(4\omega_1, 4\omega_2, 4\omega_3)|^2, \\
A_{5,3} + A_{7,3} &= E_\ell(\omega_1)\widehat{O_m}(\omega_2)\Big((E_n(0) - E_n(\omega_3))O_p(0)E_q(0)O_r(0) \\
&\quad + O_n(0) - \widehat{O_n}(\omega_3))E_p(0)O_q(0)E_r(0)\Big) \\
&= |\widetilde{P}_\ell(4\omega_1)|^2 |\widetilde{q}_m(4\omega_2)|^2 |\widetilde{P}_{23,1}(4\omega_3)|^2 =: |\widetilde{P}_{23}(4\omega_1, 4\omega_2, 4\omega_3)|^2,
\end{aligned}$$

and

$$\begin{aligned} A_{6,3}+A_{8,3} &= \widehat{O_\ell}(\omega_1)E_m(\omega_2)\Big((O_n(0)-\widehat{O_n}(\omega_3))E_p(0)E_q(0)O_r(0) \\ &\qquad + (E_n(0)-E_n(\omega_3))O_p(0)O_q(0)E_r(0)\Big) \\ &= |\widetilde{q}_\ell(4\omega_1)|^2|\widetilde{P}_m(4\omega_2)|^2|\widetilde{P}_{24,1}(4\omega_3)|^2 =: |\widetilde{P}_{24}(4\omega_1,4\omega_2,4\omega_3)|^2. \end{aligned}$$

For $A_{i,4}, i=1,\cdots,8$, we can see that they are nonnegative and can be written in terms of $\cos(4\omega_1), \cos(4\omega_2)$, $\cos(4\omega_3)$ and $\cos(4\omega_1+4\omega_2+4\omega_3)$. That is, there exist $\widetilde{P}_{25},\cdots,\widetilde{P}_{32}$ such that $A_{i,4}=|\widetilde{P}_{24+i}(4\omega_1,4\omega_2,4\omega_3)|^2, i=1,\cdots,8$. Similarly, for $A_{i,5}, i=1,\cdots,8$, we can see that they are nonnegative and can be written in terms of $\cos(4\omega_1), \cos(4\omega_2)$, $\cos(4\omega_3)$, $\cos(4\omega_1+4\omega_2+4\omega_3)$ and $\cos(4\omega_2+4\omega_3)$. That is, there exist $\widetilde{P}_{33},\cdots,\widetilde{P}_{40}$ such that $A_{i,5}=|\widetilde{P}_{32+i}(4\omega_1,4\omega_2,4\omega_3)|^2, i=1,\cdots,8$ and for $A_{i,6}, i=1,\cdots,8$, we can see that they are nonnegative and can be written in terms of $\cos(4\omega_1), \cos(4\omega_2), \cos(4\omega_3), \cos(4\omega_1+4\omega_2+4\omega_3)$, $\cos(4\omega_2+4\omega_3)$ and $\cos(4\omega_1+4\omega_3)$. That is, there exist $\widetilde{P}_{41},\cdots,\widetilde{P}_{48}$ such that $A_{i,6}=|\widetilde{P}_{40+i}(4\omega_1,4\omega_2,4\omega_3)|^2, i=1,\cdots,8$ Hence,

$$1-\sum_{j\in\{0,\pi\}^3}|P_S(2\omega+\nu)|^2=\sum_{i=1}^{48}|\widetilde{P}_i(4\omega_1,4\omega_2,4\omega_3)|^2.$$

When $n=p=q=r=1$, we can see that $E_n(\omega_3)=1=\widehat{O_n}(\omega_3)$ and $E_p(\omega_1+\omega_2+\omega_3)=1=\widehat{O_p}(\omega_1+\omega_2+\omega_3)$, $E_q(\omega_2+\omega_3)=1=\widehat{O_q}(\omega_2+\omega_3)$, $E_r(\omega_1+\omega_3)=1=\widehat{O_r}(\omega_1+\omega_3)$. Hence, $\widetilde{P}_{21}=\cdots=\widetilde{P}_{48}=0$. Thus, in this case,

$$1-\sum_{j\in\{0,\pi\}^3}|P_S(2\omega+\nu)|^2=\sum_{i=1}^{20}|\widetilde{P}_i(4\omega_1,4\omega_2,4\omega_3)|^2.$$

When any four of integers among ℓ,m,n,p,q and r are equal to 1, the number of Laurent polynomials satisfying SOS condition for trivariate box spline tight frames is 20. In particular, when the order ℓ, m, q and r are equal to 1 we have one less polynomial as we saw from the part B. Thus we have 19 Larent polynomials satisfying SOS condition. This completes the proof of Lemma 2. □