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# RECONSTRUCTION OF SPARSE POLYNOMIALS VIA QUASI-ORTHOGONAL MATCHING PURSUIT METHOD*

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**Abstract**

In this paper, we propose a Quasi-Orthogonal Matching Pursuit (QOMP) algorithm for constructing a sparse approximation of functions in terms of expansion by orthonormal polynomials. For the two kinds of sampled data, data with noises and without noises, we apply the mutual coherence of measurement matrix to establish the convergence of the QOMP algorithm which can reconstruct $s$-sparse Legendre polynomials, Chebyshev polynomials and trigonometric polynomials in $s$ step iterations. The results are also extended to general bounded orthogonal system including tensor product of these three univariate orthogonal polynomials. Finally, numerical experiments will be presented to verify the effectiveness of the QOMP method.

*Mathematics subject classification:* 41A10, 41A05 and 65D15.
*Key words:* Reconstruction of sparse polynomial, Compressive sensing, Mutual coherence, Quasi-orthogonal matching pursuit algorithm.

## 1. Introduction

Since [11], it has been an interesting research topic to accurately reconstruct functions via a sparse representation with respect to an orthogonal basis. Suppose that $\Omega \subseteq \mathbb{R}^d$ and that $\{\phi_j(x)\}_{j\in\Lambda}$ is a set of orthogonal polynomials defined on $\Omega$, where $\Lambda$ is an index set. Suppose that $|\Lambda| = n$, here $|\cdot|$ means the cardinality of $\Lambda$, and $n \gg 1$ can be finite or infinite. For any continuous function $g$ on $\Omega$, we can have an orthonormal expansion of $g$:

$$g(x) = \sum_{j\in\Lambda} c_j \phi_j(x). \tag{1.1}$$

In any practical computation, one can not have a memory to store all the coefficients $c_j$. Thus one is interested in finding a sparse representation of $g$ in the sense that

$$g(x) \approx \sum_{j\in\Lambda_s} \tilde{c}_j \phi_j(x) \tag{1.2}$$

for a given integer $s \geq 1$, where $\Lambda_s \subset \Lambda$ and $|\Lambda_s| = s \ll n$. That is, letting $\tilde{\mathbf{c}}$ be all the coefficient vectors containing all nonzero coefficients in (1.2), if $|\Lambda_s| \ll n$ and $\approx$ is $=$, the right-hand side

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of the expression in (1.2) is said to be a sparse representation of $g$. For practical purpose, we say the right-hand side of the expression in (1.2) is a sparse approximation of $g$. The problem of finding $\tilde{\mathbf{c}}$ is naturally translated into reconstructing the $s$-sparse vector $\tilde{\mathbf{c}}$.

Since [1, 7, 8, 14], compressed sensing has been a popular research topic. Its main idea is to use the sparsity of signal to reconstruct the signal by using as few observations as possible. The original model of compressed sensing is

$$\min_{\mathbf{c}\in\mathbb{C}^n} ||\mathbf{c}||_0 \quad s.t. \quad \Phi\mathbf{c} = \mathbf{b}, \tag{1.3}$$

where $||\mathbf{c}||_0$ represents the number of non-zero elements in the vector $\mathbf{c}$, that is, the sparsity of the vector $\mathbf{c}$, $\Phi \in \mathbb{C}^{m\times n}$ is a measurement matrix or sensing matrix, and $\mathbf{b} \in \mathbb{C}^{m\times 1}$ is an observation vector, such as $b_i = g(x_i), i = 1, \cdots, m$ for some locations $x_i \in \Omega$. The model (1.3) is the model of the sampled data without noise. Since there may be noise in sampled data, assume that the noise bound is $\varepsilon$, then the model (1.3) can be written as follows

$$\min_{\mathbf{c}\in\mathbb{C}^n} ||\mathbf{c}||_0 \quad s.t. \quad ||\Phi\mathbf{c} - \mathbf{b}||_2 \le \varepsilon, \tag{1.4}$$

where $||\cdot||_2$ represents Euclidean norm. If we know in advance that the sparsity of the vector to be restored is $s$, then the problem (1.4) can be rewritten as

$$\min_{\mathbf{c}\in\mathbb{C}^n} ||\Phi\mathbf{c} - \mathbf{b}||_2 \quad s.t. \quad ||\mathbf{c}||_0 \le s. \tag{1.5}$$

Many researchers have applied the technique of solving the sparse signal in compressed sensing to reconstruct the coefficient vector $\tilde{\mathbf{c}}$, see, e.g. [21], [19], [20], [23] and [10].

A greedy algorithm is one of the common methods to solve the problem (1.5). Among them, the orthogonal matching pursuit algorithm (OMP for short) proposed in [6] is an important one of the greedy algorithms. Currently, there have been many improvements of the OMP method, such as regularized Orthogonal Matching Pursuit [12], Generalized Orthogonal Matching Pursuit [15], stagewise Orthogonal Matching Pursuit [16], etc. In this paper we propose an improved method of OMP method named Quasi-Orthogonal Matching Pursuit (QOMP for short). Different from the traditional OMP method, the QOMP method selects the two columns that are most related to the space of the current redundant vector expansion in each iteration.

**Algorithm 1.1.** Quasi-Orthogonal Matching Pursuit (QOMP)

**Input:** $\Phi_{m\times n}, \mathbf{b}_{m\times 1}$, sparsity $s$, maximum number of iterations $k_{\max}(k_{\max} < m/2)$, tolerance $\varepsilon$

**Initialization:** $S_0 = \emptyset$, $\mathbf{r}_0 = (b), k = 0, \Psi_{m\times n} = \Phi_{m\times n}$

**while** $k < k_{\max}$ and $||\mathbf{r}_k||_2 > \varepsilon$

$k = k + 1$

$Res_{(i,j)}(\mathbf{r}_{k-1}) = \min_{u,v\in\mathbb{R}}\{||\Psi_i u + \Psi_j v - \mathbf{r}_{k-1}||_2\}$

$(i_k, j_k) = \text{argmin}_{1\le i,j\le n}\{Res_{(i,j)}(\mathbf{r}_{k-1})\}$

$S_k = S_{k-1} \cup \{i_k, j_k\}$

$\mathbf{r}_k = \mathbf{b} - \Phi_{S_k}\Phi_{S_k}^{\dagger}\mathbf{b}$

$\Psi_{i_k, j_k} = \mathbf{0}$

**end while**

**Output:** $S = S_k, \mathbf{c}_S = \Phi_S^{\dagger}\mathbf{b}$ and $\mathbf{c}_{S^c} = \mathbf{0}$

As shown in Algorithm 1.1, in the first iteration, we solve the best approximations

$$\min_{u,v\in\mathbb{R}} \|\Phi_i u + \Phi_j v - \mathbf{b}\|_2$$

for all $i \neq j, i, j = 1, \cdots, n$ to find the residuals. We choose the best index pair, $(i_1, j_1)$ such that the residual is the smallest:

$$\min_{i\neq j,u,v\in\mathbb{R}} \|\Phi_i u + \Phi_j v - \mathbf{b}\|_2 = \min_{u,v\in\mathbb{R}} \|\Phi_{i_1} u + \Phi_{j_1} - \mathbf{b}\|_2. \tag{1.6}$$

Once we find $(i_1, j_1)$ to solve (1.6), we let $\mathbf{r}_1 = \mathbf{b} - \Phi_{i_1} u_{i_1} - \Phi_{j_1} v_{j_1}$ and repeat the produce.

Although there have been many studies on the effectiveness of the OMP method, most results are based on the restricted isometry property(RIP) of the measurement matrix $\Phi$, such as [9, 17, 18, 22]. In this paper, we will apply the mutual coherence of the measurement matrix $\Phi$ to establish the convergence of QOMP method for function reconstruction.
The rest of this paper is organized as follows: Section 2 introduces some preliminary knowledge required for this paper. Mainly we explain three types of orthogonal system: Legendre, Chebyshev, and trigonometric systems. Section 3 gives the condition for the QOMP method to reconstruct three types of $s$-sparse univariate polynomial in $s$-step iterations in both noisy and noiseless situations and the error estimation of the QOMP method. Section 4 extends the conclusions of Section 3 to the general bounded orthogonal system including multi-dimensional orthogonal systems. The last section shows the numerical experiments to verify the effectiveness of the QOMP method.

## 2. Preparation of Manuscript

In this section, we will introduce some knowledge required for this article.

### 2.1. Preconditioned Legendre orthogonal function system

It is well known that the standard univariate Legendre polynomials (cf. [5]) $L_j(x), j = 0, 1, \cdots$ are orthogonal with respect to the uniform probability measure on $[-1, 1]$, namely,

$$\int_{-1}^{1} \frac{1}{2} L_j(x) L_k(x) dx = \delta_{j,k} = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}$$

The $L_\infty$-norm of the standard Legendre orthogonal polynomials $L_j(x)$ is

$$||L_j(x)||_\infty = |L_j(1)| = |L_j(-1)| = \sqrt{2j+1},$$

here $|\cdot|$ means to take the absolute value of a number. Obviously, as $j$ grows, the $L_\infty$-norm $||L_j(x)||_\infty$ is not uniformly bounded on $[-1, 1]$, hence consider the standard function system

$$Q_j(x) = \sqrt{\frac{\pi}{2}} (1 - x^2)^{\frac{1}{4}} L_j(x), \quad j = 0, 1, \cdots.$$

The function system $\{Q_j(x)\}_{j\in\mathbb{N}_0}$ are orthogonal with respect to the Chebyshev probability measure $v(x) = \pi^{-1}(1 - x^2)^{-\frac{1}{2}}$ on $[-1, 1]$, because

$$\int_{-1}^{1} v(x) Q_j(x) Q_k(x) dx = \frac{1}{2} \int_{-1}^{1} L_j(x) L_k(x) dx = \delta_{j,k}.$$

We call $\{Q_j(x)\}_{j\in\mathbb{N}_0}$ as preconditioned Legendre orthogonal system. Obviously, they are uniformly bounded:$||Q_j(x)||_\infty \le \sqrt{2+(1/j)} \le \sqrt{3}, j \in \mathbb{N}_0$. Assume that the sampling points $\{x_i\}_{i=1}^m$ are randomly sampled according to the Chebyshev probability measure $v(x)$ on $[-1,1]$, then the elements in the measurement matrix $\Phi$ which are generated by the values of the function system $\{Q_j(x)\}_{j=0}^{n-1}$ at the sampling point are

$$\Phi_{ij} = Q_j(x_i), \quad j = 0, 1, \cdots, n-1, \quad i = 1, \cdots, m.$$

### 2.2. Chebyshev orthogonal function system

The form of the univariate Chebyshev orthogonal polynomials (cf. [3]) is

$$T_j(x) = \cos(j \cdot \arccos(x)), \quad j = 0, 1, 2, \cdots,$$

and they are orthogonal with respect to Chebyshev probability measure $\rho(x) = (1-x^2)^{-\frac{1}{2}}$ on $[-1,1]$, namely,

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_j(x) T_k(x) dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2}, & m = n \neq 0, \\ \pi, & m = n = 0. \end{cases}$$

From its expression, obviously we have $||T_j(x)||_\infty = 1, \forall j \in \mathbb{N}_0$. Assume that the sampling points $\{x_i\}_{i=1}^m$ are randomly sampled according to the Chebyshev probability measure $\rho(x)$ on $[-1,1]$, then the elements in the measurement matrix $\Phi$ which are generated by the values of the function system $\{T_j(x)\}_{j=0}^{n-1}$ at the sampling points are:

$$\Phi_{ij} = \cos(j \cdot \arccos(x_i)), \quad j = 0, 1, \cdots, n-1, \quad i = 1, \cdots, m.$$

### 2.3. Trigonometric polynomial function system

Univariate trigonometric series (the well-known Fourier series) is another bounded orthogonal system (cf. [2]). Let $\Omega = [0,1]$, for $j \in \mathbb{Z}$ we have

$$F_j(x) = e^{2\pi i j x}, \quad x \in \Omega, \quad i = \sqrt{-1}.$$

This kind of polynomials are orthogonal with respect to Lebesgue probability measure on $[0,1]$, namely,

$$\int_0^1 F_j(x) \bar{F}_k(x) dx = \delta_{jk},$$

where $\bar{F}(x)$ represents the conjugate of $F(x)$, and obviously they are uniformly bounded: $||F_j(x)||_\infty = 1, \forall j \in \mathbb{Z}$. Using this system to expand a function $g \in L^2[0,1]$ is the standard Fourier expansion of $g$. Assume that the sampling points $\{x_i\}_{i=1}^m$ are randomly sampled according to the Lebesgue probability measure on $[0,1]$, then the elements in the measurement matrix $\Phi$ which are generated by the values of the function system $\{F_j(x)\}_{j\in\Gamma}$ at the sampling points are

$$\Phi_{lj} = e^{2\pi i j x_l}, \quad l = 1, \cdots, m, \quad j \in \Gamma.$$

Let $\Gamma = \{-\frac{n}{2}, -\frac{n}{2}+1, \cdots, \frac{n}{2}-1\}$ to make sure that the expansion truncation of the function has $n$ items. This type of matrix is usually called Fourier matrix or non-equispaced Fourier matrix.

### 2.4. Tensor product of orthonormal polynomials

It is easy to understand that for a multivariate function $g \in L^2(\Omega)$, we can use a tensor product of univariate orthonormal system to form an orthonormal system to expand $g$. For simplicity, let consider $d = 2$. That is, recalling $T_j(x)$ is Chebyshev polynomials defined on $[-1, 1]$ from a previous subsection, let

$$T_j(x)T_k(y), \quad j, k = 0, 1, 2, \cdots \tag{2.1}$$

be an orthonormal system for $L^2(\Omega)$ with $\Omega = [-1, 1]^2$. We know

$$g(x, y) = \sum_{j,k=0}^{\infty} c_{j,k} T_j(x)T_k(y)$$

with

$$c_{j,k} := \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \int_{-1}^{1} \frac{1}{\sqrt{1-y^2}} g(x, y)T_j(x)T_k(y)dxdy.$$

If $g$ has an $s$ sparse representation in terms of Chebyshev basis in (2.1) or if one is to find a best $s$-term approximation using the Chebyshev basis, one should use the compressive sensing approach in (1.5) to do. Indeed, we first randomly sample distinct points $\xi_{j',k'} = (x'_j, y'_k), j', k' = 1, \cdots, m$ over $\Omega$ subject to Chebyshev distribution $\rho(x)\rho(y)$ to obtain a sensing matrix $\Phi$ with entries $T_j(x_{j'})T_k(y_{k'})$ for $j', k' = 1, \cdots, m$ and $j, k = 0, \cdots, n-1$ with $m \ll n$. Then we solve (1.5) with the right-hand side $\mathbf{b} = [g(\xi_{j',k'}), j', k' = 1, \cdots, m]$. In this way, we obtain a best $s$ term approximation of $g$.

### 2.5. Restricted isometry property and coherence of matrix

Restricted isometry constant and mutual coherence are two important characteristic quantities of measurement matrix $\Phi$. Their definitions are standard. For convenience, let us give them as follows.

**Definition 2.1.** *The restricted isometry constant(RIC) $\delta_s > 0$ of a matrix $\Phi \in \mathbb{C}^{m\times n}$ is defined to be the smallest $\delta_s \in (0, 1)$ such that*

$$(1-\delta_s)||\mathbf{x}||_2^2 \le ||\Phi\mathbf{x}||_2^2 \le (1+\delta_s)||\mathbf{x}||_2^2$$

*for all $s$-sparse $\mathbf{x} \in \mathbb{C}^n$.*

If the restricted isometry constant $\delta_s \in (0, 1)$ of matrix $\Phi$ for a reasonable sparsity $s$, we say that the matrix $\Phi$ satisfies the restricted isometry property(RIP).

Next, we give the definition of the mutual coherence which is very important in this paper.

**Definition 2.2.** *The mutual coherence of a matrix $\Phi \in \mathbb{C}^{m\times n}$ is defined as*

$$\mu(\Phi) := \max_{j\neq k} \frac{|\langle \Phi_j, \Phi_k\rangle|}{||\Phi_j||_2 \cdot ||\Phi_k||_2},$$

*where $\Phi_j, \Phi_k$ represent the $j$th and $k$th columns of the matrix, respectively. $\langle \Phi_j, \Phi_k\rangle := \Phi_j^*\Phi_k$ is the inner product in $\mathbb{C}^n$, $\Phi_j^*$ is the conjugate transpose of $\Phi_j$.*

Obviously, if we let the $\ell_2$-norm of each column of $\Phi$ be normalized, that is $||\Phi_j||_2 = 1, j = 1, \cdots, n$, the mutual coherence of matrix $\Phi$ can be written as

$$\mu(\Phi) := \max_{j \neq k} |\langle \Phi_j, \Phi_k \rangle|.$$

It is easy to see that the normalization of columns does not change the mutual coherence of matrix $\Phi$. The following result is well-known (cf. [13]).

**Lemma 2.1.** *Let $\Phi \in \mathbb{C}^{m \times n}$ with normalized columns, coherence $\mu(\Phi)$. Then*

$$\mu(\Phi) \geq \sqrt{\frac{n-m}{m(n-1)}}. \tag{2.2}$$

For $n$ large enough, for example $n \geq 2m$, obviously there is $\mu(\Phi) \geq (2m)^{-\frac{1}{2}}$ according to (2.2). This fact will be useful later in the paper.

In the following sections of the paper, we will use the mutual coherence $\mu(\Phi)$ of the measurement matrix to explain: if the measurement matrix $\Phi$ in (1.5) is a structured random matrix formed by the above three bounded orthogonal function systems randomly sampled according to their corresponding orthogonal measures, then the QOMP method can reconstruct the sparse coefficient vector for each of the three types of orthonormal expansions in $s$ steps.

## 3. Reconstruction of Sparse Polynomial Function via QOMP Method

In this section, we will use the QOMP method to reconstruct sparse polynomial functions. It shows that when the measurement matrix meets certain conditions, the reconstruction of the $s$-sparse polynomial function can be achieved in $s$-step iterations by the QOMP method in the both settings of sampled data without noise and with noise.

### 3.1. Sparse polynomial recovery under noiseless condition

To give an estimate of the mutual coherence of the measurement matrix, we first introduce variance criterion for averages.

**Lemma 3.1 (a variance criterion for averages).** *Let $\xi_1, \xi_2, ...$ be independent random variables with mean 0, such that $\sum_n n^{-2c}\mathbb{E}(\xi_n^2) < \infty$ for some $c > 0$. Then $n^{-c}\sum_{k \leq n} \xi_k \to 0$ a.s. when $n \to \infty$.*

*Proof.* It is a strong law of large number in the sense of Kolmogorov, Marcinkiewicz and Zygmund. We refer to Theorem 3.23 in [4] for a proof. □

Next, we will apply Lemma 3.1 to estimate the upper bound of the mutual coherence of the three structured random measurement matrices $\Phi$ discussed in Section 2.

**Lemma 3.2.** *Suppose that the matrix $\Phi \in \mathbb{R}^{m \times n}$ is a measurement matrix whose entries are $\Phi_{ij} = Q_j(x_i), j = 0, 1, \cdots, n-1, i = 1, \cdots, m$, here $\{Q_j(x)\}_{j=0}^{n-1}$ is the preconditioned Legendre function system and $\{x_i\}_{i=1}^m$ are sampling points drawn independently at random from the Chebyshev density $v(x) = \pi^{-1}(1-x^2)^{-\frac{1}{2}}$ on $[-1, 1]$. Let $q(m) = o(m^{1-c})$ for $\frac{1}{2} < c < 1$. Then $q(m) \to \infty$ as $m \to \infty$ and we have $\mu(\Phi) \leq \frac{1}{q(m)}$ with high probability.*

*Proof.* Firstly because of

$$\mathbb{E}(Q_j^2(x)) = \int_{-1}^{1} v(x)Q_j(x)Q_j(x)dx = 1,$$

by the strong law of large number, we can obtain

$$\tfrac{1}{m}||\Phi_j||_2^2 = \tfrac{1}{m}\sum_{i=1}^{m} Q_j^2(x_i) \to \mathbb{E}(Q_j^2(x)) = 1 \text{ as } m \to \infty.$$

Hence, when $m \to \infty$, we have $||\Phi_j||_2 \sim \sqrt{m}$. According to the definition of coherence , with the increase of $m$, there is

$$\mu(\Phi) = \max_{0\le j,k\le n-1, j\ne k} \tfrac{|\Phi_j^\top \Phi_k|}{||\Phi_j||_2 \cdot ||\Phi_k||_2} \approx \frac{1}{m} \max_{0\le j,k\le n-1, j\ne k} |\Phi_j^\top \Phi_k|.$$

Let $g_{jk} = Q_j(x)Q_k(x), j \ne k$. Then

$$|\Phi_j^\top \Phi_k| = \left|\sum_{i=1}^{m} Q_j(x_i)Q_k(x_i)\right| = \left|\sum_{i=1}^{m} g_{jk}(x_i)\right|.$$

Since

$$\mathbb{E}(g_{jk}(x)) = \mathbb{E}(Q_j(x)Q_k(x)) = \int_{-1}^{1} v(x)Q_j(x)Q_k(x)dx = 0, \quad j \ne k,$$

we see that $g_{jk}(x)$ is a random variable with mean 0. Because of

$$\mathbb{E}(g_{jk}^2) = \int_{-1}^{1} \frac{Q_j^2(x)Q_k^2(x)}{\pi\sqrt{1-x^2}}dx \le \frac{9}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = 9 < \infty,$$

we have

$$\sum_m m^{-2c}\mathbb{E}(g_{jk}^2) \le 9\sum_m m^{-2c}.$$

Next based on the condition $\frac{1}{2} < c < 1$, we have $\sum_m m^{-2c} < \infty$. Finally, we immediately obtain $\sum_m m^{-2c}\mathbb{E}(g_{jk}^2) < \infty$.

Because the sampling points $\{x_i\}_{i=1}^m$ are independently and identically distributed, the random variables $\{g_{jk}(x_i)\}_{i=1}^m, 0 \le j,k \le n-1, j \le k$ are also independently and identically distributed. By Lemma 3.1,

$$m^{-c}\left(\sum_{i=1}^{m} g_{jk}(x_i)\right) = m^{-c}\left(\sum_{i=1}^{m} Q_j(x_i)Q_k(x_i)\right) = m^{-c}(\Phi_j^T\Phi_k) \to 0$$

holds almost surely, and then

$$m^{-c}\left|\sum_{i=1}^{m} g_{jk}(x_i)\right| = m^{-c}\left|\sum_{i=1}^{m} Q_j(x_i)Q_k(x_i)\right| = m^{-c}|\Phi_j^T\Phi_k| \to 0$$

also holds almost surely. Because of $q(m) = o(m^{1-c})$, we can obtain that

$$m^{-1}\cdot|\Phi_j^T\Phi_k|\cdot q(m) = m^{-c}\cdot|\Phi_j^T\Phi_k|\cdot m^{c-1}\cdot q(m) \to 0$$

is almost surely when $m \to \infty$. Therefore, by taking all the upper bounds of $n$, we can get that as $m \to \infty$,

$$\mu(\Phi)\cdot q(m) = \sup_{0\le j,k\le n-1, j\ne k} \tfrac{|\Phi_j^T\Phi_k|}{m}\cdot q(m) \to 0$$

holds almost surely. Hence, for large $m$, we have $\mu(\Phi)\cdot q(m) \le 1$ with high probability and the result follows. □

**Remark 3.1.** For Chebyshev orthogonal polynomials $\{T_j(x)\}_{j=0}^{n-1}$. Assume that the sampling points $\{x_i\}_{i=1}^m$ are drown independently at random from the Chebyshev probability measure $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ on $[-1,1]$, and the random variable is written as $g_{jk}(x) = T_j(x)T_k(x)$. Then we have

$$\mathbb{E}(g_{jk}^2) = \int_{-1}^{1} \frac{T_j^2(x)T_k^2(x)}{\sqrt{1-x^2}}dx \le \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = \pi < \infty.$$

Hence, following the proof of Lemma 3.2, we can immediately obtain the following lemma.

**Lemma 3.3.** *Suppose that the matrix $\Phi \in \mathbb{R}^{m\times n}$ is a measurement matrix whose entries are $\Phi_{ij} = T_j(x_i), j = 0,1,\cdots,n-1, i = 1,\cdots,m$, here $\{T_j(x)\}_{j=0}^{n-1}$ are Chebyshev orthogonal polynomials and $\{x_i\}_{i=1}^m$ are sampling points drawn independently at random from the Chebyshev density $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ on $[-1,1]$. Let $\frac{1}{2} < c < 1$, $q(m) = o(m^{1-c})$ and $q(m) \to \infty$ as $m \to \infty$. Then for a large $m$, we have $\mu(\Phi) \le \frac{1}{q(m)}$ with high probability.*

We have the similar conclusion about the Fourier matrix.

**Lemma 3.4.** *Suppose that the matrix $\Phi \in \mathbb{C}^{m\times n}$ is a measurement matrix whose entries are $\Phi_{lj} = F_j(x_l), j \in \Gamma, l = 1,\cdots,m$, here $\{F_j(x)\}_{j\in\Gamma}$ are the trigonometric orthogonal polynomials, $\Gamma = \{-\frac{n}{2}, -\frac{n}{2}+1,\cdots,\frac{n}{2}-1\}$, and $\{x_l\}_{l=1}^m$ are sampling points drawn independently at random from the Lebesgue measure on $[0,1]$. Let $\frac{1}{2} < c < 1$,$q(m) = o(m^{1-c})$ and $q(m) \to \infty$ as $m \to \infty$. Then for a large $m$, we have $\mu(\Phi) \le \frac{1}{q(m)}$ with high probability.*

*Proof.* Through the definition of coherence, we have

$$\mu(\Phi) = \max_{j,k\in\Gamma, j\neq k} \frac{|\langle \Phi_j, \Phi_k\rangle|}{||\Phi_j||_2 \cdot ||\Phi_k||_2},$$

where $\Phi_j$ and $\Phi_k$ represent the $j$th and the $k$th column of the Fourier type measurement matrix, respectively. First of all, let us consider the denominator. For any column $\Phi_j$ of the matrix $\Phi$, the square of its $\ell_2$-norm is

$$||\Phi_j||_2^2 = \sum_{l=1}^{m} (\exp(2\pi i j x_l))^2 = m, \quad i = \sqrt{-1}.$$

Through the above formula, obviously we have $||\Phi_j||_2 = \sqrt{m}$. Then the definition of coherence can be rewritten as

$$\mu(\Phi) = \frac{1}{m}|\langle \Phi_j, \Phi_k\rangle|.$$

Next, we will consider the numerator. Let $g_{jk}(x) = F_j(x)\bar{F}_k(x)$, then we have

$$|\langle \Phi_j, \Phi_k\rangle| = \left|\sum_{l=1}^{m} F_j(x_l)\bar{F}_k(x_l)\right| = \left|\sum_{l=1}^{m} g_{jk}(x_l)\right|.$$

Let $g_{jk}^l = g_{jk}(x_l) = F_j(x_l)\bar{F}_k(x_l)$. Since

$$\mathbb{E}(g_{jk}(x)) = \mathbb{E}(F_j(x)\bar{F}_k(x)) = \int_0^1 F_j(x)\bar{F}_k(x)dx = 0, \quad j \neq k,$$

the expectation of random variable $g_{jk}(x)$ is zero. Next, due to

$$(g_{jk}^m)^2 = g_{jk}^2(x_m) = \exp(4\pi i(k-j)x_m), \quad j \neq k,$$

we have

$$\mathbb{E}((g_{jk}^m)^2) = \int_0^1 \exp(4\pi i(k-j)x_m)dx = 0$$

and then we can obtain

$$\sum_m m^{-2c}\mathbb{E}(g_{jk}^m)^2 = 0 < \infty.$$

In addition, it should be noted that since the sampling points $\{x_l\}_{l=1}^m$ are independently and identically distributed, the random variables $\{g_{jk}(x_l)\}_{l=1}^m$ are also independently and identically distributed. Hence, by Lemma 3.1, we have

$$m^{-c}\sum_{l=1}^m g_{jk}(x_l) = m^{-c}\sum_{l=1}^m F_j(x_l)\bar{F}_k(x_l) \to 0.$$

Since the limit is zero, we have

$$m^{-c}\left|\sum_{l=1}^m g_{jk}(x_l)\right| = m^{-c}\left|\sum_{l=1}^m F_j(x_l)\bar{F}_k(x_l)\right| = m^{-c}|\langle\Phi_j,\Phi_k\rangle| \to 0.$$

For $0 < c < 1$, suppose that there exists a function $q(m) = o(m^{1-c})$ with $q(m) \to \infty$ as $m \to \infty$. Furthermore, by Lemma 2.1, when $n$ is large enough, that is $n \geq 2m$, the mutual coherence satisfies $\mu \geq (2m)^{-\frac{1}{2}}$. Then $\frac{1}{2} < c < 1$ obviously meets the above conditions. Therefore, when $\frac{1}{2} < c < 1$, we can obtain the result that

$$m^{-1}\cdot|\langle\Phi_j,\Phi_k\rangle|\cdot q(m) = m^{-c}\cdot|\langle\Phi_j,\Phi_k\rangle|\cdot m^{c-1}\cdot q(m) \to 0$$

holds with high probability as $m \to \infty$. Finally ,by taking all the upper bounds of $n$, we can get that as $m \to \infty$,

$$\mu(\Phi)\cdot q(m) = \sup_{j,k\in\Gamma,j\neq k}\frac{|\Phi_j^\top\Phi_k|}{m}\cdot q(m) \to 0$$

holds almost surely. Hence, for large $m$, we have $\mu(\Phi)\cdot q(m) \leq 1$ with high probability. $\square$

Lemmas 3.1, 3.2 and 3.3 show that the mutual coherence $\mu(\Phi)$ of these three kinds measurement matrix $\Phi$ mentioned before have the same upper bound. Based on these lemmas, we can immediately obtain the following conclusions about sparse polynomial reconstruction.

**Theorem 3.1.** *Suppose that $g(x) = \sum_{j\in\Lambda,|\Lambda|=n} c_j\phi_j(x)$ is an s-sparse function, i.e. the cardinality of $\{c_j \neq 0, j\in\Lambda\}$ is less than or equal to s. If $\{\phi_j(x)\}_{j\in\Lambda}$ is a preconditioned Legendre orthogonal system or a Chebyshev orthogonal system or a trigonometric polynomial orthogonal system, and the measurement matrix $\Phi \in \mathbb{C}^{m\times n}$ in* (1.5) *is generated by a certain orthogonal system based on the random sampling $\{x_i\}_{i=1}^m$ under the corresponding orthogonal probability measure. Then when the sparsity of the function satisfies*

$$2 \leq s \leq \frac{q^2(m)}{4q(m)+4},$$

*solving the problem* (1.5) *by the QOMP method can reconstruct the function $g(x)$ accurately in s-step iterations.*

*Proof.* Since $g(x)$ is an $s$-sparse function, it can be seen from the definition that the coefficient vector $\mathbf{c}$ is an $s$-sparse vector. Then we only need to explain that under the assumptions, the QOMP method can accurately reconstruct the coefficient vector $\mathbf{c}$ in $s$ steps.
Without loss of generality, assume that the support set of the coefficient vector $\mathbf{c}$ is $\Delta = \{1, \cdots, s\}$. Hence there is

$$\mathbf{b} = c_1\Phi_1 + c_2\Phi_2 + \cdots + c_s\Phi_s = \sum_{k=1}^{s} c_k\Phi_k.$$

It can be seen from the iterative process of QOMP method that in the first iteration, for each $1 \le i, j \le n$, minimizing $||\Phi_i u + \Phi_j v - \mathbf{b}||_2$ is equivalent to finding the maximum value of the projection of $\mathbf{b}$ onto the hyperplane spanned by $\Phi_i$ and $\Phi_j$. A simple computation gives

$$\begin{aligned} Proj(\mathbf{b}) =& \frac{1}{1 - |\Phi_i^\top \Phi_j|^2}(\Phi_i\Phi_i^\top + \Phi_j\Phi_j^\top - (\Phi_i^\top\Phi_j)\Phi_j\Phi_i^\top \\ & - (\Phi_j^\top\Phi_i)\Phi_i\Phi_j^\top) \cdot \left(\sum_{k=1}^{s} c_k\Phi_k\right). \end{aligned}$$

Here $Proj(\mathbf{b})$ represents the projection of $\mathbf{b}$ onto the hyperplane spanned by $\Phi_i$ and $\Phi_j$. We know that the normalization does not affect the mutual coherence of the matrix. Without loss of generality, assumed that the measurement matrix is $\ell_2$-norm standardized by column. From Lemma 3.2, Lemma 3.3 and Lemma 3.4, when the measurement matrix is a matrix that satisfies the assumptions of the theorem, then $\mu(\Phi) \le \frac{1}{q(m)}$ holds for a large $m$ with high probability.
Firstly, suppose both $i, j \notin \Delta$.By applying triangle inequality together with the assumption $\mu(\Phi) \le \frac{1}{q(m)}$, we get

$$\begin{aligned} & \|Proj(\mathbf{b})_{i,j\notin\Delta}\| \\ =& \left\| \frac{1}{1 - |\Phi_i^\top \Phi_j|^2}(\Phi_i\Phi_i^\top + \Phi_j\Phi_j^\top - (\Phi_i^\top\Phi_j)\Phi_j\Phi_i^\top - (\Phi_j^\top\Phi_i)\Phi_i\Phi_j^\top) \cdot \left(\sum_{k=1}^{s} c_k\Phi_k\right) \right\| \\ \le& \frac{1}{1 - |\Phi_i^\top \Phi_j|^2} \sum_{k=1}^{s} |c_k| \Big( |\Phi_i^\top\Phi_k| + |\Phi_i^\top\Phi_j||\Phi_j^\top\Phi_k| + |\Phi_j^\top\Phi_k| + |\Phi_j^\top\Phi_i||\Phi_i^\top\Phi_k| \Big) \\ \le& \frac{1}{1 - 1/q^2(m)}(1/q(m) + 1/q^2(m) + 1/q(m) + 1/q^2(m)) \left(\sum_{k=1}^{s} |c_k|\right) \\ =& \frac{q^2(m)}{q^2(m) - 1}\left(\frac{2}{q(m)} + \frac{2}{q^2(m)}\right)\left(\sum_{k=1}^{s} |c_k|\right) \\ =& \frac{2q(m) + 2}{q^2(m) - 1} \cdot \left(\sum_{k=1}^{s} |c_k|\right). \end{aligned}$$

Here $Proj(\mathbf{b})_{i,j\notin\Delta}$ means the projection of $\mathbf{b}$ onto the hyperplane spanned by $\Phi_i$ and $\Phi_j$ in the case that the indicators $i, j$ are not in the support set $\Delta$.

Secondly, suppose $i \in \Delta$ or $j \in \Delta$. Without loss of generality, let us assume $i \in \Delta$, $i = 1$ and $|c_1| = \max_{1\le i\le s} |c_i|$ is the one of the largest entries in absolute value. By applying triangle inequality together with the assumption $\mu(\Phi) \le \frac{1}{q(m)}$ and using one of the estimates above, we get

$$
\begin{aligned}
&\|Proj(\mathbf{b})_{i\in\Delta}\| \\
=&\left\|\frac{1}{1-|\Phi_1^\top\Phi_j|^2}(\Phi_1\Phi_1^\top+\Phi_j\Phi_j^\top-(\Phi_1^\top\Phi_j)\Phi_j\Phi_1^\top-(\Phi_j^\top\Phi_1)\Phi_1\Phi_j^\top)\cdot\left(\sum_{k=1}^{s}c_k\Phi_k\right)\right\| \\
\geq&\frac{1}{1-|\Phi_1^\top\Phi_j|^2}\left(|c_1|(\|\Phi_1\|-|\Phi_j^\top\Phi_1|\cdot\|\Phi_j\|-|\Phi_1^\top\Phi_j|\cdot\|\Phi_j\|-|\Phi_1^\top\Phi_j|^2\|\Phi_1\|)\right) \\
&-\frac{1}{1-|\Phi_1^\top\Phi_j|^2}(2/q(m)+2/q^2(m))\cdot\left(\sum_{k=2}^{s}|c_k|\right) \\
\geq&\frac{1}{1-|\Phi_1^\top\Phi_j|^2}\left[(1-2/q(m)-1/q^2(m))|c_1|-(2/q(m)+2/q^2(m))\cdot\left(\sum_{k=2}^{s}|c_k|\right)\right].
\end{aligned}
$$

It follows that to show $\|Proj(\mathbf{b})_{i\in\Delta}\|\geq\|Proj(\mathbf{b})_{i,j\notin\Delta}\|$. Obviously, if

$$
\begin{aligned}
&(1-2/q(m)-1/q^2(m))|c_1|-(2/q(m)+2/q^2(m))\cdot\left(\sum_{k=2}^{s}|c_k|\right) \\
\geq&(1-|\Phi_1^\top\Phi_j|^2)\frac{2q(m)+2}{q^2(m)-1}\cdot\sum_{k=1}^{s}|c_k|
\end{aligned}
\tag{3.1}
$$

holds, then we have $\|Proj(\mathbf{b})_{i\in\Delta}\|\geq\|Proj(\mathbf{b})_{i,j\notin\Delta}\|$. Since $|c_1|=\max_{1\leq i\leq s}|c_i|$, we divide $|c_1|$ both sides of the above inequality, it suffices to show

$$(1-2/q(m)-1/q^2(m))-(2/q(m)+2/q^2(m))(s-1)\geq(1-|\Phi_1^\top\Phi_j|^2)\frac{2q(m)+2}{q^2(m)-1}s,$$

which is

$$1-\frac{2s}{q(m)}-\frac{2s-1}{q^2(m)}\geq(1-|\Phi_1^\top\Phi_j|^2)\frac{2q(m)+2}{q^2(m)-1}\cdot s. \tag{3.2}$$

As $1-|\Phi_1^\top\Phi_j|^2\leq 1$, we can see that (3.2) will hold if

$$1-\frac{2s}{q(m)}-\frac{2s-1}{q^2(m)}\geq\frac{2q(m)+2}{q^2(m)-1}\cdot s. \tag{3.3}$$

Indeed, the above inequality will be valid when $q(m)$ large enough.

According to our assumption $2\leq s\leq\frac{q(m)^2}{4q(m)+4}$, we will have (3.3) and hence (3.2). In fact, we have

$$\|Proj(\mathbf{b})_{i\in\Delta}\|>\|Proj(\mathbf{b})_{i,j\notin\Delta}\|.$$

This means that at least one of the two indexes selected by the QOMP method at the first iteration is a correct one.

In the iteration process of the QOMP method, after each iteration, the columns corresponding to the two indexes selected in the measurement matrix are replaced by $\mathbf{0}$. Through the previous analysis, we know that in the second iteration, the QOMP method will choose at least one correct index which is different from the one selected in the first iteration. Therefore, the support set $\Delta$ can be restored in at most $s$ iterations. That is, the sparse coefficient vector $\mathbf{c}$ of function $g(x)$ can be accurately reconstructed in $s$ iterations. □

Theorem 3.1 gives a sufficient condition for the QOMP method to reconstruct $s$-sparse polynomial functions under the noiseless condition.

### 3.2. Sparse polynomial recovery under the noise setting

In this section, we will discuss the sufficient conditions for using the QOMP method to recover the $s$-sparse function when the sampling is noisy.

**Theorem 3.2.** *Suppose that $g(x)=\sum_{j\in\Lambda,|\Lambda|=n} c_j\phi_j(x)$ is an $s$-sparse function and the noise vector satisfies $||\mathbf{v}||_2\le\varepsilon$, here $0<\varepsilon\ll 1$ is the noise level. If $\{\phi_j(x)\}_{j\in\Lambda}$ is a preconditioned Legendre orthogonal system or a Chebyshev orthogonal system or a trigonometric polynomial orthogonal system, and the measurement matrix $\Phi\in\mathbb{C}^{m\times n}$ in (1.5) is generated by a certain orthogonal system based on the random sampling $\{x_i\}_{i=1}^m$ under the corresponding orthogonal probability measure. Then when the sparsity of the function satisfies $2\le s\le q(m)\left(1-(\varepsilon/c_{\max})\right)$, solving the problem (1.5) by the QOMP method can reconstruct the function $g(x)$ accurately in $s$-step iterations, here $c_{\max}=\max_{j\in\Lambda}|c_j|$.*

*Proof.* Similar to the proof of Theorem 3.1. Without loss of generality, assumed that each column of $\Phi$ is $\ell_2$-norm standardized and that the support set is $\Delta=\{1,2,\cdots,s\}$. Hence there is

$$\mathbf{b}=\mathbf{b}_0+\mathbf{v}=c_1\Phi_1+c_2\Phi_2+\cdots+c_s\Phi_s+\mathbf{v}=\sum_{k=1}^{s}c_k\Phi_k+\mathbf{v}.$$

We first consider the first iteration. Since the projection operator is a linear operator, the projection of $\mathbf{b}$ on the hyperplane spanned by $\Phi_i$ and $\Phi_j$ is

$$\begin{aligned}Proj(\mathbf{b})&=Proj(\mathbf{b}_0+\mathbf{v})\\&=Proj(\mathbf{b}_0)+\frac{1}{1-|\Phi_i^\top\Phi_j|^2}(\Phi_i\Phi_i^\top+\Phi_j\Phi_j^\top-(\Phi_i^\top\Phi_j)\Phi_j\Phi_i^\top-(\Phi_j^\top\Phi_i)\Phi_i\Phi_j^\top)\cdot\mathbf{v}.\end{aligned}$$

Firstly, suppose both $i,j\notin\Delta$. By applying triangle inequality together with the assumption $\mu(\Phi)\le\frac{1}{q(m)}$, we get

$$\begin{aligned}&\|Proj(\mathbf{b})_{i,j\notin\Delta}\|\\=&\left\|Proj(\mathbf{b_0})+\frac{1}{1-|\Phi_i^\top\Phi_j|^2}(\Phi_i\Phi_i^\top+\Phi_j\Phi_j^\top-(\Phi_i^\top\Phi_j)\Phi_j\Phi_i^\top-(\Phi_j^\top\Phi_i)\Phi_i\Phi_j^\top)\cdot\mathbf{v}\right\|\\\le&\|Proj(\mathbf{b_0})\|+\frac{1}{1-|\Phi_i^\top\Phi_j|^2}\Big((|\Phi_i^\top\mathbf{v}|+|\Phi_j^\top\Phi_i||\Phi_j^\top\mathbf{v}|)\|\Phi_i\|+(|\Phi_j^\top\mathbf{v}|+|\Phi_i^\top\Phi_j||\Phi_i^\top\mathbf{v}|)\|\Phi_j\|\Big)\\=&\|Proj(\mathbf{b_0})\|+\frac{1}{1-|\Phi_i^\top\Phi_j|^2}\Big(|\Phi_i^\top\mathbf{v}|+|\Phi_j^\top\Phi_i||\Phi_j^\top\mathbf{v}|+|\Phi_j^\top\mathbf{v}|+|\Phi_i^\top\Phi_j||\Phi_i^\top\mathbf{v}|\Big)\\\le&\frac{2q(m)+2}{q^2(m)-1}\cdot\left(\sum_{k=1}^{s}|c_k|\right)+\frac{1}{1-1/q^2(m)}\left(\|\mathbf{v}\|+\frac{1}{q(m)}\cdot\|\mathbf{v}\|+\|\mathbf{v}\|+\frac{1}{q(m)}\cdot\|\mathbf{v}\|\right)\\\le&\frac{2q(m)+2}{q^2(m)-1}\cdot\left(\sum_{k=1}^{s}|c_k|\right)+\frac{2q(m)(q(m)+1)}{q^2(m)-1}\cdot\varepsilon.\end{aligned}$$

Secondly, suppose $i\in\Delta$ or $j\in\Delta$. Without loss of generality, let us assume $i\in\Delta$, $i=1$ and $|c_1|=\max_{1\le i\le s}|c_i|$ is the one of the largest entries in absolute value. By applying triangle inequality together with the assumption $\mu(\Phi)\le\frac{1}{q(m)}$ and using one of the estimates above, we get

$$\begin{aligned}
&\|Proj(\mathbf{b})_{i\in\Delta}\| \\
=&\left\|Proj(\mathbf{b}_0) + \frac{1}{1-|\Phi_1^\top\Phi_j|^2}(\Phi_1\Phi_1^\top + \Phi_j\Phi_j^\top - (\Phi_1^\top\Phi_j)\Phi_j\Phi_1^\top - (\Phi_j^\top\Phi_1)\Phi_1\Phi_j^\top)\cdot\mathbf{v}\right\| \\
\geq&\|Proj(\mathbf{b}_0)\| - \left\|\frac{1}{1-|\Phi_1^\top\Phi_j|^2}(\Phi_1\Phi_1^\top + \Phi_j\Phi_j^\top - (\Phi_1^\top\Phi_j)\Phi_j\Phi_1^\top - (\Phi_j^\top\Phi_1)\Phi_1\Phi_j^\top)\cdot\mathbf{v}\right\| \\
\geq&\left(1-2/q(m)-1/q^2(m)\right)|c_1| - (2/q(m)+2/q^2(m))\cdot\left(\sum_{k=2}^{s}|c_k|\right) \\
&\quad - \frac{1}{1-|\Phi_1^\top\Phi_j|^2}(\|\mathbf{v}\| + \frac{1}{q(m)}\cdot\|\mathbf{v}\| + \|\mathbf{v}\| + \frac{1}{q(m)}\cdot\|\mathbf{v}\|) \\
\geq&\left(1-2/q(m)-1/q^2(m)\right)|c_1| - (2/q(m)+2/q^2(m))\cdot\left(\sum_{k=2}^{s}|c_k|\right) - (2\varepsilon + 2\varepsilon/q(m)).
\end{aligned}$$

It remains to show $\|Proj(\mathbf{b})_{i\in\Delta}\| \geq \|Proj(\mathbf{b})_{i,j\notin\Delta}\|$. Obviously, if

$$\begin{aligned}
&(1-2/q(m)-1/q^2(m))|c_1| - (2/q(m)+2/q^2(m))\cdot\left(\sum_{k=2}^{s}|c_k|\right) - (2\varepsilon+2\varepsilon/q(m)) \\
\geq&\frac{2q(m)+2}{q^2(m)-1}\cdot\left(\sum_{k=1}^{s}|c_k|\right) + \frac{2q(m)(q(m)+1)}{q^2(m)-1}\cdot\frac{\varepsilon}{|c_1|}
\end{aligned}$$

holds, we have$\|Proj(\mathbf{b})_{i\in\Delta}\| \geq \|Proj(\mathbf{b})_{i,j\notin\Delta}\|$. Since $|c_1| = \max_{1\leq i\leq s}|c_i|$, it suffices to show

$$\begin{aligned}
&1-\frac{2}{q(m)}-\frac{1}{q^2(m)}-\left(\frac{2}{q(m)}+\frac{2}{q^2(m)}\right)\cdot(s-1)-\frac{2\varepsilon}{|c_1|}-\frac{2}{q(m)\cdot|c_1|} \\
\geq&\frac{2q(m)+2}{q^2(m)-1}\cdot s + \frac{2q(m)(q(m)+1)}{q^2(m)-1}\cdot\frac{\varepsilon}{|c_1|},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&1-\frac{2s}{q(m)}-\frac{2s-1}{q^2(m)} \\
\geq&\frac{2q(m)+2}{q^2-1}\cdot s + \frac{2\varepsilon}{|c_1|} + \frac{2\varepsilon}{q(m)\cdot|c_1|} + \frac{2q(m)(q(m)+1)}{q^2(m)-1}\cdot\frac{\varepsilon}{|c_1|}.
\end{aligned} \tag{3.4}$$

According to the assumption $2 \leq s \leq q(m)(1-(\varepsilon/c_{\max}))$, we have (3.4). In fact, we have

$$\|Proj(\mathbf{b})_{i\in\Delta}\| > \|Proj(\mathbf{b})_{i,j\notin\Delta}\|$$

as $m \to \infty$. This means that at least one of the two indexes selected by the QOMP method at the first iteration is correct. Finally, through the same analysis as Theorem 3.1, we can gain the result of this theorem. □

**Remark 3.2.** The 'accurately' in Theorem 3.2. means that when the sparsity $s$ satisfies $2 \leq s \leq q(m)\,(1-(\varepsilon/c_{\max}))$, QOMP method can find the location of the nonzero elements in the coefficient vector accurately and give an approximate vector of the coefficient vector.

### 3.3. The error estimation of QOMP method

In this section, we will give the error estimation of the approximate vector $\tilde{\mathbf{c}}$ obtained by the QOMP method in the noisy setting in section 3.2.

**Theorem 3.3.** *Suppose that the noise vector* $\mathbf{v}$ *satisfies* $||\mathbf{v}||_2 \leq \varepsilon$, *where* $0 < \varepsilon \ll 1$ *is the noise level and the sparsity of the original coefficient vector* $\mathbf{c}$ *satisfies* $2 \leq s < q(m)\,(1-(\varepsilon/c_{\max}))$. *Then the error between the coefficient vector* $\tilde{\mathbf{c}}$ *obtained by the QOMP method and the original coefficient vector* $\mathbf{c}$ *is*

$$||\mathbf{c}-\tilde{\mathbf{c}}||_2 \leq M\cdot\varepsilon,$$

*here* $M$ *is the upper bound of* $||\Phi_\Theta^+||_2$ *which is independent of* $\varepsilon$. $\Phi_\Theta^+$ *denotes the pseudo-inverse of* $\Phi_\Theta$, $\Phi_\Theta$ *denotes the submatrix formed by the columns of matrix* $\Phi$ *whose indexes are in* $\Theta$ *and* $\Theta$ *denotes the index set selected by QOMP method.*

*Proof.* Without loss of generality, suppose that the support set of the coefficient vector $\mathbf{c}$ is $\Delta = \{1,2,\cdots,s\}$. According to the condition $2 \leq s < q(m)\,(1-(\varepsilon/c_{\max}))$, the QOMP method picks out at least one correct position index in each iteration. Therefore, we assume that the index set selected by the QOMP method is $\Theta = \{1,2,\cdots,2s\}$, that is the first $2s$ columns. Decompose the measurement matrix into $\Phi = [\Psi \in \Omega]$, where $\Psi$ represents the first $2s$ columns of $\Phi$, $\Omega$ represents the remaining columns. Similarly, decompose the coefficient vector and the reconstructed coefficient vector into $\mathbf{c} = [\mathbf{c}_1\ \mathbf{0}]$ and $\tilde{\mathbf{c}} = [\tilde{\mathbf{c}}_1\ \mathbf{0}]$, where $\mathbf{c}_1$ and $\tilde{\mathbf{c}}_1$ represent the first $2s$ columns of the vector $\mathbf{c}$ and $\tilde{\mathbf{c}}_1$, respectively. At this time, the non-zero part of the solution obtained by QOMP method of (1.5) is

$$\tilde{\mathbf{c}}_1 = \Psi^+\mathbf{b}.$$

Since $\mathbf{b} = \Phi\mathbf{c}+\mathbf{v}$, by substituting into the above formula we can get

$$\tilde{\mathbf{c}}_1 = \Psi^+\mathbf{b} = \Psi^+\left([\Psi\ \Omega]\mathbf{c}+\mathbf{v}\right) = \mathbf{c}_1 + \Psi^+\mathbf{v}.$$

Obviously, the reconstruction error is:

$$||\mathbf{c}-\tilde{\mathbf{c}}||_2 = ||\mathbf{c}_1-\tilde{\mathbf{c}}_1||_2 = ||\Psi^+\mathbf{v}||_2 \leq ||\Psi^+||_2||\mathbf{v}||_2 \leq M\cdot\varepsilon.$$

Thus, the conclusion of the theorem is proved. □

**Remark 3.3.** It is not difficult to see from Theorem 3.3 that since the original function is a sparse function, the reconstruction error of its coefficient vector is only related to the noise level of the sampling points. Therefore, we can obtain that when the sampled data does not contain noise, the reconstruction error of coefficient vector is 0.

## 4. Reconstruction of General Univariate Sparse Function by QOMP Method

In this section, we consider reconstructing the functions which can be expanded with respect to general univariate uniform bounded orthogonal basis. Suppose that $\{\psi_j(x)\}_{j\in\Lambda}$, $|\Lambda| = n$ is a set of standard orthogonal functions defined on $\Omega \subseteq \mathbb{R}$, which are orthogonal with respect to measure $\nu(x)$, i.e.

$$\int_\Omega \nu(x)\psi_j(x)\psi_k(x)dx = \delta_{jk} = \begin{cases} 0, & j \neq k, \\ 1, & j = k, \end{cases} \tag{4.1}$$

and the corresponding orthogonal measure is bounded on the domain, that is, there exists a constant $C$, satisfying

$$\int_\Omega \nu(x)dx \leq C. \tag{4.2}$$

In addition, let $\{\psi_j(x)\}_{j\in\Lambda}$ have an uniform upper bound $K$, i. e.

$$||\psi_j(x)||_\infty \leq K, \ \forall j \in \Lambda, \tag{4.3}$$

here $\Lambda$ is an index set known in advance, $K$ is a constant independent of $j$. Then we can draw conclusions similar to Lemma 3.2 on uniform bounded orthogonal system $\{\psi_j(x)\}_{j\in\Lambda}$.

**Lemma 4.1.** *Assume that the matrix* $\Psi \in \mathbb{C}^{m\times n}$ *is a measurement matrix generated by the bounded orthogonal system* $\{\psi_j(x)\}_{j\in\Lambda}$ *with random sampling* $\{x_i\}_{i=1}^m$ *under the corresponding orthogonal measure* $\nu(x)$*. Here the measure* $\nu(x)$ *and the orthogonal system* $\{\psi_j(x)\}_{j\in\Lambda}$ *satisfy condition* (4.2) *and* (4.3)*, respectively. Let* $\frac{1}{2} < c < 1$, $q(m) = o(m^{1-c})$ *and* $q(m) \to \infty$ *as* $m \to \infty$*. Then for a large* $m$*, we have* $\mu(\Psi) \leq \frac{1}{q(m)}$ *with high probability.*

*Proof.* First, for the denominator in $\mu(\Psi)$, similar to the proof of Lemma 3.2, according to the definition of coherence, the standard orthogonality property of the system and the strong law of large numbers, with the increase of $m$, we have

$$\mu(\Psi) = \max_{j,k\in\Lambda,\ j\neq k} \frac{|\Psi_j^T\Psi_k|}{||\Psi_j||_2 \cdot ||\Psi_k||_2} \approx \frac{1}{m} \max_{j,k\in\Lambda,\ j\neq k} |\Psi_j^T\Psi_k|.$$

Next, let us consider the numerator of $\mu(\Psi)$: let $g_{jk} = \psi_j(x)\psi_k(x)$, then we have

$$|\Psi_j^T\Psi_k| = \left|\sum_{i=1}^m \psi_j(x_i)\psi(x_i)\right| = \left|\sum_{i=1}^m g_{jk}(x_i)\right|.$$

Because of

$$\mathbb{E}(g_{jk}(x)) = \mathbb{E}(\psi_j(x)\psi_k(x)) = \int_\Omega \nu(x)\psi_j(x)\psi_k(x)dx = 0, \quad j \neq k,$$

we can immediately know that the expectation of random variable $g_{jk}(x)$ is zero. It is also not difficult to obtain that

$$\mathbb{E}(g_{jk}^2) = \int_\Omega \psi_j^2(x)\psi_k^2(x)dx \leq K^2 \int_\Omega \nu(x)dx \leq \tilde{K},$$

here $\tilde{K} = K^2C$, hence we have

$$\sum_m m^{-2c}\mathbb{E}(g_{jk}^2) \leq \tilde{K}\sum_m m^{-2c}.$$

From the assumption $\frac{1}{2} < c < 1$, $\sum_m m^{-2c} < \infty$ is available. Thus we have $\sum_m m^{-2c}\mathbb{E}(g_{jk}^2) < \infty$. Since the sampling points $\{x_i\}_{i=1}^m$ are independently and identically distributed, the random variables $\{g_{jk}(x_i)\}_{i=1}^m$ $(j \neq k)$ are also independently and identically distributed. Then from Lemma 3.1, it is obvious to get that

$$m^{-c}\left|\sum_{i=1}^m g_{jk}(x_i)\right| = m^{-c}\left|\sum_{i=1}^m \psi(x_i)\psi_k(x_i)\right| = m^{-c}|\Psi_j^T\Psi_k| \ \to 0.$$

Note that $q(m) = o(m^{1-c})$, therefore

$$m^{-1} \cdot |\Psi_j^\top \Psi_k| \cdot q(m) = m^{-c} \cdot |\Psi_j^\top \Psi_k| \cdot m^{c-1} \cdot q(m) \to 0$$

holds almost surely as $m \to \infty$. Finally, by taking all the upper bounds of $n$, we can get that as $m \to \infty$,

$$\mu(\Psi) \cdot q(m) = \sup_{j,k\in\Lambda,\ j\neq k} \frac{|\Psi_j^\top \Psi_k|}{m} \cdot q(m) \to 0$$

holds almost surely. Hence, for large $m$, we have $\mu(\Phi) \cdot q(m) \leq 1$ with high probability. □

Using the conclusion of Lemma 4.1, we can get the following two conclusions about reconstructing the $s$-sparse function using the QOMP method for the noiseless and noisy settings.

**Theorem 4.1.** *Suppose that $g(x) = \sum_{j\in\Lambda} c_j\psi_j(x)$ is an $s$-sparse function. If $\{\psi_j(x)\}_{j\in\Lambda}$ is the bounded orthogonal system which satisfies condition (4.1) – (4.3), and the measurement matrix $\Phi = \Psi \in \mathbb{C}^{m\times n}$ in (1.5) is formed by the bounded orthogonal system $\{\psi_j(x)\}_{j\in\Lambda}$ with random sampling $\{x_i\}_{i=1}^m$ under the corresponding orthogonal measure $\nu(x)$. Then if the sparsity of $g(x)$ satisfies $2 \leq s < \frac{q^2(m)}{4q(m)+4}$, the QOMP method can reconstruct the function $g(x)$ accurately in $s$ steps.*

**Theorem 4.2.** *Suppose that $g(x) = \sum_{j\in\Lambda} c_j\psi_j(x)$ is an $s$-sparse function, and the noise vector is $||\mathbf{v}||_2 \leq \varepsilon$, here $0 < \varepsilon \ll 1$ is the noise level. If $\{\psi_j(x)\}_{j\in\Lambda}$ is the bounded orthogonal system which satisfies condition (4.1) – (4.3), and the measurement matrix $\Phi = \Psi \in \mathbb{C}^{m\times n}$ in (1.5) is formed by the bounded orthogonal system $\{\psi_j(x)\}_{j\in\Lambda}$ with random sampling $\{x_i\}_{i=1}^m$ under the corresponding orthogonal measure $\nu(x)$. Then if the sparsity of $g(x)$ satisfies $2 \leq s < q(m)\left(1 - (\varepsilon/c_{\max})\right)$, the QOMP method can reconstruct the function $g(x)$ accurately in $s$ steps, here $c_{\max} = \max_{j\in\Lambda} |c_j|$.*

The proofs of Theorem 4.1 and Theorem 4.2 can be completely done by the proof processes of Theorem 3.1 and Theorem 3.2 in Section 3, and their proofs are omitted here.

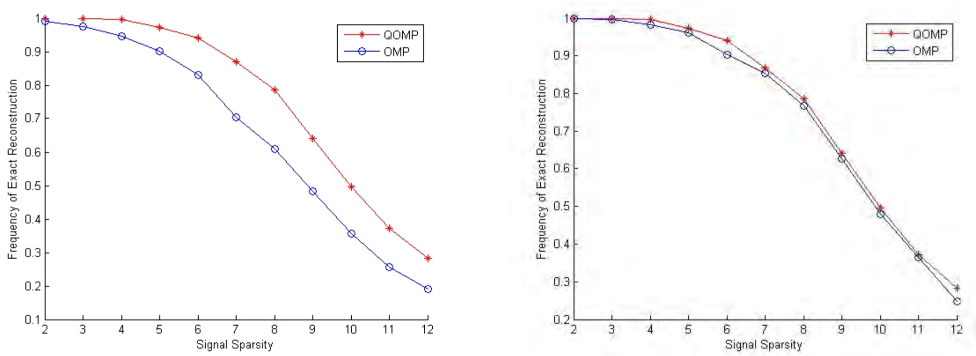
**Remark 4.1.** When the basis function is any standard bounded orthogonal system, following the proof process of Theorem 3.3 in Section 3, we can also give the error estimation of the QOMP method in the noisy setting.

## 5. Numerical Experiments

In this section, we will first demonstrate experimentally the effectiveness and robustness of the QOMP method in signal recovery, and then verify the effectiveness of the QOMP method in sparse polynomial function reconstruction.

### 5.1. The effectiveness and robustness of QOMP method in signal recovery

In this experiment, we first take the measurement matrix $\Phi \in \mathbb{C}^{m\times n}$ which is formed by preconditioned Legendre system with random sampling according to the Chebyshev probability measure as an example and compare the QOMP method with the classic OMP method (cf. [6]). For the sake of simplicity, we only test the noiseless sampled values here. Assume that the number of basis functions $n = 128$ and the number of sampling points $m = 32$, and we apply both classic OMP method and QOMP method to perform $s$ iterations to compare their frequency of exact reconstruction. The experimental results are shown in Fig. 5.1(a). Since the QOMP method selects two columns in each iteration, a total of $2s$ columns are selected in $s$ iterations. Then we also perform $2s$ iterations on the OMP method and select $2s$ columns. The experimental results are shown in Fig. 5.1(b).



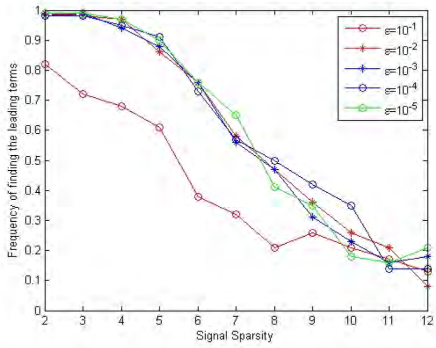

(a) Iteration number : QOMP: $s$, OMP: $s$ (b) Iteration number: QOMP: $s$, OMP: $2s$

Fig. 5.1. The comparison of OMP method and QOMP method on frequency of exact reconstruction.

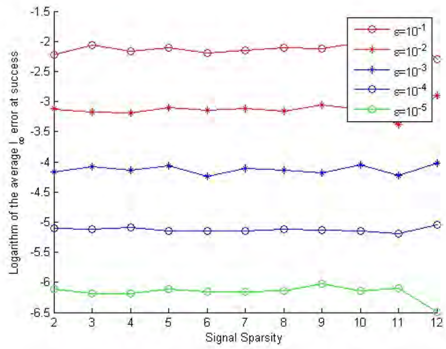
It can be seen from Fig. 5.1 that whether the OMP method performs $s$ iterations or $2s$ iterations, the success rate of the QOMP method is always higher than that of the OMP method. Meanwhile, Fig. 5.1 shows that when the number of basis functions $n$ and sampling points $m$ are fixed, the frequency of exact reconstruction of the QOMP method gradually decreases as the sparsity $s$ increases. For $s \geq 0.4m$, the frequency of exact recovery of these two algorithms are both very low and hence we do not present it in Fig. 5.1. Obviously, the CPU time of the QOMP method is greater than that of the OMP method.

The results in Fig. 5.1 illustrate the effectiveness of the QOMP method. To verify the robustness of the QOMP method, we gradually perturb the $s$-sparse signal to judge whether the QOMP method can find the leading terms of the signal, that is, the terms with relatively large absolute values. In next experiment, we set the value of the non-zero position in the original $s$-sparse signal in the range of $[1, 2]$ and the value of the noise to $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$ and $10^{-5}$, respectively. We perform 1000 independent repeated experiments on original signals with different sparsity under different noise levels. The experimental results are shown in Fig. 5.2.

Fig. 5.2(a) shows that when the sparsity $s$ is fixed, the frequency of exact reconstruction of the QOMP method increases as the noise level decreases. At the same time, we find that when the noise level is small, the frequency of exact reconstruction is almost the same as that of the reconstruction of free-noise $s$-sparse signals, which shows that the QOMP method can find the leading terms in the signal very well. Fig. 5.2(b) gives the logarithm of the average $\ell_\infty$ error when the QOMP method succeeds. It can be seen from Fig. 5.2(b) that when the noise level is relative large, even if the frequency of exact reconstruction is high, such as the sparsity $s = 2$, the reconstruction error is almost the same as the noise level.




(a) The frequency of exact reconstruction of QOMP method in different noise levels




(b) Iteration number: QOMP: $s$, OMP: $2s$

Fig. 5.2. The frequency of exact reconstruction and average $\ell_\infty$ error of QOMP method.

### 5.2. The effectiveness of QOMP method in reconstruction of sparse polynomial functions

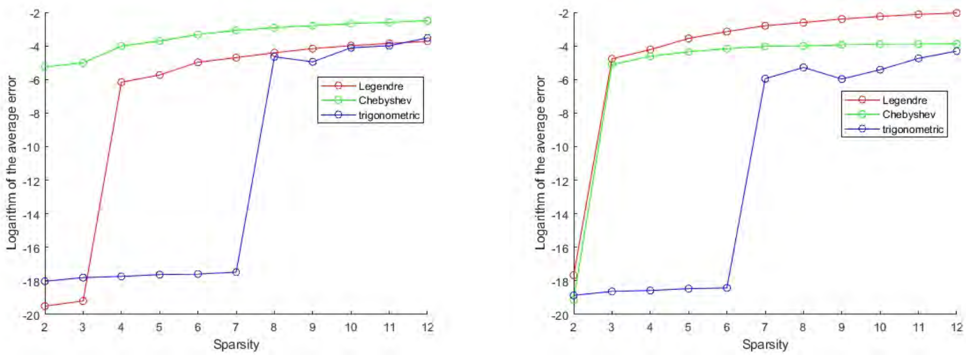
Section 5.1 has verified the effectiveness and robustness of the QOMP method in signal recovery. In this experiment, we will verify the effectiveness of the QOMP method in the reconstruction of sparse polynomial functions by the noiseless samples of univariate and bivariate sparse polynomial functions. Here, we take the number of basis functions $n = 128$ and the number of sampling points $m = 32$ for the reconstruction of univariate polynomial functions and $n = 144, m = 32$ for the reconstruction of bivariate sparse polynomial functions. Indeed, the essence of the reconstruction of a sparse polynomial function is the reconstruction of the sparse coefficient vector.

The main steps of this experiment are as follows:

**Step 1:** Randomly generate an $n$-dimensional $s$-sparse coefficient vector $\mathbf{c} \in \mathbb{C}^n$ whose support set is $\Delta$;
**Step 2:** For different systems, according to the corresponding orthogonal probability measure, randomly select $m$ sampling points $\{x_i\}_{i=1}^m$ on corresponding domain;
**Step 3:** Generate $b_i = g(x_i) = \sum_{j\in\Lambda} c_j\phi_j(x_i)$, where we express the three types of basis functions as $\{\phi_j(x)\}_{j\in\Lambda}$;
**Step 4:** Use the QOMP method to solve the problem (1.5);
**Step 5:** Compare the obtained results with the original coefficient vector and function.

Here we take the univariate polynomial functions as the examples in **Step** 3 and it is similar for bivariate polynomial functions. We will first use the QOMP method to perform 1000 independent repeated experiments for the three different types of orthogonal systems mentioned before according to the above experimental steps in the noiseless setting. Then we calculate the average error of all experiments. The results are shown in Fig. 5.3.

Fig. 5.3(a) and (b) show the average errors of the three orthogonal polynomial functions in the univariate and multivariate cases, respectively. In this experiment, the error is defined as $\|\mathbf{g} - \tilde{\mathbf{g}}\|/N$, here $N$ is the number of test points and $N = 1000$ and 40101 for univariate functions and bivariate functions, respectively. $\mathbf{g}$ are the values of the original functions at

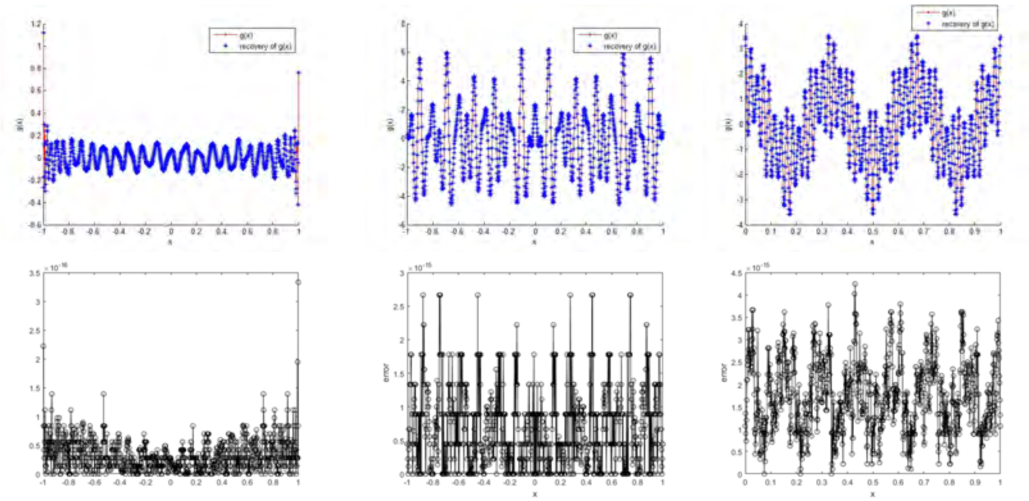



(a) Univariate (b) Bivariate

Fig. 5.3. The average error of QOMP method in reconstruction of univariate and bivariate sparse polynomial functions.

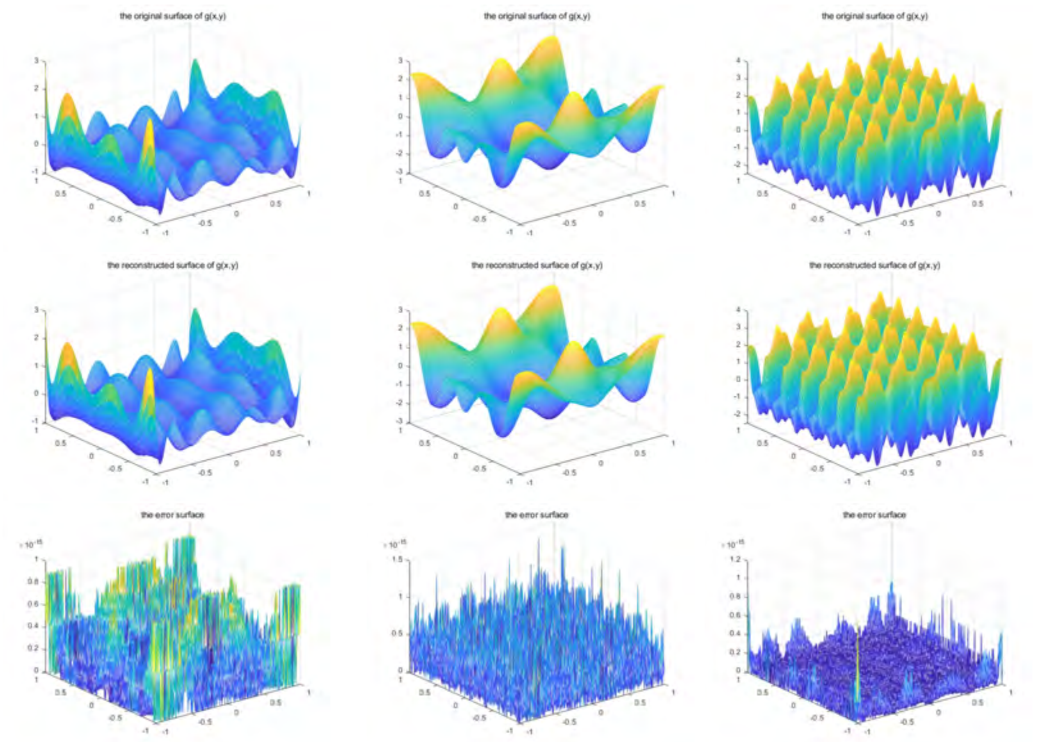
these test points and $\tilde{\mathbf{g}}$ are those of the reconstructed functions. When the sparsity $s = 2$, from Sec 5.1, the frequency of exact reconstruction of 1000 independent repeated tests is almost 1, hence the magnitude of average error is almost zero. And then as shown in these figures, the average errors increase as the sparsity increases. This is due to the gradual decrease in the frequency of exact reconstruction. When the QOMP method find the locations of the non-zero items accurately, the average error can be $10^{-14}$ (see Fig. 5.4 and Fig. 5.5).

Finally, we will give some examples of the original images and reconstruction images via QOMP method of univariate and bivariate sparse polynomial functions in the noiseless setting. Here we take $s = 5$ as an example.



(a) Legendre (b) Chebyshev (c) Trigonometric

Fig. 5.4. The images and average errors of QOMP method in reconstruction of univariate sparse polynomial functions.




(a) Legendre (b) Chebyshev (c) Trigonometric

Fig. 5.5. The surfaces and average errors of QOMP method in reconstruction of biavariate sparse polynomial functions.

The first row of Fig. 5.4 are the images of the original functions (red) and the reconstructed functions (blue) and the second row of Fig. 5.4 are the $\ell_2$ error images of those two kinds of functions. It can be seen from the second column that the errors of reconstruction are almost zero.

The first and second rows of Fig. 5.5 are the surfaces of bivariate sparse polynomial functions mentioned before and the third row are the $\ell_2$ error images respectively. We can clearly see from these figures that the smoothness of these original functions is very poor. Based on the knowledge of approximation theory, it is easy to know that if we use the traditional interpolation methods, such as polynomial interpolation, reconstructing such functions well is very difficult. However, it can be clearly seen from our experiments that even if the smoothness of the original functions is poor, we can still give an accurate reconstruction of these types of original functions by QOMP method.

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